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# A countably cellular topological group all of whose countable subsets are closed need not be $\mathbb{R}$-factorizable

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*Abstract.* We construct a Hausdorff topological group $G$ such that $\aleph_1$ is a precalibre of $G$ (hence, $G$ has countable cellularity), all countable subsets of $G$ are closed and $C$-embedded in $G$, but $G$ is not $\mathbb{R}$-factorizable. This solves Problem 8.6.3 from the book "Topological Groups and Related Structures" (2008) in the negative.

*Keywords:* $\mathbb{R}$-factorizable; cellularity; $C$-embedded; Sorgenfrey line; $P$-group; Dieudonné completion; Hewitt–Nachbin completion; Bohr topology

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## 1. Introduction

If all $G_\delta$-sets in a space are open, it is called a *P-space.* Similarly, if all $G_\delta$-sets in a topological group $H$ are open, $H$ is called a *P-group.* It is known that a $P$-group is $\mathbb{R}$-factorizable if and only if the group is pseudo-$\aleph_1$-compact, that is, every locally finite family of open sets in the group is countable, see [1, Theorem 8.6.12]. Notably, pseudo-$\aleph_1$-compactness is precisely what ensures that each continuous real-valued function on a product of spaces depends on no more than countably many coordinates, see [3]. Countable cellularity is a far more strong property than pseudo-$\aleph_1$-compactness, therefore one could hope that the former implies $\mathbb{R}$-factorizability in topological groups, see [1, Problem 8.1.1]. This hypothesis is, however, disproved in [4].

A.V. Arhangel'skii asked in 2001 if the $P$-property of a topological group could be weakened to the property that all countable subsets of the group are closed, in order to generalize the aforementioned theorem on $\mathbb{R}$-factorizability of $P$-groups. Afterwards, the question appeared as Problem 8.6.3 in [1]. We answer this question in the negative in Theorem 2.3 by constructing a topological group $G$ of countable cellularity such that all countable subgroups of $G$ are discrete (equivalently, all countable subsets of $G$ are closed), but $G$ is not $\mathbb{R}$-factorizable. Our

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construction of the group $G$ is based on the use of a separable topological Boolean group that is not $\mathbb{R}$-factorizable, as described by E. Reznichenko and O. Sipacheva in [4].

In Theorem 2.6 we strengthen the topology of $G$ and obtain a Hausdorff topological group $G^*$ of countable cellularity such that all countable subsets of $G^*$ are closed and $C$-embedded, all subgroups of $G^*$ are closed, but $G^*$ fails to be $\mathbb{R}$-factorizable. We do not know, however, whether $\aleph_1$ is a precalibre of $G^*$.

**1.1 Notation and terminology.** A group $G$ with identity $e$ is *Boolean* if $x^2 = e$ for each $x \in G$. All Boolean groups are Abelian, so we will use additive notation for Boolean (and Abelian) groups. For a subset $A$ of a group $G$, the smallest subgroup of $G$ containing $A$ is denoted by $\langle A \rangle$. Also, if $n \geq 1$ is an integer, we put $nG = \{nx \colon x \in G\}$. If the group $G$ is Abelian, $nG$ is a subgroup of $G$.

Let $X$ be a Tychonoff space and $A(X)$ the free Abelian topological group over $X$, see [1, Section 7.1]. The *free Boolean topological group* on $X$, denoted by $B(X)$, can be identified with the quotient group $A(X)/2A(X)$, see [5]. The group $B(X)$ is Hausdorff because the subgroup $2A(X)$ is closed in $A(X)$, which can be easily verified.

A topological group $G$ is $\mathbb{R}$-*factorizable* if for every continuous real-valued function $f$ on $G$, one can find a continuous homomorphism $p \colon G \to H$ onto a second-countable topological group $H$ and a continuous real-valued function $h$ on $H$ such that $f = h \circ p$. The class of $\mathbb{R}$-factorizable groups is quite wide, it includes all precompact groups, Lindelöf groups, dense subgroups of arbitrary products of Lindelöf $\Sigma$-groups, etc. see [1, Chapter 7].

According to [4, Corollary 10 (3)], the free Boolean topological group $B(\mathbb{S})$ on the Sorgenfrey line $\mathbb{S}$ is not $\mathbb{R}$-factorizable. This is a first example of separable topological group that fails to be $\mathbb{R}$-factorizable. We denote this group by $H$ and use it in the proof of Theorem 2.3.

## 2. Two examples

First we present two simple lemmas that will be applied in our recursive construction of the group $G$ in Theorem 2.3.

**Lemma 2.1.** *Let $B_a(X)$ be the abstract free Boolean group on a set $X$, let $S$ be a countable subgroup of $B_a(X)$ and $h \colon S \to K$ a homomorphism to a Boolean group $K$. Then there exist a countable set $Y \subset X$ and a mapping $f \colon Y \to K$ such that $S \subset \langle Y \rangle$ and the (unique) extension of $f$ to a homomorphism $\tilde{f} \colon \langle Y \rangle \to K$ satisfies $\tilde{f} \restriction_S = h$.*

Proof: Since $S$ is countable, there exists a countable set $Y \subset X$ such that $S \subset \langle Y \rangle$. Let $P = \langle Y \rangle$. Notice that every Boolean group can be considered as a linear space over the two-element field $F_2 = \{0,1\}$. Therefore, $h$ extends to a homomorphism $\tilde{h} \colon P \to K$. Then the restriction $f = \tilde{h}\restriction_Y$ is the required mapping. $\square$

**Lemma 2.2.** *Let $\mathbb{N}$ be the set of nonnegative integers considered as a subspace of the Sorgenfrey line $\mathbb{S}$. Then the subgroup $\langle \mathbb{N} \rangle$ of the group $H = B(\mathbb{S})$ algebraically generated by $\mathbb{N}$ is discrete.*

Proof: Let $r \colon \mathbb{S} \to \mathbb{N}$ be a continuous retraction. Denote by $\tilde{r}$ an extension of $r$ to a continuous homomorphism of $B(\mathbb{S})$ to $B(\mathbb{N})$. Clearly, the free Boolean topological group $B(\mathbb{N})$ on the discrete space $\mathbb{N}$ is also discrete. Since the restriction of $\tilde{r}$ to $\langle \mathbb{N} \rangle$ is one-to-one, we conclude that the subgroup $\langle \mathbb{N} \rangle$ of $B(\mathbb{S})$ is discrete. $\square$

Let us recall that a space $X$ is *Moscow* if the closure of every open set in $X$ is the union of a family of $G_\delta$-sets in $X$, see [1, Section 6.1].

**Theorem 2.3.** *There exists a topological group $G$ such that $\aleph_1$ is a precalibre of $G$ (hence, $G$ has countable cellularity and is a Moscow group), all countable subsets of $G$ are closed and $C$-embedded in $G$, but $G$ is not $\mathbb{R}$-factorizable.*

Proof: We construct $G$ as a dense subgroup of $H^{\mathfrak{c}}$, where $\mathfrak{c} = 2^\omega$. Since the groups $H$ and $H^{\mathfrak{c}}$ are separable, the cardinal $\aleph_1$ is a precalibre for every dense subgroup of $H^{\mathfrak{c}}$.

For every nonempty set $A \subset \mathfrak{c}$, let $p_A \colon H^{\mathfrak{c}} \to H^A$ be the projection. The group $G$ will be constructed to satisfy $p_A(G) = H^A$ for every countable set $A \subset \mathfrak{c}$. This property of $G$ guarantees that the restriction of $p_A$ to $G$ is an open continuous homomorphism for each countable set $A \subset \mathfrak{c}$. In particular, $H$ is a continuous open homomorphic image of $G$. Due to [1, Theorem 8.4.2], if $G$ was $\mathbb{R}$-factorizable, $H$ would likewise be $\mathbb{R}$-factorizable. Since $H$ is not, we deduce that neither is $G$.

In addition, we have to guarantee that all countable subsets of $G$ are closed or, equivalently, that all countable subgroups of $G$ are discrete. We will see that each of the two equivalent properties implies that the countable subsets of $G$ are $C$-embedded in $G$.

Our construction of the group $G$ requires some preliminary work of introducing and enumerating all the necessary objects. Let

$$B = \bigcup \{H^A \colon A \subset \mathfrak{c},\ |A| = \omega\}.$$

Then $|B| = \mathfrak{c}$, so we enumerate $B$ in length $\mathfrak{c}$, say, $B = \{b_\alpha\colon \omega \le \alpha < \mathfrak{c}\}$. We consider each $b_\alpha$ as a function of $A_\alpha = \operatorname{dom} b_\alpha$ to $H$. Clearly, the enumeration of $B$ can be chosen to satisfy $A_\alpha \subset \alpha$ for each infinite ordinal $\alpha < \mathfrak{c}$.

Let also $X$ be a set of cardinality $\mathfrak{c}$ and $B_a(X)$ the abstract free Boolean group on $X$. Our aim is to define a family $\mathcal{H} = \{h_\alpha\colon \alpha < \mathfrak{c}\}$ of homomorphisms of $B_a(X)$ to $H$ and take $G = h(B_a(X)) \subset H^\mathfrak{c}$, where $h$ is the diagonal of the family $\mathcal{H}$. To ensure the required properties of $G$ considered as a topological subgroup of the product group $H^\mathfrak{c}$, we define homomorphisms $h_\alpha$ in a special way, by recursion of length $\mathfrak{c}$.

For every nonempty countable subset $Y$ of $X$, denote by $\mathcal{E}_Y$ the family of all mappings of $Y$ to $H$. Clearly, $|\mathcal{E}_Y| = |H^Y| = \mathfrak{c}$. Therefore, the family

$$\mathcal{E} = \bigcup\{\mathcal{E}_Y\colon\ Y \subset X,\ 1 \le |Y| \le \omega\}$$

satisfies $|\mathcal{E}| = \mathfrak{c}$, so we enumerate it as $\mathcal{E} = \{f_\alpha\colon \alpha < \mathfrak{c}\}$.

We will define an increasing family $\{X_\nu\colon \nu < \mathfrak{c}\}$ of subsets of $X$ satisfying $X = \bigcup_{\nu<\mathfrak{c}} X_\nu$ and at each step $\alpha < \mathfrak{c}$ of our recursive construction, a family $\mathcal{G}_\alpha = \{g_{\beta,\nu}\colon\ \beta \le \nu < \alpha\}$, where each $g_{\beta,\nu}$ is a mapping of $X_\nu$ to $H$, satisfying the following conditions whenever $\beta \le \nu < \alpha$:

(i) $|X_\nu| \le |\nu+1|\cdot\omega$ and $X_\mu \subset X_\nu$ if $\mu < \nu$;
(ii) $\operatorname{dom} f_\beta \subset \operatorname{dom} g_{\beta,\beta}$ and $g_{\beta,\beta}$ extends $f_\beta$;
(iii) $g_{\beta,\nu}$ extends $g_{\beta,\mu}$ if $\mu < \nu$;
(iv) for every infinite ordinal $\beta < \alpha$, there exists $x \in X_\beta$ such that $b_\beta(\gamma) = g_{\gamma,\beta}(x)$ for each $\gamma \in A_\beta = \operatorname{dom} b_\beta$.

At the step 1, we put $X_0 = \operatorname{dom} f_0$, $g_{0,0} = f_0$ and $\mathcal{G}_1 = \{g_{0,0}\}$. Our choice of $X_0$ and $g_{0,0}$ agrees with (i) and (ii), while (iii)–(iv) are void at this step.

Assume that $0 < \alpha < \mathfrak{c}$ and that we have defined families $\{X_\gamma\colon\ \gamma < \alpha\}$ and $\mathcal{G}_\beta = \{g_{\gamma,\nu}\colon\ \gamma \le \nu < \beta\}$ for each $\beta < \alpha$ satisfying (i)–(iv). If the ordinal $\alpha$ is limit, we put $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ and $\mathcal{G}_\alpha = \{g_{\beta,\nu}\colon \beta \le \nu < \alpha\}$. This definition agrees with (i)–(iv) at the step $\alpha$.

Finally, assume that $\alpha$ is a successor ordinal, $\alpha = \beta + 1$, and that we have defined families $\{X_\gamma\colon\ \gamma < \beta\}$ and $\mathcal{G}_\beta = \{g_{\gamma,\nu}\colon\ \gamma \le \nu < \beta\}$ satisfying (i)–(iv). First we put $Y_\beta = A_\beta \cup \operatorname{dom} f_\beta \cup \bigcup_{\gamma<\beta} X_\gamma$ and choose a point $x_\beta \in X \setminus Y_\beta$ (if $\beta < \omega$, we can put $A_\beta = \emptyset$ in the above definition of $Y_\beta$). Then the set $X_\beta = Y_\beta \cup \{x_\beta\}$ satisfies $|X_\beta| \le |\beta+1|\cdot\omega$. Since $\operatorname{dom} f_\beta \subset X_\beta$, we take as $g_{\beta,\beta}$ any function in $H^{X_\beta}$ extending $f_\beta$. This guarantees (ii).

Let $\gamma$ be an arbitrary ordinal with $\gamma < \beta$. For (iv), assuming that $\omega \le \beta$, we define $g_{\gamma,\beta}(x_\beta) = b_\beta(\gamma)$ if $\gamma \in A_\beta$, and $g_{\gamma,\beta}(x_\beta) = e_H$, zero element of $H$, if $\gamma \in X_\beta \setminus A_\beta$. Notice that $A_\beta \subset \beta$, so the latter definition does not interfere with our choice of $g_{\beta,\beta}$. According to (iii), there exists a function $g_{\gamma,\beta}\colon X_\beta \to H$

extending $g_{\gamma,\nu}$ for each $\nu < \beta$. The value of $g_{\gamma,\beta}$ at $x_\beta$ has just been defined. It is easy to see that the families $\{X_\gamma : \gamma \le \beta\}$ and $\mathcal{G}_\alpha = \{g_{\gamma,\nu} : \gamma \le \nu \le \beta\}$ satisfy conditions (i)–(iv) at the step $\alpha$. This completes our recursive construction.

We claim that $X = \bigcup_{\alpha<\mathfrak{c}} X_\alpha$. Indeed, take an arbitrary point $x \in X$ and choose $\alpha < \mathfrak{c}$ such that $x \in \operatorname{dom} f_\alpha$. Then (ii) implies that $x \in \operatorname{dom} g_{\alpha,\alpha} \subset \operatorname{dom} g_\alpha = X_\alpha \subset X$. This proves our claim.

For every $\alpha < \mathfrak{c}$, let $g_\alpha = \bigcup_{\nu<\mathfrak{c}} g_{\alpha,\nu}$. By (iii), $g_\alpha$ is a function. Because $\operatorname{dom} g_{\alpha,\nu} = X_\nu$ for each $\nu$ with $\alpha \le \nu < \mathfrak{c}$, we deduce that $X$ is the domain of every function $g_\alpha$.

Every function $g_\alpha$ extends to a homomorphism $h_\alpha : B_a(X) \to H$, so we obtain the family $\mathcal{H} = \{h_\alpha : \alpha < \mathfrak{c}\}$. Let $h$ be the diagonal of the family $\mathcal{H}$, $h: B_a(X) \to H^{\mathfrak{c}}$. We claim that the homomorphism $h$ is one-to-one. Indeed, take an arbitrary element $g \in B_a(X)$ distinct from zero. Then $g = x_1 + \cdots + x_n$ for some pairwise distinct elements $x_1, \ldots, x_n \in X$. Let $f$ be a function on $X$ such that $\operatorname{dom} f = \{x_1, \ldots, x_n\}$, $f(x_1) \ne e_H$ and $f(x_k) = e_H$ for each $k > 1$. Then $f = f_\alpha$ for some $\alpha < \mathfrak{c}$. It follows from our definition of $h_\alpha$ that $h_\alpha(g) = f_\alpha(x_1) = f(x_1) \ne e_H$. Hence $h(g)$ is distinct from the identity element of $H^{\mathfrak{c}}$ and the kernel of $h$ is trivial. This proves our claim.

Since $h$ is a monomorphism, we can identify $B_a(X)$ with its image $G = h(B_a(X))$ considered as a topological subgroup of the group $H^{\mathfrak{c}}$. For every $\alpha < \mathfrak{c}$, let $p_\alpha$ be the projection of $H^{\mathfrak{c}}$ to the $\alpha$th factor $H_{(\alpha)}$. If follows from the definition of $h$ that $p_\alpha \circ h = h_\alpha$ for each $\alpha < \mathfrak{c}$. Therefore, under our identification of $B_a(X)$ and $G$, the restriction of $p_\alpha$ to $G$ is the homomorphism $h_\alpha$.

Let us show that $p_A(X) = H^A$ for each countable set $A \subset \mathfrak{c}$. Take an arbitrary element $b \in H^A$. Then $b = b_\alpha$ for some infinite ordinal $\alpha < \mathfrak{c}$. According to (iv), there exists $x \in X$ such that $b_\alpha(\gamma) = g_{\gamma,\alpha}(x) = g_\gamma(x) = h_\gamma(x)$ for each $\gamma \in A = \operatorname{dom} b_\alpha$. Since $h$ is the diagonal of the family $\mathcal{H}$, we deduce that $p_A(x) = b_\alpha = b$. This proves the equality $p_A(X) = H^A$. As $X \subset B_a(X) = G$, we see that $p_A(G) = H^A$.

The aforementioned property of the projections $p_A$ for a countable set $A \subset \mathfrak{c}$ implies that the restriction of $p_A$ to $G$ is an open homomorphism of $G$ onto the topological group $H^A$. Hence, $H$ is a continuous open homomorphic image of the group $G$. As explained above, the latter implies that the group $G$ is not $\mathbb{R}$-factorizable. The same property of the projections $p_A$ guarantees that $G$ is dense in the separable group $H^{\mathfrak{c}}$. Therefore, $\aleph_1$ is a precalibre of $G$ and the space $G$ has countable cellularity. Every topological group of countable cellularity is a Moscow space, see item 5, of Corollary 6.4.11 in [1]. Hence, the space $G$ is Moscow.

It remains to show that every countable subset of $G$ is closed and $C$-embedded in $G$. This requires the following two facts.

*Claim 1.* For every countable subgroup $C$ of $G$ and every homomorphism $\varphi\colon C \to H$, there exists $\alpha < \mathfrak{c}$ such that the projection $p_\alpha$ extends $\varphi$.

Consider an arbitrary homomorphism $\varphi\colon C \to H$, where $C$ is a countable subgroup of $G$. By Lemma 2.1, one can find a countable set $Y \subset X$ and a mapping $f\colon Y \to H$ such that the extension of $f$ to a homomorphism $\tilde{f}\colon \langle Y\rangle \to H$ satisfies $\tilde{f}\restriction_C = \varphi$. Take $\alpha < \mathfrak{c}$ such that $f = f_\alpha$. The mapping $g_\alpha$ extends $f_\alpha$, while the homomorphism $h_\alpha$ extends $g_\alpha$. We see, therefore, that $\varphi = \tilde{f}\restriction_C = h_\alpha\restriction_C$. This implies Claim 1 since $h_\alpha$ and $p_\alpha$ coincide on the group $G$.

It follows from Claim 1 that every homomorphism $\varphi\colon C \to H$ defined on a countable subgroup $C$ of $G$ is continuous and admits an extension to a continuous homomorphism of $G$ to $H$.

*Claim 2.* Every countable subgroup of $G$ is discrete.

Let $C$ be a countable subgroup of $G$. There exists a countable subset $Y$ of $X$ such that $C \subset \langle Y\rangle$. Let $Y = \{y_n\colon n \in \omega\}$ be a faithful enumeration of $Y$. Denote by $f$ the mapping of $Y$ onto the subspace $\mathbb{N}$ of the Sorgenfrey line $\mathbb{S}$ defined by $f(y_n) = n$ for each $n \in \omega$. Let $\tilde{f}$ be an extension of $f$ to a homomorphism of $\langle Y\rangle$ to the subgroup $\langle\mathbb{N}\rangle$ of $H = B(\mathbb{S})$. Then $\tilde{f}$ is one-to-one, while the group $\langle\mathbb{N}\rangle$ is discrete in virtue of Lemma 2.2. By Claim 1, the homomorphism $\tilde{f}$ is continuous. Hence, the group $\langle Y\rangle$ and its subgroup $C$ are discrete. This proves Claim 2.

Every discrete subgroup of a Hausdorff topological group is closed. Since every countable subset of $G$ is contained in a countable subgroup of $G$, Claim 2 implies that all countable subsets of $G$ are closed. Finally, let $D$ be a countable subset of $G$. Let $C$ be a countable subgroup of $G$ that contains $D$. Since $C$ is discrete, there exists an open symmetric neighborhood $U$ of zero element $e_G$ in $G$ such that $C \cap (U + U + U + U) = \{e_G\}$. A simple verification, similar to the one used in the proof of [1, Theorem 1.4.23], shows that the disjoint family of open sets $\{U + x\colon x \in C\}$ is discrete in $G$. It follows that *every* real-valued function on $C$ and, hence, on $D$ extends to a continuous real-valued function on $G$. This proves that $D$ is $C$-embedded in $G$. $\square$

**Remark 2.4.** The Dieudonné completion and Hewitt–Nachbin completion of the topological group $G$ constructed in Theorem 2.3 coincide and are naturally homeomorphic to $H^{\mathfrak{c}}$, where $H = B(\mathbb{S})$.

Indeed, let $f\colon \mathbb{S} \to \mathbb{R}$ be the identity mapping of the Sorgenfrey line to the real line. Then $f$ extends to a continuous homomorphism $\tilde{f}\colon B(\mathbb{S}) \to B(\mathbb{R})$. Clearly, the homomorphism $\tilde{f}$ is one-to-one. The group $B(\mathbb{R})$ has a countable network and, according to [1, Corollary 3.4.27], admits a continuous one-to-one homomorphism onto a second-countable Hausdorff topological group. The latter implies that the space $B(\mathbb{S})$ is Dieudonné complete, see [2, 8.5.13 (g)]. Further, for spaces of countable cellularity, Dieudonné completeness and Hewitt–Nachbin completeness coincide, see [2, 8.5.13, (h)]. So the spaces $H$ and $H^{\mathfrak{c}}$ are Hewitt–Nachbin complete.

By our construction in the proof of Theorem 2.3, $G$ fills all countable faces in $H^{\mathfrak{c}}$. Therefore, $G$ meets every nonempty $G_\delta$-set in $H^{\mathfrak{c}}$. Since the groups $H$ and $H^{\mathfrak{c}}$ are separable, it follows from [1, Corollary 6.4.11] that $H^{\mathfrak{c}}$ is a Moscow group. Applying [1, Theorem 6.1.7] we conclude that $G$ is $C$-embedded in $H^{\mathfrak{c}}$. Since the latter space is Hewitt–Nachbin complete, we deduce that it is the Hewitt–Nachbin completion of $G$.

**Remark 2.5.** All countable subsets of the group $G$ in Theorem 2.3 are closed. This property of $G$ cannot be strengthened by extending it to all subsets of size at most $\aleph_1$, without losing countable cellularity of $G$. Indeed, if all subsets of size less than or equal to $\aleph_1$ of a topological group $K$ are closed, these subsets are discrete. Take a subgroup $S$ of $K$ with $|S| = \aleph_1$ and choose a symmetric open neighborhood $U$ of the identity element $e_K$ in $K$ such that $S \cap (U+U+U+U) = \{e_K\}$. Then the disjoint family $\{U + x \colon x \in S\}$ of open sets in $K$ is discrete, so $K$ fails to be pseudo-$\aleph_1$-compact and, therefore, has uncountable cellularity.

Even though Remark 2.5 indicates some limitations for further strengthening of Theorem 2.3, there is still room for adding certain properties to the group $G$.

We recall that a continuous mapping $f\colon X \to Y$ is called *d-open* if for every open set $U$ in $X$, there exists an open set $V$ in $Y$ such that $f(U)$ is a dense subset of $V$. This is equivalent to saying that the inclusion $f(U) \subset \operatorname{Int}\overline{f(U)}$ holds for each open set $U \subset X$, see [6].

**Theorem 2.6.** *There exists a Hausdorff topological group $G^*$ of countable cellularity such that all countable subsets of $G^*$ are closed and $C$-embedded, all subgroups of $G^*$ are closed, but $G^*$ is not $\mathbb{R}$-factorizable.*

PROOF: To produce the group $G^*$, the topology of the group $G$ constructed in Theorem 2.3 is refined by declaring *all* homomorphisms of $G$ to the torus $\mathbb{T}$ to be continuous. In other words, the topology of $G^*$ is the upper bound of the topology of the group $G$ and the *Bohr* topology of the abstract group $G$, see [1, Section 9.9].

Denote by $G^\#$ the group $G$ equipped with Bohr topology, i.e., the coarsest topological group topology on $G$ that renders every homomorphism of $G$ to $\mathbb{T}$ continuous. Then the group $G^*$ can be identified with the diagonal subgroup

$$\Delta = \{(x,x) \colon x \in G\} \subset G \times G^\#,$$

where the first factor $G$ is exactly the topological group in Theorem 2.3. Let us show that the group $\Delta$ is as required.

First, the projection of $\Delta$ to the second factor is a continuous one-to-one homomorphism onto $G^{\#}$. By [1, Proposition 9.9.9 a)], all subgroups of $G^{\#}$ are closed. Hence, $G^*$ has the same property. Similarly, the projection of $\Delta$ to the first factor is a continuous one-to-one homomorphism onto $G$. Given that every countable subgroup of $G$ is discrete, the same conclusion holds for countable subgroups of $\Delta$. Hence, all countable subsets of $\Delta$ are closed and $C$-embedded.

The group $G^{\#}$ is precompact, so the Weil completion of $G^{\#}$, denoted by $bG$, is a compact Hausdorff topological group [1, Corollary 9.9.6]. Let $K$ be the closure of $\Delta$ in $G \times bG$ and $p$ be the projection of $G \times bG$ to $G$. Clearly $p$ is a continuous open homomorphism. Also, since the second factor $bG$ is compact, the homomorphism $p$ is a perfect mapping. Hence, the restriction of $p$ to the closed subgroup $K$ of the product $G \times bG$, say, $\pi$ is also a perfect homomorphism of $K$ onto $G$. It follows that $\pi$ is open. Therefore, the restriction of $\pi$ to the dense subgroup $\Delta$ of $K$ is a $d$-open homomorphism of $\Delta \cong G^*$ onto $G$. Since $G$ fails to be $\mathbb{R}$-factorizable and every $d$-open homomorphic image of an $\mathbb{R}$-factorizable topological group is also $\mathbb{R}$-factorizable (see [8, Theorem 2.1], where the required result is proved for the wider class of paratopological groups), we conclude that $G^*$ cannot be $\mathbb{R}$-factorizable.

Finally, we show that $\Delta$ has countable cellularity. We know that $\pi$ is a perfect (hence, open) homomorphism of $K$ onto $G$. Let $N$ be the kernel of $\pi$. Clearly, $N$ is a compact subgroup of $K$. Then $G \cong K/N$, so we can apply [7, Theorem 2.6] stating that $c(G) = c(K)$. Since the cellularity of $G$ is countable, the group $K$ has the same property. The density of $\Delta$ in $K$ implies that the cellularity of $\Delta$ is also countable. This completes the proof of the theorem. $\square$

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