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A note on functional tightness and minitightness of space of the G-permutation degree

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Abstract. We study the behavior of the minimal tightness and functional tightness of topological spaces under the influence of the functor of the permutation degree. Analytically:

a) We introduce the notion of τ-open sets and investigate some basic properties of them.

b) We prove that if the map $f\colon X \to Y$ is τ-continuous, then the map $SP^n f\colon SP^n X \to SP^n Y$ is also τ-continuous.

c) We show that the functor SP^n preserves the functional tightness and the minimal tightness of compacts.

d) Finally, we give some facts and properties on τ-bounded spaces. More precisely, we prove that the functor of permutation degree SP^n preserves the property of being τ-bounded.

Keywords: τ-open set; τ-bounded space; functional tightness; minimal tightness

Classification: 54C05, 54B20

1. Introduction

At the Prague Topological Symposium in 1981, V. V. Fedorchuk, see [9], posed the following general problem in the theory of covariant functors, which determined a new direction for research in the field of Topology:

- Let P be some geometric property and F be a covariant functor. If a topological space X has the property P, then whether has $F(X)$ the same property P? Or vice versa, that is, if $F(X)$ has the property P, does it follow that the topological space X has also the property P?

In our case, X is a topological T_1-space and $F \in \{SP^n_G, \exp\}$.

In [16], a functor $O\colon \text{Comp} \to \text{Comp}$ of weakly additive functionals acting in the category of compact spaces and their continuous mappings is defined. It was proved that the functor $O\colon \text{Comp} \to \text{Comp}$ satisfies the normality conditions, except the preimage preservation condition.

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The functor of Radon functionals in the category of weakly additive functionals was investigated in [7] and it was proved that this functor is normal.

In [9] and [10] V. V. Fedorchuk and V. V. Filippov investigated a functor of G-permutation degree and it was proved that a functor of G-permutation degree is a normal functor in the category of compact spaces and their continuous mappings.

In recent researches an interest in the theory of cardinal invariants and their behavior under the influence of various covariant functors is increasing fast. In [6], [2], [3], [8], [4], [12], [5] the authors investigated several cardinal invariants under the influence of some weakly normal and normal functors and hyperspaces.

The current paper is devoted to the investigation of cardinal invariants such as the functional tightness, the minimal tightness and some other topological properties of the space of permutation degree, as well as, some basic properties of τ-bounded spaces are studied.

The concept of functional tightness of a topological space was first introduced by A. Arkhangel'skii in [1]. As it turned out, cardinal invariants such as the minimal tightness and the functional tightness are in many ways similar to each other, and for many natural and classical cases they coincide. Moreover, there is an example of a topological space, the minimum tightness of which is countable, and the functional tightness is uncountable, see [17].

In [15], the action of closed and R-quotient maps on functional tightness is investigated. It is proved that the R-quotient mapping does not increase functional tightness. As well as, in [15] it is proved that the functional tightness of the product of two locally compact spaces does not exceed the product of functional tightnesses of those spaces.

Throughout the paper all spaces are assumed to be regular, and τ means an infinite cardinal number.

2. Preliminary notes

The set of all nonempty closed subsets of a topological space X is denoted by $\exp X$. The family of all sets of the form

$$O\langle U_1, U_2, \dots, U_n\rangle = \left\{F\colon F \in \exp X,\ F \subset \bigcup_{i=1}^{n} U_i,\ F \cap U_i \neq \emptyset,\ i = 1, \dots, n\right\}$$

where $U_1, U_2, \dots, U_n$ are open subsets of X, generates a base of the topology on the set $\exp X$. This topology is called the *Vietoris topology.* The set $\exp X$ with the Vietoris topology is called *exponential space* or the *hyperspace of a space* X.

We put

$$\exp_n X = \{F \in \exp X \colon |F| \le n\},$$

see [11].

It is known that a permutation group is the group of all permutations, that is one-to-one mappings $X \to X$. A permutation group of a set X is usually denoted by $S\big(X\big)$. Especially, if $X = \{1, 2, \ldots, n\}$, then $S\big(X\big)$ is denoted by S_n.

Let X^n be the nth power of a compact space X. The permutation group S_n of all permutations acts on the nth power X^n as permutation of coordinates. The set of all orbits of this action with the quotient topology is denoted by SP^nX. Thus, points of the space SP^nX are finite subsets (equivalence classes) of the product X^n.

Two points $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ are considered to be *equivalent* if there exists a permutation $\sigma \in S_n$ such that $y_i = x_{\sigma(i)}$. The space SP^nX is called the *n-permutation degree* of the space X.

Equivalent relation by which we obtain the space SP^nX is called the *symmetric equivalence relation*. The nth permutation degree is a quotient of X^n. Therefore, the quotient map, denoted by $\pi_n^s \colon X^n \to SP^nX$, is defined as

$$\pi_n^s((x_1, x_2, \ldots, x_n)) = [(x_1, x_2, \ldots, x_n)]$$

for every $(x_1, x_2, \ldots, x_n) \in X^n$.

Let G be a subgroup of the permutation group S_n and X be a compact space. The group G acts on the nth power of the space X as permutation of coordinates. The set of all orbits of this action with the quotient topology is denoted by SP^n_GX. Thus, points of the space SP^n_GX are finite subsets (equivalence classes) of the product X^n. The space SP^n_GX is called *G-permutation degree* of the space X.

Equivalence relations by which we obtained spaces SP^n_GX and $\exp_n X$ are called the *symmetric* and *hypersymmetric equivalence relations*, respectively.

Any symmetrically equivalent points in X^n are hypersymmetrically equivalent. But the inverse is not correct, in general. So, for $x \ne y$ points (x, x, y), (x, y, y) are hypersymmetrically equivalent, but not symmetrically equivalent.

Let $f \colon X \to Y$ be a continuous mapping. For an equivalence class $[(x_1, x_2, \ldots, x_n)] \in SP^n_GX$ we put

$$SP^n_Gf[(x_1, x_2, \ldots, x_n)] = [(f(x_1), f(x_2), \ldots, f(x_n))].$$

Thereby, a mapping $SP^n_Gf \colon SP^n_GX \to SP^n_GY$ is defined. It is easy to check that the operation SP^n_G so constructed is a normal functor in the category of compacts. This functor is called the *functor of G-permutation degree*.

The G-symmetric equivalence class $[(x_1, x_2, \ldots, x_n)]$ uniquely determines the hypersymmetric equivalence class $[(x_1, x_2, \ldots, x_n)]^{hc}$ containing it. Thereby, a mapping

$$\pi^h_{n,G}\colon SP^n_G X \to \exp_n X,$$

is defined representing the functor $\exp_n$ as the factor functor of the functor SP^n_G, see [9], [10].

3. On τ-open sets

Recall that a subset A of a topological space X is called τ-*closed*, see [14], if for some $B \subset A$ with $|B| \le \tau$, the closure $[B]$ in X of the set B is contained in A.

The τ -*closure* of the set A is defined as

$$[A]_\tau = \bigcup \big\{[B]\colon\, B \subset A,\ |B| \le \tau\big\}.$$

Definition 3.1. Let X be a topological space. A set $F \subset X$ is called τ-open in the space if its complement $X - F$ is τ-closed.

The τ-interior of a set $A \subset X$ is the union of all τ-open sets contained in A, i.e.

$$\mathrm{Int}_\tau A = \bigcup \{U\colon\ U \subset A \text{ and } U \text{ is } \tau\text{-open}\}.$$

It is easy to check that the τ-interior of every set is τ-open.

Proposition 3.1. *For every* $A \subset X$ *we have* $\mathrm{Int}_\tau A = X - [X - A]_\tau$.

Proof: By definition of the τ-closure operation we have

$$[X - A]_\tau = \bigcap \{M\colon\ X - A \subset M \text{ and } M \text{ is } \tau\text{-closed in } X\}.$$

Hence

$$X - [X - A]_\tau = X - \Big(\bigcap \{M\colon\ X - A \subset M \text{ and } M \text{ is } \tau\text{-closed in } X\}\Big).$$

Now by De Morgan's laws we have

$$X - [X - A]_\tau = \bigcup \{X - M\colon\ X - A \subset M \text{ and } M \text{ is } \tau\text{-closed in } X\}.$$

Therefore,

$$X - [X - A]_\tau = \bigcup \{X - M\colon\ X - M \subset A \text{ and } M \text{ is } \tau\text{-closed in } X\} = \mathrm{Int}_\tau A.$$

□

The following example shows that a τ-open set need not be open, in general.

Example 3.1. Let R be the real line and θ the family consisting of all sets whose cardinality of the complement is at most countable. It is easy to verify that (R, θ) is a topological space. Since any countable set is closed, the set of all irrational numbers is ω-closed. Then the set of all rational numbers $\mathbb{Q}$ is ω-open, but not open.

Proposition 3.2 ([4])**.** *For a mapping $f\colon X \to Y$ of arbitrary topological spaces X and Y the following conditions are equivalent:*

1) *A mapping $f\colon X \to Y$ is τ-continuous.*
2) *For every closed set F in Y, the preimage $f^{-1}(F)$ is τ-closed in X.*
3) *For every τ-closed set F in Y, the preimage $f^{-1}(F)$ is τ-closed in X.*
4) *$f([A]_\tau) \subset [f(A)]_\tau$ for an arbitrary subset $A \subset X$.*
5) *$[f^{-1}(B)]_\tau \subset f^{-1}([B]_\tau)$ for an arbitrary subset $B \subset Y$.*

We generalize Proposition 3.2 as follows.

Proposition 3.3. *For a mapping $f\colon X \to Y$ of arbitrary topological spaces X and Y the following conditions are equivalent:*

1) *A mapping $f\colon X \to Y$ is τ-continuous.*
2) *For every closed set F in Y, the preimage $f^{-1}(F)$ is τ-closed in X.*
3) *For every τ-closed set F in Y, the preimage $f^{-1}(F)$ is τ-closed in X.*
4) *$f([A]_\tau) \subset [f(A)]_\tau$ for an arbitrary subset $A \subset X$.*
5) *$[f^{-1}(B)]_\tau \subset f^{-1}([B]_\tau)$ for an arbitrary subset $B \subset Y$.*
6) *$f^{-1}(\mathrm{Int}_\tau C) \subset \mathrm{Int}_\tau f^{-1}(C)$ for an arbitrary subset $C \subset Y$.*
7) *For every τ-open set V in Y, the preimage $f^{-1}(V)$ is τ-open in X.*
8) *For every open set V in Y, the preimage $f^{-1}(V)$ is τ-open in X.*

Proof: For the implications 1) $\Rightarrow$ 2), 2) $\Rightarrow$ 3), 3) $\Rightarrow$ 4) and 4) $\Rightarrow$ 5) see Proposition 3.2.

5) $\Rightarrow$ 6) By Proposition 3.1 we have $\mathrm{Int}_\tau C = Y - [Y - C]_\tau$. Moreover,

$$f^{-1}(\mathrm{Int}_\tau C) = f^{-1}(Y - [Y - C]_\tau).$$

Clearly, $[f^{-1}(Y - C)]_\tau \subset f^{-1}([Y - C]_\tau)$. Thus

$$X - f^{-1}([Y - C]_\tau) \subset X - [f^{-1}(Y - C)]_\tau.$$

Therefore, we have

$$f^{-1}(\mathrm{Int}_\tau C) \subset X - [f^{-1}(Y - C)]_\tau = X - [X - f^{-1}(C)]_\tau = \mathrm{Int}_\tau f^{-1}(C).$$

Hence,

$$f^{-1}(\mathrm{Int}_\tau C) \subset \mathrm{Int}_\tau f^{-1}(C).$$

6) $\Rightarrow$ 7) Assume that the inclusion $f^{-1}(\mathrm{Int}_\tau C) \subset \mathrm{Int}_\tau f^{-1}(C)$ holds for any $C \subset Y$. Consider an arbitrary τ-open subset V of Y. It is known that any open

set is τ-open. In this case, we have

$$f^{-1}(V) = f^{-1}(\mathrm{Int}_\tau V) \subset \mathrm{Int}_\tau(f^{-1}(V)).$$

Therefore, $f^{-1}(V)$ is τ-open in X.

7) $\Rightarrow$ 8) It is known that any open set is τ-open. Then by 7) the preimage $f^{-1}(V)$ is τ-open in X.

8) $\Rightarrow$ 1) Consider an arbitrary closed subset E of Y. We have to show that the preimage $f^{-1}(E)$ is τ-closed in X. Since $Y-E$ is an open set in Y, then the preimage $f^{-1}(Y-E)$ is τ-open in X. By the equality $f^{-1}(Y-E) = X - f^{-1}(E)$, we directly obtain that $f^{-1}(E)$ is τ-closed in X. Proposition 3.3 is proved. □

4. The functional tightness of G-permutation degree

Definition 4.1 ([1])**.** Let X and Y be topological spaces. A function $f\colon X \to Y$ is said to be τ-continuous if for every subspace A of X such that $|A| \leq \tau$, the restriction $f|_A$ is continuous.

Definition 4.2 ([1])**.** Let X and Y be topological spaces. A function $f\colon X \to Y$ is said to be strictly τ-continuous if for every subspace A of X such that $|A| \leq \tau$, the restriction of f to A coincides with the restriction to A of some continuous function $g\colon X \to Y$.

Operation SP^n preserves τ-continuity of the mappings, i.e. the following holds.

Theorem 4.1. *If $f\colon X \to Y$ is a τ-continuous mapping, then the mapping $SP^n f\colon SP^n X \to SP^n Y$ is τ-continuous.*

Proof: Consider an arbitrary subset Ω of $SP^n X$, such that $|\Omega| \leq \tau$. Let us prove that the restriction of the mapping $SP^n f$ onto the set Ω is continuous.

We put $M = pr_1((\pi_n^s)^{-1}(\Omega))$, where $pr_1\colon X^n \to X$ is defined as

$$pr_1(z_1, z_2, \ldots, z_n) = z_1,$$

for any $(z_1, z_2, \ldots, z_n) \in X^n$ and $\pi_n^s\colon X^n \to SP^n X$. It is clear that $M \subset X$ and $|M| \leq \tau$. Take an arbitrary element $[x]$ from Ω. Let $[x] = [(x_1, x_2, \ldots, x_n)]$, where $x_i \in X$, $i = 1, 2, \ldots, n$. Then

$$SP^n f([x]) = [(f(x_1), f(x_2), \ldots, f(x_n))] \in SP^n Y.$$

Suppose W is an arbitrary neighborhood of the orbit $SP^n f([x])$ in $SP^n Y$. By definition of the quotient mapping there exist neighborhoods $V_1, V_2, \ldots, V_n$ of the points $f(x_1), f(x_2), \dots, f(x_n)$ such that $[V_1 \times V_2 \times \cdots \times V_n] \subset W$. In this case we have $x_1, x_2, \dots, x_n \in M$. Since $M \subset X$ and $|M| \le \tau$, we have that $f|_M \colon M \to Y$ is continuous. By continuity of f on M, there exist neighborhoods $U_1, U_2, \dots, U_n$ of the points $x_1, x_2, \dots, x_n$ satisfying the condition $f(U_i) \subset V_i$ for all $i = 1, 2, \dots, n$. Then

$$SP^n f[U_1 \times U_2 \times \cdots \times U_n] = [f(U_1) \times f(U_2) \times \cdots \times f(U_n)] \subset W.$$

It means that the restriction $SP^n f|_\Omega$ is continuous at the point $[x]$. Theorem 4.1 is proved. □

In [1] A. Arkhangel'skiĭ introduced cardinal invariants so called the functional tightness and the minimal tightness of a topological space as follows.

Definition 4.3 ([1])**.** The functional tightness of a space X is

$$t_0(X) = \min\{\tau\colon \tau \text{ is an infinite cardinal and every } \tau\text{-continuous real-valued function on } X \text{ is continuous}\}.$$

Definition 4.4 ([1])**.** The minimal tightness of a space X is

$$t_m(X) = \min\{\tau\colon \tau \text{ is an infinite cardinal and every strictly } \tau\text{-continuous real-valued function on } X \text{ is continuous}\}.$$

Note that we always have that $t_m(X) \le t_0(X)$ for an arbitrary topological space, since every strictly τ-continuous function is τ-continuous. Besides, in [1] it was shown that $t_m(X) = t_0(X)$ for an arbitrary normal space X.

For the function $f\colon X \to R$, where R is the set of real numbers, the operation $f_{\exp}\colon \exp_n X \to R$ is defined as follows: each set $F \in \exp_n X$ is associated with the maximum value of f on the set F, i.e.

$$f_{\exp}(F) = \max\{f(x)\colon x \in F\}.$$

This operation is defined correctly, since F is a finite set [4].

Lemma 4.1 ([4])**.** *For any (strictly) τ-continuous function $f\colon X \to R$, the function $f_{\exp}\colon \exp_n X \to R$ is (strictly) τ-continuous.*

Proposition 4.1 ([15], [14])**.** *If $\phi\colon X \to Y$ and $\psi\colon Y \to Z$ are (strictly) τ-continuous mappings, then the composition $\psi \circ \phi\colon X \to Z$ is (strictly) τ-continuous.*

For the function $f\colon X \to R$, the operation $f^n_{SP}\colon SP^n X \to R$ is defined as follows: each orbit $[x] = [(x_1, x_2, \dots, x_n)] \in SP^n X$ is associated with the maximum value of f on the orbit $[x]$, i.e.

$$f^n_{SP}([x]) = \max\{f(x_i)\colon\ i = 1, 2, \ldots, n\}.$$

Proposition 4.2. *For any (strictly) τ-continuous function $f\colon X \to R$, the function $f^n_{SP}\colon SP^nX \to R$ is (strictly) τ-continuous.*

Proof: Let $f\colon X \to R$ be a (strictly) τ-continuous function. We show that $f^n_{SP} = f_{\exp} \circ \pi^h_n$ is (strictly) τ-continuous, i.e. the following diagram is commutative:

$$\begin{array}{ccc} SP^nX & \xrightarrow{\ \pi^h_n\ } & \exp_n X \\ & \underset{f^n_{SP}}{\searrow} & \Big\downarrow f_{\exp} \\ & & R \end{array}$$

For an arbitrary orbit $[x] = [(x_1, x_2, \ldots, x_n)] \in SP^nX$ we have

$$\begin{aligned}(f_{\exp} \circ \pi^h_n)([x]) &= f_{\exp}(\pi^h_n([x])) = f_{\exp}(\{x_1, x_2, \ldots, x_n\}) \\ &= \max\{f(x_i)\colon\ i = 1, 2, \ldots, n\} = f^n_{SP}([x]).\end{aligned}$$

By Lemma 4.1 in [4] and Proposition 4.1 in [15], [14], the function $f^n_{SP}\colon SP^nX \to R$ is (strictly) τ-continuous. Proposition 4.2 is proved. □

Corollary 4.1. *The function $f\colon X \to R$ is continuous if and only if the function $f^n_{SP}\colon SP^nX \to R$ is continuous.*

Theorem 4.2. *For any infinite topological space X we have*

$$t_0(X) \le t_0(SP^nX).$$

Proof: Suppose $t_0(SP^nX) \le \tau$. Consider an arbitrary τ-continuous function $f\colon X \to R$. By Proposition 4.2, the function $f^n_{SP}\colon SP^nX \to R$ is also τ-continuous. Since $t_0(SP^nX) \le \tau$, the function $f^n_{SP}\colon SP^nX \to R$ is continuous and the function $f\colon X \to R$ is continuous as the restriction of the continuous function $f^n_{SP}\colon SP^nX \to R$ on the subspace $\Delta = \{[(x, x, \ldots, x)]\colon\ x \in X\}$, which is homeomorphic to X. Theorem 4.2 is proved. □

Theorem 4.3. *For any infinite topological space X we have*

$$t_m(X) \le t_m(SP^nX).$$

Proof: Suppose $t_m(SP^nX) \le \tau$. Consider an arbitrary strictly τ-continuous function $f\colon X \to R$. By Proposition 4.2, the function $f^n_{SP}\colon SP^nX \to R$ is also strictly τ-continuous. Since $t_m(SP^nX) \le \tau$, the function $f^n_{SP}\colon SP^nX \to R$ is continuous and the function $f\colon X \to R$ is continuous as the restriction of the continuous function $f^n_{SP}\colon SP^nX \to R$ on the subspace $\Delta = \{[(x, x, \dots, x)]\colon x \in X\}$, which is homeomorphic to X. Hence $t_m(X) \leq \tau$. Theorem 4.3 is proved. □

Proposition 4.3 ([15]). *If $p\colon X \to Y$ is a quotient mapping, then*

$$t_0(Y) \leq t_0(X).$$

Theorem 4.4 ([14], [15]). *If X is a locally compact space, then*

$$t_0(X \times Y) \leq t_0(X)t_0(Y)$$

and

$$t_m(X \times Y) \leq t_m(X)t_m(Y)$$

for any topological space Y.

Note that there exists a quotient map from X^n onto SP^nX for every topological space X and natural number n. It is known that any quotient map does not increase the functional tightness, see Proposition 4.3 in [15]. Besides, by Theorem 4.4 in [15], any finite product preserves functional tightness of locally compact spaces.

Corollary 4.2. *For an arbitrary infinite locally compact space X we have*

$$t_0(X) = t_0(SP^nX).$$

Corollary 4.3. *For an arbitrary infinite compact X we have*

$$t_0(X) = t_m(X) = t_m(SP^nX) = t_0(SP^nX).$$

5. On τ-bounded spaces

Definition 5.1 ([14]). A space X is called τ-bounded, if the closure in X of every subset of cardinality at most τ is compact.

In this section we investigate some properties of τ-bounded spaces. Some of the considered properties are related to hyperspaces.

Proposition 5.1 ([4]). *Continuous image of a τ-bounded space is τ-bounded.*

Theorem 5.1 ([4]). *The Cartesian product of nonempty spaces is τ-bounded if and only if all spaces are τ-bounded.*

Since the space SP^nX can be represented as a quotient space of the finite product X^n, from above results one can easily obtain the following.

Corollary 5.1. *A topological space X is τ-bounded if and only if SP^nX is τ-bounded, where n is a natural number.*

Remark 5.1. The above results are valid for any functor SP^n_G.

Theorem 5.2. *Let X be a topological space. Then the set*

$$\Gamma = \{F \in \exp X : \ F \cap A \neq \emptyset\}$$

is τ-closed, if $A \subset X$ is τ-closed in X.

Proof: Consider a subset $\Upsilon \subset \Gamma$ such that $|\Upsilon| \leq \tau$. Let $E \in [\Upsilon]$ such that $E \notin \Gamma$, i.e. $E \cap A = \emptyset$. Take a point x_F from each set in the form $F \cap A$, where $F \in \Upsilon$ and put

$$M = \{x_F : \ F \in \Upsilon\}.$$

In this case, it is clear that $M \subset A$, and moreover, $[M] \subseteq A$ since A is τ-closed and $|M| \leq \tau$. This implies that $[M] \cap E = \emptyset$ and that is why $E \subset X \setminus [M]$. We obtain that $E \in O\langle X \setminus [M]\rangle$. On the other side, for every $F \in \Upsilon$ we have $F \cap M \neq \emptyset$. Therefore we have $F \notin O\langle X \setminus [M]\rangle$ for every $F \in \Upsilon$, which means that $E \notin [\Upsilon]$. The obtained contradiction proves the statement. Theorem 5.2 is proved. □

Corollary 5.2. *Let X be a topological space. If $A \subset X$ is τ-open in X then the set*

$$\Gamma = \{F \in \exp X : F \subset A\}$$

is τ-open.

Proposition 5.2. *A τ-bounded subset of a Hausdorff space is τ-closed.*

Proof: Let X be a Hausdorff space and $K \subset X$ its τ-bounded subspace. We have to show that $[A] \subset K$ for any subset A of K with $|A| \leq \tau$. Indeed, since K is τ-bounded, we see that $[A]_K$ is compact. This implies that the subset $[A]_K$ is closed as a compact subset of a Hausdorff space. Therefore, $[[A]_K] = [A]_K = [A]$ and thus, $[A] \subset K$. This proves the τ-closedness of K. Proposition 5.2 is proved. □

Proposition 5.3. *A τ-closed subspace of a τ-bounded space is τ-bounded.*

Proof: Let X be a Hausdorff τ-bounded space and $M \subset X$ its τ-closed subspace. Now we show that M is τ-bounded too. Consider $A \subset M$ with $|A| \leq \tau$. Then $[A]$ is compact since X is τ-bounded. On the other side, $[A] \subset M$. We have shown that M is τ-bounded. Proposition 5.3 is proved. □

Theorem 5.3. *Let X be an infinite regular space, then $\bigcup \beta$ is τ-closed in X for every τ-bounded subspace β of the hyperspace $\exp X$.*

PROOF: Let β be an arbitrary τ-bounded subspace of $\exp X$ and let $M = \bigcup \beta$. We consider $A \subset M$ with $|A| \leq \tau$ and show that $[A] \subset M$. Clearly, for every $x \in A$ there exists $F_x \in \beta$ such that $x \in F_x$. We put

$$\gamma = \{F_x \colon x \in A\}.$$

Since β is τ-bounded and $|\gamma| \leq \tau$, we see that $[\gamma]$ is compact. Then by Theorem 2.5 in [13], the set $\bigcup[\gamma]$ is closed in X. Therefore, since $A \subset \bigcup \gamma$ and $\bigcup[\gamma] \subset \bigcup \beta = M$, M is τ-closed. Theorem 5.3 is proved. □

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