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Revisiting linear Weingarten spacelike submanifolds immersed in a locally symmetric semi-Riemannian space

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Abstract. In this paper, we deal with n-dimensional complete linear Weingarten spacelike submanifolds immersed with parallel normalized mean curvature vector field and flat normal bundle in a locally symmetric semi-Riemannian space L_p^{n+p} of index $p > 1$, which obeys some curvature constraints (such an ambient space can be regarded as an extension of a semi-Riemannian space form). Under appropriate hypothesis, we are able to prove that such a spacelike submanifold is either totally umbilical or isometric to an isoparametric submanifold of the ambient space. For this, we use three main core analytical tools: a suitable version of the Omori–Yau maximum principle, parabolicity with respect to a modified Cheng–Yau operator and a certain integrability property.

Keywords: locally symmetric semi-Riemannian space; mean curvature vector field; complete linear Weingarten spacelike submanifold; totally umbilical submanifold; isoparametric submanifold; $\mathcal{L}$-parabolicity

Classification: 53C42, 53C21, 53C50

1. Introduction

Let us denote by L_p^{n+p} an $(n+p)$-dimensional connected semi-Riemannian manifold with index p. An n-dimensional submanifold M^n immersed in L_p^{n+p} is said to be *spacelike* if the induced metric on M^n is positive definite. The study of spacelike submanifolds immersed in a semi-Riemannian space constitutes an important thematic from both physical and mathematical points of view. For instance, it was pointed out by J. Marsden and F. Tipler in [23] and S. Stumbles in [31] that spacelike hypersurfaces with constant mean curvature in an arbitrary Lorentzian space (which is a semi-Riemannian space of index $p = 1$) play an important role in the general relativity, in that they serve as convenient initial data for the Cauchy problem for Einstein's equations. Furthermore, submanifold theory provides the adequate tools to approach some important problem involving spacetime singularities and gravitational collapse. The singularity theorems

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proved in the 1960s by R. Penrose in [27] and S. W. Hawking and G. F. R. Ellis in [19] state that the formation of singularities is unavoidable, if one assumes reasonable conditions on the curvature of the spacetime, on the extrinsic geometry of certain submanifolds and on the causal structure of the Lorentzian manifold. The existence of spacelike submanifolds in the spacetime, in particular, is a key requirement in the original formulation of the singularity theorems as well as in their more recent generalizations, for more details, see G. J. Galloway and J. M. M. Senovilla in [17], Z. Liang and X. Zhang in [21], J. M. M. Senovilla in [30].

From the mathematical point of view, the interest in the study of the geometry of these submanifolds is mostly due to the fact that they exhibit nice Bernstein-type properties, and one can truly say that the first remarkable results in this branch were the rigidity theorems of E. Calabi in [7] and S. Y. Cheng and S. T. Yau in [9], who showed (the former for $n \leq 4$, and the latter for general n) that the only complete maximal spacelike hypersurfaces of the Lorentz–Minkowski space $\mathbb{L}^{n+1}$ are the spacelike hyperplanes. However, in the case that the mean curvature is a positive constant, A. E. Treibergs in [33] astonishingly showed that there are many entire solutions of the corresponding constant mean curvature equation in $\mathbb{L}^{n+1}$, which he was able to classify by their projective boundary values at infinity.

S. Nishikawa in [25] extend the results of [7] and [9] for a wide class of semi-Riemannian spaces, the so-called locally symmetric semi-Riemannian spaces. We recall that a semi-Riemannian space L_p^{n+p} is said to be *locally symmetric* when its curvature tensor $\overline{R}$ is parallel, that is, $\overline{\nabla}\,\overline{R} = 0$, where $\overline{\nabla}$ denotes the Levi–Civita connection of L_p^{n+p}. At this point, it is worth to recall that a fundamental property of curvature is its control over the relative behavior of nearby geodesics. Because a normal neighborhood $\mathcal{U}$ is filled with radial geodesics, curvature thereby gives a description of the geometry of $\mathcal{U}$. Considering only the locally symmetric case, we have that this description is so accurate that, if $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ are normal neighborhoods with the same description (and same dimension and index), then $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ must be isometric, for more details concerning locally symmetric spaces, see [26, Chapter 8]. Returning to our context, we note that the seminal paper [25] induced the appearing of several works approaching the problem of characterizing complete spacelike hypersurfaces immersed in a locally symmetric Lorentzian space, see, for instance, [4], [12], [13], [15], [16], [22]. On the other hand, considering higher codimensions, T. Ishihara in [20] applied a technique developed by S. S. Chern, M. P. do Carmo and S. Kobayashi in [11] in order to extend the results of [7] and [9] for complete maximal spacelike submanifolds in a semi-Riemannian space form of constant nonnegative sectional curvature.

More recently, J. G. Araújo et al. in [14] investigated complete maximal spacelike submanifolds immersed with flat normal bundle in a locally symmetric semi-Riemannian space obeying curvature conditions similar to those of S. Nishikawa in [25]. In this setting, they obtained a suitable Simons type formula and, as application, they showed that such a spacelike submanifold must be totally geodesic or the square norm of its second fundamental form must be bounded, extending the results of T. Ishihara in [20] and S. Nishikawa in [25]. Afterwords, J. G. Araújo et al. in [3] studied n-dimensional complete linear Weingarten spacelike submanifolds M^n with flat normal bundle and parallel normalized mean curvature vector field immersed in an $(n+p)$-dimensional locally symmetric semi-Riemannian manifold L_p^{n+p}. We also recall that a spacelike submanifold M^n of L_p^{n+p} is called *linear Weingarten* if its mean curvature H and its normalized scalar curvature R satisfy a linear relation of the type $R = aH + b$ for some real constants a and b. In this setting, they obtained sufficient conditions to guarantee that, in fact, $p = 1$ and M^n is isometric to an isoparametric hypersurface of L_1^{n+1} having two distinct principal curvatures, one of which is simple. Next, working in this same context, the authors of the present paper jointly with J. G. Araújo et al. in [2] obtained another characterization result assuming an appropriate boundedness on the square norm of the second fundamental form of M^n and considering the case that the ambient space L_p^{n+p} is also conformally flat in order to reduce the codimension to $p = 1$.

Here, our aim is also study complete linear Weingarten spacelike submanifolds with parallel normalized mean curvature vector field and flat normal bundle in a locally symmetric semi-Riemannian space L_p^{n+p} with index $p > 1$ and obeying the same set of curvature conditions assumed in [2] and [3]. Initially, we establish a more refined version of the Omori–Yau maximum principle, see Proposition 1, which enables us to prove that such a spacelike submanifold must be either totally umbilical or isometric to an isoparametric submanifold of the ambient space, see Theorem 1. Afterwords, we assume the parabolicity with respect to a modified Cheng–Yau operator and an integrability property in order to get additional characterization results, see Theorems 2 and 3.

2. Background

This section is devoted to present the necessary background to establish our characterization results for linear Weingarten submanifolds immersed in a locally symmetric semi-Riemannian space.

2.1 General facts concerning spacelike submanifolds. Let M^n be a spacelike submanifold immersed in a locally symmetric semi-Riemannian space L_p^{n+p}.

In this context, we choose a local field of semi-Riemannian orthonormal frames $e_1, \dots, e_{n+p}$ in L_p^{n+p}, with dual coframes $\omega_1, \dots, \omega_{n+p}$, such that, at each point of M^n, $e_1, \dots, e_n$ are tangent to M^n. We will use the following convention of indices

$$1 \leq A, B, C, \dots \leq n+p, \qquad 1 \leq i, j, k, \dots \leq n$$

and

$$n+1 \leq \alpha, \beta, \gamma, \dots \leq n+p.$$

In this setting, the semi-Riemannian metric of L_p^{n+p} is given by

$$\mathrm{d}\overline{s}^2 = \sum_A \varepsilon_A \, \omega_A^2,$$

where $\varepsilon_i = 1$ and $\varepsilon_\alpha = -1$. Denoting by $\{\omega_{AB}\}$ the connection forms of L_p^{n+p}, we have that the structure equations of L_p^{n+p} are given by:

$$\mathrm{d}\omega_A = \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0,$$
$$\mathrm{d}\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \varepsilon_C \varepsilon_D \overline{R}_{ABCD} \omega_C \wedge \omega_D,$$

where, $\overline{R}_{ABCD}$ is the semi-Riemannian curvature tensor of the Lorentz space L_p^{n+p}. In this configuration, the components $\overline{R}_{CD}$ of the Ricci tensor and the normalized scalar curvature $\overline{R}$ of L_p^{n+p} are defined respectively by

$$\overline{R}_{CD} = \sum_B \varepsilon_B \overline{R}_{CBDB} \qquad \text{and} \qquad \overline{R} = \sum_A \varepsilon_A \overline{R}_{AA}.$$

Moreover, the components $\overline{R}_{ABCD;E}$ of the covariant derivative of the Riemannian curvature tensor L_p^{n+p} are defined by

$$\sum_E \varepsilon_E \overline{R}_{ABCD;E} \omega_E = \mathrm{d}\overline{R}_{ABCD} - \sum_E \varepsilon_E \big(\overline{R}_{EBCD} \omega_{EA} + \overline{R}_{AECD} \omega_{EB} + \overline{R}_{ABED} \omega_{EC} + \overline{R}_{ABCE} \omega_{ED} \big).$$

Next, we restrict all the tensors to M^n. First of all,

$$\omega_\alpha = 0, \qquad n+1 \leq \alpha \leq n+p.$$

Consequently, the Riemannian metric of M^n is written as

$$\mathrm{d}s^2 = \sum_i \omega_i^2.$$

Since

$$\sum_i \omega_{\alpha i} \wedge \omega_i = \mathrm{d}\omega_\alpha = 0,$$

from Cartan's lemma we can write

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j, \qquad h^\alpha_{ij} = h^\alpha_{ji}. \tag{2.1}$$

This gives the *second fundamental form* of M^n,

$$A = \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i \otimes \omega_j e_\alpha,$$

and its square length from second fundamental form is

$$S = |A|^2 = \sum_{\alpha,i,j} (h^\alpha_{ij})^2.$$

Furthermore, we define the *mean curvature vector field* h and the *mean curvature function* H of M^n respectively by

$$h = \frac{1}{n}\sum_\alpha \left(\sum_i h^\alpha_{ii}\right) e_\alpha \qquad \text{and} \qquad H = |h| = \frac{1}{n}\sqrt{\sum_\alpha \left(\sum_i h^\alpha_{ii}\right)^2}.$$

The structure equations of M^n are given by

$$\mathrm{d}\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0,$$
$$\mathrm{d}\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$

where R_{ijkl} are the components of the curvature tensor of M^n. Using the previous structure equations, we obtain Gauss equation

$$R_{ijkl} = \overline{R}_{ijkl} - \sum_\beta (h^\beta_{ik}h^\beta_{jl} - h^\beta_{il}h^\beta_{jk}). \tag{2.2}$$

Denoting by R the *normalized scalar curvature* of M^n, from (2.2) we get

$$n(n-1)R = \sum_{i,j} \overline{R}_{ijij} - n^2H^2 + S. \tag{2.3}$$

We also state the structure equations of the normal bundle of M^n

$$\mathrm{d}\omega_\alpha = -\sum_\beta \omega_{\alpha\beta}\wedge\omega_\beta, \qquad \omega_{\alpha\beta}+\omega_{\beta\alpha}=0,$$
$$\mathrm{d}\omega_{\alpha\beta} = -\sum_\gamma \omega_{\alpha\gamma}\wedge\omega_{\gamma\beta} - \frac{1}{2}\sum_{k,l} R_{\alpha\beta kl}\omega_k\wedge\omega_l.$$

Let M^n have normal bundle flat, that is, $R^\perp = 0$ (equivalently $R_{\alpha\beta jk}=0$), we get the following Ricci equation

$$\overline{R}_{\alpha\beta ij} = \sum_k (h^\alpha_{ik}h^\beta_{kj} - h^\alpha_{kj}h^\beta_{ik}). \tag{2.4}$$

The components h^α_{ijk} of the covariant derivative ∇B satisfy

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{ik}\omega_{kj} + \sum_k h^\alpha_{jk}\omega_{ki} - \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}. \tag{2.5}$$

In this setting, from (2.1) and (2.5) we get *Codazzi equation*

$$\overline{R}_{\alpha ijk} = h^\alpha_{ijk} - h^\alpha_{ikj}.$$

The first and the second covariant derivatives of h^α_{ij} are denoted by h^α_{ijk} and h^α_{ijkl}, respectively, which satisfy

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} + \sum_l h^\alpha_{ljk}\omega_{li} + \sum_l h^\alpha_{ilk}\omega_{lj} + \sum_l h^\alpha_{ijl}\omega_{lk} - \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}.$$

Thus, taking the exterior derivative in (2.5), we obtain the following *Ricci identity*

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{im}R_{mjkl} + \sum_m h^\alpha_{mj}R_{mikl}.$$

Restricting the covariant derivative $\overline{R}_{ABCD;E}$ of $\overline{R}_{ABCD}$ on M^n, then $\overline{R}_{\alpha ijk;l}$ is given by

$$\overline{R}_{\alpha ijkl} = \overline{R}_{\alpha ijk;l} + \sum_\beta \overline{R}_{\alpha\beta jk}h^\beta_{il} + \sum_\beta \overline{R}_{\alpha i\beta k}h^\beta_{jl} + \sum_\beta \overline{R}_{\alpha ij\beta}h^\beta_{kl} + \sum_{m,k}\overline{R}_{mijk}h^\alpha_{lm},$$

where $\overline{R}_{\alpha ijkl}$ denotes the covariant derivative of $\overline{R}_{\alpha ijk}$ as a tensor on M^n.

2.2 Linear Weingarten spacelike submanifolds. For our purposes, we will consider that the mean curvature function H is positive, so that in the local orthonormal frame $\{e_1,\dots,e_{n+p}\}$ we take

$$e_{n+1} = \frac{h}{H}\,.$$

Thus, we deal with the *traceless second fundamental form* Φ, which is defined as the symmetric tensor

$$\Phi = \sum_{\alpha,i,j} \Phi_{ij}^{\alpha} \omega_i \otimes \omega_j e_{\alpha},$$

where $\Phi_{ij}^{\alpha} = h_{ij}^{\alpha} - H^{\alpha}\delta_{ij}$. Here, H^{α} denotes the mean curvature function of M^n in the direction of e_{α}, that is,

$$H^{n+1} = \frac{1}{n}\mathrm{tr}(h^{n+1}) = H \quad \text{and} \quad H^{\alpha} = \frac{1}{n}\mathrm{tr}(h^{\alpha}) = 0, \qquad \alpha \geq n+2,$$

where $h^{\alpha} = (h_{ij}^{\alpha})$ denotes the second fundamental form of M^n in direction e_{α} for every $n+1 \leq \alpha \leq n+p$. From here it is not difficult to verify that Φ is a traceless tensor, that is, $\mathrm{tr}(\Phi) = 0$ and that it holds the following relation

$$|\Phi|^2 = S - nH^2.$$

Moreover, $|\Phi|$ vanishes identically on M^n if and only if M^n is a totally umbilical spacelike submanifold. For this reason, Φ is also called the total umbilicity tensor of M^n. We also note that, by (2.3), the following relation is trivially satisfied:

$$n(n-1)R = \sum_{i,j} \overline{R}_{ijij} - n(n-1)H^2 + |\Phi|^2. \tag{2.6}$$

At this point, we will assume that M^n is a *linear Weingarten* spacelike submanifold, which means that the normalized scalar curvature and mean curvature functions are linearly related in the following way: there exist real constants $a, b \in \mathbb{R}$ such that

$$R = aH + b.$$

Related to the geometry of linear Weingarten spacelike submanifolds there exists a *Cheng–Yau type differential operator*, which recently has been considered by many authors. More precisely, let us introduce the second order linear differential operator $\mathcal{L}\colon \mathcal{C}^{\infty}(M) \to \mathcal{C}^{\infty}(M)$ defined by

$$\mathcal{L} = L + \frac{n-1}{2}\, a\Delta, \tag{2.7}$$

where Δ is the Laplacian operator on M^n and $L\colon \mathcal{C}^{\infty}(M) \to \mathcal{C}^{\infty}(M)$ denotes the standard Cheng–Yau's operator defined by S. Y. Cheng and S. T. Yau in [10], which is given by

$$L(u) = \mathrm{tr}(P \circ \nabla^2 u) \tag{2.8}$$

for every $u \in C^\infty(M)$. Here, $\nabla^2 u$ is the self-adjoint linear tensor metrically equivalent to the Hessian of u and $P\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ denotes the first Newton transformation of M^n, that is, the tensor

$$P = nHI - h^{n+1}. \tag{2.9}$$

Thus, from (2.7) and (2.8) we get

$$\mathcal{L}(u) = \mathrm{tr}(\mathcal{P} \circ \nabla^2 u), \tag{2.10}$$

where

$$\mathcal{P} = \Big(nH + \frac{n-1}{2}a\Big)I - h^{n+1}. \tag{2.11}$$

2.3 Some curvature constraints. Inspired by the configuration assumed by J. O. Baek, Q. M. Cheng and Y. J. Suh in [4], along this work we will suppose that there exist constants c_1, c_2 and c_3 such that the sectional curvature $\overline{K}$ and the curvature tensor $\overline{R}$ of the ambient space L_p^{n+p} satisfy the following set of constraints:

$$\overline{K}(u, \eta) = \frac{c_1}{n} \tag{2.12}$$

for any spacelike vector u and any timelike vector η; when $p > 1$, suppose that

$$\langle \overline{R}(\xi, u)\eta, u\rangle = 0 \tag{2.13}$$

for any spacelike vector u and timelike vectors ξ, η, with $\langle \xi, \eta\rangle = 0$;

$$\overline{K}(u, v) \geq c_2 \tag{2.14}$$

for any spacelike vectors u, v;

$$\overline{K}(\eta, \xi) = \frac{c_3}{p} \tag{2.15}$$

for timelike vectors η, ξ.

Remark 1. The curvature constraints (2.12) and (2.14) are natural extensions for higher codimension of those assumed by S. Nishikawa in [25] for the study of spacelike hypersurfaces. When the ambient space L_p^{n+p} has constant sectional curvature c, then it is a locally symmetric space satisfying all these curvature constraints with $c_1 = nc$, $c_2 = c$ and $c_3 = pc$, see [26, Corollary 8.11]. On the other hand, [2, Example 3.1] gives us a situation where these curvature constraints are satisfied but the ambient space does not have constant sectional curvature. More precisely, let be the locally symmetric semi-Riemannian space

$$L_p^{n+p} = \mathbb{R}_p^p \times \mathbb{S}^n,$$

where $\mathbb{R}^p_p$ stands for the p-dimensional semi-Euclidean space of index p and $\mathbb{S}^n$ is the n-dimensional unit Euclidean sphere. Hence, the curvature constraints (2.12), (2.13), (2.14) and (2.15) are satisfied for $c_1 = 0$, $c_2 = 1$ and $c_3 = 0$. Furthermore, we note that $\mathbb{R}^p_p \times \mathbb{S}^n$ can be regarded as a natural extension of the $(n+1)$-dimensional Einstein static universe $\mathbb{R}_1 \times \mathbb{S}^n$, see [5, Example 5.11].

Now, we denote by $\overline{R}_{CD}$ the components of the Ricci tensor of L^{n+p}_p. So, its scalar curvature $\overline{R}$ is given by

$$\overline{R} = \sum_A \varepsilon_A \overline{R}_{AA} = \sum_{i,j} \overline{R}_{ijij} - 2\sum_{i,\alpha} \overline{R}_{i\alpha i\alpha} + \sum_{\alpha,\beta} \overline{R}_{\alpha\beta\alpha\beta}.$$

Furthermore, if L^{n+p}_p satisfies conditions (2.12) and (2.15), then

$$\overline{R} = \sum_{i,j} \overline{R}_{ijij} - 2pc_1 + (p-1)c_3. \tag{2.16}$$

But, it is well known that the scalar curvature of a locally symmetric Lorentz space is constant, see [26, Proposition 8.10]. Consequently,

$$\frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijij}$$

is a constant naturally attached to a locally symmetric Lorentz space satisfying conditions (2.12) and (2.15), which will be denoted by $\overline{\mathcal{R}}$.

Considering the previous digression, we obtain the following lemma whose proof can be found in [3].

Lemma 1. *Let M^n be a linear Weingarten spacelike submanifold immersed in locally symmetric space L^{n+p}_p satisfying conditions* (2.12) *and* (2.15), *such that $R = aH + b$ for some $a, b \in \mathbb{R}$. Suppose that*

$$(n-1)a^2 + 4n(\overline{\mathcal{R}} - b) \geq 0.$$

Then,

$$|\nabla A|^2 \geq n^2 |\nabla H|^2. \tag{2.17}$$

Moreover, if the equality holds in (2.17) *on M^n, then H is constant on M^n.*

3. Characterization results

This section is dedicated to state and prove our main results concerning linear Weingarten spacelike submanifolds immersed with parallel normalized mean curvature vector field in a semi-Riemannian locally symmetric space. For this, we will use three main core analytical tools: a suitable version of the Omori–Yau maximum principle, parabolicity with respect to the modified Cheng–Yau operator defined in (2.7) and a certain integrability property.

3.1 Via Omori–Yau maximum principle. In order to prove our first result, we will make use of a generalized version of the Omori–Yau maximum principle for trace type differential operators proved by L. J. Alías, P. Mastrolia and M. Rigoli in [1]. Let M^n be a Riemannian manifold and let $\mathcal{L} = \operatorname{tr}(\mathcal{P} \circ \nabla^2)$ be a semi-elliptic operator, where $\mathcal{P} \colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a positive semi-definite symmetric tensor. Following the terminology introduced by S. Pigola, M. Rigoli and A. G. Setti in [29], we say that the Omori–Yau maximum principle holds on M^n for the operator $\mathcal{L}$ if for any function $u \in \mathcal{C}^2(M)$ with

$$u^* = \sup_{M^n} u \ll \infty,$$

there exists a sequence of points $\{p_j\} \subset M^n$ satisfying

$$u(p_j) > u^* - \frac{1}{j}, \qquad |\nabla u(p_j)| < \frac{1}{j} \qquad \text{and} \qquad \mathcal{L}u(p_j) < \frac{1}{j},$$

for all $j \in \mathbb{N}$. Equivalently, for any function $u \in \mathcal{C}^2(M)$ with

$$u_* = \inf_{M^n} u \gg -\infty,$$

there exists a sequence of points $\{p_j\} \subset M^n$ satisfying

$$u(p_j) < u_* + \frac{1}{j}, \qquad |\nabla u(p_j)| < \frac{1}{j} \qquad \text{and} \qquad \mathcal{L}u(p_j) > -\frac{1}{j}$$

for all $j \in \mathbb{N}$.

The following proposition establishes a suitable version of the Omori–Yau maximum principle for the Cheng–Yau type differential operator $\mathcal{L}$ defined in (2.7).

Proposition 1. *Let M^n be an n-dimensional linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector field in a locally symmetric semi-Riemannian space L_p^{n+p} satisfying curvature conditions* (2.12), (2.14), (2.15), *and such that $R = aH + b$ for some $a, b \in \mathbb{R}$, with $b < \overline{\mathcal{R}}$ ($b \leq \overline{\mathcal{R}}$, respectively). The following holds:*

(i) *The operator $\mathcal{L}$ defined in* (2.7) *is elliptic (semi-elliptic, respectively) or, equivalently, $\mathcal{P}$ defined in* (2.11) *is positive definite (semi-definite, respectively);*

(ii) *If*

$$\sup_{M^n} |\Phi|^2 \ll \infty,$$

then the Omori–Yau maximum principle holds on M^n for the operator $\mathcal{L}$ defined in (2.7).

PROOF: Initially, we recall that conditions (2.12) and (2.15) guarantee that $\overline{\mathcal{R}}$ is constant.

For proof of item (i), let us consider the case that $a = 0$. Since $R = b < \overline{\mathcal{R}}$, from (2.3) if we choose a (local) orthonormal frame $\{e_i\}$ on M^n such that

$$h_{ij}^{n+1} = \lambda_i \delta_{ij},$$

we have that

$$\sum_{i<j} \lambda_i \lambda_j > 0.$$

Consequently,

$$n^2 H^2 = \sum_i \lambda_i^2 + 2 \sum_{i<j} \lambda_i \lambda_j > \lambda_i^2$$

for any i and, hence, we have that

$$nH - |\lambda_i| > 0$$

for every i. Therefore, in this case, we conclude that L is elliptic.

Now, suppose that $a \neq 0$. From (2.3) we get that

$$a = \frac{1}{n(n-1)H} \left(S - n^2 H^2 + n(n-1)\overline{\mathcal{R}} - n(n-1)b \right). \tag{3.1}$$

For any i, from (3.1) we have

$$\begin{aligned} nH - \lambda_i^{n+1} + \frac{n-1}{2} a &= nH - \lambda_i^{n+1} \\ &\quad + \frac{1}{2nH}(S - n^2H^2 + n(n-1)(\overline{\mathcal{R}} - b)) \\ &= \Big(\frac{1}{2}(nH)^2 - nH\lambda_i^{n+1} + \frac{1}{2} S \\ &\quad + \frac{1}{2} n(n-1)(\overline{\mathcal{R}} - b) \Big) (nH)^{-1}. \end{aligned} \tag{3.2}$$

Since

$$\sum_j \lambda_j^{n+1} = nH \qquad \text{and} \qquad S \geq \sum_j (\lambda_j^{n+1})^2,$$

from (3.2) we have

$$\begin{aligned}
&nH - \lambda_i^{n+1} + \frac{n-1}{2}a \\
&\quad \geq \left\{ \frac{1}{2}\Big(\sum_j \lambda_j^{n+1} \Big)^2 - \lambda_i^{n+1} \sum_j \lambda_j^{n+1} + \frac{1}{2} \sum_j (\lambda_j^{n+1})^2 \right\} (nH)^{-1} \\
&\quad\quad + \frac{1}{2} n(n-1)(\overline{\mathcal{R}} - b)(nH)^{-1} \\
&\quad = \left\{ \sum_j (\lambda_j^{n+1})^2 + \frac{1}{2} \sum_{l \neq j} \lambda_l^{n+1} \lambda_j^{n+1} - \lambda_i^{n+1} \sum_j \lambda_j^{n+1} \right\} (nH)^{-1} \\
&\quad\quad + \frac{1}{2} n(n-1)(\overline{\mathcal{R}} - b)(nH)^{-1} \\
&\quad = \left\{ \sum_{i \neq j} (\lambda_j^{n+1})^2 + \frac{1}{2} n(n-1)(\overline{\mathcal{R}} - b) + \frac{1}{2} \sum_{l \neq j, l, j \neq i} \lambda_l^{n+1} \lambda_j^{n+1} \right\} (nH)^{-1} \\
&\quad = \frac{1}{2} \left\{ \sum_{i \neq j} (\lambda_j^{n+1})^2 + n(n-1)(\overline{\mathcal{R}} - b) + \Big(\sum_{j \neq i} \lambda_j^{n+1} \Big)^2 \right\} (nH)^{-1}.
\end{aligned}$$

Therefore, considering $b < \overline{\mathcal{R}}$ ($b \leq \overline{\mathcal{R}}$), we conclude that $\mathcal{L}$ is an elliptic (semi-elliptic) operator.

Now, let us proof item (ii). By (2.6) we find

$$|\Phi|^2 = n(n-1)(H^2 + aH) + n(n-1)(b - \overline{\mathcal{R}}), \tag{3.3}$$

which assures that

$$\sup_{M^n} H \ll \infty$$

because of our assumption on $|\Phi|^2$. From here and of (2.3) for every α, i, j, it holds that

$$(h_{ij}^\alpha)^2 \leq |A|^2 = n(nH^2 + (n-1)aH) + n(n-1)(b - \overline{\mathcal{R}}),$$

so that

$$\sup_{M^n} h_{ij}^\alpha \ll \infty.$$

Thus, it follows from the Gauss equation, (2.14) and (2.16) that

$$R_{ijij} = \overline{R}_{ijij} - \sum_\alpha \left(h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2 \right) \geq c_2 - \sum_\alpha h_{ii}^\alpha h_{jj}^\alpha \gg -\infty, \tag{3.4}$$

that is, the sectional curvatures of M^n are bounded from below.

Besides, from (2.11) one verifies that

$$\operatorname{tr}(\mathcal{P}) = n(n-1)H + \frac{n(n-1)a}{2}. \tag{3.5}$$

In particular, from (3.5) we get

$$\sup_{M^n} \operatorname{tr}(\mathcal{P}) \ll \infty.$$

Therefore, taking into account (2.10) and (3.4), we can apply [1, Theorem 6.13] to conclude the desired result. □

So, we apply Proposition 1 to establish the following characterization result:

Theorem 1. *Let M^n be an n-dimensional complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector field and flat normal bundle in a locally symmetric semi-Riemannian space L_p^{n+p} with $p > 1$ and satisfying conditions* (2.12), (2.13), (2.14) *and* (2.15), *such that $R = aH + b$, with $a \geq 0$ and $b \leq \overline{\mathcal{R}} < b + c$, where $c = c_1/n + 2c_2$. Suppose that there exists an orthogonal basis for TM that diagonalizes simultaneously all A_ξ, $\xi \in TM^\perp$. Then,*

(i) *either $|\Phi| \equiv 0$ and M^n is a totally umbilical submanifold,*
(ii) *or*

$$\sup_{M^n} |\Phi| \geq \alpha(n, p, a, b, c, \overline{\mathcal{R}}) > 0,$$

where $\alpha(n, p, a, b, c, \overline{\mathcal{R}})$ is a positive constant that depends only on n, p, a, b, c and $\overline{\mathcal{R}}$. Moreover, if $b < \overline{\mathcal{R}}$, the equality

$$\sup_{M^n} |\Phi| = \alpha(n, p, a, b, c, \overline{\mathcal{R}})$$

holds and this supremum is attained at some point of M^n, then M^n is an isoparametric submanifold, in the sense that its principal curvatures are constant.

Proof: Initially we must obtain a suitable lower boundedness for the operator $\mathcal{L}$ acting on the squared norm of the total umbilicity tensor Φ of M^n. To get it, let us begin observing that, since M^n is linear Weingarten, by (2.6) we get

$$\begin{aligned} \frac{n}{2(n-1)} \mathcal{L}(|\Phi|^2) &= \frac{1}{2} \mathcal{L}(n^2 H^2) + \frac{an}{2} \mathcal{L}(nH) \\ &= nH\mathcal{L}(nH) + n^2 \langle \mathcal{P} \nabla H, \nabla H \rangle + \frac{an}{2} \mathcal{L}(nH). \end{aligned} \tag{3.6}$$

By using item (i) of Proposition 1, we have that $\mathcal{P}$ is positive semi-definite. In particular, from (3.6) we find

$$\frac{1}{2(n-1)} \mathcal{L}(|\Phi|^2) \geq \Big(H + \frac{a}{2}\Big) \mathcal{L}(nH). \tag{3.7}$$

On the other hand, since we are supposing that M^n has parallel normalized mean curvature vector field, flat normal bundle and that there exists an orthogonal basis for TM that diagonalizes simultaneously all $A_\xi, \xi \in TM^\perp$, from the proof of [3, Proposition 1], see the bottom of page 75, we have the following

$$\begin{aligned}\mathcal{L}(nH) &= \frac{1}{2}\,\Delta S - n^2|\nabla H|^2 - n\sum_{i,j} h_{ij}^{n+1} H_{ij}\\ &\geq |\nabla A|^2 - n^2|\nabla H|^2 + cn|\Phi|^2\\ &\quad - nH\sum_{i,j,m,\alpha} h_{ij}^\alpha h_{mi}^\alpha h_{mj}^{n+1} + \sum_{\alpha,\beta}[\mathrm{tr}(h^\alpha h^\beta)]^2.\end{aligned} \tag{3.8}$$

Moreover, we see that

$$\begin{aligned}-nH\sum_\alpha \mathrm{tr}[h^{n+1}(h^\alpha)^2] &+ \sum_{\alpha,\beta}[\mathrm{tr}(h^\alpha h^\beta)]^2\\ &\geq \frac{-n(n-2)}{\sqrt{n(n-1)}}\,H|\Phi|^3 - nH^2|\Phi|^2 + \frac{|\Phi|^4}{p}.\end{aligned} \tag{3.9}$$

From (3.8) and (3.9), we have

$$\begin{aligned}\mathcal{L}(nH) &\geq |\nabla A|^2 - n^2|\nabla H|^2\\ &\quad + |\Phi|^2\Big(\frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}\,H|\Phi| - n(H^2-c)\Big).\end{aligned} \tag{3.10}$$

Besides, from (2.6) we have

$$H + \frac{a}{2} = \frac{1}{\sqrt{n(n-1)}}\sqrt{|\Phi|^2 + n(n-1)\Big(\frac{a^2}{4} + \overline{\mathcal{R}} - b\Big)}.$$

This jointly with (3.7), (3.10) and Lemma 1 enables us to deduce that

$$\frac{1}{2}\,\mathcal{L}(|\Phi|^2) \geq (n-1)|\Phi|^2 Q_{n,p,a,b,c,\overline{\mathcal{R}}}(|\Phi|)\sqrt{\frac{|\Phi|^2}{n(n-1)} + \frac{a^2}{4} + \overline{\mathcal{R}} - b}, \tag{3.11}$$

where the function $Q_{n,p,a,b,c,\overline{\mathcal{R}}}(x)$ is given by

$$\begin{aligned}Q_{n,p,a,b,c,\overline{\mathcal{R}}}(x) &= \frac{n-p-1}{p(n-1)}\,x^2\\ &\quad + \Big(na - \frac{n(n-2)}{\sqrt{n(n-1)}}x\Big)\sqrt{\frac{x^2}{n(n-1)} + \frac{a^2}{4} + \overline{\mathcal{R}} - b}\\ &\quad + \frac{n(n-2)a}{2\sqrt{n(n-1)}}\,x + n\Big(-\frac{a^2}{2} + b + c - \overline{\mathcal{R}}\Big).\end{aligned} \tag{3.12}$$

At this point, we will make a brief analysis of the behavior of the function $Q_{n,p,a,b,c,\overline{\mathcal{R}}}(x)$, considering $p>1$, $a\geq 0$ and $b\leq\overline{\mathcal{R}}<b+c$. Let us observe that when $x>0$, from (3.12) we get

$$\begin{aligned}\lim_{x\to\infty} Q_{n,p,a,b,c,\overline{\mathcal{R}}}(x) = {}& \lim_{x\to\infty} x^2\Big\{\frac{n-p-1}{p(n-1)}\\ &+\Big(\frac{na}{x}-\frac{n(n-2)}{\sqrt{n(n-1)}}\Big)\sqrt{\frac{1}{n(n-1)}+\frac{a^2}{4x^2}+\frac{\overline{\mathcal{R}}-b}{x^2}}\\ &+\frac{n(n-2)a}{2x\sqrt{n(n-1)}}+\frac{n}{x^2}\Big(-\frac{a^2}{2}+b+c-\overline{\mathcal{R}}\Big)\Big\}.\end{aligned}\tag{3.13}$$

Thus, taking into account that $p>1$, from (3.13) we obtain

$$\lim_{x\to\infty} Q_{n,p,a,b,c,\overline{\mathcal{R}}}(x) = \lim_{x\to\infty} x^2\Big\{\frac{n-p-1}{p(n-1)}-\frac{n-2}{n-1}\Big\}=-\infty.\tag{3.14}$$

Since we are also assuming that $b\leq\overline{\mathcal{R}}<b+c$ and $a\geq 0$, we also have that

$$\begin{aligned}Q_{n,p,a,b,c,\overline{\mathcal{R}}}(0) &= n\bigg(a\sqrt{\frac{a^2}{4}+\overline{\mathcal{R}}-b}-\frac{a^2}{2}\bigg)+n(b+c-\overline{\mathcal{R}})\\ &\geq n(b+c-\overline{\mathcal{R}})>0.\end{aligned}\tag{3.15}$$

From (3.14) and (3.15), we can define $\alpha(n,p,a,b,c,\overline{\mathcal{R}})$ as being the first positive root of the function $Q_{n,p,a,b,c,\overline{\mathcal{R}}}(x)$.

Now, we are going to finish the proof by applying our version of the Omori–Yau maximum principle to the operator $\mathcal{L}$ acting on the function $|\Phi|^2$. Before, we note that if

$$\sup_{M^n}|\Phi|=\infty,$$

then the claim (ii) of Theorem 1 trivially holds and there is nothing to prove.

So, let us assume without loss of generality that

$$\sup_{M^n}|\Phi|\ll\infty.$$

In this case, from item (ii) of Proposition 1 we obtain a sequence $\{p_j\}$ in M^n satisfying

$$\lim_j|\Phi|(p_j)=\sup_{M^n}|\Phi| \qquad\text{and}\qquad \mathcal{L}(|\Phi|^2)(p_j)<\frac{1}{j},$$

for all $j \in \mathbb{N}$, which jointly with (3.11) gives

$$\begin{aligned}\frac{1}{j} &> \mathcal{L}(|\Phi|^2)(p_j)\\ &\geq (n-1)|\Phi|^2(p_j)Q_{n,p,a,b,c,\overline{\mathcal{R}}}(|\Phi|(p_j))\sqrt{\frac{|\Phi|^2(p_j)}{n(n-1)}+\frac{a^2}{4}+\overline{\mathcal{R}}-b}\end{aligned}$$

for all $j \in \mathbb{N}$. Taking the limit as $j \to \infty$, we infer

$$\left(\sup_{M^n}|\Phi|\right)^2 Q_{n,p,a,b,c,\overline{\mathcal{R}}}\left(\sup_{M^n}|\Phi|\right)\sqrt{\frac{(\sup_{M^n}|\Phi|)^2}{n(n-1)}+\frac{a^2}{4}+\overline{\mathcal{R}}-b} \leq 0.$$

It follows from here that either

$$\sup_{M^n}|\Phi| = 0,$$

which means that $|\Phi| \equiv 0$ on M^n and the submanifold is totally umbilical, or

$$\sup_{M^n}|\Phi| > 0$$

and then

$$Q_{n,p,a,b,c,\overline{\mathcal{R}}}\left(\sup_{M^n}|\Phi|\right) \leq 0.$$

Thus, from the behavior of the function $Q_{n,p,a,b,c,\overline{\mathcal{R}}}(x)$ and according to our choice of the positive constant $\alpha(n,p,a,b,c,\overline{\mathcal{R}})$, we deduce that

$$\sup_{M^n}|\Phi| \geq \alpha(n,p,a,b,c,\overline{\mathcal{R}}).$$

Finally, let us assume that

$$\sup_{M^n}|\Phi| = \alpha(n,p,a,b,c,\overline{\mathcal{R}}).$$

In this case, from (3.11) and taking into account once more the behavior of the function $Q_{n,p,a,b,c,\overline{\mathcal{R}}}(x)$, we get that $\mathcal{L}(|\Phi|^2) \geq 0$. But, since we are assuming that $b < \overline{\mathcal{R}}$, item (i) of Proposition 1 guarantees that $\mathcal{L}$ is elliptic. Consequently, since we are also supposing that the supremum of $|\Phi|$ on M^n is attained at some point of M^n, we conclude that $|\Phi|$ is constant on M^n and, from (3.3), the same holds for H. Hence, returning to (3.10) we obtain

$$\sum_{i,j,k}(h^\alpha_{ijk})^2 = |\nabla A|^2 = n^2|\nabla H|^2 = 0,$$

that is, $h^\alpha_{ijk} = 0$ for all $i, j \in \{1, \ldots, n\}$. Therefore, we conclude that M^n is an isoparametric submanifold of L^{n+p}_p. $\square$

Remark 2. For the spacelike submanifold $M^n = \{0\}\times\mathbb{S}^n$ in the locally symmetric semi-Riemannian space $L_p^{n+p} = \mathbb{R}_p^p \times \mathbb{S}^n$, in the example mentioned in Remark 1, we note that besides checking the assumptions (2.12), (2.13), (2.14) and (2.15), the hypothesis $b \le \overline{\mathcal{R}} < b + c$ is also satisfied and it is such that there exists an orthogonal basis for TM that diagonalizes simultaneously all A_ξ, $\xi \in TM^\perp$, the conditions required in the statement of Theorem 1. Indeed, we have that

$$\overline{\mathcal{R}} = \frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijij} = \frac{1}{n(n-1)} \sum_{i,j} \overline{K}(e_i, e_j) = 1.$$

Consequently, since $\overline{\mathcal{R}} = 1$, $b = 1$ and $c = 2$, we conclude that $b \le \overline{\mathcal{R}} < b + c$. Furthermore, as the immersion $M^n = \{0\} \times \mathbb{S}^n \hookrightarrow L_p^{n+p} = \mathbb{R}_p^p \times \mathbb{S}^n$ is totally geodesic, we get that $A_\xi \equiv 0$ for all $\xi \in TM^\perp$.

Remark 3. In Theorem 1, since the normal bundle of linear Weingarten spacelike submanifold M^n is assumed to be flat, from Ricci equation (2.4) we observe that the existence of an orthonormal basis for TM that diagonalizes simultaneously all A_ξ, with $\xi \in TM^\perp$, is guaranteed when the components $\overline{R}_{\alpha\beta ij}$ of the curvature tensor $\overline{R}$ of L_p^{n+p} vanish identically. Indeed, in this case, the commutator of any two shape operators is identically zero and, therefore, an orthonormal basis that diagonalizes one of these shape operators will also diagonalizes the other ones. This geometric configuration is more evident when the ambient space L_p^{n+p} has constant sectional curvature, but it also happens in other types of ambient spaces, see Remark 2.

3.2 Via $\mathcal{L}$-parabolicity. We recall that a Riemannian manifold M^n is said to be parabolic (with respect to the Laplacian operator) if the constant functions are the only subharmonic functions on M^n which are bounded from above; that is for a function $u \in \mathcal{C}^2(M)$

$$\Delta u \ge 0 \text{ and } u \le u^* \ll \infty \text{ implies } u = \text{constant}.$$

From a physical viewpoint, parabolicity is closely related to the recurrence of the Brownian motion. Roughly speaking, the parabolicity is equivalent to the property that all particles will pass through any open set at an arbitrarily large time, for more details see [18].

Extending this previous concept for the operator $\mathcal{L}$ defined in (2.10), M^n is said to be $\mathcal{L}$-parabolic if the constant functions are the only functions $u \in \mathcal{C}^2(M)$ which are bounded from above and satisfies $\mathcal{L}u \ge 0$, that is, for a function $u \in \mathcal{C}^2(M)$,

$$\mathcal{L}u \ge 0 \text{ and } u \le u^* \ll \infty \text{ implies } u = \text{constant}.$$

In this setting, we obtain the following gap result:

Theorem 2. *Let M^n be an n-dimensional complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector field and flat normal bundle in a locally symmetric semi-Riemannian space L_p^{n+p} with $p > 1$ and satisfying conditions* (2.12), (2.14) *and* (2.15), *such that $R = aH + b$, with $a \geq 0$ and $b \leq \overline{\mathcal{R}} < b + c$, where $c = c_1/n + 2c_2$. Suppose that there exists an orthogonal basis for TM that diagonalizes simultaneously all A_ξ, $\xi \in TM^\perp$. Assume in addition that $0 \leq |\Phi| \leq \alpha(n, p, a, b, c, \overline{\mathcal{R}})$, where $\alpha(n, p, a, b, c, \overline{\mathcal{R}})$ is the positive constant which was obtained in Theorem* 1*. If M^n is a $\mathcal{L}$-parabolic submanifold, then either $|\Phi| \equiv 0$ and M^n is totally umbilical, or $|\Phi| \equiv \alpha(n, p, a, b, c, \overline{\mathcal{R}})$ and M^n is an isoparametric submanifold.*

Proof: Suppose that M^n is not totally umbilical. Since we are assuming that $0 \leq |\Phi| \leq \alpha(n, p, a, b, c, \overline{\mathcal{R}})$, we obtain

$$0 < \sup_{M^n} |\Phi|^2 \leq \alpha(n, p, a, b, c, \overline{\mathcal{R}}).$$

In this case, from item (ii) of Theorem 1 we get that

$$\sup_{M^n} |\Phi|^2 = \alpha(n, p, a, b, c, \overline{\mathcal{R}}).$$

Furthermore, since estimate (3.11) jointly with our restriction on $|\Phi|$ implies $\mathcal{L}(|\Phi|^2) \geq 0$ on M^n, from the $\mathcal{L}$-parabolicity of M^n we conclude that $|\Phi|$ must be constant and identically equal to $\alpha(n, p, a, b, c, \overline{\mathcal{R}})$. Therefore, at this point we can proceed as in the last part of the proof of Theorem 1 to conclude the result. $\square$

When the ambient space L_p^{n+p} is supposed to be Einstein, reasoning as in the first part of the proof of [12, Theorem 1.1], from (2.7) and (2.8) it is not difficult to verify that

$$\mathcal{L}(f) = \operatorname{div}(\mathcal{P}(\nabla f)), \tag{3.16}$$

where $\mathcal{P}$ is just the operator defined in (2.11). Taking into account this fact, we obtain the following criterion for $\mathcal{L}$-parabolicity of complete linear Weingarten spacelike submanifolds:

Proposition 2. *Let M^n be an n-dimensional complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector field in a locally symmetric Einstein semi-Riemannian space L_p^{n+p} satisfying conditions* (2.12), (2.14) *and* (2.15), *such that $R = aH + b$, with $a \geq 0$ and $b \leq \overline{\mathcal{R}} <$*

$b+c$, where $c=c_1/n+2c_2$. If

$$\sup_{M^n}|\Phi|^2 \ll \infty$$

and for some reference point $o\in M^n$,

$$\int_0^\infty \frac{\mathrm{d}r}{\mathrm{vol}(\partial B_r)}=\infty, \tag{3.17}$$

then M^n is $\mathcal{L}$-parabolic. Here B_r denotes the geodesic ball of radius r in M^n centered at the origin o.

Proof: We consider on M^n the symmetric $(0,2)$ tensor field ξ given by

$$\xi(X,Y)=\langle \mathcal{P}X,Y\rangle,$$

or equivalently,

$$\xi(\nabla u,\cdot)^\sharp=\mathcal{P}(\nabla u),$$

where $\mathcal{P}$ is defined in (2.9) and $^\sharp\colon T^*M\to TM$ denotes the musical isomorphism. Thus, from (3.16) we get

$$\mathcal{L}(u)=\mathrm{div}\left(\xi(\nabla u,\cdot)^\sharp\right).$$

On the other hand, since we are assuming that

$$\sup_{M^n}|\Phi|^2 \ll \infty$$

and $a\geq 0$, from (3.3) we get that

$$\sup_{M^n} H \ll \infty.$$

So, we can define a positive continuous function ξ_+ on $[0,\infty)$, by

$$\xi_+(r)=2n\left(\sup_{\partial B_r} H\right)+(n-1)a. \tag{3.18}$$

Thus, from (3.18) we have

$$\xi_+(r)=2n\left(\sup_{\partial B_r} H\right)+(n-1)a\leq 2n\left(\sup_{M^n} H\right)+(n-1)a\ll\infty. \tag{3.19}$$

Hence, from (3.17) and (3.19) we get

$$\int_0^\infty \frac{\mathrm{d}r}{\xi_+(r)\mathrm{vol}(\partial \mathrm{B_r})}=\infty.$$

Therefore, we can apply [28, Theorem 2.6] to conclude the proof. $\square$

Remark 4. Taking into account Proposition 2, it is natural to ask oneself about the existence of Einstein manifolds which are locally symmetric. In this direction,

K. P. Tod in [32] showed that four-dimensional Einstein manifolds which are also D'Atri spaces are necessarily locally symmetric. Later on, S. Brendle in [6] proved that a compact Einstein manifold of dimension $n \geq 4$ having nonnegative isotropic curvature must be locally symmetric, extending a previous result of M. J. Micallef and M. Y. Wang for $n = 4$, see [24, Theorem 4.4]. See also [34] for another sufficient conditions for an Einstein manifold to be locally symmetric.

3.3 Via integrability property. In [35], S. T. Yau established the following version of Stokes' theorem on an n-dimensional complete noncompact Riemannian manifold M^n: *If $\omega \in \Omega^{n-1}(M)$ is an $(n-1)$-differential form on M^n, then there exists a sequence $\{B_i\}$ of domains on M^n such that*

$$B_i \subset B_{i+1}, \qquad M^n = \bigcup_{i\geq 1} B_i \qquad \text{and} \qquad \lim_{i\to\infty} \int_{B_i} \mathrm{d}\omega = 0.$$

Supposing that M^n is oriented by the volume element $\mathrm{d}M$ and considering the contraction of $\mathrm{d}M$ in the direction of a smooth vector field X on M^n, that is, $\mathrm{d}\omega = \iota_X \mathrm{d}M$, A. Caminha in [8] obtained a suitable consequence of Yau's result, which is described below (specifically, see [8, Proposition 2.1]). In what follows, $\mathcal{L}^1(M)$ stands for the space of Lebesgue integrable functions on M^n.

Lemma 2. *Let X be a smooth vector field on the n-dimensional complete oriented Riemannian manifold M^n, such that* $\mathrm{div}X$ *does not change sign on M^n. If $|X| \in \mathcal{L}^1(M)$, then* $\mathrm{div}X = 0$.

We close our paper applying Lemma 2 in order to obtain the following characterization result.

Theorem 3. *Let M^n be an n-dimensional complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector field and flat normal bundle in a locally symmetric Einstein semi-Riemannian space L_p^{n+p} with $p > 1$ and satisfying conditions* (2.12), (2.13), (2.14) *and* (2.15), *such that $R = aH + b$, with $a \geq 0$ and $b \leq \overline{\mathcal{R}} < b + c$, where $c = c_1/n + 2c_2$. Suppose that there exists an orthogonal basis for TM that diagonalizes simultaneously all A_ξ, $\xi \in TM^\perp$. Assume in addition that $0 \leq |\Phi| \leq \alpha(n, p, a, b, c, \overline{\mathcal{R}})$, where $\alpha(n, p, a, b, c, \overline{\mathcal{R}})$ is the positive constant which was obtained in Theorem 1. If $|\nabla H| \in \mathcal{L}^1(M)$, then either $|\Phi| \equiv 0$ and M^n is totally umbilical, or $|\Phi| \equiv \alpha(n, p, a, b, c, \overline{\mathcal{R}})$ and M^n is an isoparametric submanifold.*

Proof: Since $R = aH + b$ and taking into account that (3.3) gives that H is bounded on M^n, from (2.3) we have that A is bounded on M^n. Consequently, from (2.11) we conclude that the operator $\mathcal{P}$ is bounded, that is, there exists a positive constant C_1 such that $|\mathcal{P}| \leq C_1$. Since we are also assuming that

$|\nabla H| \in \mathcal{L}^1(M)$ and (3.3), we obtain that

$$|P(\nabla H)| \leq |P||\nabla H| \leq C_1|\nabla H| \in \mathcal{L}^1(M). \tag{3.20}$$

Thus, taking into account (3.16) and (3.20), we can apply Lemma 2 to obtain

$$\mathcal{L}(nH) = \operatorname{div}(\mathcal{P}(nH)) = 0. \tag{3.21}$$

Hence, using the fact that $0 \leq |\Phi| \leq \alpha(n, p, a, b, c, \overline{\mathcal{R}})$, from (3.10) and (3.21) we conclude that

$$0 = \mathcal{L}(nH) \geq |\nabla A|^2 - n^2|\nabla H|^2 + |\Phi|^2 Q_{n,p,a,b,c,\overline{\mathcal{R}}}(|\Phi|) \geq 0. \tag{3.22}$$

Thus, from (3.22) we get that $|\nabla A|^2 = n^2|\nabla H|^2$ and, consequently, Lemma 1 guarantees that H is constant. Hence,

$$\sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 = |\nabla A|^2 = n^2|\nabla H|^2 = 0,$$

that is, $h_{ijk}^\alpha = 0$ for all i, j, and we obtain that M^n is isoparametric. Therefore, the result follows once more as in the last part of the proof of Theorem 1. $\square$

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