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GENERALIZATION OF THE S-NOETHERIAN CONCEPT

ABDELAMIR DABBABI AND ALI BENHISSI

Abstract. Let $A$ be a commutative ring and $S$ a multiplicative system of ideals. We say that $A$ is $S$-Noetherian, if for each ideal $Q$ of $A$, there exist $I \in S$ and a finitely generated ideal $F \subseteq Q$ such that $IQ \subseteq F$. In this paper, we study the transfer of this property to the polynomial ring and Nagata’s idealization.

1. Introduction

In this paper a ring means a commutative ring with unit element. Let $A$ be an integral domain with quotient field $K$. E. Hamann, E. Houston and J. Johnson in [3] defined an ideal $I$ of $A[X]$ to be almost principal, if there exist an $s \in A \setminus \{0\}$ and an $f \in I$ such that $sI \subseteq fA[X]$, and they called the polynomial ring $A[X]$ an almost principal ideal domain if each ideal of $A[X]$ that extends to a proper ideal of $K[X]$ is almost principal. In [1], D.D. Anderson and T. Dumitrescu have defined the concept of $S$-Noetherian rings as follows. Let $A$ be a ring and $S \subseteq A$ a multiplicative set. The ring $A$ is called $S$-Noetherian, if for each ideal $I$ of $A$, there exist $s \in S$ and a finitely generated ideal $F \subseteq I$ of $A$ such that $sI \subseteq F$. They have shown that if $A$ is $S$-Noetherian, then so is $A[X]$, provided $\left( \bigcap_{n=1}^{\infty} s^n A \right) \cap S \neq \emptyset$ for each $s \in S$. These results have been extended in [1], [4] and [5]. We extend this definition using an arbitrary multiplicative system of ideals.

Let $S$ be a multiplicative system of ideals of a ring $A$. We shall call $A$ to be $S$-Noetherian, if for each ideal $Q$ of $A$, there exist an ideal $I \subseteq S$ and a finitely generated ideal $F \subseteq Q$ of $A$ such that $IQ \subseteq F$. In the case when $S$ consists of principal ideals, the notions $S$-Noetherian and $S$-Noetherian are equivalent, where $S = \{ s \in A \mid sA \in S \}$. But in general we can not present a multiplicative system of ideals by a multiplicative set. In this paper, we investigate some properties of the $S$-Noetherian concept. For instance, we give a Cohen-type theorem for $S$-Noetherian rings. Also, we study the transfer of this property from $A$ to the polynomial ring $A[X]$ and Nagata idealization $A(+)^{M}$, where $M$ is an $A$-module. In fact, we show that the ring $A(+)^{M}$ is $S_1$-Noetherian if and only if the ring $A$ is $S$-Noetherian.

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and the $A$-module $M$ is $S$-finite, where $S_1 = \{ I(+)IM, \ I \in S \}$. We give examples of $S$-Noetherian rings $A$ with $S$ a multiplicative system of nonprincipal ideals of $A$.

2. MAINS RESULTS

We introduce the main concept of this paper as follows.

**Definition 2.1.** Let $A \subseteq B$ be a rings extension, $M$ an $A$-module and $S$ a multiplicative system of ideals of $A$.

1. An $A$-submodule $N$ of $M$ is said to be $S$-finite, if there exist $a_1, \cdots, a_n \in N$ and $I \in S$ such that $IN \subseteq \langle a_1, \cdots, a_n \rangle$.

2. We say that $M$ is $S$-Noetherian, if each submodule of $M$ is $S$-finite.

3. An ideal $Q$ of $B$ is called $S$-finite, if there exist $a_1, \ldots, a_n \in Q$ and $I \in S$ such that $IQ \subseteq \langle a_1, \ldots, a_n \rangle B$.

4. We say that $B$ is an $S$-Noetherian ring, if each ideal of $B$ is $S$-finite.

With the same notations of the previous definition, clearly $B$ is $S$-Noetherian if and only if it is $S'$-Noetherian, where $S' = \{ IB \mid I \in S \}$. It is clear that if $IM = 0$ for some $I \in S$, then $M$ is an $S$-Noetherian $A$-module.

Obviously a Noetherian ring $A$ is $S$-Noetherian for every multiplicative system of ideals $S$ of $A$.

**Example 2.2.** Let $A = \prod_{i=1}^{\infty} \mathbb{Z}/p^j\mathbb{Z}$ where $p$ is a prime number, $a_1, \ldots, a_n \in A$ some finite support nonzero elements (i.e, if $a_i = (a_i,j)_{j \in \mathbb{N}}$, then $a_i,j = 0$ except for a finite number of indices $j$), $I = \langle a_1, \ldots, a_n \rangle$ and $S = \{ I^n, \ n \geq 1 \}$. For each ideal $Q$ of $A$, the ideal $IQ$ has a finite cardinality. Hence $IQ \subseteq \langle IQ \rangle \subseteq Q$, thus $Q$ is $S$-finite.

So $A$ is an example of an $S$-Noetherian ring which is not Noetherian.

**Proposition 2.3.** Let $A$ be a ring, $M$ an $A$-module, $N$ a submodule of $M$ and $S$ a multiplicative system of ideals of $A$. The following assertions are equivalent:

1. The $A$-module $M$ is $S$-Noetherian.

2. The $A$-modules $N$ and $M/N$ are $S$-Noetherian.

**Proof.** (1) $\implies$ (2) Trivial.

(2) $\implies$ (1) Let $L$ be a submodule of $M$. Denote $\tilde{L} = \{ \bar{x} \in M/N \mid x \in L \}$. It is easy to check that $\tilde{L}$ is a submodule of $M/N$, then it is $S$-finite. Therefore, there exist $x_1, \ldots, x_n \in L$ and $I \in S$ such that $IL \subseteq \langle \bar{x}_1, \ldots, \bar{x}_n \rangle$.

Let $T = L \cap N$. It is clear that $T$ is a submodule of $N$, so it is $S$-finite. Hence there exist $y_1, \ldots, y_k \in T$ and $J \in S$ such that $JT \subseteq \langle y_1, \ldots, y_k \rangle$. For $x \in L$ fixed, we have $a\bar{x} \in \langle \bar{x}_1, \ldots, \bar{x}_n \rangle$ for each $a \in I$. Let $a \in I$, write $a\bar{x} = \sum_{i=1}^{n} \alpha_i \bar{x}_i$ with $\alpha_i \in A$, $i = 1, \ldots, n$. Then $ax - \sum_{i=1}^{n} \alpha_i x_i \in N \cap L = T$. Thus $J(ax - \sum_{i=1}^{n} \alpha_i x_i) \subseteq \langle y_1, \ldots, y_k \rangle$. It
Theorem 2.7. Let \( y \) with \( y \) yields that \( (JI)L \subseteq \langle y_1, \ldots, y_k, x_1, \ldots, x_n \rangle \) with \( y_1, \ldots, y_k, x_1, \ldots, x_n \in L \) and \( IJ \in S \). □

Corollary 2.4. A finite direct sum of modules is \( S \)-Noetherian if and only if so is every term. In particular, \( A^n \) is \( S \)-Noetherian for each \( n \geq 1 \) provided that \( A \) is an \( S \)-Noetherian ring.

Corollary 2.5. Let \( A \) be a ring, \( M \) an \( A \)-module and \( S \) a multiplicative system of ideals of \( A \). If \( A \) is \( S \)-Noetherian and \( M \) a finitely generated \( A \)-module, then \( M \) is an \( S \)-Noetherian \( A \)-module.

Proof. The \( A \)-module \( M \) is an epimorphic image of some \( A^n \). By Corollary 2.4, the \( A \)-module \( M \) is \( S \)-Noetherian.

Corollary 2.6. Let \( A \) be a ring, \( S \) a multiplicative system of ideals of \( A \) and \( M \) an \( S \)-finite \( A \)-module. If \( A \) is an \( S \)-Noetherian ring, so is the \( A \)-module \( M \).

Proof. There exist a finitely generated submodule \( N \) of \( M \) and \( I \in S \) such that \( IM \subseteq N \). By Corollary 2.5, \( N \) is a \( S \)-Noetherian \( A \)-module. Thus \( IM \) is an \( S \)-Noetherian \( A \)-module. Hence, the \( A \)-module \( M \) is \( S \)-Noetherian by the exact sequence \( 0 \rightarrow IM \rightarrow M \rightarrow M/IM \rightarrow 0 \). □

Theorem 2.7. Let \( A \) be a ring and \( S \) a multiplicative system of ideals of \( A \) such that for each \( I \in S \), \( \bigcap_{n=1}^{\infty} I^n \) contains some ideal of \( S \). If \( A \) is \( S \)-Noetherian, so is \( A[X] \).

Proof. Let \( L \) be an ideal of \( A[X] \) and \( L_0 \) the set of leading coefficients of polynomials of \( L \). It is easy to check that \( L_0 \) is an ideal of \( A \). Since \( A \) is \( S \)-Noetherian, there exist \( a_1, \ldots, a_n \) and \( I \in S \) such that \( IL_0 \subseteq \langle a_1, \ldots, a_n \rangle A \). For \( 1 \leq i \leq n \), let \( f_i \in L \) such that \( a_i \) is the leading coefficient of \( f_i \). We can assume that \( d = \deg(f_1) = \cdots = \deg(f_n) \) (it suffices to multiply by some \( X^{l_i} \), \( 1 \leq i \leq n \)). Let \( M = A + AX + \cdots + AX^d \). Let \( f \in L \) of degree \( r + d \). Let \( a_1, \ldots, a_r \) be arbitrary elements of \( I \). Substracting repeatedly from \( f \) suitable combinations of \( f_1, \ldots, f_n \) we get that \( a_1 \ldots a_r f \) belongs to \( \langle f_1, \ldots, f_n \rangle + L \cap M \). It follows that \( I^r f \subseteq \langle f_1, \ldots, f_n \rangle + L \cap M \), thus \( JL \subseteq \langle f_1, \ldots, f_n \rangle + L \cap M \) where \( J \) is some ideal of \( S \) contained in \( \bigcap_{k=1}^{\infty} I^k \). Since \( M \) is a finitely generated \( A \)-module, it is \( S \)-Noetherian, by Corollary 2.5. Consequently, \( L \cap M \) is \( S \)-finite. Then there exist \( g_1, \ldots, g_m \in L \cap M \) and \( J' \in S \) such that \( J'(L \cap M) \subseteq \langle g_1, \ldots, g_m \rangle A \subseteq \langle g_1, \ldots, g_m \rangle A[X] \). It yields that \( (J'J)f \subseteq \langle f_1, \ldots, f_n, g_1, \ldots, g_m \rangle A[X] \). Therefore, \( (J'J)L \subseteq \langle f_1, \ldots, f_n, g_1, \ldots, g_m \rangle \) with \( J'J \in S \) and \( f_1, \ldots, f_n, g_1, \ldots, g_m \in L \). Hence \( A[X] \) is an \( S \)-Noetherian ring. □

Corollary 2.8. Let \( A \) be a ring and \( S \) a multiplicative system of ideals of \( A \) such that for every \( I \in S \), \( \bigcap_{n=1}^{\infty} I^n \) contains some ideal of \( S \). If \( A \) is \( S \)-Noetherian, so is \( A[X_1, \ldots, X_n] \) for each \( n \geq 1 \).
Theorem 2.9. Let \( \mathcal{A} = (A_n)_{n \geq 0} \) be an increasing sequence of rings, \( A = \bigcup_{n=0}^{\infty} A_n \) and \( X \) an indeterminate over \( A \). Recall from [4] that \( \mathcal{A}[X] = \{ f = \sum_{i=0}^{n} a_i X^i \in A[X] \mid n \geq 0, a_i \in A_i, \ i = 0, 1, \ldots, n \} \).

Proof. By induction using Theorem 2.7.

Let \( \mathcal{A} = (A_n)_{n \geq 0} \) be an increasing sequence of rings, \( A = \bigcup_{n=0}^{\infty} A_n \) and \( X \) an indeterminate over \( A \). Recall from [4] that \( \mathcal{A}[X] = \{ f = \sum_{i=0}^{n} a_i X^i \in A[X] \mid n \geq 0, a_i \in A_i, \ i = 0, 1, \ldots, n \} \).

Theorem 2.9. Let \( \mathcal{A} = (A_n)_{n \geq 0} \) be an increasing sequence of rings and \( \mathcal{S} \) a multiplicative system of ideals of \( A_0 \) such that for every \( I \in \mathcal{S}, \bigcap_{n=1}^{\infty} I^n \) contains some ideal of \( \mathcal{S} \). The following conditions are equivalent:

1. The ring \( \mathcal{A}[X] \) is \( \mathcal{S} \)-Noetherian.
2. The ring \( A_0 \) is \( \mathcal{S} \)-Noetherian and the \( A_0 \)-module \( A = \bigcup_{n=0}^{\infty} A_n \) is \( \mathcal{S} \)-finite.

Proof. (1) \( \Rightarrow \) (2) Let \( Q \) be an ideal of \( A_0 \). Then \( QA[X] \) is an \( \mathcal{S} \)-finite ideal of \( \mathcal{A}[X] \). Hence, there exist \( a_1, \ldots, a_n \in Q \) and \( I \in \mathcal{S} \) such that \( I(QA[X]) \subseteq \langle a_1, \ldots, a_n \rangle \mathcal{A}[X] \). Thus \( IQ \subseteq \langle a_1, \ldots, a_n \rangle A_0 \). Hence \( A_0 \) is \( \mathcal{S} \)-Noetherian.

Let \( n \geq 1 \) be an integer. The ideal \( X^n A_n \mathcal{A}[X] \) of \( \mathcal{A}[X] \) is \( \mathcal{S} \)-finite. Then there exist \( a_1, \ldots, a_k \in A_n \) and \( I \in \mathcal{S} \) such that \( I(X^n A_n \mathcal{A}[X]) \subseteq \langle a_1 X^n, \ldots, a_k X^n \rangle \). Let \( a \in A_n \) and \( b \in I \). There exist \( f_1(X), \ldots, f_k(X) \in \mathcal{A}[X] \) such that \( b(aX^n) = \sum_{i=1}^{k} f_i(aX^n) \). Identifying coefficients of \( X^n \), we obtain \( ba = \sum_{i=1}^{k} f_i(0)a_i \) with \( f_i(0), \ldots, f_k(0) \in A_0 \). Therefore, \( A_n \) is an \( \mathcal{S} \)-finite \( A_0 \)-module.

The ideal \( Q \) of \( \mathcal{A}[X] \) generated by \( \{ aX^i, \ i \in \mathbb{N}^* \} \) is \( \mathcal{S} \)-finite, then there exist \( I \in \mathcal{S}, a_1 X^{a_1}, \ldots, a_r X^{a_r}, \ a_i \in A_0, \ a_i \geq 1 \) such that,
\[
IQ \subseteq \langle a_k X^{a_k}, 1 \leq k \leq r \rangle \mathcal{A}[X].
\]

Let \( m = \text{max}(a_1, \ldots, a_r) \). Then \( a_1, \ldots, a_r \in A_m \). For a fixed \( i > m \). Let \( b \in I \) and \( y \in A_i \). By definition of \( Q \), \( yX^i \in Q \). Thus
\[
byX^i \in \langle a_k X^{a_k}, 1 \leq k \leq r \rangle \mathcal{A}[X].
\]

It yields that \( byX^i = \sum_{k=1}^{r} a_k X^{a_k} g_k \) with \( g_k = \sum_{j=0}^{n_k} g_{k,j} X^j \in \mathcal{A}[X] \). By identification, we get \( by = \sum_{k=1}^{r} a_k g_{k,i} X^{a_k} \) with \( g_{k,i} \in A_{i-a_k} \subseteq A_{i-1} \). Hence
\[
bA_i \subseteq a_1 A_{i-1} + \cdots + a_r A_{i-1} \subseteq A_{i-1}.
\]

It follows that \( IA_i \subseteq A_{i-1} \). Iterating we get \( I^{m-i} A_i \subseteq A_m \). It follows that \( JA_i \subseteq A_m \) for some ideal \( J \) of \( \mathcal{S} \) contained in \( \bigcap_{n=0}^{\infty} I^n \). Consequently, \( JA_n \subseteq A_m \) for every \( n \geq m \). It yields that \( JA = \bigcup_{n=0}^{\infty} A_n = \bigcup_{n=m}^{\infty} A_n = \bigcup_{n=m}^{+\infty} JA_n \subseteq A_m \). Thus \( A \) is
an $S$-finite $A_0$-module.

(2) $\implies$ (1) Since the $A_0$-module $A$ is $S$-finite, there exist $a_1, \ldots, a_n \in A$ and $C \in S$ such that $CA \subseteq \langle a_1, \ldots, a_n \rangle A_0$. Thus $CA[X] \subseteq \langle a_1, \ldots, a_n \rangle A_0[X]$. Hence the $A_0[X]$-module $A[X]$ is $S$-finite. On the other hand, $A_0$ is $S$-Noetherian and for each $I \in S$, $\bigcap_{k=1}^{\infty} I^k$ contains some ideal of $S$. By Theorem 2.7, the ring $A_0[X]$ is $S$-Noetherian. By Corollary 2.6, the $A_0[X]$-module $A[X]$ is $S$-Noetherian, and so is the submodule $A[X]$. Thus the ring $A[X]$ is $S$-Noetherian. \hfill $\square$

Lemma 2.10. Let $A$ be a ring, $S$ a multiplicative system of ideals of $A$ and $M$ an $S$-finite $A$-module. If $N$ is a submodule of $M$ maximal among the non-$S$-finite submodules of $M$, then $[N : M]$ is a prime ideal of $A$.

Proof. Denote $P = [N : M]$. Assume that $P$ is not a prime ideal. Let $a, b \in A \setminus P$ such that $ab \in P$. By maximality of $N$, $N + aM$ is $S$-finite. Consequently, there exist $n_1, \ldots, n_k \in N$, $m_1, \ldots, m_k \in M$ and $I \in S$ such that $I(N + aM) \subseteq \langle n_1 + am_1, \ldots, n_k + am_k \rangle$. Since $aN \subseteq N$ and $bx \in [N : a]$ for each $x \in M$ ($N \neq M$), $N \subseteq [N : a]$. Then $[N : a]$ is $S$-finite. It yields that there exist $q_1, \ldots, q_t \in [N : a]$ and $J \in S$ such that $J[N : a] \subseteq \langle q_1, \ldots, q_t \rangle$. Let $x \in N$, $\alpha \in I$ and $\beta \in J$. We have $\alpha x = \sum_{i=1}^{k} \alpha_i (n_i + am_i)$ with $\alpha_1, \ldots, \alpha_k \in A$. Thus $a \sum_{i=1}^{k} \alpha_i m_i = \alpha x - \sum_{i=1}^{k} \alpha_i n_i \in N$. Hence $y = \sum_{i=1}^{k} \alpha_i m_i \in [N : a]$. Therefore, $\beta y = \sum_{j=1}^{t} \beta_j q_j$ with $\beta_1, \cdots, \beta_t \in A$. Thus $\beta \alpha x = \sum_{i=1}^{k} (\beta \alpha_i) n_i + \beta ay = \sum_{i=1}^{k} (\beta \alpha_i) n_i + \sum_{j=1}^{t} \beta_j (aq_j) \in \langle n_1, \ldots, n_k, aq_1, \ldots, aq_t \rangle$. Hence $JIN \subseteq \langle n_1, \ldots, n_k, aq_1, \ldots, aq_t \rangle \subseteq N$ with $JI \in S$, so $N$ is $S$-finite, contradiction. Therefore, $P$ is a prime ideal of $A$. \hfill $\square$

Let $A$ be a ring, $S$ a multiplicative system of finitely generated ideals of $A$, $P$ a prime ideal of $A$ and $M$ an $S$-finite $A$-module. It is clear that $P$ and $PM$ are $S$-finite when $P$ contains some ideal in $S$.

Theorem 2.11. Let $A$ be a ring, $S$ a multiplicative system of finitely generated ideals of $A$ and $M$ an $S$-finite $A$-module. Then $M$ is an $S$-Noetherian $A$-module if and only if for each prime ideal $P$ of $A$ not containing any ideal in $S$, the submodule $PM$ is $S$-finite.

Proof. $\implies$ Trivial.

$\Leftarrow$ Assume that $M$ is not $S$-Noetherian. Let $F$ be the set of submodules of $M$ which are not $S$-finite. We order $F$ by inclusion. Let $(H_\alpha)_{\alpha \in \Lambda}$ be a totally ordered family of $F$ and $H = \bigcup_{\alpha \in \Lambda} H_\alpha$. Assume that $H \notin F$. Then there exist $a_1, \ldots, a_n \in H$ and $I \in S$ such that $IH \subseteq \langle a_1, \ldots, a_n \rangle$. Since the family $(H_\alpha)_{\alpha \in \Lambda}$ is totally ordered, there exists $\alpha \in \Lambda$ such that $a_1, \ldots, a_n \in H_\alpha$. Hence $IH_\alpha \subseteq IH \subseteq \langle a_1, \ldots, a_n \rangle$. Therefore, $H_\alpha$ is $S$-finite, absurd. Thus $H \notin F$. Therefore $F$ is inductively ordered.
By Zorn’s lemma, \( \mathcal{F} \) has a maximal element \( N \). By Lemma 2.10, \( P = [N : M] \) is a prime ideal of \( A \). Let \( m_1, \ldots, m_k \in M \) and \( J \in \mathcal{S} \) such that \( JM \subseteq \langle m_1, \ldots, m_k \rangle \).

If there exists \( I \in \mathcal{S} \) such that \( IM \subseteq N \), then \( IJN \subseteq I(\langle m_1, \ldots, m_k \rangle) \subseteq N \), contradiction (since \( I \) is finitely generated, so is the submodule \( I(\langle m_1, \ldots, m_k \rangle) \)). Therefore, for each \( I \in \mathcal{S} \), \( IM \nsubseteq N \). Thus \( P = [N : M] \subseteq [N : \langle m_1, \ldots, m_k \rangle] \subseteq [N : JM] = P : J = P \). Hence, \( P = [N : \langle m_1, \ldots, m_k \rangle] = [N : m_1] \cap \cdots \cap [N : m_k] = [N : m_{i_0}] \) for some \( 1 \leq i_0 \leq k \). Since \( P \neq A \), so \( m_{i_0} \notin N \), hence \( N + Am_{i_0} \) is \( \mathcal{S} \)-finite by the maximality of \( N \). There exist then \( n_1, \ldots, n_t \in N, a_1, \ldots, a_t \in A \) and \( I \in \mathcal{S} \) such that \( I(N + Am_{i_0}) \subseteq \langle n_1 + a_1m_{i_0}, \ldots, n_t + a_tm_{i_0} \rangle \). Let \( x \in N, b \in A \) and \( a \in I \).

There exist \( \alpha_1, \ldots, \alpha_t \in A \) such that \( \alpha(x + bm_{i_0}) = \sum_{i=1}^t (\alpha_i n_i + \alpha_i a_i m_{i_0}) \). Hence
\[
(\alpha b - \sum_{i=1}^t \alpha_i a_i) m_{i_0} = \sum_{i=1}^t \alpha_i n_i - \alpha x \in N. \text{ Thus } \alpha b - \sum_{i=1}^t \alpha_i a_i \in P. \text{ It yields that } \alpha x = \sum_{i=1}^t \alpha_i n_i + (\sum_{i=1}^t \alpha_i a_i - \alpha b)m_{i_0} \in \langle n_1, \ldots, n_t \rangle + PM. \text{ Since } PM \text{ is } \mathcal{S} \text{-finite, there exist } \beta_1, \ldots, \beta_r \in PM \text{ and } L \in \mathcal{S} \text{ such that } L(PM) \subseteq \langle \beta_1, \ldots, \beta_r \rangle \subseteq PM \subseteq N. \text{ Consequently, } (LI)N \subseteq \langle n_1, \ldots, n_t, \beta_1, \ldots, \beta_r \rangle \subseteq N. \text{ Hence } N \text{ is } \mathcal{S} \text{-finite, absurd. Therefore, } M \text{ is an } \mathcal{S} \text{-Noetherian } A \text{-module.} \]

**Corollary 2.12.** Let \( A \) be a ring and \( \mathcal{S} \) a multiplicative system of finitely generated ideals of \( A \). Then the ring \( A \) is \( \mathcal{S} \)-Noetherian, if and only if, each prime ideal of \( A \) not containing any ideal in \( \mathcal{S} \) is \( \mathcal{S} \)-finite.

The next example shows that for each \( n \geq 1 \), there exists an \( n \)-dimensional \( \mathcal{S} \)-Noetherian ring which is not Noetherian.

**Example 2.13.** Let \( A \) be a finite dimensional valuation domain, \( P \) its height one prime ideal, \( I \subseteq P \) a finitely generated ideal and \( \mathcal{S} = \{I^n, n \geq 1\} \). Then \( A \) is \( \mathcal{S} \)-Noetherian. Indeed, let \( Q \) be a nonzero prime ideal of \( A \). Thus \( IQ \subseteq I \subseteq P \subseteq Q \). Hence \( Q \) is \( \mathcal{S} \)-finite.

**Example 2.14.** The hypothesis that \( \mathcal{S} \) consists of finitely generated ideals is necessary. Indeed, let \( X_1, X_2, \ldots \) be a countably family of indeterminates over a field \( K \), \( A = K[X_n, n \geq 1]/(X_n^n, n \geq 1) \), \( M = \langle X_n, n \geq 1 \rangle A \) and \( \mathcal{S} = \{M^n, n \geq 1\} \).

The only prime ideal of \( A \) is \( M \). Assume that \( A \) is \( \mathcal{S} \)-Noetherian. Then \( M \) is \( \mathcal{S} \)-finite. Hence there exist \( k, m \in \mathbb{N}^* \) such that \( M^k M \subseteq \langle X_1, \ldots, X_m \rangle \). Then \( M^l = 0 \) for some \( l \geq 1 \), absurd. Hence the ring \( A \) is not \( \mathcal{S} \)-Noetherian.

**Corollary 2.15.** Let \( A \subseteq B \) be a rings extension and \( \mathcal{S} \) a multiplicative system of finitely generated ideals of \( A \) such that \( B \) is an \( \mathcal{S} \)-finite \( A \)-module. Then the ring \( A \) is \( \mathcal{S} \)-Noetherian if and only if \( B \) is \( \mathcal{S} \)-Noetherian.

**Proof.** \( \Rightarrow \) The \( A \)-module \( B \) is \( \mathcal{S} \)-finite. By Corollary 2.5, the \( A \)-module \( B \) is \( \mathcal{S} \)-Noetherian. Hence, the ring \( B \) is \( \mathcal{S} \)-Noetherian.

\( \Leftarrow \) By Theorem 2.11, the \( A \)-module \( B \) is \( \mathcal{S} \)-Noetherian. Therefore, the ring \( A \) is \( \mathcal{S} \)-Noetherian (as an \( A \)-submodule of \( B \)). \( \square \)
Let $A$ be a ring and $M$ an $A$-module. Recall that Nagata introduced the extension ring of $A$ called the idealization of $M$ in $A$, denoted here by $A(+)M$, whose underlying abelian group is $A \times M$ and multiplication defined by:

$$(a, x)(a', x') = (aa', ax' + a'x), \text{ for every } (a, x), (a', x') \in A(+)M.$$ 

It is well known that $A(+)M$ is a commutative ring with identity element $(1, 0)$. (It is also called the trivial extension of $A$ by $M$.) For more details see [2] and [4].

Let $A$ be an ideal of $A$. Note that $I(+)IM$ is the extension of $I$ in $A(+)M$, so $S_1 = \{I(+)IM, I \in S\}$ is clearly a multiplicative system of ideals of $A(+)M$. As $A \subseteq A(+)M$, we get $A(+)M$ is $S$-Noetherian if and only if $A(+)M$ is $S_1$-Noetherian.

**Proposition 2.16.** Let $A$ be a ring, $S$ a multiplicative system of finitely generated ideals of $A$ and $M$ an $A$-module. Denote $S_1 = \{I(+)IM, I \in S\}$. Then the ring $A(+)M$ is $S_1$-Noetherian if and only if the ring $A$ is $S$-Noetherian and the $A$-module $M$ is $S$-finite.

**Proof.** $\implies$ The map $\phi: A(+)M \rightarrow A$ defined by $\phi(a, x) = a$ for every $(a, x) \in A(+)M$ is a surjective homomorphism of rings. Since $A(+)M$ is $S_1$-Noetherian, the ring $A$ is $\phi(S_1) = S$-Noetherian. The ideal $\{0\}(+)M$ of $A(+)M$ is $S_1$-finite. Then there exist $m_1, \ldots, m_k \in M$ and $I \in S$ such that $(I(+)IM)(\{0\}(+)M) \subseteq ((0, m_1), \ldots, (0, m_k))A(+)M$. Therefore, $IM \subseteq (m_1, \ldots, m_k)A$. It yields that the $A$-module $M$ is $S$-finite.

$\iff$ It is clear that the extension $A \subseteq A(+)M$ is $S$-finite. Then $A$ is $S$-Noetherian if and only if $A(+)M$ is $S$-Noetherian by Corollary 2.15. Thus the ring $A(+)M$ is $S_1$-Noetherian. $\square$

**Example 2.17.** Let $A$ be an $n$-dimensional non-Noetherian integral domain. Assume that $P = \cap\{Q \mid (0) \neq Q \in \text{Spec}(A)\}$ is a nonzero ideal of $A$ and let $I \subseteq P$ be a finitely generated nonprincipal ideal of $A$. Set $S = \{I^k, k \geq 1\}$. Clearly $A$ is an $S$-Noetherian ring (since each nonzero prime ideal of $A$ contains $I$). Then for each $S$-finite $A$-module $M$, the ring $A(+)M$ is $S_1$-Noetherian, by Proposition 2.16, where $S_1 = \{I(+)IM, I \in S\}$.

Let $A$ be a ring and $P \in \text{Spec}(A)$. Denote $S_P = \{I \text{ ideal of } A \text{ such that } I \not\subseteq P\}$. $S_P$ is clearly a multiplicative system of ideals of $A$.

**Theorem 2.18.** The following assertions are equivalent for an $A$-module $E$:

1. The module $E$ is Noetherian.
2. The module $E$ is $S_P$-Noetherian for every $P \in \text{Spec}(A)$.
3. The module $E$ is $S_M$-Noetherian for every $M \in \text{Max}(A)$.

**Proof.** The implications (1) $\implies$ (2) $\implies$ (3) are simple. (3) $\implies$ (1) Let $N$ be a submodule of $E$. For each $M \in \text{Max}(A)$, there exist $I_M \in S_M$ and a finitely generated submodule $F_M \subseteq N$ of $E$ such that $I_MN \subseteq F_M$. Let $Q = \langle I_M \mid M \in \text{Max}(A) \rangle$. Since $I_M \not\subseteq M$ for each maximal ideal $M$ of $A$, we get $Q = A$. Therefore there exist $M_1, \ldots, M_r \in \text{Max}(A)$ such that $A = \langle I_{M_1}, \ldots, I_{M_r} \rangle$. Hence $N = AN = \langle I_{M_1}, \ldots, I_{M_r} \rangle N = I_{M_1}N + \cdots + I_{M_r}N \subseteq F_{M_1} + \cdots + F_{M_r} \subseteq N$. Thus $N = F_{M_1} + \cdots + F_{M_r}$ is finitely generated. $\square$
Corollary 2.19. The following assertions are equivalent for a ring $A$:

1. The ring $A$ is Noetherian.
2. The ring $A$ is $S_P$-Noetherian for every $P \in \text{Spec}(A)$.
3. The ring $A$ is $S_M$-Noetherian for every $M \in \text{Max}(A)$.

Questions. We end this paper by posing two questions.

1. Let $A$ be an integral domain with quotient field $K$ and $S$ a multiplicative system of ideals of $A$ such that $A$ is $S$-Noetherian. Does it follow that the generalized fraction ring $A_S = \{x \in K; \ xH \subseteq A \text{ for some } H \in S\}$ is Noetherian?

2. Under the hypothesis of Theorem 2.7, is the power series ring $A[[X]]$ $S$-Noetherian?

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References


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