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Rodrigo Hidalgo Linares, Oleg Okunev

Abstract. We study the relation of L-equivalence defined between Tychonoff spaces, that is, we study the topological isomorphisms of their respective free locally convex spaces. We introduce the concept of an L-retract in a Tychonoff space in terms of the existence of a special kind of simultaneous extensions of continuous functions, explore the relation of this concept with the Dugundji extension theorem, and find some conditions that allow us to identify L-retracts in various classes of topological spaces. As applications, we present a method for constructing examples of L-equivalent mappings and L-equivalent spaces and in particular, we show that the properties of being an open mapping or a perfect mapping are not L-invariant.

Keywords: free locally convex space; L-equivalence; retraction

Classification: 46A03

1. Basic properties of free locally convex spaces

In what follows, every topological space is assumed to be Tychonoff, that is, T_1 and completely regular. Likewise, all topological vector spaces are assumed to be Hausdorff and are over $\mathbb{R}$. The weak topological dual of a locally convex space E will be denoted by E'. We say that E is *weak* if E is topologically isomorphic to $(E')'$ (equivalently, the topology of E is projective with respect to E').

The free locally convex space (in the Markov sense) over a topological space X is a pair $(\delta_X, L(X))$ formed by a continuous injection $\delta_X\colon X \to L(X)$ and a locally convex space $L(X)$ such that $L(X)$ is the linear span of $\delta_X(X)$ and for every continuous function $f\colon X \to E$ to a locally convex space E, there exists a unique continuous linear mapping $f_{\#}\colon L(X) \to E$ such that $f = f_{\#} \circ \delta_X$.

Similarly to M. I. Graev, we define the free locally convex space in the Graev sense over the topological space with a distinguished point (X, x_0) as a pair $(\delta_X, GL(X, x_0))$ formed by a continuous injection $\delta_X\colon X \to GL(X, x_0)$ with $\delta_X(x_0) = 0$ and a locally convex space $GL(X, x_0)$ such that $GL(X, x_0)$ is the linear span of $\delta_X(X)$ and for every continuous function $f\colon X \to E$ to a locally

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convex space E such that $f(x_0) = 0$, there exists a unique continuous linear mapping $f_\#\colon GL(X,x_0) \to E$ such that $f = f_\# \circ \delta_X$.

The mapping δ_X is known as the *Dirac embedding,* and for each $x \in X$, $\delta_X(x) = \delta_x$ is a linear functional that assigns to each $f \in \mathbb{R}^X$ its value at x, that is, $\delta_x(f) = f(x)$. In this sense, we can view the set $L(X)$ ($GL(X,x_0)$) as the set of finite linear combinations $\lambda_1\delta_{x_1} + \cdots + \lambda_n\delta_{x_n}$, with $n \in \mathbb{N}$, $\lambda_i \in \mathbb{R}$ and $x_i \in X$ ($x_i \in X \setminus \{x_0\}$). The following facts are well known, see [8].

Theorem 1.1. *Let X be a topological space and $x_0, x_1 \in X$ two different points. Then:*

(1) *The spaces $L(X)$ and $GL(X,x_0)$ always exist and are unique up to a topological isomorphism.*
(2) *$\delta_X(X)$ is a Hamel base for $L(X)$, and $\delta_X(X \setminus \{x_0\})$ is a Hamel base for $GL(X,x_0)$.*
(3) *The topologies of $L(X)$ and $GL(X,x_0)$ are Hausdorff and make the Dirac embedding a topological embedding, so that X is embedded in $L(X)$ and $GL(X,x_0)$ as closed subspace.*
(4) *For any $x_0, x_1 \in X$, the spaces $GL(X,x_0)$ and $GL(X,x_1)$ are topologically isomorphic.*

To simplify notation, we will assume that X is a subset of $L(X)$. The next statement is immediate from the definition.

Corollary 1.2. *A linear mapping $f\colon L(X) \to E$ to a locally convex space E is continuous if and only if the restriction $f|X$ is continuous.*

Corollary 1.3. *Let X and Y be topological spaces, x_0 a point of X, and $X \oplus Y$ their topological sum. Then $GL(X \oplus Y, x_0) = GL(X,x_0) \oplus L(Y)$.*

Let us show a more explicit relationship between $L(X)$ and $GL(X,x_0)$. Consider the function $e_X\colon X \to \mathbb{R}$ such that $e_X(x) = 1$ for all $x \in X$, and let $(e_X)_\#\colon L(X) \to \mathbb{R}$ be the unique linear mapping that extends e_X. Denote the kernel of $(e_X)_\#$ by $L^0(X)$. Observe that

$$L^0(X) = \left\{ \sum_{i=1}^{n} \lambda_i\delta_{x_i} \colon n \in \mathbb{N},\ \lambda_i \in \mathbb{R},\ x_i \in X,\ 1 \le i \le n,\ \sum_{i=1}^{n} \lambda_i = 0 \right\}.$$

We will say that a topological isomorphism $\varphi\colon L(X) \to L(Y)$ is *special* if the composition $(e_Y)_\# \circ \varphi\colon L(X) \to \mathbb{R}$ is constant on X.

As shown in [13], if there exists a topological isomorphism between $L(X)$ and $L(Y)$, then there exists always a special topological isomorphism between them. Furthermore, we can deduce the following statements.

Lemma 1.4. *For each nonzero continuous linear functional $\psi\colon L(X) \to \mathbb{R}$ there exists a linear topological isomorphism $u\colon L(X) \to L(X)$ such that $\psi \circ u(X) = \{1\}$.*

Proof: Let $x_0 \in X$ be such that $\psi(x_0) = \lambda \neq 0$. We define the mappings $u_0\colon X \to L(X)$ and $v_0\colon X \to L(X)$ as $u_0(x) = x - (1/\lambda)(\psi(x)-1)x_0$ and $v_0(x) = x + (\psi(x) - 1)x_0$. It is clear that u_0 and v_0 are continuous and $\psi(u_0(x)) = 1$ for each $x \in X$. Let u and v be the linear mappings that extend to u_0 and v_0, respectively. Then u is the required linear homeomorphism and v is its inverse mapping. □

Theorem 1.5. *Given a topological isomorphism $\psi\colon L(X) \to L(Y)$, there exists always a topological isomorphism $\varphi\colon L(X) \to L(Y)$ such that $(e_Y)_\# \circ \varphi = (e_X)_\#$.*

Proof: Applying Lemma 1.4 to the nonzero continuous linear functional $(e_Y)_\# \circ \psi$, we get a topological isomorphism $u\colon L(X) \to L(X)$ such that $\varphi = \psi \circ u$ is special and $(e_Y)_\# \circ \varphi = (e_X)_\#$. □

Corollary 1.6. *Let x_0 be a point of X. The spaces $L^0(X)$ and $GL(X, x_0)$ are topologically isomorphic.*

The last corollary demonstrates the fact that the free locally convex space, in the sense of Graev, is independent (up to a topological isomorphism) of the choice of the distinguished point. Henceforth, we will refer to this space simply as $GL(X)$.

Corollary 1.7. *Let X and Y be topological spaces. The spaces $L(X)$ and $L(Y)$ are topologically isomorphic if and only if $GL(X)$ and $GL(Y)$ are topologically isomorphic.*

Proof: If $L(X)$ and $L(Y)$ are topologically isomorphic, then Theorem 1.5 implies the existence of a topological isomorphism $\varphi\colon L(X) \to L(Y)$ such that $(e_Y)_\# \circ \varphi = (e_X)_\#$. It follows that $\varphi|L^0(X)\colon L^0(X) \to L^0(Y)$ is a topological isomorphism, which establishes the topological isomorphism between $GL(X)$ and $GL(Y)$. On the other hand, if the spaces $GL(X)$ and $GL(Y)$ are topologically isomorphic, then $L(X) = GL(X) \oplus \mathbb{R}$ and $L(Y) = GL(Y) \oplus \mathbb{R}$ are also topologically isomorphic. □

Given the close relationship between the spaces $L(X)$ and $GL(X)$ we can define the L-equivalence relation as follows: the spaces X and Y are called *L-equivalent* ($X \overset{L}{\sim} Y$) if their free locally convex spaces $L(X)$ and $L(Y)$ are topologically isomorphic. Furthermore, following [12] we can extend this relation to continuous mappings between topological spaces. We say that two continuous mappings $f\colon X \to Y$ and $g\colon Z \to T$ are *L-equivalent* ($f \overset{L}{\sim} g$) if there exists topological isomorphisms $\varphi\colon L(X) \to L(Z)$ and $\psi\colon L(Y) \to L(T)$ such that $\psi \circ f_{\#} = g_{\#} \circ \varphi$. Clearly, these are equivalence relations.

Likewise, any topological property of spaces or mappings that is preserved by the L-equivalence relation will be called *L-invariant*. It is worth noting that the L-equivalence between the identity mappings $\mathrm{id}_X\colon X \to X$ and $\mathrm{id}_Y\colon Y \to Y$ is equivalent to the L-equivalence between the spaces X and Y.

In a similar order of ideas, we can define the free weak topological vector space $L_p(X)$ over the topological space X as a pair $(\delta_X, L_p(X))$ formed by a continuous injection $\delta_X\colon X \to L_p(X)$ and a weak topological vector space $L_p(X)$ such that for every continuous function $f\colon X \to E$ to a weak topological vector space E, there exists a unique continuous linear mapping $f_{\#}\colon L_p(X) \to E$ such that $f = f_{\#} \circ \delta_X$, see [2]. In addition, Theorem 1.1, as well as all subsequent statements, remain valid for the space $L_p(X)$.

Naturally, this leads us to establish the concept of spaces and functions L_p-equivalent, and the notion of L_p-invariant properties. It should be noted that the concept of L_p-equivalence relation is often linked to the functor $\mathbf{C}_p$, in which case we say that two spaces X and Y are l-equivalent if their spaces of continuous real functions $C_p(X)$ and $C_p(Y)$ are topologically isomorphic, see [2]. This should not worry us, since the spaces $C_p(X)$ and $L_p(X)$ are in duality, so $C_p(X)$ is topologically isomorphic to $C_p(Y)$ if and only if $L_p(X)$ is topologically isomorphic to $L_p(Y)$. Therefore, following the notation already established, the L_p-equivalence relation is the same as the l-equivalent relation, and the properties that are L_p-invariant are l-invariant.

Finally, we will briefly describe the relation between the topologies of the spaces $L(X)$ and $L_p(X)$. First, from the definitions of these objects, it is easy to see that the identity $(\mathrm{id}_X)_{\#}\colon L(X) \to L_p(X)$ is a continuous linear mapping, therefore, the underlying sets of the spaces $L(X)$ and $L_p(X)$ are identical, and it is also clear that the topology of $L_p(X)$ is the $*$-weak topology of $L(X)$. Second, there exists a relatively simple way to describe its topology: since the spaces $L(X)$ and $C(X)$ are in algebraic duality, and any locally convex topology over a space E is the topology of uniform convergence on the equicontinuous sets of its topological dual E', the topology of $L(X)$ is the topology of uniform convergence on the equicontinuous pointwise bounded sets of $C(X)$, see [7]. Similarly, since the topology of $L_p(X)$ is weak, and we can embed $L_p(X)$ in $C_p(C_p(X))$, whose topology is also weak, we conclude that the topology of $L_p(X)$ is inherited from $C_p(C_p(X))$. Thus, a local neighborhood base of zero in $L(X)$ ($L_p(X)$) is the family of sets of the form

$$V[0, F, \varepsilon] = \{\alpha \in L(X)\colon\ |\alpha(f)| = |f_{\#}(\alpha)| < \varepsilon\},$$

where $F \subset C(X)$ is an equicontinuous pointwise bounded set (respectively, a finite set) and $\varepsilon > 0$.

2. L-retracts

As mentioned at the beginning, we will see what useful properties l-embedded sets have, and then we will try to find similar properties in the context of free locally convex spaces.

We start with a definition, let X be a topological space and Y a subspace of X. An extender is a mapping $\phi\colon C(Y) \to C(X)$ such that $\phi(f)|Y = f$ for every $f \in C(Y)$. An extender may be linear or not, but what really matters to us is its continuity. If there exists a continuous (linear and continuous) extender $\phi\colon C_p(Y) \to C_p(X)$, we will say that Y is *t-embedded* (*l-embedded*) in X.

A basic fact about t-embedded sets is that they are always closed. Clearly, every l-embedded set is also t-embedded, and it is easy to verify that X is always l-embedded in $L_p(X)$. The following statement is also easy to prove.

Proposition 2.1. *Let Y be a subset of X. The following statements are equivalent:*

(1) *Y is l-embedded in X.*
(2) *There exists a continuous linear retraction $r\colon L_p(X) \to L_p(Y)$.*
(3) *There exists a continuous function $f\colon X \to L_p(Y)$ such that $f|Y = \delta_Y$.*
(4) *Every continuous function from Y to a weak topological vector space E extends to a continuous function from X to E.*

It is generally not true that if Y is a subspace of X, then the subspace $L(Y, X)$ of $L(X)$ spanned by Y is $L(Y)$, even if Y is closed. Therefore, if Y is a subspace of X such that $L(Y, X)$ coincides with $L(Y)$, we will say that Y is *L-embedded* in X.

If Y is a subspace of X, Y is *P-embedded* in X if every continuous pseudometric on Y can be extended to a continuous pseudometric on X. The concept of a P-embedded set has several characterizations; the one given by K. Yamazaki [17, Theorem 3.1] is the one used in the proof of the following statement.

Proposition 2.2. *Let Y be a subset of X. The following statements are equivalent:*

(1) *Y is L-embedded in X.*
(2) *Any equicontinuous pointwise bounded subset of $C(Y)$ can be extended to an equicontinuous pointwise bounded subset of $C(X)$.*
(3) *Y is P-embedded in X.*

Taking into account that the concept of an L-embedded set is related to simultaneous extension of equicontinuous pointwise bounded sets, we can ask, of course, what relationship exists between the notions of an l-embedded set and an L-embedded set.

Example 2.3. An L-embedded set need not be l-embedded.

Consider the space $X = \omega_1 + 1$ with the order topology, and let Y be the dense subspace ω_1. Recall that Y is a pseudocompact non-compact space, and that X is the Stone–Čech compactification of Y. Since the square of Y is pseudocompact and X^2 is the Stone–Čech compactification of Y^2, Y is P-embedded in X, that is, Y is L-embedded in X. Considering that Y is not a closed set in X, Y is not l-embedded.

Example 2.4. An l-embedded set need not be L-embedded.

Let Y be an uncountable discrete space, and let $X = L_p(Y)$. Then Y is l-embedded in X, and X has the Souslin property. By [9, Theorem 1.2], Y cannot be P-embedded in X, and hence Y is not L-embedded.

As we have observed, the l-embedded sets and free locally convex space do not have a direct relationship; this is another reason to study L-retracts, as they possess all the desirable qualities of both l-embedded and L-embedded sets. Specifically, if Y is L-embedded in X and the subspace $L(Y)$ of $L(X)$, spanned by Y, is a linear retract of $L(X)$, then Y is an *L-retract* of X. As we will see, this combinations of concepts improves their properties.

Proposition 2.5. *Every L retract is an L-embedded and l-embedded set. In particular, every L-retract is a closed set.*

We still do not know if the reverse of the previous propositions holds, that is, in which cases the L-embedded and l-embedded set are an L-retract. We only can guarantee the following.

Theorem 2.6. *Let Y be a subspace of X. Then Y is an L-retract of X if and only if there exists a continuous linear extender $\phi\colon C_p(Y) \to C_p(X)$ such that if $B \subset C(Y)$ is an equicontinuous pointwise bounded set, then $\phi(B)$ also is an equicontinuous pointwise bounded set.*

Proof: Suppose that Y is an L-retract of X. Then there exists a continuous linear retraction $r\colon L(X) \to L(Y)$. Define $\phi\colon C_p(Y) \to C_p(X)$ by $\phi(f) = (f_\# \circ r)|X$, where $f_\#\colon L(Y) \to \mathbb{R}$ is the linear extension of the function f to $L(Y)$. Consequently ϕ is a continuous linear extender.

Let $B \subset C(Y)$ be an equicontinuous pointwise bounded set; let us verify that the set $\phi(B) = \{f_\# \circ r\colon f \in B\}$ is equicontinuous and pointwise bounded set in $C(X)$. By the definition of equicontinuity in a topological vector space, see [14], just note that given an $\varepsilon > 0$, the set

$$\bigcap_{f\in B}(f_\# \circ r)^{-1}(-\varepsilon,\varepsilon) = r^{-1}\Big(\bigcap_{f\in B} f_\#^{-1}(-\varepsilon,\varepsilon)\Big)$$

is a neighborhood of zero. Thus, $\phi(B)$ is an equicontinuous pointwise bounded subset of $C(X)$.

It only remains to prove that if such a continuous linear extender exists, then Y is an L-retract of X. Define $q\colon X \to L(Y)$ by $q(x) = \delta_x \circ \phi$, and let $r\colon L(X) \to L(Y)$ be the linear extension of q. Note that $q(x)$ is a continuous linear function on $C_p(Y)$, so $q(x) \in L_p(Y)$; therefore $q(x)$ also is an element of $L(Y)$, that is, q is well-defined.

Moreover, the restriction $r|Y$ coincides with the Dirac embedding of Y in $L(Y)$, implying that r is a retraction. To prove the continuity of q, consider $U = U[0, A, \varepsilon]$ as a neighborhood of zero in $L(Y)$, where $A \subset C(Y)$ is an equicontinuous pointwise bounded set and $\varepsilon > 0$. Since $\phi(A)$ is an equicontinuous pointwise bounded subset of $C(X)$, the set $V = V[0, \phi(A), \varepsilon]$ is a neighborhood of zero in $L(X)$ and $r(V) \subset U$. As r is linear and continuous, we conclude that $r\circ\delta_X = q$ is also continuous, and hence Y is an L-retract of X. □

If $\phi\colon C_p(Y) \to C_p(X)$ is a continuous linear mapping such that for every equicontinuous pointwise bounded set A in $C(Y)$ the image $\phi(A)$ is an equicontinuous pointwise bounded set in $C(X)$, we will say that ϕ *preserves equicontinuous pointwise bounded sets.*

Corollary 2.7. *The spaces X and Y are L-equivalent if and only if there exists a topological isomorphism $\phi\colon C_p(Y) \to C_p(X)$ such that both ϕ and ϕ^{-1} preserve equicontinuous pointwise bounded sets.*

Proof: First let us suppose that X and Y are L-equivalent, that is, there exists a topological isomorphism $\psi\colon L(X) \to L(Y)$. Consider the mapping $\phi\colon C_p(X) \to C_p(Y)$ defined by the rule $\phi(f) = f_\# \circ \psi^{-1} \circ \delta_Y$. It is clear that ϕ is continuous, linear and has the inverse topological isomorphism $\phi^{-1}(g) = g_\# \circ \psi \circ \delta_X$. It remains to show that given $\varepsilon > 0$ and an equicontinuous pointwise bounded set $A \subset C(X)$ the set

$$\bigcap_{f\in A}(f_\# \circ \psi^{-1})^{-1}(-\varepsilon,\varepsilon) = \psi\Big(\bigcap_{f\in A} f_\#^{-1}(-\varepsilon,\varepsilon)\Big)$$

is a neighborhood of zero, but this is straightforward.

Conversely, if there exists a topological isomorphism $\phi\colon C_p(Y) \to C_p(X)$ such that both ϕ and ϕ^{-1} preserve equicontinuous pointwise bounded sets, we can consider the map $\psi\colon L(X) \to L(Y)$ defined by $\psi(\alpha) = \alpha \circ \phi^{-1}$. Recall that $\alpha\circ\phi^{-1}$ is a continuous linear function on $C_p(Y)$, so $\alpha\circ\phi^{-1}$ is in $L(Y)$. Of course, ψ has an inverse topological isomorphism given by $\psi^{-1}(\beta) = \beta\circ\phi$. Since both ϕ and ϕ^{-1} preserve equicontinuous pointwise bounded sets, both ψ and ψ^{-1} are continuous. □

We say that a set A in a space X is *bounded* if every real continuous function on X is bounded on A. Recall that a function $f\colon X \to \mathbb{R}$ is *b-continuous* if for every bounded set A in X there exists a continuous function $g\colon X \to \mathbb{R}$ such that $g|A = f|A$. A space X is called a b_f-space if every b-continuous real function is continuous. The class of b_f-spaces contains all k-spaces.

$C_b(X)$ is the space $C(X)$ endowed with the topology of uniform convergence on the bounded sets of X. If X is a b_f-space, then a set $B \subset C_b(X)$ is compact if and only if B is closed, equicontinuous and pointwise bounded, see [16].

Corollary 2.8. *Let X and Y be two b_f-spaces that are l-equivalent. Then X and Y are L-equivalent.*

Proof: Let $\varphi\colon C_p(X) \to C_p(Y)$ be a topological isomorphism. According to [1] we can see that $\varphi\colon C_b(X) \to C_b(Y)$ is also a topological isomorphism. Now, let us take a set $A \subset C_p(X)$ which is equicontinuous and pointwise bounded. Since X is a b_f-space we have that $[A]_b$, the closure of A in $C_b(X)$ is compact. Hence $\varphi(A) \subset \varphi([A]_b)$ is equicontinuous and pointwise bounded. But $[A]_b = [A]_p$, the closure in $C_p(X)$, that is, φ preserves equicontinuous pointwise bounded sets. □

Example 2.9. It is known that if X is an uncountable discrete space, then the spaces $L_p(X)$ and $L_p(X) \oplus X$ are l-equivalent and they are not L-equivalent, that is, there does not exist a topological isomorphism between $C_p(L_p(X))$ and $C_p(L_p(X) \oplus X)$ that preserves equicontinuous pointwise bounded sets.

Returning to the consequences of Theorem 2.6, we have the following statement.

Corollary 2.10. *The following assertions are equivalent:*

(1) *Y is an L-retract of X.*
(2) *There exists a continuous linear retraction $r\colon L(X) \to L(Y)$.*
(3) *There exists a continuous linear extender $\varphi\colon C_p(Y) \to C_p(X)$ such that φ preserves equicontinuous pointwise bounded sets.*
(4) *Every continuous function from Y to a locally convex space E extends to a continuous function from X to E.*

From this proposition it follows immediately that, in the same way that X is l-embedded in $L_p(X)$ (X is an l-retract of $L_p(X)$), X is an L-retract of $L(X)$. Also, note that in view of Example 2.4, X is not always an L-retract of $L_p(X)$.

Now it is the time to apply our results. First, we can apply the Dugundji theorem to obtain the following.

Theorem 2.11. *Let X be a metric space. The following statements are equivalent:*

(1) *Y is a closed subset of X;*
(2) *Y is an L-retract of X;*
(3) *Y is l-embedded in X.*

Proof: Let Y be a closed subset of a metric space X and $\delta_Y : Y \to L(Y)$ the Dirac embedding of Y in $L(Y)$. Applying the Dugundji extension theorem we get a continuous function $f : X \to L(Y)$ such that $f|Y = \delta_Y$, then Y is an L-retract of X. The other implications are clear. $\square$

The Dugundji extension theorem has been subject to various generalizations, specifically, C. R. Borges generalized it to stratifiable spaces, and I. S. Stares did the same for the decreasing (G) spaces (a decreasing (G) space is a T_1 topological space with a countable decreasing local base that satisfies the (G) condition of [5]). On the other hand, note that each stratifiable space is a decreasing (G) space, and each decreasing (G) space is hereditarily paracompact, so we could ask if for the hereditarily paracompact spaces it is true that every closed set is an L-retract. However, the answer is “no”.

Example 2.12. There exists a hereditarily paracompact space X and a closed subset Y such that Y is not an L-retract of X.

Let X be the Michael line, see [6, Example 5.1.32], and let $E = C_k(P)$, where P is the set of irrational numbers equipped with the topology inherited from the Euclidean metric. Consider the subset Y of X consisting of all rational numbers with the subspace topology. In this context, Y is a closed and P-embedded set, but [15] indicates that the continuous function $f : Y \to C_k(P)$ defined by $f(x)(p) = 1/(x - p)$, $x \in Y$ and $p \in P$, cannot be continuously extended to the space X. That is, Y is not an L-retract of X.

The previous example shows that, in general, we must impose stronger conditions on the subset Y to make sure that Y be an L-retract of X. For instance, we will see that some of them need metrizability as an additional condition.

A set $A \subset X$ is called *strongly discrete* if there exists a discrete family $\{U_a : a \in A\}$ of disjoint open sets in X such that $a \in U_a$ for every $a \in A$. Taking into account the final observation of [11] we easily get the following.

Corollary 2.13. *Let Y be a subspace of X. Then:*

(1) *If X is paracompact and Y is closed and metrizable, then Y is an L-retract of X.*

(2) *If X is normal and Y is closed, metrizable and separable, then Y is an L-retract of X.*
(3) *If X is Tychonoff and Y is compact and metrizable, then Y is an L-retract of X.*
(4) *If X is Tychonoff and Y is strongly discrete, then Y is an L-retract of X.*

Proof: The first three statements are obvious. In [3] it was shown that if Y is a strongly discrete subspace, then Y is l-embedded in X. We will present the original proof, emphasizing the preservation of equicontinuous pointwise bounded sets by the defined extender. Let $\mathcal{U} = \{U_y : y \in Y\}$ be a discrete family of disjoint open sets in X such that $y \in U_y$ for every $y \in Y$, also, for every $y \in Y$ let $h_y \in C(X)$ be a function such that $h_y(X) \subset [0,1]$, $h_y(y) = 1$ and $h_y(X \setminus U_y) \subset \{0\}$. As the family $\mathcal{U}$ is discrete, the function $\sum_{y\in Y} h_y$ is defined on X and is continuous. Therefore, we can define a linear extender $\varphi\colon C_p(Y) \to C_p(X)$ by the rule $\varphi(f) = \sum_{y\in Y} f(y) \cdot h_y$. Since at every point of Y only finitely many functions h_y are distinct from 0, φ is continuous.

Let $\mathcal{F} \subset C_p(Y)$ be an equicontinuous and pointwise bounded family of functions. We will verify that $\varphi(\mathcal{F}) = \{\varphi(f) : f \in \mathcal{F}\}$ is equicontinuous and pointwise bounded. For each $y \in Y$ let $M_y \in \mathbb{R}$ be such that $\{f(y) : f \in \mathcal{F}\} \subset [-M_y, M_y]$. Given $\varepsilon > 0$ and $x \in X$, if x has a neighborhood disjoint from $\bigcup\mathcal{U}$, we have $\varphi(f)(x) = 0$ for every $f \in C_p(Y)$. Otherwise, there exists a neighborhood U of x such that $U \cap U_y \neq \emptyset$ for a unique $y \in Y$. Put $V = h_y^{-1}(h_y(x) - \varepsilon/M_y, h_y(x) + \varepsilon/M_y)$ and $W = U \cap V$. Then W is an open neighborhood of x, and for each $z \in W$ and $f \in \mathcal{F}$ we have

$$|\varphi(f)(x) - \varphi(f)(z)| = |f(y)(h_y(x) - h_y(z)| \leq M_y|h_y(x) - h_y(z)| < M_y \cdot \frac{\varepsilon}{M_y} = \varepsilon,$$

that is, $\varphi(\mathcal{F})$ is an equicontinuous set that clearly is pointwise bounded. □

Note that although in the class of metric spaces the L-retracts and the l-embedded sets are equivalent, in general, in the generalizations of the Dugundji extension theorem, we cannot change the condition that Y is an L-retract and replace it with the weaker condition that Y is an l-embedded set.

Example 2.14. In Corollary 2.13, it is important to note that the conditions of "being a strongly discrete set" or "being a compact and metrizable subspace" cannot be weakened to "being a discrete set" or "being a compact subspace", respectively.

Consider the discrete space Y of cardinality ω_1 and let $X = L_p(Y)$. Although Y is an l-embedded set, the function $\delta_Y : Y \to L(Y)$ does not have a continuous extension to X. If it had, then Y would be an L-retract of X.

Even, if both X and Y are compact spaces, it does not necessarily imply that Y is an L-retract of X. For example, let $X = \beta\mathbb{N}$ and $Y = \beta\mathbb{N} \setminus \mathbb{N}$. In this case, Y is not t-embedded in X, which implies that there exists no continuous mapping $\varphi\colon C_p(Y) \to C_p(X)$, see [3].

3. A method for constructing examples of L-equivalent spaces

Now we will concentrate on finding a method that generates examples of L-equivalence spaces. The method described by O. Okunev in [12, Theorem 2.4] already generates examples of L-equivalent spaces, however, the notion of a retract used in this method is quite restrictive, and as we see, every retract is an L-retract. Thus, we will show that the notion of an L-retract is sufficient to modify the method for L-equivalence.

Let K_1 and K_2 be two L-retracts of a space X, we will say that K_1 and K_2 are *parallel* if there exists continuous linear retractions $r_1\colon L(X) \to L(K_1)$ and $r_2\colon L(X) \to L(K_2)$ such that $r_1 \circ r_2 = r_1$ and $r_2 \circ r_1 = r_2$.

Proposition 3.1. *The sets K_1 and K_2 are parallel L-retracts of X if and only if there exists a continuous linear retraction $r_1\colon L(X) \to L(K_1)$ such that the restriction $r_1|L(K_2)$ is a topological isomorphism from $L(K_2)$ onto $L(K_1)$. In particular, K_1 and K_2 are L-equivalent.*

Proof: Suppose K_1 and K_2 are parallel L-retracts of X. Let $r_1\colon L(X) \to L(K_1)$ and $r_2\colon L(X) \to L(K_2)$ be continuous linear retractions such that $r_1 \circ r_2 = r_1$ and $r_2 \circ r_1 = r_2$. Then $i = r_1|L(K_2)$ is a topological isomorphism of $L(K_2)$ onto $L(K_1)$ with the inverse $j = r_2|L(K_1)$.

Conversely, suppose there exists a continuous linear retraction $r_1\colon L(X) \to L(K_1)$ such that the restriction $r_1|L(K_2)$ is a topological isomorphism from $L(K_2)$ onto $L(K_1)$, let $j = i^{-1}$ and put $r_2 = j \circ r_1$. Then r_2 is a continuous linear retraction from $L(X)$ to $L(K_2)$, $r_1 \circ r_2 = r_1$ and $r_2 \circ r_1 = r_2$. □

Recall that a continuous mapping $p\colon X \to Y$ is called $\mathbb{R}$-quotient if $p(X) = Y$ and whenever f is a real function on Y such that the composition $f \circ p\colon X \to \mathbb{R}$ is continuous, f is continuous, see [10]. The following statement is Proposition 1.10 in [12].

Proposition 3.2. *If $p\colon X \to Y$ is an $\mathbb{R}$-quotient mapping, Z is a completely regular space and $f\colon Y \to Z$ is a function such that the composition $f \circ p$ is continuous, then f is continuous.*

Proposition 3.3. *A mapping $p\colon X \to Y$ is $\mathbb{R}$-quotient if and only if its extension $p_\#\colon L(X) \to L(Y)$ is open.*

Proof: Suppose that $p_\#$ is open and let $f\colon Y \to \mathbb{R}$ be a function such that $f \circ p$ is continuous. Let $p_\#\colon L(X) \to L(Y)$ and $f_\#\colon L(Y) \to \mathbb{R}$ be the continuous linear extensions of f and p. Then $f_\# \circ p_\# = (f \circ p)_\#$ is continuous, and since $p_\#$ is open, $f_\#$ is continuous. Thus, $f = f_\# \circ \delta_Y$ is continuous.

Conversely, if p is $\mathbb{R}$-quotient, then the subspace $H = \ker p_\#$ is closed by continuity. Let $L = L(X)/H$ be the quotient space. This space L is locally convex and Hausdorff, hence Tychonoff. Furthermore, there exists a continuous bijection $i\colon L \to L(Y)$ such that $p_\# = i \circ \pi$, where $\pi\colon L(X) \to L$ is the natural projection. We now verify that the mapping $j = i^{-1}\colon L(Y) \to L$ is continuous. To do this, we only need to show that the restriction $f = j|Y$ is continuous. Since $f \circ p = (j \circ p_\#)|X = \pi|X$, we have that $f \circ p$ is continuous. Since p is $\mathbb{R}$-quotient, it follows that f is also continuous. Thus, j is continuous, and therefore i is a topological isomorphism. As π is open, it follows that $p_\#$ is open. $\square$

There exists a simple characterization of L-equivalence of $\mathbb{R}$-quotient mappings.

Proposition 3.4. *Two $\mathbb{R}$-quotient mappings $f\colon X \to Y$ and $g\colon Z \to T$ are L-equivalent if and only if there exists a topological isomorphism $i\colon L(X) \to L(Z)$ such that $i(\ker f_\#) = \ker g_\#$.*

Proof: If f and g are L-equivalent, then there exist topological isomorphisms $i\colon L(X) \to L(Z)$ and $j\colon L(Y) \to L(T)$ such that $j \circ f_\# = g_\# \circ i$. Let $A = \ker f_\#$ and $B = \ker g_\#$. Then $\{0\} = j \circ f_\#(A) = g_\# \circ i(A) = g_\#(i(A))$, that is, $i(A) \subset \ker g_\#$. Since $g_\# = j \circ f_\# \circ i^{-1}$, it follows that $\{0\} = g_\#(B) = j \circ f_\# \circ i^{-1}(B)$. Considering that j is bijective we have that $f_\# \circ i^{-1}(B) = \{0\}$, we can conclude that $i^{-1}(B) \subset A$, and this is enough to establish the equality.

Conversely, suppose that there exists a topological isomorphism $i\colon L(X) \to L(Z)$ such that $i(\ker f_\#) = \ker g_\#$. Then there exists an (algebraic) isomorphism $j\colon L(Y) = L(X)/\ker f_\# \to L(T) = L(Z)\ker g_\#$ such that $j \circ f_\# = g_\# \circ i$. Since $g_\#$ and i are continuous, and $f_\#$ is open, j is continuous. Similarly, $j^{-1} \circ g_\# = f_\# \circ i^{-1}$, $f_\#$ and i^{-1} are continuous, and $g_\#$ is open, so j^{-1} is continuous. Therefore, we can conclude that i and j are topological isomorphisms, as required in the definition of L-equivalent mapping. $\square$

We will now define the $\mathbb{R}$-quotient spaces. Let $p\colon X \to Y$ be a mapping of the topological space X onto a set Y. Then there exists a unique completely regular topology on the set Y that makes p an $\mathbb{R}$-quotient mapping. This topology can be described as the weakest topology with respect to which all real functions on Y, whose composition with p is continuous, are continuous. This topology is called the $\mathbb{R}$-*quotient topology*, and Y endowed with this topology is the $\mathbb{R}$-*quotient space* *with respect to the mapping* p, or simply the $\mathbb{R}$-quotient space if the mapping p is clear from the context. In this situation we say that p is the *natural mapping*.

Now, if X is a space and K is a closed subset in X, let us denote $X/K = (X \setminus K) \cup \{K\}$, and let $p(x) = x$ for $x \in X \setminus K$, and $p(x) = K$ for each $x \in K$. Therefore, there exists only one completely regular topology on X/K that makes it the $\mathbb{R}$-quotient space with respect to p. It is shown in [12] that this space is Tychonoff. Also note that $p|(X \setminus K)\colon X \setminus K \to X/K \setminus p(K)$ is a homeomorphism [12, Corollary 1.7].

With all of this, we can establish our method.

Theorem 3.5. *If K_1 and K_2 are parallel L-retracts of X, then the $\mathbb{R}$-quotient mappings $p_1\colon X \to X/K_1$ and $p_2\colon X \to X/K_2$ are L-equivalent. In particular, the spaces X/K_1 and X/K_2 are L-equivalent.*

Proof: Suppose that $r_1\colon L(X) \to L(K_1)$ and $r_2\colon L(X) \to L(K_2)$ are parallel L-retractions. We define a mapping $i\colon L(X) \to L(X)$ by the rule $i(\alpha) = r_1(\alpha) + r_2(\alpha) - \alpha$ for all $\alpha \in L(X)$. Clearly, i is a continuous linear mapping such that $i \circ i(\alpha) = \alpha$, that is, i is its own inverse, so i is a topological isomorphism.

Let us put $s_2 = r_2|L(K_1)$, then s_2 is a topological isomorphism and satisfies $s_2 \circ r_1 = r_2 \circ i$. It follows that $i(L(K_1)) = L(K_2)$ and $i(\ker r_1) = \ker r_2$. It is also evident that $\ker(p_i)_\# = L^0(K_i) = \ker(e_{K_i})_\#$, $i = 1, 2$.

As K_1 and K_2 are L-equivalent, there exists a special topological isomorphism $k\colon L(K_1) \to L(K_2)$ such that $(e_{K_2})_\# \circ k = (e_{K_1})_\#$. Let $g = k \times j$, where $j = i|\ker r_1$, and $\alpha \in L(X)$. The mappings $\eta_i\colon L(X) \to L(K_i) \times \ker r_i$, $i = 1, 2$, defined by $\eta_i(\alpha) = (r_i(\alpha), \alpha - r_i(\alpha))$ are topological isomorphisms with inverses $\xi_i\colon L(K_i) \times \ker r_i \to L(X)$, where $\xi_i(\alpha, \beta) = \alpha + \beta$, $i = 1, 2$.

Defining a mapping ψ by

$$\begin{aligned}\psi(\alpha) &= \xi_2 \circ g \circ \eta_1(\alpha) = \xi_2 \circ g(r_1(\alpha), \alpha - r_1(\alpha))\\ &= \xi_2\big(k(r_1(\alpha)), j(\alpha - r_1(\alpha))\big) = k(r_1(\alpha)) + j(\alpha) - j(r_1(\alpha)),\end{aligned}$$

we obtain a topological isomorphism such that $\psi(L^0(K_1)) = L^0(K_2)$. Thus, by Proposition 3.4, p_1 is L-equivalent to p_2. $\square$

We denote X^+ the space $X \oplus \{a\}$ where $a \notin X$.

Corollary 3.6. *Let X be a topological space and $K \subset X$ be an L-retract of X. Then the spaces X^+ and $X/K \oplus K$ are L-equivalent.*

Proof: Let K' be a homeomorphic copy of K that is disjoint from X, let $\varphi\colon K \to K'$ be a homeomorphism and $r\colon L(X) \to L(K)$ be an L-retraction. Recall that if $Z = X \oplus K'$, then $L(Z)$ is topologically isomorphic to $L(X) \oplus L(K')$. Define $r_1\colon L(Z) \to L(K)$ and $r_2\colon L(Z) \to L(K')$ as follows: $r_1|L(X) = r$,

$r_1|L(K') = \varphi_\#^{-1}$, $r_2|L(X) = \varphi_\# \circ r$ and $r_2|L(K') = \mathrm{id}_{L(K')}$. Then we have $(r_1 \circ r_1)|L(X) = r_1 \circ r = r = r_1|L(X)$ and $(r_1 \circ r_1)|L(K') = r_1 \circ \varphi_\#^{-1} = \varphi_\#^{-1} = r_1|L(K')$ (because $r_1|L(K)$ is the identity), so r_1 is a retraction. Similarly r_2 is also a retraction.

Moreover, it can be shown that $(r_1 \circ r_2)|L(X) = r_1 \circ \varphi_\# \circ r = \varphi_\#^{-1} \circ \varphi_\# \circ r = r_1|L(X)$ and $(r_1 \circ r_2)|L(K') = r_1|L(K')$. Therefore $r_1 \circ r_2 = r_1$. Similarly, we can show that $r_2 \circ r_1 = r_2$, which implies that r_1 and r_2 are parallel L-retracts. By Theorem 3.5, the spaces Z/K and Z/K' are L-equivalent. It is also evident that Z/K is homeomorphic to $X/K \oplus K$ and Z/K' is homeomorphic to X^+. $\square$

Note that in the proof of Theorem 3.5 the fact that the L-retracts are parallel served to guarantee the existence of a pair of topological isomorphisms s_2 and i such that $s_2 \circ r_1 = r_2 \circ i$. Therefore, in the case that two sets K_1 and K_2 are L-retracts of X and there exist topological isomorphisms $i\colon L(X) \to L(X)$, $j\colon L(K_1) \to L(K_2)$ and continuous linear retractions $r_1\colon L(X) \to L(K_1)$ and $r_2\colon L(X) \to L(K_2)$ such that $j \circ r_1 = r_2 \circ i$ we will say that these sets are *equivalent L-retracts.*

Proposition 3.7. *Let $r\colon L(X) \to L(K)$ be a continuous linear retraction, where $K \subset X$. Then $L(X)$ is topologically isomorphic to $GL(X/K) \times L(K)$ and $GL(X/K)$ is topologically isomorphic to $\ker r$.*

Proof: We will write $L \cong E$ if the topological linear spaces L and E are topologically isomorphic. The first part of the proof follows from Theorem 3.5, Corollaries 1.7 and 1.3, and the following chain of topological isomorphisms

$$L(X) \cong GL(X^+) \cong GL(X/K \oplus K) \cong GL(X/K) \oplus L(K).$$

The second part of the proof is due to the observation that if $r\colon L(X) \to L(K)$ is a continuous linear retraction, then $L(X) \cong L(K) \times \ker r$. Thus

$$L(K) \times \ker r \cong L(K) \oplus GL(X/K) \cong L(K) \times GL(X/K).$$

To conclude the proof, we note that the natural mapping $p\colon X \to X/K \subset GL(X/K)$ is $\mathbb{R}$-quotient. Therefore, the linear continuous extension $p_\#\colon L(X) \to GL(X/K)$ is open and onto. Moreover, we can easily see that $\ker p_\# = L(K)$, which implies that $L(X)/L(K) \cong GL(X/K)$. On the other hand, consider the continuous linear mapping $\psi\colon L(X) \to \ker r$ given by $\psi(\alpha) = \alpha - r(\alpha)$. This mapping is open, its kernel is $L(K)$ and $L(X)/L(K) \cong \ker r$. Hence, we can conclude that $GL(X/K) \cong \ker r$. $\square$

In a way, if we have an L-retraction $r\colon L(X) \to L(K)$, we can obtain enough information about $L(X)$ since, as a corollary of the previous proposition, we can see that $L(X)$ is topologically isomorphic to $\ker p_{\#} \oplus \ker r \oplus \mathbb{R}$, where $p_{\#}$ is the linear continuous extension of the natural mapping p.

Proposition 3.8. *Let K_1 and K_2 be two L-retracts of X. If the natural mappings $p_1\colon X \to X/K_1$ and $p_2\colon X \to X/K_2$ are L-equivalent, then K_1 and K_2 are equivalent L-retracts.*

Proof: Let r_1 and r_2 be a pair of retractions associated with K_1 and K_2, respectively. Since the natural mappings p_1 and p_2 are L-equivalent, there exists topological isomorphisms $i\colon L(X) \to L(X)$ and $j\colon L(X/K_1) \to L(X/K_2)$ such that $j \circ (p_1)_{\#} = (p_2)_{\#} \circ i$. In view of the assumption that X/K_1 is L-equivalent to X/K_2, we have that $\ker r_1$ and $\ker r_2$ are topologically isomorphic; let us denote such topological isomorphism by t.

Using the equality $i(L^0(K_1)) = i(\ker(p_1)_{\#}) = \ker(p_2)_{\#} = L^0(K_2)$, we obtain that $L(K_1)$ is topologically isomorphic to $L(K_2)$. Let us denote by k such a topological isomorphism. Then $w = k \times t$ is a topological isomorphism between $L(K_1) \times \ker r_1$ and $L(K_2) \times \ker r_2$. Therefore, we have a topological isomorphism $\varphi\colon L(X) \to L(X)$, which is defined by the formula

$$\begin{aligned}\varphi(\alpha) &= \xi_2 \circ w \circ \eta_1(\alpha) = \xi_2 \circ w(r_1(\alpha), \alpha - r_1(\alpha)) \\ &= \xi_2\big(k(r_1(\alpha)), t(\alpha - r_1(\alpha))\big) = k(r_1(\alpha)) + t(\alpha) - t(r_1(\alpha)).\end{aligned}$$

The mappings ξ_i and η_i, $i = 1, 2$, are defined as in the proof of Theorem 3.5.

We quickly notice that under this isomorphism, $\varphi(\ker r_1) = \ker r_2$, and therefore, defining $\psi\colon L(K_1) \to L(K_2)$ by $\psi(\alpha) = r_2 \circ \varphi(r_1^{-1}(\alpha))$, we obtain a topological isomorphism such that $\psi \circ r_1 = r_2 \circ \varphi$. This proves that K_1 and K_2 are equivalent L-retracts. ☐

Corollary 3.9. *Let K_1 and K_2 be L-retracts of X, and $p_1\colon X \to X/K_1$, $p_2\colon X \to X/K_2$ the corresponding natural mappings. The following statements are equivalent:*

(1) *K_1 and K_2 are equivalent L-retracts;*
(2) *p_1 and p_2 are L-equivalent;*
(3) *K_1 is L-equivalent to K_2, and X/K_1 is L-equivalent to X/K_2.*

Proof: The equivalence between items (1) and (2) is evident. To demonstrate the remaining equivalences, it suffices to prove that item (3) implies item (1). First, according to the hypothesis, we have $GL(X/K_1) \cong GL(X/K_2)$, and thus, by Proposition 3.7 we conclude that $\ker r_1$ and $\ker r_2$ are topologically isomorphic. Then, using the technique described in the previous propositions, we obtain topological isomorphism $i\colon L(X) \to L(X)$ and $j\colon L(K_1) \to L(K_2)$ such that $i(\ker r_1) = \ker r_2$ and $j \circ r_1 = r_2 \circ i$. Therefore, K_1 and K_2 are equivalent L-retracts. ☐

Corollary 3.10. *Let $r_1\colon X \to K_1$ and $r_2\colon X \to K_2$ be two retractions in X, and $p_1\colon X \to X/K_1$, $p_2\colon X \to X/K_2$ the respective natural mappings. The following statements are equivalent:*

(1) *r_1 is L-equivalent to r_2;*
(2) *p_1 is L-equivalent to p_2;*
(3) *K_1 is L-equivalent to K_2, and X/K_1 is L-equivalent to X/K_2.*

Corollary 3.11. *Two retractions onto the same retract are L-equivalent.*

Corollary 3.12. *Let X and Y be two L-equivalent spaces, K_1 and K_2 be retracts respectively of X and Y, which are L-equivalent and such that X/K_1 is L-equivalent to Y/K_2. Then any two retractions $X \to K_1$ and $Y \to K_2$ are L-equivalent, moreover, the corresponding natural mappings are also L-equivalent.*

Example 3.13. Consider the retractions, r_1 and r_2, defined on the interval $[0,1]$ to $[0,1/2]$ as follows: $r_1(x) = x$ for $x \in [0,1/2]$ and $r_1(x) = 1-x$ for $x \in [1/2,1]$, $r_2(x) = x$ for $x \in [0,1/2]$ and $r_2(x) = 1/2$ for $x \in [1/2,1]$. These retractions are L-equivalent. It is worth noting that r_1 is both perfect and open, whereas r_2 is perfect but not open.

Corollary 3.14. *The property of being an open mapping is not preserved under the relation of L-equivalence, even within the class of perfect retractions.*

Example 3.15. Let X be the topological product of two disjoint copies of the integers $\mathbb{Z}$. Consider the set $K = (\mathbb{N}\cup\{0\})\times\{0\}$ and the retractions $r_1, r_2\colon X \to K$ defined as $r_1(n,m) = (\max\{|n|,|m|\},0)$, and $r_2(n,m) = (|n+m|,0)$. By Corollary 3.11 these retractions are L-equivalent. Furthermore, r_1 is perfect and finite-to-one, while r_2 is closed but not perfect (since r_2 has no compact fibers) and not finite-to-one.

Corollary 3.16. *The property of being a perfect function is not an L-invariant of continuous functions, even within the class of retractions. In particular, the property of being a function with compact fibers is not L-invariant of continuous functions.*

Corollary 3.17. *The property of being a finite-to-one function is not an L-invariant of continuous functions, even within the class of retractions.*

Example 3.15 can be modified to show that the property of being a perfect function is not preserved by means of the M-equivalence relation (a relation that derives from the construction of the free topological group $F(X)$), in particular, the property of being a function with compact fibers is not an M-invariant either. In this way, Example 3.15 provides a solution to Question 3.25 of [12]. Moreover, in view of Corollary 3.14 and Example 3.15, the following question arises: Is the property of being a closed function preserved through any of these algebraic-topological equivalence relations?

On the other hand, $L_n(X)$ denotes the subsets of $L(X)$ consisting of linear combinations of at most n elements, and we will say that two spaces X and Y are *strongly L-equivalent* if there exists a topological isomorphism $\varphi\colon L(X) \to L(Y)$ so that $\varphi(X) \subset L_n(Y)$ and $\varphi^{-1}(Y) \subset L_m(X)$ for some integers m and n.

Example 3.18. Let X be a compact space with uncountable cellularity. If we take the set ω with the discrete topology, then the spaces $X\times\omega$ and $(X\times\omega)\oplus X$ are homeomorphic and σ-compact. Since $L(X\times\omega)$ is an L-retract of $L(L(X\times\omega))$ there exists a continuous linear retraction $r\colon L(L(X\times\omega)) \to L(X\times\omega)$ and we can decompose $L(L(X\times\omega))$ as $L(X\times\omega)\oplus\ker r$. But $L(X\times\omega)\oplus\ker r$ is topologically isomorphic to

$$L((X \times \omega) \oplus X) \oplus \ker r \cong L(X) \oplus L(X \times \omega) \oplus \ker r \cong L(X) \oplus L(L(X \times \omega)),$$

that is, $L(X \times \omega)$ and $L(X \times \omega) \oplus X$ are L-equivalent.

We know that cellularity is an (l-invariant) L-invariant in the strong sense, and every σ-compact topological group has the Souslin property. Therefore $L(X \times \omega)$ has the Souslin property, however it is not true that $L(X\times\omega)\oplus X$ has the Souslin property, that is, these spaces are not L-equivalent in the strong sense.

Corollary 3.19. *There exist spaces that are L-equivalent and not strongly L-equivalent.*

Example 3.20. Let $X = [0, 1]$ and $K = \{0, 1\}$, then K is an L-retract of X and X^+ is L-equivalent to $X/K \oplus K$, however, passing to the free locally convex spaces in the sense of Graev we have that $L([0, 1])$ is topologically isomorphic to $L((\mathbb{S}^1)^+)$, where X/K is homeomorphic to the unit circle $\mathbb{S}^1$. The unit interval $[0, 1]$ is connected, but $(\mathbb{S}^1)^+$ is not, so these spaces are not A-equivalent (their free topological Abelian groups are not topologically isomorphic).

Corollary 3.21. *There exist L-equivalent compact metrizable spaces that are not A-equivalent.*

Corollary 3.22. *Connectedness is not an L-invariant property, even in the class of compact metrizable spaces.*

All definitions and propositions in this section can be adapted to be applicable to free topological Abelian groups and free topological vector spaces. This means that the material in this section can be used to develop a method for constructing examples of spaces that are A-equivalent or V-equivalent. However, it is necessary to establish characterizations of A-retracts and V-retracts.

The free topological vector space (in the Markov sense) over a topological space X is a pair $(\delta_X, V(X))$ formed by a continuous injection $\delta_X\colon X \to V(X)$ and a topological vector space $V(X)$ such that $V(X)$ is the linear span of $\delta_X(X)$ and for every continuous function $f\colon X \to E$ to a topological vector space E, there exists a unique continuous linear mapping $f_\#\colon V(X) \to E$ such that $f = f_\# \circ \delta_X$. The concepts of V-equivalent spaces and functions, V-invariant properties, V-retracts, among others, are entirely similar to those already defined in previous sections. It is important to note that the sets $V(X)$ and $L(X)$ are essentially the same, with the difference being in the topology. Therefore, the identity map $(\mathrm{id}_X)_\#\colon V(X) \to L(X)$ is continuous.

To characterize those sets that are V-retracts, we must consider that R. Cauty constructed a metrizable and σ-compact topological vector space V in [4], which is not an absolute extensor for the class of metric spaces. That is, not every closed subset of a metric space will be a V-retract, even if the spaces are compact. Although this limits our options, we can at least ensure the following:

Theorem 3.23. *If K is a strongly discrete and at most countable subset of a topological space X, then K is a V-retract of X.*

Proof: For countable and strongly discrete subsets $K \subset X$, we have that $L(K) = V(K)$, in this sense, we have that $L(K) = \mathbb{R}^n$ for some $n \in \mathbb{N}$ or $L(K) = L(\mathbb{N})$ is the inductive limit of the spaces $\mathbb{R}^n$, $n \in \mathbb{N}$. $\square$

If we modify Examples 3.18 and 3.20, we can obtain the following statements.

Corollary 3.24. *There exist spaces that are V-equivalent but not strongly V-equivalent.*

Corollary 3.25. *There exist V-equivalent compact metrizable spaces that are not A-equivalent.*

Corollary 3.26. *Connectedness is not a V-invariant property, even in the class of compact metrizable spaces.*

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