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Applications of Mathematics, Vol. 68 (2023), No. 1, 15–33

Persistent URL: `http://dml.cz/dmlcz/151494`

SPECTRAL ESTIMATES OF VIBRATION FREQUENCIES OF ANISOTROPIC BEAMS

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Received March 15, 2021. Published online June 29, 2021.

Abstract. The use of one theorem of spectral analysis proved by Bordoni on a model of linear anisotropic beam proposed by the author allows the determination of the variation range of vibration frequencies of a beam in two typical restraint conditions. The proposed method is very general and allows its use on a very wide set of problems of engineering practice and mathematical physics.

Keywords: theory of beams; deformation of cross section; spectral geometry; comparison of spectra

MSC 2020: 74B05, 74K10, 35P15, 47A75

1. Introduction

The motion dynamic equations of an anisotropic beam with deformable cross section have been obtained by the author in a previous article [12]. These equations depend on the kinematic descriptors of the beam such as Lagrangian variables. These equations are also maps of a Hilbert space H into itself; these Hilbert spaces are modeled on a manifold $\mathcal{M}$ in which the kinematic descriptors of the beam take values and are appropriate Sobolev spaces, consistent with the order of equations. The model of beam is expressed by the triplet $(\mathcal{Q}, \mathcal{M}, \eta\colon \mathcal{M} \to \mathcal{Q})$, where $\mathcal{Q}$ (the base) is the rectangle $[0, L] \times [0, t_f]$, L is the length of the beam and t_f the final instant of the motion, $\mathcal{M}$ is the manifold on which the kinematic descriptors take values and $\eta\colon \mathcal{M} \to \mathcal{Q}$ is the *fibration map.* The mathematical structure is that of a *fiber bundle.* It is possible to see two different models of beams as two different fiber bundles: $(\mathcal{Q}', \mathcal{M}', \eta'\colon \mathcal{M}' \to \mathcal{Q}')$ and $(\mathcal{Q}, \mathcal{M}, \eta\colon \mathcal{M} \to \mathcal{Q})$. These models lead to more complex analyses, for example $(\mathcal{Q}', \mathcal{M}', \eta'\colon \mathcal{M}' \to \mathcal{Q}')$ to the simplest $(\mathcal{Q}, \mathcal{M}, \eta\colon \mathcal{M} \to \mathcal{Q})$. The map between these fiber bundles is a projec-

DOI: 10.21136/AM.2021.0057-21

tion $\Pi\colon \mathcal{M}' \to \mathcal{M}$ on a selected coordinate of the first space. We compare the spectrum of a differential operator $\Phi'\colon H' \to H'$, describing the dynamics of the used model of beam, with the spectrum of the operator $\Phi\colon H \to H$, sketching the most simple model of beam consistent with the selected behavior. The comparison is done thanks to some theorems of spectral analysis proposed by Mordoni ([1], [2] and [3]). Section 2 presents mathematical tools necessary for the determination of the lower bounds of the frequencies: all definitions and specifications culminate in Theorem 2.4 and applied in Section 3. In each subsection we specify the spaces where the operators act and we test all assumptions of Theorem 2.4; the comparison is done using the standard second order *d'Alembert Operator* for the problem of axial extension in Subsection 3.2 and the fourth order *Euler Operator* for the bending in Subsection 3.4. The use of Theorem 2.4 follows easily in this frame: we determine the lower bounds of the beam vibration frequencies. The upper bound is given, as usual, using the Rayleigh ratio with weighted functions. However, the structure of the problem does not provide the minimum but only a maximum. It coincides with the vibration frequency of the beam model used for the comparison, it is thus necessary to find an approximate estimate of the frequencies upper bounds. This is done by exploiting the conventional inequalities of the elastic constants. We present the exact determination of vibration frequencies of a beam subjected to torsion-warping only at the end of Subsection 3.6; this computation is allowed by the mathematical structure of the examined problem.

2. Mathematical preliminaries

A measure space (M, μ) is a nonempty set M endowed with a measure μ (see e.g. [4]). We assume that μ is positive and that $\mu(x) = 0$ for every $x \in M$. We shall write briefly $L^2(M)$ for $L^2(M, \mu)$. Let (M', μ') and (M, μ) be two measure spaces and let $\Pi\colon M' \to M$ be a surjective map compatible with the measures μ' and μ. Then for every $x \in M$ the fiber $\Pi^{-1}(x)$ is μ-measurable. The map Π is said to satisfy Fubini's property if and only if there exists a measure ν_x on $\Pi^{-1}(x)$ such that for any μ'-measurable function on M' one has

$$\int_{M'} f(y)\,\mathrm{d}\mu'(y) = \int_M \left(\int_{\Pi^{-1}} f(x)\,\mathrm{d}\nu_x \right) \mathrm{d}\mu(x). \tag{2.1}$$

In other words, the measure $\mathrm{d}\mu'$ splits and is equal to $\nu_x \otimes \mu$. An example is given using two Riemannian manifolds (M', g') and (M, g) with a regular surjective map $\Pi\colon M' \to M$. In this case, μ' and μ are the canonical measures induced on M' and M by the metrics g' and g, respectively. The metric g' gives by restriction a Riemannian metric g_x on the fiber $\Pi^{-1}(x)$ for all $x \in M$. Then Fubini's property (2.1) is automatically satisfied when ν_x is the canonical measure induced on $\Pi^{-1}(x)$ by g_x.

For every function $f \in L^2(M')$, its restriction $f_{\Pi^{-1}}(x)$ is defined for almost every $x \in M$. Let $\mathcal{E}$ be a vector subspace of $L^2(M')$ and let $\mathcal{E}_x$ be its image in $L^2(\Pi^{-1}(x))$ by restriction; we assume $\mathcal{E}_x = \{0\}$ when the restriction on $\mathcal{E}$ is not defined on $\Pi^{-1}(x)$. We give:

Definition 2.1. The *rank of* $\mathcal{E}$ is the essential supremum of the dimension of $\mathcal{E}_x$, i.e.

$$\text{rank of } \mathcal{E} = \inf_{A \in \mathcal{A}} \Big(\sup_{x \in M \setminus A} \dim \mathcal{E}_x \Big),$$

where $\mathcal{A}$ is the class of all subsets in M of μ-measure equal to zero.

Remark 2.2. Notice that this is not the usual definition of the *rank* and that the *rank* of $\mathcal{E}$ may be much smaller than the dimension of $\mathcal{E}$. (There is an example given in [2], where $\mathcal{E}$ has infinite dimension and finite *rank*, see Notice at page 696.)

Let $\Pi\colon (M', \mu') \to (M, \mu)$ be any surjective map which satisfies Fubini's property (2.1) and we define the map

$$\varpi\colon\ L^2(M') \to L^2(M)$$

by setting for all $f \in L^2(M')$ and for all $x \in M$:

$$\varpi f(x) = \| f|_{\Pi^{-1}(x)} \|. \tag{2.2}$$

Notice that by Fubini's property, ϖ preserves the L^2-norm

$$\|\varpi f\|_{L^2(M)} = \|f\|_{L^2(M')} \quad \forall\, f \in L^2(M').$$

As ϖ is not linear, the image $\varpi(\mathcal{E})$ of a vector subspace $\mathcal{E} \subset L^2(M')$ is not a vector subspace of $L^2(M')$, but a half-cone.

For a given vector subspace $\mathcal{K} \subset L^2(M')$ we shall denote by $\varpi f^{\mathcal{K}}$ and $\varpi f^{\perp}$ the orthogonal projections of ϖf on $\mathcal{K}$ and the component orthogonal to $\mathcal{K}$, respectively. One can show (see [2] and [3]) the following:

Lemma 2.3. *For every positive integer N let $\mathcal{E}$ be any N-dimensional vector subspace in $L^2(M')$ and let p be the rank of $\mathcal{E}$ (Definition 2.1). Let $\mathcal{K}$ be any k-dimensional vector subspace in $L^2(M)$ such that $k \leqslant N$ and let q be any fixed positive integer. Then there exists a function $u \in \mathcal{E} \setminus \{0\}$ such that*

$$\|(\varpi u)^{\perp}\|_{L^2(M)} \geqslant C(p,q) \|u\|_{L^2(M')}$$

or equivalently,

$$\|(\varpi u)^{\mathcal{K}}\|_{L^2(M)} \leqslant (1 - C(p,q)) \|u\|_{L^2(M')},$$

where $C(p,q)$ is an explicit universal constant given by

$$C(p,q) = \frac{1}{2(p+1)}\Big(\frac{q-1}{p+q} + \frac{1}{4(p+1)}\Big). \tag{2.3}$$

Proof. See [2], Theorem 2.4, p. 696. □

In other terms, the cone $\varpi(\mathcal{E})$ goes enough away from $\mathcal{K}$ to ensure that there exists at least a nonvanishing function u in $\mathcal{E}$ such that the distance of $\varpi(u)$ from $\mathcal{K}$ is not too small (does not go to zero). Lemma 2.3 gives a general comparison theorem between the spectra of two operators acting on different Hilbert spaces as a consequence.

A linear operator Φ' acting on $L^2(M')$ is said to be *semi-bounded* if there exists a real constant c such that

$$\langle\langle \Phi' f, f\rangle\rangle_{L^2(M)} \geqslant c\|f\|^2_{L^2(M)}$$

for every $f \in L^2(M')$ (we may assume $c = 0$ by a shift).

Let Φ' and Φ be two self-adjoint semi-bounded operators acting on the Hilbert spaces $L^2(M')$ and $L^2(M)$, respectively. We say that Φ' and Φ satisfy Kato's inequality with respect to the map $\varpi\colon L^2(M') \to L^2(M)$ defined by (2.2) if ϖ does not increase the energy of the operators, i.e.

$$\langle\langle \Phi(\varpi f), \varpi f\rangle\rangle_{L^2(M')} \leqslant \langle\langle \Phi'(f), f\rangle\rangle_{L^2(M)} \tag{2.4}$$

for every $f \in L^2(M')$.

Lemma 2.3, Fubini's property (2.1) and Kato's inequality (2.4), min-max and max-min principles, see [6], [7] and [8], all combined together imply the following theorem (see [1], [2] and [3]).

Theorem 2.4. *Let ϖ be any map from a measure space (M',μ') onto a measure space (M,μ) which satisfies Fubini's property* (2.1). *Let Φ' and Φ be two self-adjoint semi-bounded operators which satisfy Kato's inequality* (2.4) *with respect to the map $\varpi\colon L^2(M') \to L^2(M)$ defined by* (2.2). *Then for all positive integers N and q, the eigenvalues of Φ' and Φ are related by*

$$\lambda_N(\Phi') \geqslant (1 - C(p,q))\lambda_1(\Phi) + C(p,q)\lambda_{k+1}(\Phi) \tag{2.5}$$

and when $q = 1$,

$$\frac{1}{N}\sum_1^N \lambda_i(\Phi') \geqslant (1 - a(p) - b(p))\lambda_1(\Phi) + b(p)\bigg(\frac{1}{k}\sum_i^k \lambda_i(\Phi)\bigg) + a(p)\lambda_{k+1}(\Phi), \tag{2.6}$$

where:

▹ *p is the rank of the subspace of $L^2(M')$ spanned by the first N eigenvalues of Φ' (cf. Definition 2.1);*
▹ *$k = [N/(p+q)]$ ($[x]$ is the integer part of $x \in \mathbb{R}$);*
▹ *$C(p,q)$ is given by formula (2.3);*
▹ *$a(p) = \frac{1}{8}(p+1)^{-2}$ and $b(p) = \frac{1}{2}(p+1)^{-1}$.*

When the spectra are not discrete, the above inequalities reduce to the discrete part of the spectra (i.e. the part which lies under the essential spectra).

P r o o f. See [2], Theorem 3.2, p. 700. □

We apply the previous Theorem to estimate the vibration frequencies of anisotropic linear elastic beam using inequalities (2.5) and (2.6) in the next sections. The space M' will be represented by the space of configurations of the kinematic descriptors; the map ϖ will be the projection onto a chosen co-ordinate representing the kinematics of a simplest model of beam, featured by the space M. The spaces L^2 will be replaced by suitable Sobolev spaces $H^{2,r}$ consistent with order r of the differential operators, the results remain obviously the same.

3. Estimation of vibration frequencies

3.1. Generalities. The use of Theorem 2.4 implies the choice of the two Hilbert spaces on which the operators Φ' and Φ act and the map ϖ between them. Moreover, the whole spectrum of operator Φ must be known in such a way to get the estimates as accurate as possible. The splitting of the general dynamical problem of anisotropic beams with deformable cross section in three independent sub-problems, proved in [12], is providential in this frame. The comparison between operators is done using the standard operators typical for the beam theories:

(1) The operator sketching the (extension of beam axis)–(resizing of transversal section) problem is compared with its natural reduced one dimensional *d'Alembert operator*

$$\Phi_e(u) = \Big(\frac{\partial^2}{\partial \zeta^2} - \frac{1}{a_e^2}\frac{\partial^2}{\partial t^2}\Big)u, \tag{3.1}$$

which describes the axial motion of the beam.

(2) The operator drawing the (bending-shear of beam axis)–(reshaping of transversal section) problem is compared with the reduced one-dimensional fourth order *Euler operator*

$$\Phi_b(u) = \Big(\frac{\partial^4}{\partial \zeta^4} - \frac{1}{a_f^2}\frac{\partial^2}{\partial t^2}\Big)u, \tag{3.2}$$

which depicts the problem of the axial bending.

(3) The (torsion)-(warping) problem has a formulation such that the direct computation of the vibration frequencies is possible.

Both operators Φ_e and Φ_b work naturally in the Sobolev space $H^{2,r}(M)$ of functions $f\colon M \to \mathbb{R}$ equipped with the norm

$$\|f\|_{2,r,M} = \Big[\int_{\eta(M)} \sum_{\alpha\leqslant r} (\partial^\alpha f)\cdot(\partial^\alpha f)\Big]^{1/2},$$

$\eta(M)$ being the rectangle $[0,L]\times[0,t_f]$ and $r=2$ for the extension and $r=4$ for the bending.

The constants a_e and a_f depend on geometrical and mechanical characteristics of the model, their complete structure is not important here and will be exhibited in the next developments.

The operators whose eigenvalues have to be estimated have the shape

$$\Phi'(\mathbf{u}) = \Big[\mathbf{A}\frac{\partial^2}{\partial\zeta^2} + \mathbf{B}\frac{\partial}{\partial\zeta} + \mathbf{C} - \mathbf{M}\frac{\partial^2}{\partial t^2}\Big]\cdot(\mathbf{u}), \tag{3.3}$$

$\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{M}$ being constant matrices which will be specified for every subproblem. The operator Φ' operates naturally in the Sobolev space $H^{2,2}(M')$ of functions $\mathbf{f}\colon M' \to \mathbb{R}^m$ equipped with the norm

$$\|f\|_{2,2,M'} = \Big[\int_{\eta'(M')} \sum_{\alpha\leqslant m} (\partial^\alpha \mathbf{f})\cdot(\partial^\alpha \mathbf{f})\Big]^{1/2},$$

where $\eta'(M') = \eta(M)$ are still the rectangle $[0,L]\times[0,t_f]$ and $m=3$ for the first problem, because there are 3 the kinematic descriptors: one variable is necessary for the axis extension, two for the re-sizing. The general problem of the bending-shear-reshaping involves several variables. We go to particularize our problem to the right bending—shear so that the re-shaping requires only two parameters, as shows in [12]. The complete kinematic is consequently scanned with 4 functions, so we fix $m=4$.

The map ϖ is the projection on a selected coordinate of the first space.

Lemma 3.1. *Operators* (3.1), (3.2), *and* (3.3) *are*

(1) *semi-bounded, in particular nonnegative,*
(2) *self-adjoint.*

P r o o f of (1). The quadratic forms $\langle\langle \Phi(u), u\rangle\rangle$ represent the global power developed by internal and inertial forces during the motion. It is always finite and nonnegative, since there are no dissipative phenomena.

P r o o f of (2). Operators (3.1) and (3.2) are trivially self-adjoint. Let $\mathbf{v}\colon \eta'(M') \to \mathbb{R}^m$ be a test function ($m = 3$ for the axial extension, $m = 4$ for the bending). We have to prove that $\langle\langle \mathbf{v}, \Phi'(\mathbf{u})\rangle\rangle_{L^2(M')} - \langle\langle \mathbf{u}, \Phi'(\mathbf{v})\rangle\rangle_{L^2(M')} = 0$. By explicitly writing the previous equality

$$\int_0^{t_f}\int_0^L \Big[\mathbf{v}^\top\Big(\mathbf{A}\frac{\partial^2\mathbf{u}}{\partial\zeta^2}+\mathbf{B}\frac{\partial\mathbf{u}}{\partial\zeta}+\mathbf{C}\mathbf{u}-\mathbf{M}\frac{\partial^2\mathbf{u}}{\partial t^2}\Big)-\mathbf{u}^\top\Big(\mathbf{A}\frac{\partial^2\mathbf{v}}{\partial\zeta^2}+\mathbf{B}\frac{\partial\mathbf{v}}{\partial\zeta}+\mathbf{C}\mathbf{v}-\mathbf{M}\frac{\partial^2\mathbf{v}}{\partial t^2}\Big)\Big]\,\mathrm{d}\zeta\,\mathrm{d}t,$$

some algebraic manipulation, see [13], and the use of Stokes formula transform previous relation into

$$\int_0^L \Big(\mathbf{v}^\top\mathbf{A}\frac{\partial\mathbf{u}}{\partial\zeta}-\frac{\partial\mathbf{v}}{\partial\zeta}\mathbf{A}\mathbf{u}+\mathbf{v}\widehat{\mathbf{B}}\mathbf{u}\Big)\,\mathrm{d}\zeta+\int_0^{t_f}\Big(\mathbf{v}^\top\mathbf{M}\frac{\partial\mathbf{u}}{\partial t}-\frac{\partial\mathbf{v}^\top}{\partial t}\mathbf{M}\mathbf{u}\Big)\,\mathrm{d}t. \tag{3.4}$$

We emphasize here that $\mathbf{A}$ and $\mathbf{M}$ are symmetric matrices (see [12]) and $\widehat{\mathbf{B}}$ is the skew symmetric part of the matrix $\mathbf{B}$. Thanks to this remark we conclude that both integrals in (3.4) are equal to zero if the beam is in the same undeformed position and with the same velocity in the initial and final time instant of motion. $\square$

The conservation of L^2-norm is trivially proved for the projection Π, the validity of Kato's inequality is given later for each tackled sub-problem.

The upper bound of each frequency ν_N, $N \in \mathbb{N}$, is computed utilizing the Rayleigh ratio as usual; it is written as

$$\nu_N^2 \leqslant \frac{1}{\int_0^{t_f}\mathbf{v}_N^\top\mathbf{M}\mathbf{v}_N\,\mathrm{d}t}\int_0^L \mathbf{v}_N^\top\Big(\mathbf{A}\frac{\partial^2\mathbf{v}_N}{\partial\zeta^2}+\mathbf{B}\frac{\partial\mathbf{v}_N}{\partial\zeta}+\mathbf{C}\mathbf{v}_N\Big)\,\mathrm{d}\zeta,$$

where $\mathbf{v}_N$ is a column matrix of suitable test functions consistent with restraint conditions.

R e m a r k 3.2. It is important to emphasize that the eigenvalues of the operators Φ and Φ' are the squares of the beam vibration frequencies, i.e. $\nu_N^2 = \lambda_N$.

3.2. Extension-reduction of area of transversal section. We compare the spectra of the operator drawing the extension-resizing of the beam

$$\Phi' = \mathbf{A}\frac{\partial^2}{\partial\zeta^2}+\mathbf{B}\frac{\partial}{\partial\zeta}+\mathbf{C}-\mathbf{M}\frac{\partial^2}{\partial t^2} \tag{3.5}$$

with that one of the *d'Alembert operator* associated to the axial motion of a beam of mass density ϱ and Young modulus E:

$$\Phi = E\frac{\partial^2}{\partial\zeta^2}-\varrho\frac{\partial^2}{\partial t^2}. \tag{3.6}$$

The explicit definition of each term of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{M}$ is not necessary here, it will be shown next. Since operator (3.6) is nonnegative, its spectrum consists of an increasing sequence of positive numbers given by

$$\text{(3.7)} \qquad \text{(a)} \quad \lambda_N(\Phi) = \frac{E\pi^2}{\varrho L^2} N^2 \quad \text{or} \quad \text{(b)} \quad \lambda_N(\Phi) = \frac{E\pi^2}{4\varrho L^2}(2N-1)^2,$$

respectively, for simply posed and clamped end beam. The projection on the first coordinate $\Pi(\Phi') = \Phi$ makes trivial the proof of the conservation of the L^2-norm and of the Kato inequality cause the structure of the involved matrics. The use of Theorem 2.4 needs the specifications of the constant p as the rank (Definition 2.1) of the sub-space spanned by the first N eigenfunctions of Φ', which is unfortunately unknown. We fix $p = 3$ because 3 is the dimension of the matrices appearing in operator (3.5). To eschew the banal estimations $\lambda_N(\Phi') \geqslant \lambda_1(\Phi)$ it is necessary that

$$\left[\frac{N}{3+q}\right] > 1, \quad \text{namely} \quad q < N - 3.$$

With these assumptions, the terms k, $a(p)$, $b(p)$ and $C(p,q)$ appearing in relations (2.5) and (2.6) are given by:

$$\text{(3.8)} \qquad k = \left[\frac{N}{4}\right], \quad a(3) = \frac{1}{128}, \quad b(3) = \frac{1}{8}, \quad C(3,q) = \frac{1}{8}\Big(\frac{q-1}{q+3} + \frac{1}{16}\Big).$$

As application of the cited theorem, we present the performed procedure for the determination of the lower bound of the 8th eigenvalue; we consider the simply posed beam, see (3.7a) for the explicit formulation of eigenvalues. The clamped end beam will be analyzed in the second instance since its spectrum is only approximated. The lower bound is obtained keeping the maximal value computed with (2.5) for the single eigenvalue, or with (2.6) for the medium value of the first N eigenvalues.

▷ *Estimation of single eigenvalue using* (2.5): setting $k = 2$ and remembering that $\lambda_{k+1}(\Phi) = (k+1)^2\lambda_1(\Phi)$, we get

$$\begin{aligned} \lambda_8(\Phi') &\geqslant \Big(1 - \frac{1}{8}\Big(\frac{q-1}{q+3} + \frac{1}{16}\Big)\Big)\lambda_1(\Phi) + \frac{1}{8}\Big(\frac{q-1}{q+3} + \frac{1}{16}\Big)\lambda_{k+1}(\Phi) \\ &= \Big(1 + \Big(\frac{q-1}{q+3} + \frac{1}{16}\Big)\Big)\lambda_1(\Phi) = \Big(\frac{33}{16} - \frac{4}{q+3}\Big)\lambda_1(\Phi), \end{aligned}$$

which is an increasing sequence, running q from 1 to 10; it is enough to compute the value for $q = 10$ getting

$$\lambda_8(\Phi') \geqslant \frac{365}{208}\lambda_1(\Phi).$$

▷ *Estimation of single eigenvalue using* (2.6):

$$\frac{1}{8}\sum_{i=1}^{8}\lambda_i(\Phi') \geqslant \Big(1-\frac{1}{128}-\frac{1}{8}\Big)\lambda_1(\Phi)+\frac{1}{8}\cdot\frac{1}{2}(\lambda_1(\Phi)+\lambda_2(\Phi))+\frac{1}{128}\lambda_3(\Phi) \\ = \frac{5}{4}\lambda_1(\Phi), \tag{3.9}$$

so we get $\sum_{i=1}^{8}\lambda_i(\Phi') = \lambda_8(\Phi') + \sum_{i=1}^{7}\lambda_i(\Phi') \geqslant 10\lambda_1(\Phi)$; we got also $\sum_{i=1}^{7}\lambda_i(\Phi') \geqslant \frac{917}{128}\lambda_1(\Phi)$ with the same computations. Subtracting the previous inequality from (3.9), we obtain

$$\lambda_8(\Phi') \geqslant \frac{363}{128}\lambda_1(\Phi) \geqslant \frac{365}{208}\lambda_1(\Phi),$$

which is the best lower bound of the 8th eigenvalue.

The upper bound of each eigenvalue is computed using the Rayleigh ratio. It is therefore necessary put particular attention to restraints conditions for extreme bases in this case: the motion in their own plane may be allowed or hampered. We shall limit our discussion to a simply posed beam with extreme sections clamped in their owe plane in the next developments. We use three weighted sinusoidal functions

$$\mathbf{v} = \begin{pmatrix} 1 \\ l \\ m \end{pmatrix} \sin\frac{N\pi\zeta}{L},$$

l and m being two appropriate real constants, which can be established searching the minimum of the Rayleigh ratio. It has the following aspect for the selected restraint conditions:

$$\Lambda_N(\Phi') \leqslant \frac{N^2\pi^2(EA+l^2G_aJ_{aa}+m^2G_bJ_{bb})-AL^2(C_1l^2+C_2m^2)}{\varrho L^2(A+l^2J_{aa}+m^2J_{bb})} \\ \leqslant \frac{N^2\pi^2(EA+l^2G_aJ_{aa}+m^2G_bJ_{bb})}{\varrho L^2(A+l^2J_{aa}+m^2J_{bb})}, \tag{3.10}$$

where A is the area of the section, J_{aa} and J_{bb} its main moments of inertia, G_a, G_b, C_1 and C_2 are the material elastic constants. We search the minimum of the previous function solving the vectorial equation $\nabla\Lambda_N = \mathbf{0}$; this equation has three solutions:

(1) $l = 0$, $m = 0$, corresponding to the maximum of the frequencies of the simple beam (only one degree of freedom of d'Alembert equation);
(2) $l = 0$ and $m = \pm\sqrt{(E-G_a)A/((G_b-G_a)J_{aa})}$ if $G_b > G_a$ and
(3) $l = \pm\sqrt{(E-G_b)A/((G_a-G_b)J_{bb})}$ and $m = 0$ if $G_b < G_a$.

The Hessian of conditions 2 and 3 is negative, they correspond neither to a minimum nor to a maximum. Therefore, the function $\Lambda_N(l,m)$ assumes the typical "squared-off witch's hat" shape (see Figure 1) in a 3-dimensional space. We search the inf of the function when $l\to\infty$ and $m\to\infty$; the passage to polar coordinates gets

$$\inf\Lambda_N=\frac{N^2\pi^2}{\varrho L^2}\min\{G_a,G_b\}.$$

Setting $\min\{G_a,G_b\}\simeq E/3$, we obtain $\Lambda_N\leqslant E\pi^2N^2/(3\varrho L^2)$. The square roots of all estimates get the upper bounds of frequencies, reported in the following table:

$\nu_1(\Phi')\leqslant\frac{1}{\sqrt3}\nu_1(\Phi)$	$\sqrt{\frac{363}{128}}\nu_1(\Phi')\leqslant\nu_9(\Phi')\leqslant3\sqrt3\nu_1(\Phi)$
$\nu_2(\Phi')\leqslant\frac{2}{\sqrt3}\nu_1(\Phi)$	$\sqrt{\frac{363}{128}}\nu_1(\Phi')\leqslant\nu_{10}(\Phi')\leqslant\frac{10}{\sqrt3}\nu_1(\Phi)$
$\nu_3(\Phi')\leqslant\sqrt3\nu_1(\Phi)$	$\sqrt{\frac{363}{128}}\nu_1(\Phi')\leqslant\nu_{11}(\Phi')\leqslant\frac{11}{\sqrt3}\nu_1(\Phi)$
$\sqrt{\frac{131}{128}}\nu_1(\Phi)\leqslant\nu_4(\Phi)\leqslant\frac{4}{\sqrt3}\nu_1(\Phi)$	$\frac{3\sqrt7}{2}\nu_1(\Phi)\leqslant\nu_{12}(\Phi')\leqslant4\sqrt3\nu_1(\Phi)$
$\sqrt{\frac{703}{640}}\nu_1(\Phi)\leqslant\nu_5(\Phi)\leqslant\frac{5}{\sqrt3}\nu_1(\Phi)$	$\frac{3\sqrt7}{2}\nu_1(\Phi)\leqslant\nu_{13}(\Phi')\leqslant\frac{13}{\sqrt3}\nu_1(\Phi)$
$\sqrt{\frac{441}{384}}\nu_1(\Phi)\leqslant\nu_6(\Phi)\leqslant2\sqrt3\nu_1(\Phi)$	$\frac{3\sqrt7}{2}\nu_1(\Phi)\leqslant\nu_{14}(\Phi')\leqslant\frac{14}{\sqrt3}\nu_1(\Phi)$
$\sqrt{\frac{441}{384}}\nu_1(\Phi)\leqslant\nu_7(\Phi)\leqslant\frac{7}{\sqrt3}\nu_1(\Phi)$	$\frac{3\sqrt7}{2}\nu_1(\Phi)\leqslant\nu_{15}(\Phi')\leqslant5\sqrt3\nu_1(\Phi)$
$\sqrt{\frac{441}{384}}\nu_1(\Phi)\leqslant\nu_8(\Phi)\leqslant\frac{8}{\sqrt3}\nu_1(\Phi)$	$\frac{3\sqrt7}{2}\nu_1(\Phi)\leqslant\nu_{16}(\Phi')\leqslant\frac{16}{\sqrt3}\nu_1(\Phi)$

Table 1: Range of frequencies, simply posed beam with clamped end sections.

The use of the same proceeding to a beam with a clamped end section gets similar results, in this case we use three weighted test functions proportional to $\sin((2N-1)\pi\zeta/(2L))$. The Rayleigh ratio assumes a more complicated shape even its representation in the (l,m,Λ) is still a "squared-off witch's hat". Without going into computational details, we get the following range of frequencies for a clamped end beam:

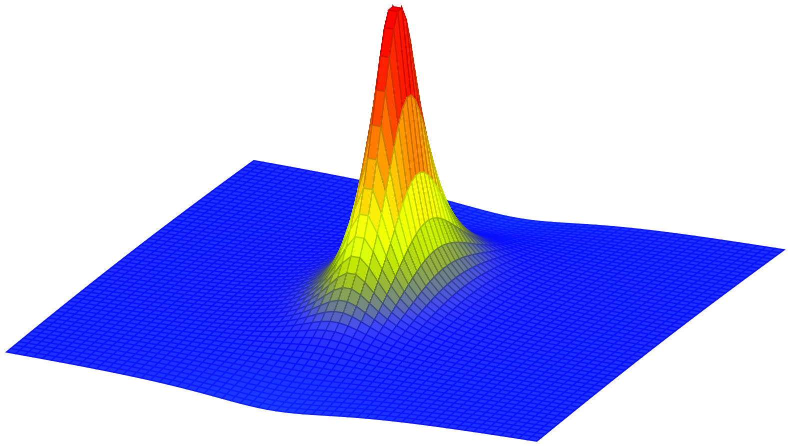

Figure 1. $\Lambda_N(l,m)$, simply posed beam with extreme sections clamped in their own plane.

$\nu_1(\Phi') \leqslant \frac{\sqrt{3}}{3}\nu_1(\Phi)$	$\frac{\sqrt{89}}{8}\nu_1(\Phi) \leqslant \nu_9(\Phi') \leqslant \frac{17\sqrt{3}}{3}\nu_1(\Phi)$
$\nu_2(\Phi') \leqslant \sqrt{3}\nu_1(\Phi)$	$\frac{\sqrt{89}}{8}\nu_1(\Phi) \leqslant \nu_{10}(\Phi') \leqslant \frac{19\sqrt{3}}{3}\nu_1(\Phi)$
$\nu_3(\Phi') \leqslant \frac{5\sqrt{3}}{3}\nu_1(\Phi)$	$\frac{\sqrt{89}}{8}\nu_1(\Phi) \leqslant \nu_{11}(\Phi') \leqslant 7\sqrt{3}\nu_1(\Phi)$
$\sqrt{\frac{17}{12}}\nu_1(\Phi) \leqslant \nu_4(\Phi') \leqslant \frac{7\sqrt{3}}{3}\nu_1(\Phi)$	$\frac{\sqrt{23}}{4}\nu_1(\Phi) \leqslant \nu_{12}(\Phi') \leqslant \frac{23\sqrt{3}}{3}\nu_1(\Phi)$
$\sqrt{\frac{17}{12}}\nu_1(\Phi) \leqslant \nu_5(\Phi') \leqslant 3\sqrt{3}\nu_1(\Phi)$	$\frac{\sqrt{23}}{4}\nu_1(\Phi) \leqslant \nu_{13}(\Phi') \leqslant \frac{25\sqrt{3}}{3}\nu_1(\Phi)$
$\sqrt{\frac{17}{12}}\nu_1(\Phi) \leqslant \nu_6(\Phi') \leqslant \frac{11\sqrt{3}}{3}\nu_1(\Phi)$	$\frac{\sqrt{23}}{4}\nu_1(\Phi) \leqslant \nu_{14}(\Phi') \leqslant 9\sqrt{3}\nu_1(\Phi)$
$\sqrt{\frac{17}{12}}\nu_1(\Phi) \leqslant \nu_7(\Phi') \leqslant \frac{13\sqrt{3}}{3}\nu_1(\Phi)$	$\frac{\sqrt{23}}{4}\nu_1(\Phi) \leqslant \nu_{15}(\Phi') \leqslant \frac{29\sqrt{3}}{3}\nu_1(\Phi)$
$\frac{\sqrt{89}}{8}\nu_1(\Phi) \leqslant \nu_8(\Phi') \leqslant 5\sqrt{3}\nu_1(\Phi)$	$3\sqrt{2}\nu_1(\Phi) \leqslant \nu_{16}(\Phi') \leqslant \frac{31\sqrt{3}}{3}\nu_1(\Phi)$

Table 2: Range of frequencies, clamped end beam with clamped end section.

3.3. First conclusions. The lower estimate of each frequency often ties itself to a fixed value for a small set of eigenvalues. This anchorage of the lower bound highlights the reorganization of the modal shapes of each degree of freedom, the passage to the higher bound endorses the presence of a new modal shape. Three sinusoidal functions correspond to the first eigenvalue in the first modal function of the simple model of beam $v(\zeta, t)$, they vibrate consistently; the shape of the functions may be still the same but they might vibrate in an unconsistent pattern in the second and third frequencies. The fourth frequency emphasizes the passage to the second modal shape, and so on, see Tables 1 and 2. The lower bound jumping every 4 limited frequencies indicates precisely the modal reorganization of the shape functions: given the shape of the axial deformation v, the resizing parameters can oscillate consistently or unconsistently with v and between them.

3.4. Bending-Reshaping of transversal section. The same procedure shown for the coupling extension-reduction of the area of transversal section is used for the problem of bending-reshaping of transversal section; the estimates are unfortunately not accurate at the maximal generality in this case. The geometry of the problem couples all kinematic descriptors of the problem: the two angles of bending with the two spatial derivatives of displacement of the axes (shears), joined with the four parameters of distortion of the section. The dynamics is represented via a hyperbolic linear partial differential equations system of 8 unknown functions in this case. A simplification of the problem is obtained relinquishing the maximal generality, restricting the analyses to a beam with transversal axes of anisotropy coincident with that ones of inertia of transversal section and limiting the study of the plane motion of the beam (pure and right bending). The problem with 8 unknowns splits as the product of two uncoupled sub-problems, each one with 4 unknown functions. We compare the operator

$$\Phi' = \mathbf{A}\frac{\partial^2}{\partial\zeta^2} + \mathbf{B}\frac{\partial}{\partial\zeta} + \mathbf{C} - \mathbf{M}\frac{\partial^2}{\partial t^2}$$

and the operator

$$\Phi = EJ_{aa}\frac{\partial^4}{\partial\zeta^4} - \varrho A\frac{\partial^2}{\partial t^2}$$

which expresses the flexural vibrations of the linear Euler's beam. The explicit expressions of the matrices **A**, **B**, **C**, and **M** can be found in [12].

Lemma 3.3. *Let $\Phi'\colon H^{2,2}(M') \to H^{2,2}(M')$ and $\Phi\colon H^{2,4}(M) \to H^{2,4}(M)$ be the above defined operators and let $\Pi\colon H^{2,2}(M') \to H^{2,4}(M)$ be the projection on the first coordinate of the first space and $\varpi\colon L^2(\Phi') \to L^2(\Phi)$ be a map between the norms of the two operators. Then for every $\mathbf{v} \in H^{2,2,}(M', \mathbb{R}^4)$ and $v \in H^{2,4}(M, \mathbb{R})$ we have*

(1) *the conservation of the L^2-norm, i.e.*

$$\|\varpi(\mathbf{v})\|^2_{L^2} = \|v\|^2_{L^2},$$

(2) *the validity of the Kato's inequality*

$$\langle\langle \Phi(\varpi f), \varpi f\rangle\rangle_{L^2(M')} \leqslant \langle\langle \Phi'(f), f\rangle\rangle_{L^2(M)}$$

for every $f \in L^2(M')$.

P r o o f of (1). The equality follows by definitions:

$$\|\varpi(\mathbf{v})\|^2_{L^2} = \left(\int_{\eta'(M')} \varpi(\mathbf{v}) \cdot \varpi(\mathbf{v})\right)^2 = \left(\int_{\eta(M)} v^2\right)^2 = \|v\|^2_{L^2}.$$

P r o o f of (2). Let $\mathcal{Q}_{\Phi'}(\varpi(\mathbf{v}))$ and $\mathcal{Q}_{\Phi}(v)$ be the quadratic forms generated by the two operators; Kato's inequality requires that $\mathcal{Q}_{\Phi'}(\varpi(\mathbf{v})) \leqslant \mathcal{Q}_{\Phi}(v)$. We have

$$\mathcal{Q}_{\Phi'}(\varpi(\mathbf{v})) = \int_{\eta'(M')} \left(AG_a \frac{\partial^2 v}{\partial \zeta^2} - \varrho A \frac{\partial^2 v}{\partial t^2}\right) \text{ and } \mathcal{Q}_{\Phi'}(v) = \int_{\eta(M)} \left(EJ_{aa} \frac{\partial^4 v}{\partial \zeta^4} - \varrho A \frac{\partial^2 v}{\partial t^2}\right).$$

Neglecting the inertial terms, equal for both the operators, we prove that

$$\int_0^L \frac{\partial^2 v}{\partial \zeta^2} \leqslant \frac{EJ_{aa}}{\varrho A} \int_0^L \frac{\partial^4 v}{\partial \zeta^4}.$$

Integrating both sides of the previous inequality by parts and performing some algebraic manipulations, we get

$$\left[\inf_{\zeta \in [0,L]} \frac{\partial v}{\partial \zeta}\right]^2 + \frac{EJ_{aa}}{\varrho A}\left[\inf_{\zeta \in [0,L]} \frac{\partial^2 v}{\partial \zeta^2}\right]^2 \leqslant \left\|\frac{\partial v}{\partial \zeta}\right\|^2_{L^2} + \frac{EJ_{aa}}{\varrho A}\left\|\frac{\partial^2 v}{\partial \zeta^2}\right\|^2_{L^2},$$

proving Katos's inequality. □

The upper bound of each eigenvalue is still done using the Rayleigh ratio. Taking

$$\begin{pmatrix} v \\ \varphi \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} l \\ 1 \\ m \\ n \end{pmatrix} \sin \frac{N\pi\zeta}{L}$$

for a simply posed beam with extreme sections clamped in their own plane, the Rayleigh ratio has a very complicated shape involving all geometric and inertial terms of the beam. Without going into details, it is the ratio of two second order polynomials as in the case of extension-resizing. It is clear that an analytical solution of this minimum problem presents big difficulties and, supposing that it exists, the form of the solution may be so complicated that it cannot afford an easy tool of discussion. Some simplification is possible and necessary for these reasons. We do follow the same way used for the previous problem, still keeping $\min\{G_a, G_b\} \simeq E/3$.

A long but straightforward computation, involving the passage to ellipsoidal coordinates and pushing the radius to infinity, yields the upper bound of each free frequency

$$\nu_N(\Phi') \leqslant 2N\varsigma\nu_1(\Phi),$$

$\varsigma = AL^2/(\pi J)$ being the beam *slenderness*. Thanks to Lemma 3.3, we are able to use Theorem 2.4 to get lower estimates of the eigenvalues of the operator Φ'. The results are reported in Table 3.

$\nu_1(\Phi') \leqslant 2\varsigma\nu_1(\Phi)$	$\sqrt{\frac{483}{200}}\nu_1(\Phi) \leqslant \nu_{14}(\Phi') \leqslant 28\varsigma\nu_1(\Phi)$
$\nu_2(\Phi') \leqslant 4\varsigma\nu_1(\Phi)$	$\sqrt{\frac{1091}{200}}\nu_1(\Phi) \leqslant \nu_{15}(\Phi') \leqslant 30\varsigma\nu_1(\Phi)$
$\nu_3(\Phi') \leqslant 6\varsigma\nu_1(\Phi)$	$\sqrt{\frac{1091}{200}}\nu_1(\Phi) \leqslant \nu_{16}(\Phi') \leqslant 32\varsigma\nu_1(\Phi)$
$\nu_4(\Phi') \leqslant 8\varsigma\nu_1(\Phi)$	$\sqrt{\frac{1091}{200}}\nu_1(\Phi) \leqslant \nu_{17}(\Phi') \leqslant 34\varsigma\nu_1(\Phi)$
$\sqrt{\frac{203}{200}}\nu_1(\Phi) \leqslant \nu_5(\Phi') \leqslant 10\varsigma\nu_1(\Phi)$	$\sqrt{\frac{1091}{200}}\nu_1(\Phi) \leqslant \nu_{18}(\Phi') \leqslant 36\varsigma\nu_1(\Phi)$
$\sqrt{\frac{213}{200}}\nu_1(\Phi) \leqslant \nu_6(\Phi') \leqslant 12\varsigma\nu_1(\Phi)$	$\sqrt{\frac{1091}{200}}\nu_1(\Phi) \leqslant \nu_{19}(\Phi') \leqslant 38\varsigma\nu_1(\Phi)$
$\sqrt{\frac{223}{200}}\nu_1(\Phi) \leqslant \nu_7(\Phi') \leqslant 14\varsigma\nu_1(\Phi)$	$\sqrt{\frac{1261}{120}}\nu_1(\Phi) \leqslant \nu_{20}(\Phi') \leqslant 40\varsigma\nu_1(\Phi)$
$\sqrt{\frac{451}{400}}\nu_1(\Phi) \leqslant \nu_8(\Phi') \leqslant 16\varsigma\nu_1(\Phi)$	$\sqrt{\frac{1261}{120}}\nu_1(\Phi) \leqslant \nu_{21}(\Phi') \leqslant 42\varsigma\nu_1(\Phi)$
$\sqrt{\frac{2067}{1800}}\nu_1(\Phi) \leqslant \nu_9(\Phi') \leqslant 18\varsigma\nu_1(\Phi)$	$\sqrt{\frac{1261}{120}}\nu_1(\Phi) \leqslant \nu_{22}(\Phi') \leqslant 44\varsigma\nu_1(\Phi)$
$\sqrt{\frac{483}{200}}\nu_1(\Phi) \leqslant \nu_{10}(\Phi') \leqslant 20\varsigma\nu_1(\Phi)$	$\sqrt{\frac{1261}{120}}\nu_1(\Phi) \leqslant \nu_{23}(\Phi') \leqslant 46\varsigma\nu_1(\Phi)$
$\sqrt{\frac{483}{200}}\nu_1(\Phi) \leqslant \nu_{11}(\Phi') \leqslant 22\varsigma\nu_1(\Phi)$	$\sqrt{\frac{1261}{120}}\nu_1(\Phi) \leqslant \nu_{24}(\Phi') \leqslant 48\varsigma\nu_1(\Phi)$
$\sqrt{\frac{483}{200}}\nu_1(\Phi) \leqslant \nu_{12}(\Phi') \leqslant 24\varsigma\nu_1(\Phi)$	$\sqrt{\frac{34479}{200}}\nu_1(\Phi) \leqslant \nu_{25}(\Phi') \leqslant 50\varsigma\nu_1(\Phi)$
$\sqrt{\frac{483}{200}}\nu_1(\Phi) \leqslant \nu_{13}(\Phi') \leqslant 26\varsigma\nu_1(\Phi)$	$\sqrt{\frac{34479}{200}}\nu_1(\Phi) \leqslant \nu_{26}(\Phi') \leqslant 52\varsigma\nu_1(\Phi)$

Table 3: Range of frequencies, simple end beam with clamped end sections.

The computation of vibration frequencies is approximate for a clamped and beam, see [13] for details. It is possible to put

$$\lambda_2(\Phi) \approx 6.267\lambda_1(\Phi), \quad \lambda_3(\Phi) \approx 17.548\lambda_1(\Phi),$$
$$\lambda_4(\Phi) \approx 34.399\lambda_1(\Phi), \quad \lambda_5(\Phi) \approx 56.849\lambda_1(\Phi)$$

and

$$\lambda_N \approx \sqrt{\frac{EJ}{\varrho A}}\frac{\pi^2(2N-1)^2}{4L^2}$$

if $N > 5$. With the same notations as in the previous case we get the bounds of the first 26 free frequencies of a beam with the initial base clamped in such a way to impede its whole motion:

$\nu_1(\Phi') \leqslant \frac{1}{2}\varsigma\nu_1(\Phi)$	$2.045\nu_1(\Phi) \leqslant \nu_{14}(\Phi') \leqslant \frac{27}{2}\varsigma\nu_1(\Phi)$
$\nu_2(\Phi') \leqslant \frac{3}{2}\varsigma\nu_1(\Phi)$	$3.115\nu_1(\Phi) \leqslant \nu_{15}(\Phi') \leqslant \frac{29}{2}\varsigma\nu_1(\Phi)$
$\nu_3(\Phi') \leqslant \frac{5}{2}\varsigma\nu_1(\Phi)$	$3.115\nu_1(\Phi) \leqslant \nu_{16}(\Phi') \leqslant \frac{31}{2}\varsigma\nu_1(\Phi)$
$\nu_4(\Phi') \leqslant \frac{7}{2}\varsigma\nu_1(\Phi)$	$3.115\nu_1(\Phi) \leqslant \nu_{17}(\Phi') \leqslant \frac{33}{2}\varsigma\nu_1(\Phi)$
$1.238\nu_1(\Phi) \leqslant \nu_5(\Phi') \leqslant \frac{9}{2}\varsigma\nu_1(\Phi)$	$3.115\nu_1(\Phi) \leqslant \nu_{18}(\Phi') \leqslant \frac{35}{2}\varsigma\nu_1(\Phi)$
$1.238\nu_1(\Phi) \leqslant \nu_6(\Phi') \leqslant \frac{11}{2}\varsigma\nu_1(\Phi)$	$3.115\nu_1(\Phi) \leqslant \nu_{19}(\Phi') \leqslant \frac{37}{2}\varsigma\nu_1(\Phi)$
$1.238\nu_1(\Phi) \leqslant \nu_7(\Phi') \leqslant \frac{13}{2}\varsigma\nu_1(\Phi)$	$4.147\nu_1(\Phi) \leqslant \nu_{20}(\Phi') \leqslant \frac{39}{2}\varsigma\nu_1(\Phi)$
$1.238\nu_1(\Phi) \leqslant \nu_8(\Phi') \leqslant \frac{15}{2}\varsigma\nu_1(\Phi)$	$4.147\nu_1(\Phi) \leqslant \nu_{21}(\Phi') \leqslant \frac{41}{2}\varsigma\nu_1(\Phi)$
$1.238\nu_1(\Phi) \leqslant \nu_9(\Phi') \leqslant \frac{17}{2}\varsigma\nu_1(\Phi)$	$4.147\nu_1(\Phi) \leqslant \nu_{22}(\Phi') \leqslant \frac{43}{2}\varsigma\nu_1(\Phi)$
$2.045\nu_1(\Phi) \leqslant \nu_{10}(\Phi') \leqslant \frac{19}{2}\varsigma\nu_1(\Phi)$	$4.147\nu_1(\Phi) \leqslant \nu_{23}(\Phi') \leqslant \frac{45}{2}\varsigma\nu_1(\Phi)$
$2.045\nu_1(\Phi) \leqslant \nu_{11}(\Phi') \leqslant \frac{21}{2}\varsigma\nu_1(\Phi)$	$4.147\nu_1(\Phi) \leqslant \nu_{24}(\Phi') \leqslant \frac{47}{2}\varsigma\nu_1(\Phi)$
$2.045\nu_1(\Phi) \leqslant \nu_{12}(\Phi') \leqslant \frac{23}{2}\varsigma\nu_1(\Phi)$	$6.358\nu_1(\Phi) \leqslant \nu_{25}(\Phi') \leqslant \frac{49}{2}\varsigma\nu_1(\Phi)$
$2.045\nu_1(\Phi) \leqslant \nu_{13}(\Phi') \leqslant \frac{25}{2}\varsigma\nu_1(\Phi)$	$6.358\nu_1(\Phi) \leqslant \nu_{26}(\Phi') \leqslant \frac{51}{2}\varsigma\nu_1(\Phi)$

Table 4: Range of frequencies, simple end beam with clamped end section.

3.5. Second conclusions. We get similar conclusions on the distribution of the lower bound of the free vibration frequencies for the beam subject to bending-reshaping to the case of extension-resizing. It is possible to see in Figure 2 that the lower bound jumps to the next level each five frequencies, see also Tables 3 and 4. We can also interpret this case thinking to a re-organization of the reshaping in consistent or, conversely, unconsistent way to the main bending-share modal shape:

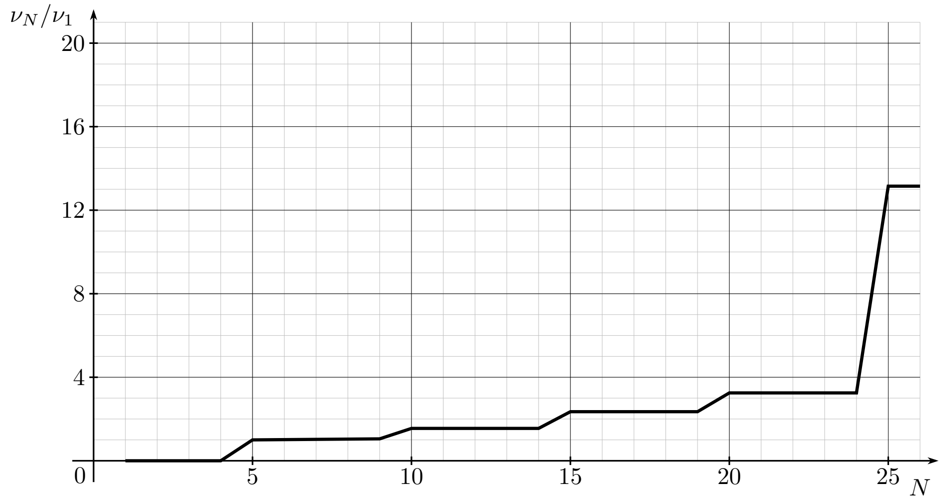

Figure 2. The frequencies lower bounds for the problem bending-reshape.

3.6. Torsion-warping. The mathematical structure of the problem of torsion-warping allows an explicit determination of the beam vibration frequencies as functions of geometric and mechanic characteristics of the beam. The Lagrangian variables are the angle of torsion θ and the amplitude of the warping ω. The equations governing the problem are:

$$\begin{cases} \mathbf{J}^* \cdot \mathbf{G}\dfrac{\partial^2\theta}{\partial\zeta^2} + \mathbf{G}\cdot\mathbf{L}_1\dfrac{\partial\omega}{\partial\zeta} = \varrho\,\mathrm{tr}\,\mathbf{J}^*\dfrac{\partial^2\theta}{\partial t^2}, \\ ED_2\dfrac{\partial^2\omega}{\partial\zeta^2} - \mathbf{G}\cdot\mathbf{L}_1\dfrac{\partial\theta}{\partial\zeta} + \mathbf{G}\cdot\mathbf{L}_2\omega = \varrho D_2\dfrac{\partial^2\omega}{\partial t^2}, \end{cases}$$

or in a compact form:

$$\mathtt{A}\partial^2_{\zeta\zeta}\mathbf{v} + \mathtt{B}\partial_\zeta\mathbf{v} + \mathtt{C}\mathbf{v} = \mathtt{M}\partial^2_{tt}\mathbf{v} \tag{3.11}$$

with $\mathbf{v} = (\theta, \omega)^\top$ and

$$\mathtt{A} = \begin{pmatrix} \mathbf{J}^*\cdot\mathbf{G} & 0 \\ 0 & ED_2 \end{pmatrix}, \quad \mathtt{B} = \begin{pmatrix} 0 & \mathbf{G}\cdot\mathbf{L}_1 \\ -\mathbf{G}\cdot\mathbf{L}_1 & 0 \end{pmatrix},$$
$$\mathtt{C} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{G}\cdot\mathbf{L}_2 \end{pmatrix}, \quad \mathtt{M} = \begin{pmatrix} \varrho\,\mathrm{tr}\,\mathbf{J}^* & 0 \\ 0 & \varrho D_2 \end{pmatrix},$$

where $\mathbf{J}^* = \int_{\mathcal{S}}(\mathbf{e}\times\mathbf{x})\otimes(\mathbf{e}\times\mathbf{x})$ is the Euler inertia tensor ($\mathbf{e}$ is the unit direction vector of the beam whose section $\mathcal{S}$ has a coordinate $\mathbf{x}$), $\mathbf{G}$ is a part of the elastic tensor with kernel $\mathbf{e}$ and $\mathbf{L}_i$, $i = 1, 2$, are inertia tensors related to torque and warping as well as $D_2 = \int_{\mathcal{S}} \Phi^2$ is the Vlasov constant related to the warping shape function $\Phi\colon\ \mathcal{S} \to \mathbb{R}$ (see [12] for complete description). As usual in the theory of linear P.D.E., we search the uncoupled solutions $\mathbf{v}(\zeta, t) = \mathbf{y}(\zeta)\gamma(t)$. System (3.11) becomes

$$(\mathtt{A}\partial^2_{\zeta\zeta}\mathbf{y} + \mathtt{B}\partial_\zeta\mathbf{y} + \mathtt{C}\mathbf{y})\gamma = \mathtt{M}\partial^2_{tt}\gamma\mathbf{y}.$$

A left-multiplication for $\mathbf{y}^\top$ and a division for $\mathbf{y}^\top\mathtt{M}\mathbf{y}$ allow to write the classical eigenvalues-eigenfuntcions problem as

$$\frac{\mathbf{y}^\top(\mathtt{A}\partial^2_{\zeta\zeta}\mathbf{y} + \mathtt{B}\partial_\zeta\mathbf{y} + \mathtt{C}\mathbf{y})}{\mathbf{y}^\top\mathtt{M}\mathbf{y}} = \frac{\partial^2\gamma}{\partial t^2} = -\lambda^2, \tag{3.12}$$

where λ is a real unknown scalar constant. This assumption splits equation (3.12) into another ordinary differential decoupled equations, the first one with unknown $\gamma(t)$ and the second one with unknown $\mathbf{y}(\zeta)$:

$$\begin{cases} \partial^2_{tt}\gamma + \lambda\gamma = 0, \\ \mathbf{y}^\top[\mathtt{A}\partial^2_{\zeta\zeta}\mathbf{y} + \mathtt{B}\partial_\zeta\mathbf{y} + (\mathtt{C} + \lambda^2\mathtt{M})\mathbf{y}] = 0, \end{cases} \tag{3.13}$$

they can be solved separately. We search nontrivial solutions of (3.13.2) of the form $\mathbf{y} = \mathbf{y}_0\exp(\mu\zeta)$ getting

$$[\mathtt{A}\mu^2 + \mathtt{B}\mu + (\mathtt{C} + \lambda^2\mathtt{M})]\mathbf{y}_0 = \mathbf{0};$$

they exist if and only if $\det[\mathtt{A}\mu^2 + \mathtt{B}\mu + (\mathtt{C} + \lambda^2\mathtt{M})] = 0$. We obtain the biquadratic characteristic equation $a\mu^4 + 2b\mu^2 + c = 0$ being $a = ED_2\mathbf{G}\cdot\mathbf{J}^*$, $2b = \mathbf{G}\cdot\mathbf{J}^*\mathbf{G}\cdot\mathbf{L}_2 + (ED_2\mathrm{tr}\,\mathbf{J}^* + \mathbf{J}^*\cdot\mathbf{G}D_1)\lambda\varrho + (\mathbf{G}\cdot\mathbf{L}_1)^2$ and $c = \lambda\varrho\mathrm{tr}\,\mathbf{J}^*(\mathbf{G}\cdot\mathbf{L}_2 + \lambda\varrho D_2)$. The characteristic equation has two negative solutions in μ^2 for Descartes's rule of signs; they correspond to four complex roots forming two purely imaginary couples. The solutions are given by

$$\mathbf{x}(\zeta) = \mathbf{x}_{01}\cos(\mu^+\zeta) + \mathbf{x}_{02}\sin(\mu^+\zeta) + \mathbf{x}_{03}\cos(\mu^-\zeta) + \mathbf{x}_{04}\sin(\mu^-\zeta),$$

where $\mu^\pm = \sqrt{(b \pm \sqrt{b^2 - ac})/a}$. We analyze the condition under which both bases are clamped to impede torsional rotation and warping, i.e. $\theta(0) = \theta(L) = 0$ and $\omega(0) = \omega(L) = 0$. The unique plausible solution, consistent with boundary conditions and with the positiveness of frequencies ν, is given by $\mathbf{x}_{0i} = 0$, $i = 1, 2, 3$, and

$\mathbf{x}_{04}\sin(\mu^- L) = 0$, the nontrivial solution forces $\mu^- L = N\pi$, or equally $(\mu^-)^2 = N^2\pi^2/L^2$. We get the equation of frequencies going backward starting from the last equality and coming up to (3.13.2):

$$
\begin{aligned}
(3.14)\qquad \varrho^2 D_2 \operatorname{tr}\mathbf{J}^*\lambda^4 - \varrho\Big[\frac{N^2\pi^2}{L^2}D_2(E\operatorname{tr}\mathbf{J}^* + \mathbf{J}^*\cdot\mathbf{G}) - \mathbf{G}\cdot\mathbf{L}_2\operatorname{tr}\mathbf{J}^*\Big]\lambda^2 \\
+ \frac{N^2\pi^2}{L^2}\Big[\frac{N^2\pi^2}{L^2}ED_2\mathbf{J}^*\cdot\mathbf{G} - \mathbf{G}\cdot\mathbf{L}_1(\mathbf{G}\cdot\mathbf{L}_2 + \mathbf{G}\cdot\mathbf{J}^*)\Big] = 0.
\end{aligned}
$$

This equation has always two positive roots in λ^2 for each natural N for Descarte's rule of signs, since each term is positive for physical reasons. Moreover, each root is an increasing function of N (see e.g. [5], Th. 15 III and Th. 15 IV, p. 66). If $\overline{\lambda}_N$ is the minimal solution of (3.14) for each natural N, we get the corresponding frequency extracting the square root, as usual: $\nu_N = \sqrt{\overline{\lambda}_N}$.

4. Concluding remarks

The proposed procedure, even if applied to a pure mathematical model of beam, allows the computation of the lower and upper bounds for each frequency of a vibrating system for which a complete spectral analysis is not possible, because the complexity of the operator describing its free dynamics prevents it. It is only necessary to find a "reduced" model which refers to the spectra comparison. It is possible to forecast lower and upper bounds of each frequency and the behavior of the complete system in this manner. These predictions are very important for every situation for which the spectral properties of excitation are known. The seismic and wind-excitations are known in terms of spectra, got with several experimental measurements, rather than in terms of time history. It is possible, with the proposed method, to see around which frequency the excitation lies and foresee any eventual dangerous resonance phenomena. Estimates of vibrations for linear elastic membranes have been obtained, for the same reason, by the author in (2018) [9]. This paper shows how to apply some theoretical result as the extension to Riemmanian manifold with non empty boundary several estimates existent in the literature for Riemmanian manifolds with empty boundary got from the author in [10] and [11].

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