Kybernetika

Alexander Guterman; Elena Kreines; Alexander Vlasov

Non-surjective linear transformations of tropical matrices preserving the cyclicity index

Kybernetika, Vol. 58 (2022), No. 5, 691–707

Persistent URL: `http://dml.cz/dmlcz/151299`

KYBERNETIKA — VOLUME *58* (2022), NUMBER 5, PAGES 691–707

NON-SURJECTIVE LINEAR TRANSFORMATIONS OF TROPICAL MATRICES PRESERVING THE CYCLICITY INDEX

Alexander Guterman, Elena Kreines, and Alexander Vlasov

To the memory of Professor Martin Gavalec.

The cyclicity index of a matrix is the cyclicity index of its critical subgraph, namely, the subgraph of the adjacency graph which consists of all cycles of the maximal average weight. The cyclicity index of a graph is the least common multiple of the cyclicity indices of all its maximal strongly connected subgraphs, and the cyclicity index of a strongly connected graph is the least common divisor of the lengths of its (directed) cycles. In this paper we obtain the characterization of linear, possibly non-surjective, transformations of tropical matrices preserving the cyclicity index. It appears that non-bijective maps with these properties exist and all maps are exhausted by transposition, renumbering of vertices, Hadamard multiplication with a matrix of a certain special structure, and certain diagonal transformation. Moreover, only diagonal transformation can be non-bijective.

Keywords: tropical linear algebra, cyclicity index, linear transformations

Classification: 05C22, 05C38, 05C50

1. INTRODUCTION

We recall that a *semiring* is a set $\mathcal{S}$ together with two binary operations, addition and multiplication, such that $\mathcal{S}$ is a commutative monoid under addition (with identity denoted by $0_{\mathcal{S}}$); $\mathcal{S}$ is a semigroup under multiplication (with identity, if any, denoted by $1_{\mathcal{S}}$); multiplication is distributive over addition on both sides; and $s \cdot 0_{\mathcal{S}} = 0_{\mathcal{S}} \cdot s = 0_{\mathcal{S}}$ for all $s \in \mathcal{S}$. The semiring $\mathcal{S}$ is said to be *commutative* if the multiplication is commutative. It is said to be *anti-negative* if $a + b = 0_{\mathcal{S}}$ implies that both a and b are equal to $0_{\mathcal{S}}$. It is a *semifield* if $\mathcal{S} \setminus \{0_{\mathcal{S}}\}$ is an abelian group under multiplication. A semiring $\mathcal{S}$ is called *idempotent* if $a + a = a$ for all $a \in \mathcal{S}$. An idempotent semiring is necessarily anti-negative; indeed if $a + b = 0_{\mathcal{S}}$ then $0_{\mathcal{S}} = a + b = a + a + b = a + 0_{\mathcal{S}} = a$ and a dual argument gives $0_{\mathcal{S}} = b$.

Here we consider the *tropical semifield* $\mathbb{R}_{\max} := \mathbb{R} \cup \{-\infty\}$ together with addition given by taking the maximum and multiplication given by extending *addition* of real

DOI: 10.14736/kyb-2022-5-0691

numbers so as to make $-\infty$ a zero element. $\mathbb{R}_{\max}$ is an idempotent and hence antinegative semifield.

Let $\mathcal{S}$ be a semiring. We write $\mathcal{S}^{i\times j}$ for the set of $i \times j$ matrices over $\mathcal{S}$, which forms an $\mathcal{S}$-*module* in the obvious way. We write $M_n(\mathcal{S})$ for $\mathcal{S}^{n\times n}$ viewed as a semigroup under the matrix multiplication induced by the operations in $\mathcal{S}$. If $\mathcal{S}$ contains a multiplicative identity element $1_{\mathcal{S}}$, then $M_n(\mathcal{S})$ is a monoid, with obvious identity element.

There is a number of works on tropical linear algebra and its applications, see for example [11] and references therein.

Given an invariant defined on a certain algebraic system, it is natural to ask: what are the transformations that can be performed on the system that leave this invariant unchanged? In the case where the algebraic system under consideration is a matrix algebra over a field, the investigation of such transformations dates back to the following result of Frobenius [4], which gives a characterization of the bijective complex linear transformations preserving the determinant:

Theorem 1.1. [4, Frobenius, 1897] Let $\mathbb{C}$ be the field of complex numbers, and let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a bijective linear transformation such that $\det T(X) = \det X$ for all matrices $X \in M_n(\mathbb{C})$. Then there exist invertible matrices $U, V \in M_n(\mathbb{C})$, with $\det(UV) = 1$, such that either $T(X) = UXV$ for all matrices $X \in M_n(\mathbb{C})$, or $T(X) = UX^TV$ for all $X \in M_n(\mathbb{C})$, where X^T denotes the transposed matrix.

This result was generalized by Schur [16], who characterized the maps preserving all subdeterminants of any fixed order r. Later Dieudonné [3] obtained a characterization of bijective linear maps preserving the set of singular matrices over arbitrary fields. The approaches of Frobenius and Schur were combinatorial and in some sense ad-hoc, Dieudonné proposed a new approach based on the fundamental theorem of projective geometry.

Starting from these initial investigations, many authors have studied the structure of linear operators on the $n \times n$ matrix algebra $M_n(F)$ over a field F that leave certain matrix relations, subsets, or properties invariant (see the surveys [14, 15] for the details). In the last two decades much attention has been paid to the investigation of maps preserving different invariants for matrices over various *semirings*, where completely different techniques are necessary to obtain a classification of linear transformations with certain preserving properties; see [14, Section 9.1] for more details and [8, 10] and references therein for more recent results.

A cycle is a simple closed path (i.e. it does not have repeated edges) and a *dicycle* is a cycle in digraph. Further we consider only directed cycles, so by cycle we mean a dicycle. In the present paper we consider linear maps of tropical matrices preserving cyclicity index which is an important invariant in combinatorial matrix theory and its applications and is defined as follows.

Definition 1.2. [11, Definition 2.3] Let G be a directed graph. We define the *cyclicity index* (or just *cyclicity*) σ_G as follows:

1. If G is strongly connected and it contains 2 vertices at least, then σ_G is the greatest common divisor of lengths of all cycles in G.

2. If G contains only 1 vertex, then $\sigma_G = 1$.

3. If G is not strongly connected, then σ_G is the least common multiple of cyclicity indices of all maximal strongly connected subgraphs of G.

Definition 1.3. Let G be a weighted digraph, C be a cycle in G. The *average weight* of C is the geometric mean of the weights of its edges in the tropical sense or, equivalently, the usual arithmetic mean. Namely, if the length of C equals l and the weights of its edges are $a_1, a_2, \ldots, a_l$, then:

$$w(C) = \sqrt[\otimes l]{a_1 \otimes a_2 \otimes \ldots \otimes a_l} = \frac{a_1 + a_2 + \ldots + a_l}{l}$$

Definition 1.4. Let G be a weighted digraph. A cycle C in the graph G is called a *critical* cycle if $w(C)$ is equal to the maximal average weight among all cycles in G. The *critical subgraph* G_c is the union of all critical cycles in G.

Definition 1.5. [11, Definition 3.5] Let $A \in M_n(\mathbb{R}_{\max})$ and G be its adjacent graph. The cyclicity index $\sigma(A)$ of the matrix A is defined by: $\sigma(A) = \sigma_{G_c}$, if G contains at least one cycle, and $\sigma(A) = 1$ otherwise.

For detailed and self-contained information on the cyclicity index we refer the reader to [1, 2, 6, 7, 11]. Below we mention just a few applications of this invariant.

The cyclicity index is usually applied to investigate periodic behavior of solutions of max-linear systems of type $x(k+1) = A \otimes x(k)$ for $k \geq 0$, where A is an $n \times n$ tropical matrix and $x(0) = x_0$ is the initial condition. In particular, it is used for determination of regular regimes in scheduling and other network problems, see [11, 12, 13, 17]. Namely, it is well known that according to the celebrated cyclicity theorem, see [11, Theorem 3.9], if A is an irreducible tropical $n \times n$ matrix with the eigenvalue λ and cyclicity σ, then there is a positive integer N such that $A^{\otimes(k+\sigma)} = \lambda^{\otimes\sigma} \otimes A^{\otimes k}$ for all integers $k \geq N$, where $\lambda^{\otimes\sigma} = \underbrace{\lambda \otimes \ldots \otimes \lambda}_{\sigma \text{ times}}$.

In [12, 13] the authors unify and extend existing techniques for deriving upper bounds on the transient behaviour of max-plus matrix powers and the cyclicity index plays an important role in these considerations as well.

The cyclicity index is applied in [5] to prove certain properties of matrix roots of tropical matrices.

To investigate the properties of the cyclicity index and to produce new matrices with a given value of cyclicity index, it is important to find out various classes of maps preserving the cyclicity. In this paper we deal with linear, possibly non-surjective, cyclicity preserving maps.

We remark that in the paper [9] bijective linear maps preserving the cyclicity index have been studied for the first time and partially characterized. It was shown there that non-bijective linear maps preserving cyclicity do exist by providing a concrete example for $n = 2$. However, this example does not admit a generalization for bigger dimensions and an existence problem for such examples was posed, see [9, Section 5 and Abstract]. Here we provide an example of a non-surjective linear map $T : M_n \to M_n$ preserving the cyclicity index for each $n \geq 2$, which solves this problem. Moreover, we obtain a complete characterization for possibly non-surjective linear maps preserving the cyclicity index that clarify and generalize the original characterization in [9, Theorem 4.1].

Our paper is organized as follows. In Section 2 we investigate some general properties of linear (possibly non-bijective) transformations preserving the cyclicity index. Then in Section 3 we characterize these transformations. In Section 4 we discuss additional properties of maps under consideration.

2. MAPS PRESERVING CYCLICITY

Below we consider only the matrices with tropical entries. Thus to shorten the notations we denote $M_n = M_n(\mathbb{R}_{\max})$. By E_{ij} we denote the (i,j)-th matrix unit, i.e. the matrix with 0 in (i,j)-th position and $-\infty$ everywhere else. By $\nu(A)$ we denote the number of finite entries of $A \in M_n$. Further we need the following sets:

$\mathcal{E}_n = \{\alpha \otimes E_{ij} \in M_n | 1 \le i,j \le n,\ \alpha \in \mathbb{R}\}$ is the set of matrices whose adjacency graph contains exactly one edge.

$\mathcal{N}_n = \{\alpha \otimes E_{ij} \in M_n | 1 \le i,j \le n,\ i \ne j,\ \alpha \in \mathbb{R}\}$ is the subset of matrices $\mathcal{E}_n$ whose adjacency graph isn't a loop.

$\mathcal{D}_n = \{\alpha \otimes E_{ii} \in M_n | 1 \le i \le n,\ \alpha \in \mathbb{R}\}$ is the subset of matrices $\mathcal{E}_n$ whose adjacency graph is a loop.

We begin with an example of a non-bijective cyclicity index preserver in each dimension. For $n = 2$ an example of such a map is provided in [9, Section 5] but that example does not allow a straightforward generalization to higher dimensions.

Example 2.1. Let the map $T : M_n \to M_n$ be defined for any $i,j,\ 1 \le i,j \le n$, as:

$$T(E_{ij}) = \begin{cases} E_{ij}, & \text{if} \quad (i,j) \ne (1,1) \\ (-1) \otimes E_{22} \oplus E_{11}, & \text{if} \quad (i,j) = (1,1) \end{cases}$$

and extended by linearity, i.e., $T(\bigoplus_{i=1}^n \bigoplus_{j=1}^n a_{ij} \otimes E_{ij}) = \bigoplus_{i=1}^n \bigoplus_{j=1}^n a_{ij} \otimes T(E_{ij})$. Then T is a non-surjective linear map and preserves cyclicity index.

To prove that the map T in Example 2.1 indeed satisfies the required properties we verify the following more general statement:

Lemma 2.2. Let k be fixed, $1 \le k \le n$, and let $c_1, \dots, c_n \in \mathbb{R}_{\max}$ be such that $c_i < 0$ for all $i = 1, \dots, n$, and there exists l, $1 \le l \le n$ such that $c_l \ne -\infty$. Assume that the map $T_0 : M_n \to M_n$ is defined for any $i,j,\ 1 \le i,j \le n$, by the formula

$$T_0(E_{ij}) = \begin{cases} E_{ij}, & \text{if} \quad (i,j) \ne (k,k) \\ E_{kk} \oplus \bigoplus_{l=1}^n c_l \otimes E_{ll}, & \text{if} \quad (i,j) = (k,k) \end{cases}$$

and extended by the linearity. Then the transformation T_0 is a non-surjective linear map and preserves the cyclicity index.

Proof. 1. Up to the renumeration of indices, without loss of generality, we assume that $k = 1$.

2. Note that T_0 acts on any matrix in the following way. Let $A = (a_{ij}) \in M_n$. Then by definition we have that if $T_0(A) = A' = (a'_{ij})$ then $a'_{ll} = a_{ll} \oplus c_l \otimes a_{11}$, $l = 2, \dots, n$, and $a'_{ij} = a_{ij}$ for all other (i,j). Indeed, since $c_1 < 0$ we have $a'_{11} = c_1 \otimes a_{11} \oplus a_{11} = a_{11}$.

3. T_0 is linear by definition. Let us show that T_0 is non-surjective. Indeed, assume that $B = (b_{ij}) \in M_n$ lies in the image of T_0. Then there is $A = (a_{ij}) \in M_n$ such that $T_0(A) = B$. From $a_{11} = b_{11}$ and the existence of $c_l \neq -\infty$ it follows that if $b_{11} \neq -\infty$ then $b_{ll} = c_l \otimes a_{11} \oplus a_{ll} \neq -\infty$. Therefore E_{11} does not lie in the image of T_0.

4. Let us show that T_0 preserves the cyclicity index. We denote by $G(X)$ a graph, adjacent to a matrix $X = (x_{ij}) \in M_n$ and by λ_X the average weight of each critical cycles of $G(X)$. Note that $\lambda_X \geq x_{ll}$ for all l since a loop is a cycle of length 1. Let $A \in M_n$ be arbitrary, and $B = T_0(A)$. For each l, $2 \leq l \leq n$, such that $c_l \neq -\infty$ the following two cases appear:

a) $a_{ll} < \lambda_A$. Then $b_{ll} = c_l \otimes a_{11} \oplus a_{ll} = max\{c_l \otimes a_{11}, a_{ll}\} < \lambda_A$. Indeed, $c_l \otimes a_{11} \leq c_l \otimes \lambda_A < \lambda_A$ since $c_l < 0$. Hence in both graphs $G(A)$ and $G(B)$ the loop in the vertex l does not belong to the critical subgraph.

b) $a_{ll} = \lambda_A$. Then $a_{11} \leq \lambda_A = a_{ll}$. Therefore, $b_{ll} = c_l \otimes a_{11} \oplus a_{ll} = max\{c_l \otimes a_{11}, a_{ll}\} = a_{ll}$ since $c_l < 0$. Hence in both graphs $G(A)$ and $G(B)$ the loop in the vertex l belongs to the critical subgraph.

If $l = 1$ or $c_l = -\infty$, then $b_{ll} = a_{ll}$. Since $T_0(E_{ij}) = E_{ij}$ for all $(i, j) \neq (1, 1)$, all cycles of $G(A)$ and $G(B)$ coincide and have the same average weights, except the loops. For each $l, 1 \leq l \leq n$, the loop in the vertex l either belongs to critical subgraphs of both graphs $G(A)$ and $G(B)$ or does not belong to any of them by Items a) and b), correspondingly. It follows that $G_c(B) = G_c(A)$ and $\lambda_B = \lambda_A$. So in both cases T_0 preserves cyclicity index. □

Corollary 2.3. For any integer $n \geq 2$ there exists a non-surjective linear map $T : M_n \to M_n$ preserving cyclicity index.

Lemma 2.4. Let $n \geq 3$, $T : M_n \to M_n$ be linear and preserve cyclicity. Then for all $\alpha, \beta \in \mathbb{R}$ and all $1 \leq i, j, k, l \leq n$ the equality $T(\alpha \otimes E_{ij}) = T(\beta \otimes E_{kl})$ implies that $i = k$, $j = l$, and $\alpha = \beta$.

P r o o f. Note that if $i = k$ and $j = l$, then $\alpha = \beta$ by linearity. Hence, the converse to the statement of the lemma means that either $i = k$ or $j = l$.

So, suppose the converse, namely, there are $\alpha, \beta \in \mathbb{R}$ and the indices i, j, k, l, $1 \leq i, j, k, l \leq n$ and either $i \neq k$ or $j \neq l$ such that $T(\alpha \otimes E_{ij}) = T(\beta \otimes E_{kl})$. The general situation splits into the following two cases.

Case 1: $i = j$ and $k = l$. Without loss of generality we assume that $\alpha \geq \beta$.

Consider an integer $0 < m \leq n$, $m \notin \{i, k\}$. Then

$$\begin{aligned} T(\alpha \otimes (E_{ii} \oplus E_{im} \oplus E_{mi})) = T(\alpha \otimes E_{ii}) \oplus T(\alpha \otimes (E_{im} \oplus E_{mi})) = \\ = T(\beta \otimes E_{kk}) \oplus T(\alpha \otimes (E_{im} \oplus E_{mi})) = T(\beta \otimes E_{kk} \oplus \alpha \otimes (E_{im} \oplus E_{mi})). \end{aligned} \tag{1}$$

Note that $\sigma(\alpha \otimes (E_{ii} \oplus E_{im} \oplus E_{mi})) = 1$, since the graph $G(\alpha \otimes (E_{ii} \oplus E_{im} \oplus E_{mi}))$ is strongly connected. At the same time $\sigma(\beta \otimes E_{kk} \oplus \alpha \otimes (E_{im} \oplus E_{mi})) = 2$ since $\alpha \geq \beta$ and the graph $G(\beta \otimes E_{kk} \oplus \alpha \otimes (E_{im} \oplus E_{mi}))$ is not connected. Since T preserves cyclicity we obtain a contradiction with (1).

Case 2: either $i \neq j$ or $k \neq l$. Without loss of generality we consider the case $i \neq j$. Then

$$T(\alpha \otimes (E_{ij} \oplus E_{ji})) = T(\alpha \otimes E_{ij}) \oplus T(\alpha \otimes E_{ji}) = T(\beta \otimes E_{kl} \oplus \alpha \otimes E_{ji}). \qquad (2)$$

Similar to the previous case we note that $\sigma(\alpha \otimes (E_{ij} \oplus E_{ji})) = 2$ since this matrix corresponds to the cycle of length 2. Also $\sigma(\beta \otimes E_{kl} \oplus \alpha \otimes E_{ji}) = 1$, since the graph of this matrix does not contain a cycle. Since T preserves cyclicity we obtain a contradiction with (2). Thus, T does not preserve cyclicity, a contradiction. □

Lemma 2.5. Let $T : M_n \to M_n$, $n \geq 3$, be a non-surjective linear map preserving cyclicity. Then there exist i, j, $1 \leq i, j \leq n$, such that for all $A \in M_n$ it holds that $T(A) \neq E_{ij}$.

P r o o f. Assume the converse. Then for any k, m, $1 \leq k, m \leq n$ there exists $A_{km} \in M_n$ such that $T(A_{km}) = E_{km}$. Then for all $B \in M_n$

$$B = \bigoplus_{1 \leq k,m \leq n} b_{km} \otimes E_{km} = \bigoplus_{1 \leq k,m \leq n} b_{km} \otimes T(A_{km}) = T(\bigoplus_{1 \leq k,m \leq n} b_{km} \otimes A_{km})$$

Hence, T is surjective, a contradiction. □

Corollary 2.6. Let $n \geq 3$, $T : M_n \to M_n$ be a non-surjective linear map preserving cyclicity. Then there exist i, j, $1 \leq i, j \leq n$, such that for all $\alpha \in \mathbb{R}$ and for all $A \in M_n$ it holds that $T(A) \neq \alpha \otimes E_{ij}$, i.e., $\{\alpha \otimes E_{ij} | \alpha \in \mathbb{R}\} \cap Im(T) = \emptyset$.

P r o o f. Follows immediately from Lemma 2.5 by linearity of T. □

Corollary 2.7. Let $n \geq 3$, $T : M_n \to M_n$ be a non-surjective linear map preserving cyclicity. Then there exist i, j, $1 \leq i, j \leq n$, such that $T(E_{ij}) \notin \mathcal{E}_n$.

P r o o f. Assume the converse, i.e., for any pair i, j, $1 \leq i, j \leq n$ we have $T(E_{ij}) \in \mathcal{E}_n$. By Corollary 2.6 there is a pair (p, q), $1 \leq p, q \leq n$ such that the matrices $\alpha \otimes E_{pq}$ are not in the image of T for all $\alpha \in \mathbb{R}$. Then by pigeonhole principle there exist pairs $(i, j), (k, l), (s, t)$, $1 \leq i, j, k, l, s, t \leq n$ and $\alpha, \beta \in \mathbb{R}$ such that $(i, j) \neq (k, l)$ but $T(E_{ij}) = \alpha \otimes E_{st}$ and $T(E_{kl}) = \beta \otimes E_{st}$.

Therefore, $T(\alpha^{-1} \otimes E_{ij}) = T(\beta^{-1} \otimes E_{kl})$, which contradicts Lemma 2.4. □

Lemma 2.8. Let $T : M_n \to M_n$ be a linear map preserving cyclicity. Assume that the matrix $A \in M_n$ is such that the graph $G(A)$ does not contain cycles. Then $G(T(A))$ does not contain cycles.

P r o o f. 1. If $A = 0$, i.e., the matrix with all entries $-\infty$, then $T(A) = 0$ by linearity. So, $T(A)$ does not contain cycles either.

2. Assume now that $A \neq 0$. Suppose, the graph $G_1 = G(T(A))$ contains at least one cycle, and the maximal average weight of the cycles in G_1 is λ. Since there are no cycles

in $G(A)$ and $A \neq 0$, there is a vertex, say i, such that there is an edge outgoing from i and there is no edge ingoing to i. Let the outgoing edge be from i to j. Then $i \neq j$ since there is no cycle in $G(A)$ and $A \oplus E_{ji}$ contains a cycle $i - j - i$ of length 2. So, $\sigma(G(A \oplus E_{ji})) = 2$. We want to construct two additional graphs now.

3. Let us denote $D = (d_{kl}) = T(E_{ji})$ and $B = (b_{kl}) = T(A)$. We consider

$$\alpha_0 = \min_{\substack{1 \leq k,l \leq n \\ b_{kl} \neq -\infty \\ d_{kl} \neq -\infty}} \{b_{kl} - d_{kl}\} \in \mathbb{R},$$

here we use usual subtraction, i.e. tropical division. Let $\alpha \in \mathbb{R}$, $\alpha \leq \alpha_0$. Let e be an edge in G_1 from s to t of the weight $w_e = b_{st}$. Then the weight of e in $G(T(A \oplus \alpha \otimes E_{ji}))$ equals $w'_e = b_{st} \oplus \alpha \otimes d_{st} = b_{st} = w_e$ for all $1 \leq s, t \leq n$.

4. Let $M = \bigoplus\limits_{1 \leq k,l \leq n} b_{kl} \oplus d_{kl}$ be the maximum weight of the edges in the graphs $G(D)$ and $G(B)$. Consider $\beta < \lambda - M$. Then for any cycle C_1 which belongs to $G(T(A \oplus \beta \otimes E_{ji}))$ and does not belong to G_1 we obtain that $w(C_1) \leq \beta \otimes M < \lambda$.

5. Let $\gamma = \min\{\alpha, \beta\}$ and $G_2 = G(T(A \oplus \gamma \otimes E_{ji}))$. Then from Items 3 and 4 it follows that the average weight of any cycle of G_2 is at most λ.

6. Therefore, the maximal average weight of cycles in G_2 is λ, since by the linearity of T the graph G_1 is a subgraph of the graph G_2.

7. Let us consider the cycles of G_2 which are not contained in G_1. By Item 4 the average weight of any such cycle is less than λ.

8. Consider a cycle C_2 in G_2. If C_2 belongs to the critical subgraph of G_1, then $w(C_2) = \lambda$ by Item 6. Thus C_2 is contained in the critical subgraph of G_2. If C_2 does not belong to the critical subgraph of G_1, then $w(C_2) < \lambda$ by the Item 7. Hence C_2 is not contained in the critical subgraph of G_2. Hence, critical subgraphs of G_1 and G_2 are the same.

9. Consider the graph $G_3 = G(A \oplus \gamma \otimes E_{ji})$. Then $\sigma_{G_1} = \sigma_{G(A)} = 1$, $\sigma_{G_3} = \sigma_{G_2} = 2$. From Item 8 it follows that the graphs G_1 and G_2 have the same critical subgraph. Hence, $\sigma_{G_1} = \sigma_{G_2}$, a contradiction. Thus G_1 does not contain a cycle. □

Corollary 2.9. Let $T : M_n \to M_n$ be a linear map preserving cyclicity. Then the following properties hold:

1. $G(T(E_{ij}))$ does not contain loops, if $i \neq j$,

2. if $G(A)$ does not contain a loop, then $G(T(A))$ does not contain a loop.

Proof. Item 1 follows directly from Lemma 2.8. Item 2 follows from Item 1 by additivity. □

Lemma 2.10. Let $T : M_n \to M_n$ be a linear map preserving cyclicity. Then

1. For each $A \in \mathcal{N}_n$ the matrix $T(A) \in \mathcal{N}_n$, so the restriction map $T^* = T|_{\mathcal{N}_n}$ is defined.

2. The map $T^* : \mathcal{N}_n \to \mathcal{N}_n$ is bijective.

Proof. 1. For any $A \in \mathcal{N}_n$ there exist $E_1, \ldots, E_{n-1} \in \mathcal{N}_n$ such that $G(E_1 \oplus \ldots \oplus E_{n-1} \oplus A)$ is a cycle of length n. We denote $E_n = A$, $C = E_1 \oplus \ldots \oplus E_n$, and $C_i = E_1 \oplus \ldots \oplus E_{i-1} \oplus E_{i+1} \oplus \ldots \oplus E_n$, $i = 1, \ldots, n$. Then $G(C_i)$ does not contain a cycle.

2. From $\sigma_C = \sigma_{T(C)} = n$ it follows that the graph $G(T(C))$ contains at least 1 cycle.

3. Consider any cycle in $G(T(C))$ and denote it by γ. By Lemma 2.8 we know that $G(T(C_i))$ does not contain a cycle. In particular, $G(T(C_i))$ does not contain the cycle γ. So, for each i, $1 \leq i \leq n$, there is an edge e_i of γ which is not an edge of $G(T(C_i))$. Since e_i is an edge of $G(T(C))$, it follows that e_i is an edge of $G(T(E_i))$ and e_i is not an edge of $G(T(E_j))$ for all $j \neq i$. Therefore the number of edges $|\gamma| \geq n$. Then $|\gamma| = n$ since there are no cycles of length greater than n on n vertices.

4. Assume, $G(T(C)) \neq \gamma$. Since $|\gamma| = n$, the vertex sets of $G(T(C))$ and γ coincide. Then there exists an edge e of $G(T(C))$ which is not an edge of γ. It follows that either e is a loop or e divides γ into two cycles of smaller length. This is a contradiction with Item 3. Hence, $G(T(C)) = \gamma$.

5. It follows from Item 3 that $|G(T(E_i))| = 1$, i.e., $\nu(T(E_i)) = 1$ for each $i = 1, \ldots, n$. Hence, $\nu(T(A)) = 1$, i.e., $T(A) \in \mathcal{N}_n$ as desired. This proves the first part of the lemma.

6. Transformation T is linear, so it is sufficient to prove the bijectivity of T^* on the set of matrix units $\mathcal{E} = \{E_{ij} | 1 \leq i \neq j \leq n\}$. At first, consider the case $n = 2$. The cycle of length 2 is mapped to the cycle of length 2, and 2 distinct edges are mapped to 2 distinct edges, so T^* is bijective. Now, let $n \geq 3$. Then it follows from Lemma 2.4 that T^* is injective. Since the set $\mathcal{E}$ is finite, T^* is bijective. □

Corollary 2.11. Let $n \geq 3$, $T : M_n \to M_n$ be a linear map preserving cyclicity. Assume that T is non-surjective. Then there exists i, $1 \leq i \leq n$, such that for all $A \in M_n$ it holds that $T(A) \neq E_{ii}$.

Proof. By Lemma 2.5 there is a matrix $X \in \mathcal{E}_n$ which is not in the image of T. By Lemma 2.10 $X \notin \mathcal{N}_n$. Then $X = \alpha E_{ii}$ for some $\alpha \in \mathbb{R}$ and i, $1 \leq i \leq n$. Hence by linearity of T we have that $E_{ii} \notin Im(T)$. □

Corollary 2.12. Let $n \geq 3$, $T : M_n \to M_n$ be a linear map preserving cyclicity. Assume that T is non-surjective. Then there exists E_{ii} such that $T(E_{ii}) \notin \mathcal{E}_n$.

Proof. By Corollary 2.7 there exist indices i, j, $1 \leq i, j \leq n$ such that $T(E_{ij}) \notin \mathcal{E}_n$. By Lemma 2.10 it follows that $i = j$. □

Lemma 2.13. Let $T : M_n \to M_n$ be a linear map preserving cyclicity and $A \in \mathcal{D}_n$. Then all edges of $G(T(A))$ are loops.

Proof. Assume the converse, i.e. there are i, j, $1 \leq i \neq j \leq n$, such that the graph $G(T(A))$ contains the edge from i to j. By Lemma 2.10 there exists $E \in \mathcal{N}_n$ such that $T(E) = E_{ji}$. Then $G(T(A \oplus E))$ contains the cycle $i \to j \to i$. We denote this cycle of the length 2 by C. Let us show that there exists β such that $\sigma_{T(A \oplus \beta \otimes E)} = 2$. In order to do this we denote the maximal average weight of cycles in the graph $G(T(A \oplus E))$ by λ and $w(C)$ by μ. Then $\mu \leq \lambda$, since C is a cycle of $G(T(A \oplus E))$. Hence, the general situation splits into the following two cases.

1. Let $\lambda > \mu$. We consider $\beta = 12(\lambda - \mu)$, here all operations are usual, not tropical. Then $w(C) = \mu + \beta/2 = 6\lambda - 5\mu$. Other cycles containing E_{ji} have at least 3 edges, so the average weight of each of them is less than or equal to $\lambda + \beta/3 = 5\lambda - 4\mu$. From $\lambda > \mu$ it follows that $6\lambda - 5\mu > 5\lambda - 4\mu$. Thus C is a critical subgraph of $G(T(A \oplus \beta \otimes E))$.

2. Let $\lambda = \mu$. Then we can take any $\beta > 0$. Hence $w(C)$ increases by $\beta/2$, the average weight of each other cycle containing E_{ji} increases by $\beta/3$ and the average weight of other cycles does not change. Thus C is a critical subgraph of $G(T(A \oplus \beta \otimes E))$.

Hence in each case for the chosen value β the critical subgraph of $G(T(A \oplus \beta \otimes E))$ is equal to C, hence, $\sigma_{T(A \oplus \beta \otimes E)} = 2$. But $\sigma_{A \oplus \beta \otimes E} = 1$, which is a contradiction. □

3. CHARACTERIZATION OF MAPS PRESERVING CYCLICITY

The map T^* defined in Lemma 2.10 induces the bijection $\widehat{T}$ on the set of edges of the complete graph without loops. Namely, for any $1 \le i \ne j \le n$ if e_{ij} is the edge from i to j, then $\hat{T}(e_{ij})$ is the unique edge of the graph $G(T(E_{ij}))$.

Denote by $\bar{e}$ the edge with the same vertices as e and such that $\bar{e}$ and e have opposite directions.

Lemma 3.1. Let $T : M_n \to M_n$ be a linear map preserving cyclicity. Then $\widehat{T}(\bar{e}) = \overline{\widehat{T}(e)}$ for each edge e.

Proof. Since T preserves cyclicity and $\hat{T}$ is a bijection on the set of edges, we obtain that any cycle of the length 2 is mapped to a cycle of the length 2. So, the counter-directional edges are mapped to counter-directional edges, and the result follows. □

Proposition 3.2. Let $T : M_n \to M_n$ be a linear map preserving cyclicity. Then $\widehat{T}$ maps pairs of adjacent edges to pairs of adjacent edges.

Proof. By Lemma 3.1 the result follows for pairs of counter-directional edges. Thus we may further assume that the edges e_1, e_2 are adjacent and do not constitute a cycle of the length 2. Then without loss of generality we can assume that there exist $1 \le i, j, k \le n$, $k \ne i$, such that $e_1 = e_{ij}$ and either $e_2 = e_{jk}$ or $e_2 = e_{kj}$. In both cases consider the cycle of the length three $G(E_{ij} \oplus E_{jk} \oplus E_{ki})$. Then $G(T(E_{ij} \oplus E_{jk} \oplus E_{ki})) = G(T(E_{ij}) \oplus T(E_{jk}) \oplus T(E_{ki}))$ is a cycle of the length 3 with the edges $\hat{T}(e_1)$, $\hat{T}(e_{ki})$, and either $\hat{T}(e_2)$ or $\hat{T}(\bar{e}_2)$. Therefore, $\hat{T}(e_1)$ and $\hat{T}(e_2)$ are adjacent. □

Lemma 3.3. Let $T : M_n \to M_n$ be a linear map preserving cyclicity. Then there exists a permutation $\tau \in S_n$ such that either $T(E_{ij}) = b_{ij} \otimes E_{\tau(i)\tau(j)}$ for all $1 \le i \ne j \le n$ or $T(E_{ij}) = b_{ij} \otimes E_{\tau(j)\tau(i)}$ for all $1 \le i \ne j \le n$. In other words, if $G(A)$ does not contain loops, then $G(T(A))$ is obtained from $G(A)$ by renumbering the vertices, changing the weights of the edges and possibly reversing all edges.

Proof. By Proposition 3.2 it is sufficient to prove that one of the following two statements holds:

1. For any matrices $E_1, E_2 \in \mathcal{N}_n$ it holds that:

   a) If the head vertex of $G(E_1)$ is the tail vertex of $G(E_2)$, then the head vertex of $G(T(E_1))$ is the tail vertex of $G(T(E_2))$.

   b) If $G(E_1)$ and $G(E_2)$ are outgoing edges from the same vertex, then $G(T(E_1))$ and $G(T(E_2))$ are outgoing edges from the same vertex.

   c) If $G(E_1)$ and $G(E_2)$ are ingoing edges to the same vertex, then $G(T(E_1))$ and $G(T(E_2))$ are ingoing edges to the same vertex.

2. For any matrices $E_1, E_2 \in \mathcal{N}_n$ it holds that:

   a) If the head vertex of $G(E_1)$ is the tail vertex of $G(E_2)$, then the tail vertex of $G(T(E_1))$ is the head vertex of $G(T(E_2))$.

   b) If $G(E_1)$ and $G(E_2)$ are outgoing edges from the same vertex, then $G(T(E_1))$ and $G(T(E_2))$ are ingoing edges to the same vertex.

   c) If $G(E_1)$ and $G(E_2)$ are ingoing edges to the same vertex, then $G(T(E_1))$ and $G(T(E_2))$ are outgoing edges to the same vertex.

By Lemma 2.10 and due to the linearity of T it is sufficient to consider the matrix units only. For each pair $E_{ij}, E_{jk} \in \mathcal{N}_n$ by Proposition 3.2 either 1a) or 2a) holds, since the cycle containing the edges $(i,j),\ (j,k)$ is mapped to a cycle. Let us prove that for any two distinct pairs of the edges the same item holds. If $n = 2$ then the result follows by Lemma 3.1. So, assume that $n \geq 3$ and there are indices $i, j, k,\ i \neq j \neq k \neq i$ such that the matrices $E_{ij}, E_{jk} \in \mathcal{N}_n$ satisfy 1a). From Lemma 3.1 it follows that the pair of reversed edges, corresponding to the matrices E_{kj}, E_{ji} also satisfies 1a). Now for any $E_{ml} \in \mathcal{N}_n$ we have the following alternative.

i. If $\{m, l\} \cap \{i, j, k\} = \emptyset$ then $E_{ij} \oplus E_{jk} \oplus E_{km} \oplus E_{ml} \oplus E_{li}$ is a cycle. Hence its image is a cycle of the same length, and 1a) is satisfied for any pair of adjacent edges in the cycle.

ii. If $m = i$, then $E_{kj} \oplus E_{ji} \oplus E_{il} \oplus E_{lk}$ is a cycle showing that E_{ml} and E_{ji} satisfy the same condition 1a).

iii. If $m = j$ we consider the cycle $E_{kj} \oplus E_{jl} \oplus E_{lk}$ showing that E_{ml} and E_{kj} satisfy the condition 1a).

iv. If $m = k$ then we similarly consider $E_{ij} \oplus E_{jk} \oplus E_{kl} \oplus E_{li}$ to obtain the desired result.

v. Similarly the cases $l \in \{i, j, k\}$ and $\{l, m\} \subseteq \{i, j, k\}$ can be considered.

Thus all edges E_{ml} satisfy 1a).

Since any cycle of length 2 is mapped to a cycle of length 2, it follows from Lemma 3.1 that in the both cases Items b) and c) follow from Item a). This concludes the proof. □

The following is an immediate consequence:

Corollary 3.4. Let $T : M_n \to M_n$ be a linear map preserving cyclicity. Then for each $l > 1$ the graph $G(T(A))$ contains the same number of cycles of length l as A.

Lemma 3.5. Let $T : M_n \to M_n$ be a linear map preserving cyclicity. Then there exists $\lambda \in \mathbb{R}$ such that for all $C \in M_n$ satisfying $G(C)$ is a cycle of length at least two it holds that $w(G(T(C))) = \lambda \otimes w(G(C))$.

P r o o f. 1. For $n = 2$ the result follows directly from the fact that there is only one cycle of length at least 2. So, further we assume that $n \geq 3$. We prove this lemma by induction on cycle length. We are going to prove the lemma for cycles with all edges having the weight 0, then consider the general case. Let us at first show that there exists $\lambda \in \mathbb{R}$ such that for all cycles $C \in M_n$ of the length two it holds that $w(G(T(C))) = \lambda \otimes w(G(C))$.

2. The base of induction. We consider two cycles of length 2 having one vertex in common. Up to the renumbering of vertices without loss of generality we can assume that these cycles are $1 \to 2 \to 1$ and $1 \to 3 \to 1$. Let us denote them by C_1 and C_2, correspondingly. Denote by C_3 the cycle $1 \to 2 \to 3 \to 1$. Let $A_1 = E_{12} \oplus E_{21}$, $A_2 = E_{13} \oplus E_{31}$ and $A_3 = E_{12} \oplus E_{23} \oplus E_{31}$. Then $C_1 = G(A_1)$, $C_2 = G(A_2)$, and $C_3 = G(A_3)$. We set the weight of each edge in these cycles to be equal to 0. From Lemma 3.3 it follows that $G(T(E_{12} \oplus E_{21} \oplus E_{23} \oplus E_{31}))$ also consists of two cycles of lengths 2 and 3, correspondingly. Since T preserves the cyclicity index, $\sigma(T(E_{12} \oplus E_{21} \oplus E_{23} \oplus E_{31})) = 1$. Therefore both cycles $G(T(A_1))$ and $G(T(A_3))$ are contained in the critical subgraph of $G(T(E_{12} \oplus E_{21} \oplus E_{23} \oplus E_{31}))$, i.e. $w(G(T(A_1))) = w(G(T(A_3)))$. Similarly for A_2 and A_3. Therefore, $w(G(T(A_1))) = w(G(T(A_2)))$. Since $w(C_1) = w(C_2)$, they are increased by the same $\lambda \in \mathbb{R}$. Thus, for all cycles C of length 2 with a common vertex there is $\lambda \in \mathbb{R}$ such that $w(G(T(C))) = \lambda \otimes w(G(C))$.

Now consider two arbitrary cycles of length 2. Without loss of generality we can assume that these cycles are $1 \to 2 \to 1$ and $3 \to 4 \to 3$. Then average weight of both of them increase by the same number as a cycle $1 \to 3 \to 1$. Hence, the statement holds for any two cycles of length 2.

3. The induction step. Assume that the result holds for all cycles of length less than or equal to k. Consider an arbitrary cycle of length $k+1$ with the edges of zero weight and denote it C_{k+1}. Up to the renumbering of vertices without loss of generality we may assume that $C_{k+1} = 1 \to 2 \to \ldots \to k \to (k+1) \to 1$. Then $C_{k+1} = G(A_{k+1})$, where $A_{k+1} = E_{12} \oplus E_{23} \oplus \ldots \oplus E_{k\,k+1} \oplus E_{k+1\,1} \in M_n$.

Let us consider two auxiliary graphs $C_k = 1 \to 2 \to \ldots \to k \to 1 = G(A_k)$ and $C = G(E_{12} \oplus E_{23} \oplus \ldots \oplus E_{k-1\,k} \oplus E_{k\,k+1} \oplus E_{k+1\,1} \oplus E_{k\,1}) = G(A)$, where $A, A_k \in M_n$ are $(0, -\infty)$-matrices uniquely determined by their graphs. Then C is a cycle of the length $k+1$ with the chord between k and 1. Hence $\sigma(C) = 1$. Since all edges have the same weight, C is the critical subgraph of $G(A)$, and thus $\sigma(A) = 1$. Therefore, $\sigma(T(A)) = 1$. By Lemma 3.3 the graphs $G(T(A_k))$ and $G(T(A_{k+1}))$ are cycles of the lengths k and $k+1$, respectively.

Recall that e_{ij} denotes the edge from i to j. Since T is linear and the edges $e_{12}, \ldots, e_{k-1\,k}$ are the common edges of C_{k+1} and C_k, by Lemma 3.1 it follows that $\hat{T}(e_{12}), \ldots, \hat{T}(e_{k-1\,k})$ are the common edges of $G(T(A_k))$ and $G(T(A_{k+1}))$, both of which are subgraphs of the graph $G(T(A))$ with $k+2$ edges. By the induction hypothesis $w(G(T(A_k))) = \lambda \otimes w(C_k) = \lambda$. Since $\sigma(T(A)) = 1$, it follows that the critical subgraph of $G(T(A))$ contains at least two cycles. Hence, $w(G(T(A_k))) = w(G(T(A_{k+1})))$, and therefore $w(G(T(A_{k+1}))) = \lambda = \lambda \otimes w(C_{k+1})$. This proves the induction step, therefore, the lemma holds for all cycles of length at least 2, in which all edges have weight 0.

4. Let us consider an arbitrary cycle C. Let $C = G(A)$ and $C' = G(A')$, where $A = \alpha_1 \otimes E_{12} \oplus \alpha_2 \otimes E_{23} \oplus \cdots \oplus \alpha_{k-1} \otimes E_{k-1\,k} \otimes \alpha_k \otimes E_{k\,1}$ and $A' = E_{12} \oplus E_{23} \oplus \cdots \oplus E_{k-1\,k}$. By Lemma 3.3 $G(T(A))$ and $G(T(A'))$ are cycles. They have common vertices and

edges, but different weights. Let $T(E_{i\,i+1}) = b_i \otimes E_{j_i\,j_{i+1}}$ for $1 \le i \le k-1$ and $T(E_{k\,1}) = b_k \otimes E_{j_{k-1}\,j_1}$. By Item 3, $w(T(A')) = \frac{1}{k}\sum\limits_{1\le i\le k} b_k = \lambda$. Note that $T(A) = \bigoplus\limits_{1\le i\le k-1}(\alpha_i \otimes T(E_{i\,i+1})) \oplus \alpha_k \otimes T(E_{k\,1}) = \bigoplus\limits_{1\le i\le k-1}(\alpha_i \otimes b_i \otimes E_{j_i\,j_{i+1}}) \oplus \alpha_k \otimes b_k \otimes E_{j_k\,1}$. Then $w(T(A)) = \frac{1}{k}\sum\limits_{1\le i\le k}(\alpha_k + b_k) = \frac{1}{k}\sum\limits_{1\le i\le k}\alpha_k + \frac{1}{k}\sum\limits_{1\le i\le k} b_k = w(C) + \lambda$. Hence, $w(T(A)) = \lambda \otimes w(C)$, and the lemma follows. □

By linearity Lemma 3.5 immediately implies

Corollary 3.6. Let $T : M_n \to M_n$ be a linear map preserving cyclicity. Then there exists $\lambda \in \mathbb{R}$ such that for all $A \in M_n$ for each cycle C of length at least 2 in the graph $G(A)$ it holds that $w(C) = \lambda \otimes w(C')$, where C' is the corresponding cycle in $G(T(A))$.

Let us remind that by Lemma 2.13 the image of a diagonal matrix is diagonal, i.e., there are $c_{ij} \in \mathbb{R}$ such that $T(E_{ii}) = \bigoplus\limits_{1\le j\le n} c_{ij} \otimes E_{jj}$ for all i, $1 \le i \le n$.

Lemma 3.7. Let $n \ge 3$ and $T : M_n \to M_n$ be a linear map preserving cyclicity. Let $\lambda = \lambda(T) \in \mathbb{R}$ be the value determined in Lemma 3.5 and $\tau = \tau(T) \in S_n$ be the permutation determined in Lemma 3.3. Then $c_{i\tau(i)} = \lambda$ and $c_{i\tau(j)} < \lambda$ for all $1 \le j \ne i \le n$.

Proof. Up to renumbering of vertices without loss of generality we assume that $\tau = id$.

1. Let us prove that $\max\limits_{1\le j\le n} c_{ij} = \lambda$ for all i.

Let $X = E_{12} \oplus E_{23} \oplus \ldots \oplus E_{n-1\,n} \oplus E_{n1} \in M_n$ and $A_i = E_{ii} \oplus X$. Then $\sigma(A_i) = 1$, and hence $\sigma(T(A_i)) = 1$. By Lemma 3.3 it follows that $G(T(X))$ is a cycle of length n and by Lemma 3.5 $w(G(T(X)) = \lambda$. By Lemma 2.13 all edges of $T(E_{ii})$ are loops. If an average weight of each of these loops is less than λ, then the critical subgraph of $G(T(A_i))$ contains only one cycle $G(T(X))$ and $\sigma(T(A_i)) = n$, which is a contradiction. Therefore the average weight of at least one loop in $G(T(E_{ii}))$ is bigger than or equal to λ. So, $\max\limits_{1\le j\le n} c_{ij} \ge \lambda$. If $\alpha = \max\limits_{1\le j\le n} c_{ij} > \lambda$, then there exists $\beta \in \mathbb{R}$, $\lambda - \alpha < \beta < 0$. Consider $B = \beta \otimes E_{ii} \oplus X$. Observe that $\sigma(B) = n$ and $\sigma(T(B)) = 1$, which is a contradiction. Hence, $\max\limits_{1\le j\le n} c_{ij} = \lambda$.

2. Let us show that $c_{ij} < \lambda$ for all $1 \le j \ne i \le n$.

For each $1 \le i \le n$ we consider

$$Y = E_{12} \oplus \ldots \oplus E_{i-2\,i-1} \oplus E_{i-1\,i+1} \oplus E_{i+1\,i+2} \oplus \ldots \oplus E_{n-1\,n} \oplus E_{n\,1} \in M_n,$$

where the indices of the matrix units are taken module n, i.e., the cycle, containing all vertices except i-th, where $1 \le i \le n$. Let $D = E_{ii} \oplus Y$. If there exists j, $1 \le j \le n$, $j \ne i$ such that $c_{ij} \ge \lambda$ then $\sigma(T(D)) = 1$, as far as $\sigma(D) = n-1$, since $G(D)$ is not connected. This contradiction concludes the proof of this step.

3. From $\max\limits_{1\le j\le n} c_{ij} = \lambda$ and $c_{ik} < \lambda$ for all $k \ne i$ it follows that $c_{ii} = \lambda$. □

Definition 3.8. Let $A = (a_{ij}),\ B = (b_{ij}) \in M_n$. Let us say $A \succeq B$ if for all $i, j,\ 1 \leq i, j \leq n$ the inequality $b_{ij} \neq -\infty$ implies the inequality $a_{ij} \neq -\infty$. Otherwise $A \not\succeq B$.

Corollary 3.9. Let $T : M_n \to M_n$, $n \geq 3$, be a linear map preserving cyclicity. Then for each matrix $E \in \mathcal{E}_n$ there exists $A \in \mathcal{E}_n$ such that $T(A) \succeq E$, i.e. $G(T(A))$ contains the edge $G(E)$.

Proof. It follows from Lemma 2.10 for $E \in \mathcal{N}_n$ and from Lemma 3.7 for $E \in \mathcal{D}_n$. □

Let $A \circ B$ denote entrywise product of matrices (also known as the Schur product or Hadamard product), i.e. $(A \circ B)_{ij} = a_{ij} \otimes b_{ij}$.

Theorem 3.10. Let $n \geq 3$ and $T : M_n \to M_n$ be a linear map. Then T preserves the cyclicity index if and only if there exist

1. a permutation matrix $P \in M_n$,
2. a matrix $B = (b_{ij}) \in M_n$ such that $b_{ij} \neq -\infty$ for all $1 \leq i, j \leq n$ and the average weight of each cycle in $G(B)$ is 0,
3. diagonal matrices $C_i = \mathrm{diag}(c_{i1}, \ldots, c_{in}) \in M_n$, $1 \leq i \leq n$, such that $c_{ij} < 0$ for all $1 \leq i, j \leq n$,
4. $\lambda \in \mathbb{R}$,

such that T is a composition of the following transformations:

- $T_1(A) = A^T$ for all $A \in M_n$,
- $T_2(A) = P \otimes A \otimes P^T$ for all $A \in M_n$,
- $T_3(A) = \lambda \otimes A$ for all $A \in M_n$,
- $T_4(A) = A \circ B$ for all $A \in M_n$,
- $T_5(A) = A \oplus \bigoplus_{i=1}^{n} (a_{ii} \otimes C_i)$ for all $A = (a_{ij}) \in M_n$.

Proof. The "if" part holds because each of these transformations preserves cyclicity. It is obvious for T_1, T_2, and T_3. The map T_4 preserves the average weight of each cycle and does not add new edges, therefore, it also preserves the cyclicity index. The map T_5 is a composition of the maps $T_{0k}(A) = A \oplus a_{kk} \otimes C_k$. Since each of them preserves cyclicity by Lemma 2.2, it follows that T_5 preserves cyclicity.

Now we are going to prove the second part of the theorem.

By Lemma 2.13 we have $T(\mathcal{D}_n)$ is a subset of the set of diagonal matrices, i.e., lies in the linear span of $\mathcal{D}_n$ and by Lemma 3.3 $T(\mathcal{N}_n) \subset \mathcal{N}_n$. Thus by linearity we may consider the action of T on the set of diagonal matrices and on the set $\mathcal{N}_n$ separately. From Lemma 3.3 it follows that any linear map preserving cyclicity acts on the set $\mathcal{N}_n$ of scalar multiples of non-diagonal matrix units as a composition of T_1, T_2 and scalar multiplication by certain constants. Let us consider a map T' such that T is a composition $T = T_1 \circ T_2 \circ T'$. Then $T'(\alpha \otimes E_{ij}) = \alpha \otimes \alpha_{ij} \otimes E_{ij}$ for all $1 \le i \ne j \le n$, all $\alpha \in \mathbb{R}$ and appropriate $\alpha_{ij} \in \mathbb{R}$. By Corollary 3.6, the average weight of each cycle is increased by the same value $\lambda \in \mathbb{R}$. Hence, T' is a composition of T_3 and T_4.

By Lemma 3.7 and linearity of T for each $i, 1 \le i \le n$ there exists $C_i = \mathrm{diag}(c_{i1}, \dots, c_{in})$, such that $T(E_{ii}) = \lambda \otimes E_{ii} \oplus \bigoplus_{j \ne i} c_{ij} \otimes E_{jj}$, where $c_{ij} < \lambda$. Then for any $A \in \mathcal{D}_n$ it holds that

$$T(A) = \lambda \otimes A \oplus \bigoplus_{i=1}^{n} (a_{ii} \otimes \bigoplus_{j \ne i} c_{ij} \otimes E_{jj}) = \lambda \otimes A \oplus \bigoplus_{i=1}^{n} (a_{ii} \otimes C_i).$$

By applying a bijective cyclicity preserving map T_2 we may consider the case $\lambda = 0$. Therefore, $c_{ij} < 0$ for all $j \ne i$. We can assume that also $c_{ii} < 0$, since the map T does not depend on them. The required form is obtained. □

Theorem 3.11. Let $T : M_2 \to M_2$ be a linear map. Then T preserves the cyclicity index if and only if T is a composition of the following transformations

1. $T_1 \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} c_{11} \otimes a_{11} \oplus c_{21} \otimes a_{22} & \alpha \otimes a_{12} \\ \beta \otimes a_{21} & c_{12} \otimes a_{11} \oplus c_{22} \otimes a_{22} \end{pmatrix}$, where $\alpha, \beta \ne -\infty$ and $c_{11} \oplus c_{12} = c_{21} \oplus c_{22} = \sqrt[\otimes 2]{\alpha \otimes \beta} = \frac{\alpha+\beta}{2}$,
2. $T_2(A) = A^T$, for all $A \in M_2$.

P r o o f. It is obvious that the transformation T_2 preserves cyclicity index. Now we are going to prove that T_1 preserves cyclicity too. Note that $(c_{11} \otimes a_{11} \oplus c_{21} \otimes a_{22}) \oplus (c_{12} \otimes a_{11} \oplus c_{22} \otimes a_{22}) = (c_{11} \oplus c_{12}) \otimes a_{11} \oplus (c_{21} \oplus c_{22}) \otimes a_{22} = \sqrt[\otimes 2]{\alpha \otimes \beta} \otimes (a_{11} \oplus a_{22})$ and $\sqrt[\otimes 2]{\alpha \otimes a_{12} \otimes \beta \otimes a_{21}} = \sqrt[\otimes 2]{\alpha \otimes \beta} \otimes \sqrt[\otimes 2]{a_{12} \otimes a_{21}}$. It follows that the maximum weight of loops and weight of the cycle of length 2 increase by the same number and T_1 preserves cyclicity. Therefore the "if" part holds.

Let us prove the "only if" part of the theorem.

By Lemma 2.10 $T(E_{12}) = \alpha \otimes E_{12}$ and $T(E_{21}) = \beta \otimes E_{21}$ or $T(E_{12}) = \beta \otimes E_{21}$ and $T(E_{12}) = \alpha \otimes E_{21}$. From Lemma 2.13 it follows that $T(E_{11}) = c_{11} \otimes E_{11} \oplus c_{12} \otimes E_{22}$ and $T(E_{22}) = c_{21} \otimes E_{11} \oplus c_{21} \otimes E_{22}$. By linearity T has the same form as T_1 or $T_2 \circ T_1$.

Now we want to show that $c_{11} \oplus c_{12} = c_{21} \oplus c_{22} = \sqrt[\otimes 2]{\alpha \otimes \beta}$. We consider matrices $A = E_{11} \oplus E_{12} \oplus E_{22}$ and $B = \gamma \otimes E_{11} \oplus E_{12} \oplus E_{21}$, where $\gamma < 0$. Since $\sigma_{T(G(A))} = \sigma_{G(A)} = 1$, $c_{11} \oplus c_{12} \ge \sqrt[\otimes 2]{\alpha \otimes \beta}$. Suppose that $(c_{11} \oplus c_{12}) \otimes \lambda = \sqrt[\otimes 2]{\alpha \otimes \beta}$, where $\lambda < 0$. Then we take $\gamma = \lambda/2$. From $\sigma_{T(G(B))} = \sigma_{G(B)}$ it follows that $\gamma \otimes (c_{11} \oplus c_{12}) < \sqrt[\otimes 2]{\alpha \otimes \beta}$. But $\frac{\lambda}{2} \otimes (c_{11} \oplus c_{12}) > \lambda \otimes (c_{11} \oplus c_{12})$, a contradiction. Hence, $c_{11} \oplus c_{12} = \sqrt[\otimes 2]{\alpha \otimes \beta}$. Similarly, $c_{21} \oplus c_{22} = \sqrt[\otimes 2]{\alpha \otimes \beta}$. □

4. ADDITIONAL PROPERTIES

Lemma 4.1. Let $n \ge 3$, $T_5 : M_n \to M_n$ be defined in Theorem 3.10. Then T_5 is bijective if and only if $c_{ij} = -\infty$ for all i, j, $1 \le i \ne j \le n$. If T_5 is not bijective, then T_5 is both non-injective and non-surjective.

Proof. Let us consider T_5 as a composition of the following maps.

Fix an index i, $1 \le i \le n$, and define the transformation T_{5i} by $T_{5i}(A) = A \oplus (a_{ii} \otimes C_i)$, where $C_i = \operatorname{diag}(c_{i1}, \dots, c_{in}) \in M_n$, $i = 1, \dots, n$. If $c_{ij} = -\infty$ for all $j \ne i$, then the map T_{5i} is identity and hence bijective.

If T_{5i} is bijective for all $i = 1, \dots, n$, then T_5 is bijective as a composition of bijective maps.

If there exists i, j, $1 \le i \ne j \le n$, with $c_{ij} \ne -\infty$, then consider the map T_{5i}. We have T_{5i} is non-injective, since $T_{5i}(E_{ii}) = E_{ii} \oplus \bigoplus_{k=1}^{n} c_{ik} \otimes E_{kk} = T_{5i}(E_{ii} \oplus \bigoplus_{k=1}^{n} c_{ik} \otimes E_{kk})$, but $E_{ii} \ne E_{ii} \oplus \bigoplus_{k=1}^{n} c_{ik} \otimes E_{kk}$ since $c_{ij} \ne -\infty$. Also T_{5i} is non-surjective, since $T_{5i}(A) \ne E_{ii}$ for all $A \in M_n$.

This proves the lemma. □

Corollary 4.2. Let $T : M_n \to M_n$ be a linear map preserving cyclicity. Then T is injective if and only if T is surjective.

Proof. By Theorem 3.10 if $n \ge 3$ then T is a composition of the maps T_1, T_2, T_3, T_4, T_5 introduced in Theorem 3.10. Note that the maps T_k, $k = 1, 2, 3, 4$, are bijective. So, it remains to consider the bijectivity of T_5. By Lemma 4.1 the map T_5 is either bijective or both non-injective and non-surjective.

Similarly to the proof of Lemma 4.1, we can define the transformation T_{1i} by $T_{1i}(A) = A \oplus (a_{ii} \otimes C_i)$, where $C_i = \operatorname{diag}(c_{i1}, c_{i2})$. Consideration of this map conclude the proof for $n = 2$. □

Corollary 4.3. Let $n \ge 3$, $T : M_n \to M_n$ be a bijective linear map. Then T preserves cyclicity if and only if it is a composition of transformations T_1, T_2, T_3, T_4 from Theorem 3.10.

Proof. The transformations T_1, T_2, T_3, T_4 defined in Theorem 3.10 are bijective. By Lemma 4.1 the transformation T_5 is bijective if and only if $c_{ij} = -\infty$ for all $1 \le j \ne i \le n$, and in this case T_5 is the identity. □

Corollary 4.4. Let $n \ge 3$, $T : M_n \to M_n$ be a non-surjective linear map of tropical matrices. Then T preserves cyclicity index if and only if it is a composition of transformations T_1, T_2, T_3, T_4, T_5 from Theorem 3.10, and there exist $1 \le j \ne i \le n$ with $c_{ij} \ne -\infty$.

Proof. This statement is a direct consequence of Corollary 4.3. □

Lemma 4.5. Let $B = (b_{ij}) \in M_n$, $b_{ij} \ne -\infty$ for all i, j. Then the average weight of each cycle in $G(B)$ is 0 if and only if there exist $d_1, \dots, d_n \in \mathbb{R}$ and the vectors $x = (d_1, \dots, d_n)^T$ and $y = (-d_1, \dots, -d_n)^T$ such that $B = xy^T$.

Proof. Let $J = (0) \in M_n$ be the matrix with all zero entries. We are going to prove that the matrix B is obtained from J by several tropical multiplications of the i-th column by β_i and the i-th row by $\beta_i^{\otimes -1}$, i.e. the matrix B is of the form $b_{ij} = d_i - d_j$, where $d_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, which is equivalent to the desired condition.

Consider a graph $G(B)$. The tropical multiplications of the i-th column by β_i and the i-th row by $\beta_i^{\otimes -1}$ increase the weights of edges outgoing from i-th vertex by β_i, and decrease the weights of edges ingoing to i-th vertex by β_i. Therefore, the average weight of each cycle does not change. This proves the "if" part of the lemma.

Now let the average weight of each cycle in $G(B)$ be 0. We prove the "only if" part of the lemma by induction.

The base of induction is $n = 2$. In this case the conditions of the lemma imply that $B = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$, and the result follows.

The induction step. Let us show that if the statement holds for $n = k$, then it also holds for $n = k+1$. Let $B \in M_{k+1}$. Consider the graph $G = G(B)$ and its subgraph G^* induced by the vertices $1, \ldots, k$ of G. Let $B^* \in M_k$ be the adjacency matrix of G^*. By the induction hypothesis, B^* is obtained from $J \in M_n$ by several tropical multiplications of the i-th column by β_i and the i-th row by $\beta_i^{\otimes -1}$. Applying inverse multiplications in the reversed order to the matrix B, we get the matrix B'. The average weight of each cycle does not change, hence, B' is of the form

$$B' = \begin{pmatrix} 0 & \ldots & 0 & b'_{1,k+1} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & b'_{k,k+1} \\ -b'_{1,k+1} & \ldots & -b'_{k,k+1} & 0 \end{pmatrix}$$

Consider the cycle $i \to k+1 \to j \to i$. The average weight of it equals $\frac{b'_{i,k+1} - b'_{j,k+1} + 0}{3} = 0$, hence, $b'_{i,k+1} = b'_{j,k+1}$ for all i, j. Then, multiplying the last column by $-b'_{1,k+1}$, and the last row by $b'_{1,k+1}$, we obtain the matrix $J \in M_{n+1}$. Since this operation is invertible, the lemma is proved. $\square$

ACKNOWLEDGEMENT

The authors are grateful to the referees for careful reading and valuable suggestions. The first and the second authors wish to thank Technion – Israel Institute of Technology for the warm hospitality. The work of the first author was funded by the European Union (ERC, GENERALIZATION, 101039692). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

(Received December 20, 2021)

REFERENCES

[1] F. Baccelli, G. Cohen, G.J. Olsder, J.-P. Quadrat: Synchronization and Linearity. Wiley 1992.

[2] P. Butkovič: Max-algebra: the linear algebra of combinatorics? Linear Algebra and its Applications *367* (2003), 313–335. DOI:10.1016/s0024-3795(02)00655-9

[3] D.J. Dieudonné: Sur une généralisation du groupe orthogonal á quatre variables. Arch. Math. *1* (1949), 282–287.

[4] G. Frobenius: Über die Darstellung der endlichen Gruppen durch lineare Substitutionen. Sitzungsber Deutsch. Akad. Wiss., Berlin 1997.

[5] D. Jones: Matrix roots in the max-plus algebra. Linear Algebra and its Applications *631* (2021), 10–34. DOI:10.1016/j.laa.2021.08.008

[6] M. Gavalec: Periodicity in Extremal Algebras. Hradec Králové: Gaudeamus 2004.

[7] M. Gavalec: Linear matrix period in max-plus algebra. Linear Algebra and its Applications *307* (2000), 167–182. DOI:10.1016/s0024-3795(00)00020-3

[8] A. Guterman, M. Johnson, M. Kambites: Linear isomorphisms preserving Green's relations for matrices over anti-negative semifields. Linear Algebra and its Applications *545* (2018), 1–14. DOI:10.1016/j.laa.2018.01.023

[9] A. Guterman, E. Kreines and C. Thomassen: Linear transformations of tropical matrices preserving the cyclicity index. Special Matrices *9* (2021), 112–118. DOI:10.1515/spma-2020-0128

[10] A. Guterman and A. Maksaev: Maps preserving scrambling index. Linear and Multilinear Algebra *66* (2018), 840–851. DOI:10.1080/03081087.2017.1329814

[11] B. Heidergott, G.J. Olsder and J. van der Woude: Max Plus at Work. Princeton Series in Applied Mathematics 2006.

[12] A. Kennedy-Cochran-Patrick, G. Merlet, T. Nowak, S. Sergeev: New bounds on the periodicity transient of the powers of a tropical matrix: Using cyclicity and factor rank. Linear Algebra and its Applications *611* (2021) 279–309. DOI:10.1016/j.laa.2020.10.032

[13] G. Merlet, T. Nowak, S. Sergeev: Weak CSR expansions and transience bounds in max-plus algebra. Linear Algebra and its Applications *461* (2014) 163-199. DOI:10.1016/j.laa.2014.07.027

[14] S. Pierce, et al.: A survey of linear preserver problems. Linear Multilinear Algebra *33* (1992), 1–119. DOI:10.1080/03081089208818176

[15] L. Rodman and P. Šemrl: A localization technique for linear preserver problems Linear Algebra and its Applications *433* (2010), 2257–2268. DOI:10.1016/j.laa.2010.07.032

[16] I. Schur: Einige Bemerkungen zur Determinantentheorie. Akad. Wiss. Berlin: S.-Ber. Preuss., (1925) 454–463.

[17] S. Sergeev: Max algebraic powers of irreducible matrices in the periodic regime: An application of cyclic classes. Linear Algebra and its Applications *431* (2009), 1325–339. DOI:10.1016/j.laa.2009.04.027

Alexander Guterman, Technion – Israel Institute of Technology, Haifa, 3200003, Israel; Faculty of Mathematics and Mechanics, Lomonosov Moscow State University, Moscow, 119991, Russia

e-mail: alexander.guterman@gmail.com

Elena Kreines, Department of Mathematics, School of Mathematical Sciences, Tel Aviv University, Tel Aviv, 6997801, Israel; Moscow Center of Fundamental and Applied Mathematics, Moscow, 119991, Russia

e-mail: kreines@tauex.tau.ac.il

Alexander Vlasov, Faculty of Mathematics and Mechanics, Lomonosov Moscow State University, Moscow, 119991, Russia; Moscow Center of Fundamental and Applied Mathematics, Moscow, 119991, Russia

e-mail: aleksandr.vlasov@math.msu.ru