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UNIQUE SOLVABILITY OF FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATION ON THE BASIS OF VALLÉE-POUSSIN THEOREM

Satyam Narayan Srivastava, Alexander Domoshnitsky, Seshadev Padhi, and Vladimir Raichik

Abstract. We propose explicit tests of unique solvability of two-point and focal boundary value problems for fractional functional differential equations with Riemann-Liouville derivative.

1. Introduction

In this paper we consider the fractional functional differential equation

\[(D^\alpha_{0+}x)(t) + \sum_{i=0}^{m} (T_i x^{(i)})(t) = f(t), \quad t \in [0, 1], \quad m \leq n - 2, \quad n \geq 2,\]

where \(D^\alpha_{0+}\) is the Riemann-Liouville fractional derivative of the order \(n - 1 < \alpha \leq n\) (see [11], [14]), \(n\) is integer, the operators \(T_i: C \rightarrow L_\infty\) are linear continuous operators acting from the space of the continuous functions \(C\) to the space of essentially bounded functions \(L_\infty\), \(i = 0, \ldots, m\), and \(f \in L_\infty\).

We consider also the auxiliary equation

\[(D^\alpha_{0+}x)(t) + \sum_{i=0}^{m} (|T_i|x^{(i)})(t) = f(t), \quad t \in [0, 1], \quad m \leq n - 2, \quad n \geq 2,\]

where the positive operator \(|T_i|\) is such that the following inequalities hold:

\[-(|T_i|1)(t) \leq (T_i1)(t) \leq (|T_i|1)(t), \quad t \in [0, 1].\]

Of course, it will be clear below, that we are interested in the operators \(|T_i|\) with the minimal norms in the space of continuous functions \(C\).

The operators \(T_i: C \rightarrow L_\infty\) and \(|T_i|: C \rightarrow L_\infty\) can be, for example, of the following forms:

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\
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\
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\
\text{DOI: 10.5817/AM2023-1-117}\]
1) Operators with deviations

\[
(T_i x^{(i)})(t) = \sum_{j=0}^{m_i} q_{ij}(t)x^{(i)}(t - \tau_{ij}(t)),
\]

\[
(|T_i| x^{(i)})(t) = \sum_{j=0}^{m_i} |q_{ij}(t)|x^{(i)}(t - \tau_{ij}(t)),
\]

where \(\tau_{ij} : [0, 1] \rightarrow \mathbb{R}\), \(q_{ij} : [0, 1] \rightarrow \mathbb{R}\), are measurable bounded functions, \(\mathbb{R} = (-\infty, +\infty)\). To complete the description of these operators, we have to define what has to be substituted into (1.4) instead of \(x^{(i)}(t - \tau_{ij}(t))\) in the case of \(t - \tau_{ij}(t) \notin [0, 1]\). Let us assume that

\[
x^{(i)}(\xi) = 0 \quad \text{for} \quad \xi \notin [0, 1], \quad i = 0, \ldots, m,
\]

that allows us to preserve the \(n\)-dimensional fundamental system for the homogeneous equation

\[
(D_{0+}^\alpha x)(t) + \sum_{j=0}^{m_i} q_{ij}(t)x^{(i)}(t - \tau_{ij}(t)) = 0.
\]

2) Integral operators

\[
(T_i x^{(i)})(t) = \int_0^1 K_i(t, s)x^{(i)}(s)\, ds,
\]

\[
(|T_i| x^{(i)})(t) = \int_0^1 |K_i(t, s)|x^{(i)}(s)\, ds,
\]

under the standard assumptions on the kernels \(K_i(t, s)\) implementing that \(T_i : C \rightarrow L_{\infty}\), for example, \(K_i(t, s)\) is a continuous function \([0, 1] \times [0, 1] \rightarrow \mathbb{R}\) (see, [12]).

3) Linear combinations and superpositions of the deviations and integral operators, for example, the operators

\[
(T_i x^{(i)})(t) = \int_0^1 \sum_{j=1}^{m_i} K_{ij}(t, s)x^{(i)}(s - \tau_{ij}(s))\, ds.
\]

\[
(|T_i| x^{(i)})(t) = \int_0^1 \sum_{j=1}^{m_i} |K_{ij}(t, s)|x^{(i)}(s - \tau_{ij}(s))\, ds.
\]

We consider the boundary value problem consisting of equation (1.1) and the boundary conditions

\[
x^{(i)}(0) = 0 \quad \text{for} \quad i = 0, 1, \ldots, n - 2, \quad x^{(k)}(1) = 0,
\]
where \( k \) is an integer which is between 0 and \( n - 1 \). In the case of \( k = 0 \), we have the classical two-point \((n - 1, 1)\)-problem. In the case of \( k \leq n - 1 \), we have the sort of focal problems. We assume below that \( m \leq k \).

We consider equation (1.1) in the space \( D \) of functions \( x: [0, 1] \rightarrow \mathbb{R} \) such that \( x^{(n-1)} \) is absolutely continuous on every interval \([\varepsilon, 1]\), where \( \varepsilon > 0 \) and summable on \([0, 1]\) and \( x^{(n)} \) such that \( tx^{(n)} \) is summable. The norm in the space \( D \) define as

\[
\|x\|_D = \sum_{i=0}^{n-2} \max_{0 \leq t \leq 1} |x^{(i)}(t)| + \int_0^1 \left| x^{(n-1)}(t) \right| dt + \int_0^1 t \left| x^{(n)}(t) \right| dt.
\]

Considering this space \( D \) looks naturally when fractional equations with the Riemann-Liouville derivatives and the boundary conditions (1.9) are considered. We say that \( x \in D \) is a solution of (1.1) if it satisfies this equation for almost every \( t \in [0, 1] \).

If the problem consisting of the homogeneous equation

\[
(D_0^\alpha x)(t) + m \sum_{i=0}^m (T_i x^{(i)})(t) = 0
\]

and condition (1.9) has only the trivial solution, then problem (1.1), (1.9) has a unique solution which can be represented in the form

\[
(1.10) \quad x(t) = \int_0^1 G(t, s)f(s)ds.
\]

For applications of fractional differential equations in various field of science and engineering one can refer the classical books [11, 14].

The main reason for the study of fractional functional differential equations could be, in our opinion, around the following idea for the study of systems of fractional equations. Consider a boundary value problem consisting, for example, of a system of two “ordinary fractional differential equations”. For its analysis, we can use the integral representations of solutions of the first equation and obtain \( x_1(t) \) through \( x_2(t) \). Then we substitute this representation instead of \( x_1(t) \) into the second equation and obtain a scalar fractional functional differential equation. In the simplest case of a system of “ordinary” fractional equations, the equation, we get, includes the integral operator of type 2). If we start with a system of delay fractional differential equations, the equation, we get after the substitution into the second equation, is a fractional functional differential equation that includes the superpositions of deviation and integral operators. Thus, operators of type 3) appear. Examples of such systems can be found in [7, 8, 9].

Positivity of solutions is one of the most important properties in applications (see, for example, the book by Henderson and Luca [7]). Concerning problem (1.4), (1.9), in the case of so called ordinary linear equations, (i.e. \( \tau_{ij}(t) \equiv 0, t \in [0, 1], j = 0, \ldots, m_i, i = 1, \ldots, m \) in (1.4)) and its nonlinear generalizations, we can note the following papers [3, 8, 9, 10, 13, 15].

One of the motivations for our research is Lyapunov’s inequalities for fractional differential equations which have been presented in Chapter 5 of the recent book by Agarwal, Bohner, and Ozbekler [1]. Note the following assertion was presented for the first time in [5]. Actually, the result in [5] is more general than Theorem 1.1 as the solution need not be assumed to be different from zero on \((0, 1)\).
\[ \int_0^1 |q_0(t)| \, dt > \Gamma(\alpha)4^{\alpha-1} \]

holds.

Note that in [5], it was not assumed that \( x(t) \neq 0 \) for \( t \in (0, 1) \). For (1.11) with a constant coefficient \( q_0(t) = q_0 \), we have (1.13) in the form

\[ |q_0| \geq \Gamma(\alpha)4^{\alpha-1}. \]

Using Corollary 2.3 (one can refer [4] for proof), we get that the inequality

\[ |q_0| < \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \Gamma(\alpha + 1) \]

guarantees that the problem (1.11) has only the trivial solution. Note that the part on unique solvability coincides with the known result of [6]. Inequality (1.14) means that in the case of zeros of solution \( x(t) \) at the points 0 and 1, we obtain that

\[ |q_0(t)| \geq \frac{\alpha^{\alpha}}{(\alpha - 1)^{\alpha-1}} \Gamma(\alpha + 1) \]

since in the case of the coefficient \( q_0 \) satisfying inequality (1.11) we exclude the existence of zero at the point 1, i.e. \( x(1) \neq 0 \). Let us compare (1.13) and (1.15), computing the right-hand sides in them, we have values in Table 1.

<table>
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<th>In inequality (1.15)</th>
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<td>1.7724538</td>
<td>3.45372767</td>
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</table>

Table 1 demonstrates the advances of our results if we compare the results of [1, 5] and ours.
2. Main Results

Lemma 2.1. Using the technique of [13], one can obtain the uniqueness of solution to the problem

\[\begin{align*}
D_{0+}^\alpha x(t) &= f(t), \\
x(0) &= x'(0) = \ldots = x^{(n-2)}(0) = 0, \\
x^{(k)}(1) &= 0,
\end{align*}\]

where \(k\) is an integer number which is between 0 and \(n-1\), in the form

\[x(t) = \int_0^1 G_k(t, s)f(s) \, ds,
\]

where \(G_k(t, s)\) is Green's function of problem (2.1) defined by

\[G_k(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (t-s)^{\alpha-1}-t^{\alpha-1}(1-s)^{\alpha-1-k}, & 0 \leq s \leq t \leq 1, \\ -t^{\alpha-1}(1-s)^{\alpha-1-k}, & 0 \leq t < s \leq 1 \end{cases},
\]

and its \(j\)-th derivative is defined by

\[\frac{\partial^j}{\partial t^j} G_k(t, s) = \frac{(\alpha-1)(\alpha-2) \cdots (\alpha-j)}{\Gamma(\alpha)} \begin{cases} (t-s)^{\alpha-j-1}-t^{\alpha-j-1}(1-s)^{\alpha-1-k}, & 0 \leq s \leq t \leq 1, \\ -t^{\alpha-j-1}(1-s)^{\alpha-1-k}, & 0 \leq t < s \leq 1 \end{cases}.
\]

Let us define the operator \(K: L_\infty \to L_\infty\) and \(|K|: L_\infty \to L_\infty\) by the equalities

\[\begin{align*}
(Kz)(t) &= -\sum_{i=0}^m T_i \left[ \int_0^1 \frac{\partial^i}{\partial t^i} G_k(t, s) z(s) \, ds \right] (t) = f(t), \\
(|K|z)(t) &= -\sum_{i=0}^m |T_i| \left[ \int_0^1 \frac{\partial^i}{\partial t^i} G_k(t, s) z(s) \, ds \right] (t) = f(t).
\end{align*}\]

We use the notation \(T_i[\gamma(t)]\), \(|T_i[\gamma(t)]|\) meaning that the operator \(T_i\) and \(|T_i|\) acts on the continuous function \(\gamma(t)\), i.e. \(T_i[\gamma(t)] = (T_i\gamma)(t), |T_i[\gamma(t)]| = (|T_i|\gamma)(t)\).

Theorem 2.2. Assume that there exist a function \(v \in D\) such that \(v(t) > 0, v'(t) > 0, \ldots, v^{(k)}(t) > 0\) for \(t \in (0, 1)\), \(v(0) = v'(0) = \ldots = v^{(n-2)}(0) = 0\) and

\[\begin{align*}
(D_{0+}^\alpha v)(t) + \sum_{i=0}^m |T_i| |v^{(i)}(t)| &\equiv \psi(t) \leq -\varepsilon < 0 \quad \text{for} \quad t \in (0, 1);
\end{align*}\]

then the problem (1.1), (1.9) is uniquely solvable for any essentially bounded \(f\) and the spectral radius of \(|K|: L_\infty \to L_\infty\) is less than one.

Proof. Consider the auxiliary problem

\[\begin{align*}
(D_{0+}^\alpha x)(t) &= z(t), \\
x^{(i)}(0) &= v^{(i)}(0), \quad x^{(k)}(1) = v^{(k)}(1), \quad i = 0, 1, \ldots, n-2.
\end{align*}\]
where \( z(t) \) is a function in \( L_\infty \) and such that there exists a positive number \( \delta \) such that \( z(t) \leq -\delta \) for \( t \in [0,1] \). It is clear that

\[
\begin{align*}
    x(t) &= \int_0^t G_k(t,s)z(s)\,ds + u_k(t), \\
    x'(t) &= \int_0^t \frac{\partial}{\partial t} G_k(t,s)z(s)\,ds + u'_k(t), \\
    x''(t) &= \int_0^t \frac{\partial^2}{\partial t^2} G_k(t,s)z(s)\,ds + u''_k(t), \\
    \vdots \\
    x^{(m)}(t) &= \int_0^t \frac{\partial^m}{\partial t^m} G_k(t,s)z(s)\,ds + u^{(m)}_k(t),
\end{align*}
\]

(2.8)

where \( u(t) \) is a solution of the homogeneous equation \( D^{\alpha}_{0+}u(t) = 0 \) satisfying the conditions \( u^{(i)}(0) = v^{(i)}(0), i = 0, \ldots, n-2, u^{(k)}(1) = v^{(k)}(1) \). Let us substitute these representations instead of \( u(t) \) and its derivatives into inequality (2.6):

\[
    z(t) + \sum_{i=0}^m T_i \left[ \int_0^1 \frac{\partial^i}{\partial t^i} G_k(t,s)z(s)\,ds \right] + \sum_{i=0}^m (T_i u^i)(t)) = \psi(t).
\]

(2.9)

It is clear that \( |T_i|: C \to L_\infty \) are positive operators for \( i = 0, 1, \ldots, m \), and this imply that the operator \( |K|: L_\infty \to L_\infty \) defined by equality (2.5) is positive. Thus, we have the equation

\[
    z(t) - (|K| z)(t) = \Psi(t), \quad t \in [0,1],
\]

(2.10)

where

\[
    \Psi(t) \equiv \psi(t) - \sum_{i=0}^m (|T_i| u^i)(t).
\]

(2.11)

It is clear that \( u^{(i)}(t) > 0 \) for \( t \in (0,1] \). This implies that \( \Psi(t) \leq -\varepsilon < 0 \). The function \( w(t) = -z(t) \) satisfies the inequality \( w(t) - (|K| w)(t) = -\Psi(t) > 0 \) for \( t \in [0,1] \). From equality (2.10), according to [12, Theorem 5.3 on page 76] it follows that \( \rho(|K|) < 1 \). This completes the proof of the theorem. \( \square \)

**Corollary 2.3.** If \( n-1 < \alpha \leq n \) and the following inequality is fulfilled

\[
    \left| T_0 \right| t^{\alpha-1} \left( \frac{\alpha}{\alpha-k} - t \right) + \sum_{i=1}^m \alpha(\alpha-1) \cdots (\alpha-i+1) |T_i| t^{\alpha-i-1} \left( \frac{\alpha-i}{\alpha-k} - t \right) < \Gamma(\alpha+1), \quad t \in [0,1],
\]

then problem (1.1), (1.9) is uniquely solvable for any \( f \in L_\infty \).

**Proof.** The proof follows from Corollary 4 of [3]. \( \square \)

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References


