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DEGREES OF COMPATIBLE L-SUBSETS AND COMPATIBLE MAPPINGS

Fu-Gui Shi and Yan Sun

Based on a completely distributive lattice L, degrees of compatible L-subsets and compatible mappings are introduced in an L-approximation space and their characterizations are given by four kinds of cut sets of L-subsets and L-equivalences, respectively. Besides, some characterizations of compatible mappings and compatible degrees of mappings are given by compatible L-subsets and compatible degrees of L-subsets. Finally, the notion of complete L-sublattices is introduced and it is shown that the product of complete L-sublattices is still a complete L-sublattice and the compatible degree of an L-subset is a complete L-sublattice.

Keywords: L-approximation spaces, compatible L-subsets, compatible mappings, complete L-sublattices

Classification: 06B75, 06D10, 06D72

1. INTRODUCTION

In 1965, Zadeh [42] introduced the notion of fuzzy sets, which plays an important role in the development of fuzzy set theory. Many mathematical structures were endowed with fuzzy set theory, such as fuzzy ideals and fuzzy filters [41], fuzzy convergence structures [12, 47, 48], fuzzy convex structures [24, 26, 33] and so on. A nonempty set with an equivalence relation on it is called an approximation space [28]. Motivated by fuzzy set theory, the notion of approximation spaces has also been extended to the fuzzy case, which leads to the concept of an L-approximation space where L denotes the lattice background. For research related to "compatible" (or so called extensional in [19]), Ganter and Wille [11] proposed the definition of compatible subcontexts and studied that the concept lattice of a compatible subcontext of a context is always a homomorphic image of the concept lattice of the context, and the concepts to a compatible subcontext yields a map between the concept lattices, which necessarily has to be structure-preserving. Compatible subsets as the union of some equivalence classes, which shows compatibility between a subset and an equivalence relation, are a special class of subsets in an approximation space. As an example of compatible subsets, the upper and lower approximations of a rough set are used to solve uncertainty problems [34, 39] in rough set theory [13, 35]. Klawonn et al. [19] proposed the concepts of a compatible L-subset and a compatible mapping under the framework of an L-approximation space by generalizing those in the

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classical case. Konecny and Krupka [20] generalized the notion of a complete binary relation on a complete lattice to a residuated lattice valued ordered sets, considered power structures of L-ordered sets with compatible L-subsets and compatible L-relations, and stated that it can be used for reducing dimensionality of concept lattices. The research of "compatible" was not only limited to rough set theory and formal concept analysis [4], but also widely applied in graph theory [10], optimization [5] and computational science [29].

The degree approach that equips each set or mathematical structure with some degree description is an essential character of fuzzy set theory. In 1980, Bandler and Kohout [2] discussed that the degree to which X can be contained in Y is (estimated as) some number from 0 to 1 inclusive, which may not sound like fuzzy possibility theory. They thought that entirely different operations are performed on these fuzzy degrees than are performed on probabilities, and this reflects a deeper semantic and epistemological difference. Ying [40] used a fuzzy method to propose a topology whose logical fundament is fuzzy. This reflects fuzzification of the specific mathematical structure with the idea of degree. Then the degree approach was applied to different mathematical environments, such as fuzzy logic [3, 16], fuzzy algebra [8, 32], fuzzy topology [21, 23, 45], fuzzy convex structure [22, 27, 49], fuzzy convergence [25, 38, 46]. Motivated by this, we aim to propose a fuzzy approach to study the collection of all compatible L-subsets, this approach acquires the best answer to the question of whether an L-subset belongs to this collection is a certain truth degree from an appropriate ordered scale of truth degrees. Meanwhile, in [9], Fang studied that if a mapping is compatible, then the image of a compatible L-subset is still compatible. Furthermore, we find that compatible mappings can be characterized by compatible L-subsets. In order to explore more approaches to characterize compatible mappings, we will first introduce the notion of degree to which an L-subset is compatible and the notion of degree to which a mapping between two L-approximation spaces is compatible.

The theory of fuzzy lattices and fuzzy sublattices is also an important part of fuzzy set theory. Yuan and Wu [41] were the first to introduce the concepts of fuzzy sublattices and proved that the lattice of fuzzy ideals is isomorphic to the lattice of fuzzy congruences on a generalized Boolean algebra. Ajmal and Thomas [1] provided a general development of the theory of (convex) fuzzy sublattices, and gave characterizations of (convex) fuzzy sublattices by cut sets. Tepavčević and Trajkovski [36] introduced the notion of L-fuzzy sublattices based on L-subsets in the sense of Goguen [14], and gave characterizations of this notion in terms of one kind of cut sets of L-subsets. By this motivation, we generalize complete sublattices to complete fuzzy sublattices, which will be called complete L-sublattices in this paper. Considering the connections with the first aim, we will construct complete L-sublattices by using the compatible degree of L-subsets in an L-approximation space, and further we will introduce the product of complete L-sublattices.

This paper is organized as follows. In Section 2, we recall some necessary concepts about completely distributive lattices, four kinds of cut sets and compatible L-subsets. In Section 3, we propose the notion of degree to which an L-subset is a compatible subset, which is called compatible degree of an L-subset and give some characterizations of the compatible degree of L-subsets. In Section 4, we generalize compatible mappings, and introduce the definition of degree to which a mapping is compatible, which is called compatible degree of a mapping. We give a new characterization of the compatible degree of mappings under considering the relationship between compatible L-subsets and compatible mappings, and explore the properties of the compatible degree of the composition of mappings. In Section 5, we present the notion of complete L-sublattices and study the product of complete L-sublattices, and obtain that the compatible degree of L-subsets is a complete L-sublattice of $(L^X, \leq)$.

2. PRELIMINARIES

Throughout this paper, unless otherwise stated, L denotes a completely distributive lattice and M denotes a complete lattice. The largest and smallest elements in L are denoted by 1 and 0, respectively. An element a in L is called co-prime if $a \leq b \vee c$ implies $a \leq c$ or $a \leq b$. The set of nonzero co-prime elements in L is denoted by $J(L)$. An element a in L is called prime if $b \wedge c \leq a$ implies $b \leq a$ or $c \leq a$. The set of non-unit prime elements in L is denoted by $P(L)$. Each element of L is the sup of co-prime elements and the inf of prime elements [37].

For each $a, b \in L$, a is wedge below b in L, in symbols $a \prec b$, which means that for each $D \subseteq L$, $b \leq \bigvee D$ implies $a \leq d$ for some $d \in D$. A complete lattice L is completely distributive if and only if $b = \bigvee\{a \in L \mid a \prec b\}$ for each $b \in L$. The set $\{a \in L \mid a \prec b\}$ denoted by $\beta(b)$, is called the greatest minimal family of b. Moreover, define a binary relation $\prec^{op}$ as follows: for each $a, b \in L$, $a \prec^{op} b$ if and only if for each $T \subseteq L$, $\bigwedge T \leq a$ implies $t \leq b$ for some $t \in T$. The set $\{b \in L \mid a \prec^{op} b\}$, denoted by $\alpha(a)$, is called the greatest maximal family of a [37].

Theorem 2.1. Let L be a completely distributive lattice. Then for each subfamily $\{a_i \mid i \in I\} \subseteq L$, we have

(1) $\beta(\bigvee_{i\in I} a_i) = \bigcup_{i\in I} \beta(a_i)$, i. e., β is a union-preserving mapping.

(2) $\alpha\Big(\bigwedge_{i\in I} a_i\Big) = \bigcup_{i\in I} \alpha(a_i)$, i. e., α is a $\bigwedge$–$\bigcup$ mapping.

Lemma 2.2. (Sji et al. [32]) Let L be a completely distributive lattice and $a, b \in L$. Then the following statements are equivalent:

(1) $a \leq b$.

(2) $\forall p \in L$, $p \leq a \Longrightarrow p \leq b$.

(3) $\forall p \in P(L)$, $p \notin \alpha(a) \Longrightarrow p \notin \alpha(b)$.

(4) $\forall p \in P(L)$, $a \not\leq p \Longrightarrow b \not\leq p$.

(5) $\forall p \in \beta(1)$, $p \in \beta(a) \Longrightarrow p \in \beta(b)$.

In a completely distributive lattice L, there exists an implication operation $\rightarrow$: $L \times L \longrightarrow L$ as the right adjoint for the meet operation $\wedge$ denoted by $a \rightarrow b = \bigvee\{c \in L \mid a \wedge c \leq b\}$ [6].

Next, we list some properties of the implication operation in the following lemma.

Lemma 2.3. (Birkhoff [6], Dong and Shi [8], Goguen [15]) Let L be a completely distributive lattice and $\to$ be the implication operation corresponding to $\wedge$. Then for each $a, b, c \in L$ and $\{a_i \mid i \in I\} \subseteq L$, the following statements hold.

(1) $a \leq b \to c \Longleftrightarrow a \wedge b \leq c$.

(2) $a \to \bigwedge_{i\in I} b_i = \bigwedge_{i\in I}(a \to b_i)$.

(3) $\bigvee_{j\in J} a_j \to b = \bigwedge_{j\in J}(a_j \to b)$.

(4) $a \to b \to c = (a \wedge b) \to c$.

(5) $(a \to b) \wedge (b \to c) \leq (a \to c)$.

(6) $\bigwedge_{i\in I}(a_i \to b_i) \leq \bigvee_{i\in I} a_i \to \bigvee_{i\in I} b_i$.

(7) $\bigwedge_{i\in I}(a_i \to b_i) \leq \bigwedge_{i\in I} a_i \to \bigwedge_{i\in I} b_i$.

(8) $a \to b \leq (b \to c) \to (a \to c)$.

(9) $a \to b \leq a \wedge c \to b \wedge c$.

Definition 2.4. (Ajmal and Thomas [1]) Let M be a complete lattice and A be a nonempty subset of M. A is called a complete sublattice of M provided that $\bigvee B \in A$ and $\bigwedge B \in A$ for each $B \subseteq A$.

For a nonempty set X, the power set of X is denoted by 2^X and the set of all L-subsets of X is denoted by L^X. The largest and smallest elements in L^X are denoted by $\underline{1}$ and $\underline{0}$, respectively. All algebraic operations $\{\vee, \wedge, \to\}$ of L can be extended to L^X in a pointwise way [15].

Definition 2.5. (Shi [30], Huang and Shi [17]) Let $A \in L^X$ and $a \in L$. Then we define

$$A_{[a]} = \{x \in X \mid a \leq A(x)\}, \qquad A^{(a)} = \{x \in X \mid A(x) \not\leq a\},$$
$$A_{(a)} = \{x \in X \mid a \in \beta(A(x))\}, \qquad A^{[a]} = \{x \in X \mid a \notin \alpha(A(x))\}.$$

Definition 2.6. (Zadeh [43]) Let R: $X \times X \longrightarrow L$ be a binary L-relation on X. If it satisfies

(1) $R(x, x) = 1$ for each $x \in X$ (Reflexive);

(2) $R(x, y) = R(y, x)$ for each $x, y \in X$ (Symmetric);

(3) $R(x, y) \wedge R(y, z) \leq R(x, z)$ for each $x, y, z \in X$ (Transitive),

then R is called an L-equivalence (also called a fuzzy similarity relation or fuzzy equality) on X. The pair (X, R) is called an L-approximation space.

When $L = \{0, 1\}$, $R \subseteq X \times X$ is a classical equivalence relation on X and the pair (X, R) is called an approximation space.

Theorem 2.7. (Shi [31]) Let R be a binary L-relation on X. Then the following statements are equivalent:

(1) R is an L-equivalence on X.

(2) $R_{[a]}$ is an equivalence on X for each $a \in L$.

(3) $R_{[a]}$ is an equivalence on X for each $a \in J(L)$.

(4) $R^{[a]}$ is an equivalence on X for each $a \in L$.

(5) $R^{[a]}$ is an equivalence on X for each $a \in P(L)$.

(6) $R^{(a)}$ is an equivalence on X for each $a \in P(L)$.

Theorem 2.8. (Shi [31]) Let R be a binary L-relation on X. If $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ for each $a, b \in L$, then the following statements are equivalent:

(1) R is an L-equivalence on X.

(2) $R_{(a)}$ is an equivalence on X for each $a \in \beta(1)$.

Let (X, R) be an approximation space and $A \subseteq X$. Then A is called a compatible subset on (X, R) provided that

$$\forall x, y \in X,\ (x \in A \text{ and } (x, y) \in R) \Longrightarrow (y \in A).$$

For convenience, let $2^{(X,R)}$ denote the set of all compatible subsets on (X, R).

In the following definition, the notion of a compatible subset is fuzzified.

Definition 2.9. (Fang [9], Klawonn and Castro [19], Konecny and Krupka [20]) Let (X, R) be an L-approximation space and $A \in L^X$. If A satisfies

$$\forall x, y \in X,\ A(x) \wedge R(x, y) \leq A(y), \tag{1}$$

then A is called a compatible L-subset on (X, R), or is called compatible (extensional) with R. For convenience, let $L^{(X,R)}$ denote the set of all compatible L-subsets on (X, R).

Example 2.10. (Fang [9] and Zadeh [43]) Let (X, R) be an L-approximation space and $x \in X$. Define an L-subset R_x as follow:

$$\forall y \in X,\ R_x(y) = R(x, y).$$

Then R_x is a compatible L-subset on (X, R).

Definition 2.11. (Demirci [7], Fang [9], Klawonn [18]) Let (X, R_X) and (Y, R_Y) be two L-approximation spaces and $f\colon X \longrightarrow Y$ be a mapping. Then f is called a compatible mapping (or so called extensional function) from (X, R_X) to (Y, R_Y) provided that

$$\forall x, y \in X,\ R_X(x, y) \leq R_Y(f(x), f(y)).$$

When $L = \{0, 1\}$, R_X and R_Y are two equivalences, then f: $X \longrightarrow Y$ is called a classical compatible mapping if it satisfies $(x, y) \in R_X$ implies $(f(x), f(y)) \in R_Y$ for each $x, y \in X$.

Let f: $X \longrightarrow Y$ be a mapping. We define $f^{\rightarrow}$: $L^X \longrightarrow L^Y$ and $f^{\leftarrow}$:$L^Y \longrightarrow L^X$ by $f^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)$ for each $A \in L^X$ and $y \in Y$, and $f^{\leftarrow}(B)(x) = B(f(x))$ for each $B \in L^Y$ and $x \in X$. $f^{\rightarrow}$ and $f^{\leftarrow}$ are called Zadeh extension mappings [44].

Definition 2.12. (Fang [9]) Let (X, R_X) and (Y, R_Y) be two L-approximation spaces and f: $X \longrightarrow Y$ be a mapping. Then f is called a generalized injective mapping from (X, R_X) to (Y, R_Y) provided that

$$\forall x, y \in X,\ R_Y(f(x), f(y)) \leq R_X(x, y).$$

3. THE COMPATIBLE DEGREE OF L-SUBSETS AND ITS CHARACTERIZATIONS

In this section, we will generalize the notion of a compatible L-subset. Then we will give four characterizations under the influence of characterizations of L-equivalences.

First of all, we should transform "A is a compatible L-subset on (X, R)" (or "$A \in L^{(X,R)}$") into some degree formula. We know

$$A \in L^{(X,R)} \Longleftrightarrow (1)\,\text{in Definition 2.9.}$$

Gottwald [16] pointed out that the language for fuzzy logic is the same for many-valued logic. Inspired by this, we introduce the notion of a compatible degree of L-subsets by means of the implication operation of L.

Definition 3.1. Let (X, R) be an L-approximation space and $A \in L^X$. Then $\mathscr{C}(A)$ defined by

$$\mathscr{C}(A) = \bigwedge_{x,y \in X} (A(x) \wedge R(x, y) \to A(y))$$

is called the degree to which A is a compatible L-subset, or is called the compatible degree of A.

When $\mathscr{C}(A) = 1$, then A is a compatible L-subset in the sense of Definition 2.9.

Example 3.2. Let $L = [0, 1]$, $X = \{x_1, x_2, x_3\}$, $R = \begin{bmatrix} 1 & 0.7 & 0.5 \\ 0.7 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix}$ and define $\wedge$ and $\to$ as follows: $a \wedge b = \min\{a, b\}$ and

$$a \to b = \begin{cases} 1, \text{ if } a \leq b, \\ b, \text{ if } b \leq a. \end{cases}$$

If an L-subset A is defined by

$$A(x) = \begin{cases} 0.4, \text{ if } x = x_1, \\ 0.3, \text{ if } x = x_2, \\ 0.8, \text{ if } x = x_3, \end{cases}$$

then $\mathscr{C}(A) = 0.3$, that is, the degree to which A is a compatible L-subset is 0.3.

Definition 3.3. (Fang [9], Klawonn and Castro [19], Konecny and Krupka [20]) Let (X, R) be an L-approximation space and $A \in L^X$. Then $\hat{A}$ defined by

$$\forall y \in X,\ \hat{A}(y) = \bigvee_{x \in X} (A(x) \wedge R(x, y))$$

is called the smallest compatible L-subset containing A, or called a compatible closure of A.

Definition 3.4. (Goguen [14]) Let X be a nonempty set and L be a completely distributive lattice. Define a binary L-relation S: $L^X \times L^X \longrightarrow L$ on L^X as follow:

$$\forall A, B \in L^X,\ S(A, B) = \bigwedge_{x \in X} (A(x) \to B(x)).$$

Then S is called an L-inclusion relation, and $S(A, B)$ is called the subsethood degree of A contains in B.

According to the notion of the subsethood degree in Defininition 3.4, we can give another characterization of Definition 3.1.

Proposition 3.5. Let (X, R) be an L-approximation space. Then $\mathscr{C}(A) = S(\hat{A}, A)$ for each $A \in L^X$.

Proof. For each $A \in L^X$, it follows from Definitions 3.3 and 3.4 that

$$\begin{aligned} S(\hat{A}, A) &= \bigwedge_{y \in X} (\hat{A}(y) \to A(y)) \\ &= \bigwedge_{y \in X} (\bigvee_{x \in X} (A(x) \wedge R(x, y)) \to A(y)) \\ &= \bigwedge_{x, y \in X} ((A(x) \wedge R(x, y)) \to A(y)) \quad \text{(By Lemma 2.3)} \\ &= \mathscr{C}(A) \end{aligned}$$

as desired. □

Next, we will present some characterizations of the compatible degree of L-subsets.

Lemma 3.6. Let (X,R) be an L-approximation space and $A\in L^X$. Then for each $a\in L$,

$$a\le\mathscr{C}(A)\Longleftrightarrow\forall x,y\in X,\ A(x)\wedge R(x,y)\wedge a\le A(y).$$

Theorem 3.7. Let (X,R) be an L-approximation space and $A\in L^X$. Then

$$\mathscr{C}(A)=\bigvee\{a\in L\mid A(x)\wedge R(x,y)\wedge a\le A(y),\ \forall x,y\in X\}.$$

Proof. It follows immediately from Lemma 3.6. □

Theorem 3.8. Let (X,R) be an L-approximation space and $A\in L^X$. Then

(1) $\mathscr{C}(A)=\bigvee\{a\in L\mid\forall b\le a,\ A_{[b]}\ \text{is a compatible subset on}\ (X,R_{[b]})\}$.

(2) $\mathscr{C}(A)=\bigvee\{a\in L\mid\forall b\in P(L),\ a\not\le b,\ A^{(b)}\ \text{is a compatible subset on}\ (X,R^{(b)})\}$.

(3) $\mathscr{C}(A)=\bigvee\{a\in L\mid\forall b\notin\alpha(a),\ A^{[b]}\ \text{is a compatible subset on}\ (X,R^{[b]})\}$.

(4) $\mathscr{C}(A)=\bigvee\{a\in L\mid\forall b\in\beta(a),\ A_{(b)}\ \text{is a compatible subset on}\ (X,R_{(b)})\}$, if $\beta(a\wedge b)=\beta(a)\cap\beta(b)$ for each $a,b\in L$.

Proof. (1) Assume that $c\in L$ with the property of $A(x)\wedge R(x,y)\wedge c\le A(y)$ for each $x,y\in X$. For each $b\le c$, let $x\in A_{[b]}$ and $(x,y)\in R_{[b]}$, i.e., $b\le A(x)$ and $b\le R(x,y)$. Then it follows that $b\le A(x)\wedge R(x,y)\wedge c\le A(y)$, which implies $y\in A_{[b]}$. Thus, $A_{[b]}$ is a compatible subset on $(X,R_{[b]})$. This means

$$c\in\{a\in L\mid\forall b\le a,\ A_{[b]}\ \text{is a compatible subset on}\ (X,R_{[b]})\}.$$

By the arbitrariness of c, we have

$$\mathscr{C}(A)\le\bigvee\{a\in L\mid\forall b\le a,\ A_{[b]}\ \text{is a compatible subset on}\ (X,R_{[b]})\}.$$

Conversely, assume that $d\in L$ and $A_{[b]}$ is a compatible subset on $(X,R_{[b]})$ for each $b\le d$. For each $x,y\in X$, let $c\in L$ such that $c\le A(x)\wedge R(x,y)\wedge d$. Then it follows that $c\le d$, which implies $A_{[c]}$ is a compatible subset on $(X,R_{[c]})$. Since $c\le A(x)$ and $c\le R(x,y)$, i.e. $x\in A_{[c]}$ and $(x,y)\in R_{[c]}$, we have $y\in A_{[c]}$, i.e., $c\le A(x)$. By the arbitrariness of c and Lemma 2.2, we can obtain $A(x)\wedge R(x,y)\wedge d\le A(y)$. This means

$$d\in\{a\in L\mid A(x)\wedge R(x,y)\wedge a\le A(y),\quad\forall x,y\in X\}.$$

This implies

$$\bigvee\{a\in L\mid\forall b\le a,\ A_{[b]}\ \text{is a compatible subset on}\ (X,R_{[b]})\}\le\mathscr{C}(A).$$

(2) Assume that $c\in L$ with the property of $A(x)\wedge R(x,y)\wedge c\le A(y)$ for each $x,y\in X$. For each $b\in P(L)$ with $c\not\le b$, let $x\in A^{(b)}$ and $(x,y)\in R^{(b)}$, i.e., $A(x)\not\le b$ and $R(x,y)\not\le b$. Then it follows from the definition of prime elements that $A(x)\wedge R(x,y)\wedge c\not\le b$,

which implies $A(y) \not\leq b$, i.e., $y \in A^{(b)}$. Thus, $A^{(b)}$ is a compatible subset on $(X, R^{(b)})$. This means

$$c \in \{a \in L \mid \forall b \in P(L),\ a \not\leq b,\ A^{(b)} \text{ is a compatible subset on } (X, R^{(b)})\}.$$

By the arbitrariness of c, we have

$$\mathscr{C}(A) \leq \bigvee\{a \in L \mid \forall b \in P(L),\ a \not\leq b,\ A^{(b)} \text{ is a compatible subset on } (X, R^{(b)})\}.$$

Conversely, assume that $d \in L$ and $A^{(b)}$ is a compatible subset on $(X, R^{(b)})$ for each $b \in P(L)$ with $d \not\leq b$. For each $x, y \in X$, let $c \in P(L)$ such that $A(x) \wedge R(x, y) \wedge d \not\leq c$. Then it follows that $d \not\leq c$, which implies $A^{(c)}$ is a compatible subset on $(X, R^{(c)})$. Since $A(x) \not\leq c$ and $R(x, y) \not\leq c$, i.e, $x \in A^{(c)}$ and $(x, y) \in R^{(c)}$, we have $y \in A^{(c)}$, i.e., $A(y) \not\leq c$. By the arbitrariness of c, we can obtain $A(x) \wedge R(x, y) \wedge d \leq A(y)$. This means

$$d \in \{a \in L \mid A(x) \wedge R(x, y) \wedge a \leq A(y),\quad \forall x, y \in X\}.$$

This implies

$$\bigvee\{a \in L \mid \forall b \in P(L),\ a \not\leq b,\ A^{(b)} \text{ is a compatible subset on } (X, R^{(b)})\} \leq \mathscr{C}(A).$$

(3) Assume that $c \in L$ with the property of $A(x) \wedge R(x, y) \wedge c \leq A(y)$ for each $x, y \in X$. For each $b \notin \alpha(c)$, let $x \in A^{[b]}$ and $(x, y) \in R^{[b]}$, i.e., $b \notin \alpha(A(x))$ and $b \notin \alpha(R(x, y))$. Then it follows from Theorem 2.1 that

$$b \notin \alpha(A(x)) \cup \alpha(R(x, y)) \cup \alpha(c) = \alpha(A(x) \wedge R(x, y) \wedge c).$$

Since $\alpha(A(y)) \subseteq \alpha(A(x) \wedge R(x, y) \wedge c)$, we have $b \notin \alpha(A(y))$, i.e., $y \in A^{[b]}$. Thus, $A^{[b]}$ is a compatible subset on $(X, R^{[b]})$. This means

$$c \in \{a \in L \mid \forall b \notin \alpha(a),\ A^{[b]} \text{ is a compatible subset on } (X, R^{[b]})\}.$$

By the arbitrariness of c, we have

$$\mathscr{C}(A) \leq \bigvee\{a \in L \mid \forall b \notin \alpha(a),\ A^{[b]} \text{ is a compatible subset on } (X, R^{[b]})\}.$$

Conversely, assume that $d \in L$ and $A^{[b]}$ is a compatible subset on $(X, R^{[b]})$ for each $b \notin \alpha(d)$. For each $x, y \in X$, let $c \in L$ such that $c \notin \alpha(A(x) \wedge R(x, y) \wedge d)$. Then it follows from $\alpha(d) \subseteq \alpha(A(x) \wedge R(x, y) \wedge d)$ that $c \notin \alpha(d)$, which implies $A^{[c]}$ is a compatible subset on $(X, R^{[c]})$. Since $c \notin \alpha(A(x))$ and $c \notin \alpha(R(x, y))$, we have $c \notin \alpha(A(y))$. By the arbitrariness of c, we can obtain $A(x) \wedge R(x, y) \wedge d \leq A(y)$. This means $d \in \{a \in L \mid A(x) \wedge R(x, y) \wedge a \leq A(y),\quad \forall x, y \in X\}$. This implies

$$\bigvee\{a \in L \mid \forall b \notin \alpha(a),\ A^{[b]} \text{ is a compatible subset on } (X, R^{[b]})\} \leq \mathscr{C}(A).$$

(4) Assume that $c \in L$ with the property of $A(x) \wedge R(x, y) \wedge c \leq A(y)$ for each $x, y \in X$. For each $b \in \beta(c)$, let $x \in A_{(b)}$ and $(x, y) \in R_{(b)}$, i.e., $b \in \beta(A(x))$ and $b \in \beta(R(x, y))$. Then it follows from Theorem 2.1 that

$$b \in \beta(A(x)) \cap \beta(R(x, y)) \cap \beta(c) = \beta(A(x) \wedge R(x, y) \wedge c) \subseteq \beta(A(y)),$$

which implies $y \in A_{(b)}$. Thus, $A_{(b)}$ is a compatible subset on $(X, R_{(b)})$. This means $c \in \{a \in L \mid \forall b \in \beta(a),\ A_{(b)} \text{ is a compatible subset on } (X, R_{(b)})\}$. By the arbitrariness of c, we have

$$\mathscr{C}(A) \leq \bigvee\{a \in L \mid \forall b \in \beta(a),\ A_{(b)} \text{ is a compatible subset on } (X, R_{(b)})\}.$$

Conversely, assume that $d \in L$ and $A_{(b)}$ is a compatible subset on $(X, R_{(b)})$ for each $b \in \beta(d)$. For each $x, y \in X$, let $c \in L$ such that $c \in \beta(A(x) \wedge R(x,y) \wedge d)$. Then it follows from $\beta(A(x) \wedge R(x,y) \wedge d) \subseteq \beta(d)$ that $c \in \beta(d)$, which implies $A_{(c)}$ is a compatible subset on $(X, R_{(c)})$. Since $c \in \beta(A(x))$ and $c \in \beta(R(x,y))$, we have $c \in \beta(A(y))$. By the arbitrariness of c, we can obtain $A(x) \wedge R(x,y) \wedge d \leq A(y)$. This means $d \in \{a \in L \mid A(x) \wedge R(x,y) \wedge a \leq A(y),\quad \forall x, y \in X\}$. This implies

$$\bigvee\{a \in L \mid \forall b \in \beta(a),\ A_{(b)} \text{ is a compatible subset on } (X, R_{(b)})\} \leq \mathscr{C}(A).$$

□

From Theorems 2.7, 2.8, 3.8 and Definition 2.9, we can obtain the following corollaries.

Corollary 3.9. Let (X, R) be an L-approximation space and $A \in L^X$. Then the following statements are equivalent:

(1) A is a compatible L-subset on (X, R).

(2) $A_{[a]}$ is a compatible subset on $(X, R_{[a]})$ for each $a \in L$.

(3) $A_{[a]}$ is a compatible subset on $(X, R_{[a]})$ for each $a \in J(L)$.

(4) $A^{[a]}$ is a compatible subset on $(X, R^{[a]})$ for each $a \in L$.

(5) $A^{[a]}$ is a compatible subset on $(X, R^{[a]})$ for each $a \in P(L)$.

(6) $A^{(a)}$ is a compatible subset on $(X, R^{(a)})$ for each $a \in P(L)$.

Corollary 3.10. Let (X, R) be an L-approximation space and $A \in L^X$. If $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ for each $a, b \in L$, then the following statements are equivalent:

(1) A is a compatible L-subset on (X, R).

(2) $A_{(a)}$ is a compatible subset on $(X, R_{(a)})$ for each $a \in \beta(1)$.

4. THE COMPATIBLE DEGREE OF MAPPINGS AND ITS CHARACTERIZATIONS

In this section, we will present the definition of the compatible degree of mappings, look for its a new characterization in terms of the compatible degree of L-subsets and give eight characterizations of the compatible degree of mappings by four kinds of cut sets of L-equivalences and the compatible degree of L-subsets, respectively. Then, we will explore the relationships between the compatible degree of the composition of two mappings and compatible degrees of the two mappings.

4.1. The compatible degree of mappings and its characterizations in terms of cut sets of L-equivalences

In this subsection, we will fuzzify compatible mappings, introduce the notion of the compatible degree of mappings and give its four characterizations by four kinds of cut sets of L-equivalences.

Definition 4.1. Let (X, R_X) and (Y, R_Y) be two L-approximation spaces and $f\colon X \longrightarrow Y$ be a mapping. $\mathscr{D}(f)$ defined by

$$\mathscr{D}(f) = \bigwedge_{x,y\in X} (R_X(x,y) \to R_Y(f(x), f(y)))$$

is called the degree to which f is a compatible mapping.

Similarly, we can give another characterization of Definition 4.1 by using the definition of the subsethood degree.

Proposition 4.2. Let (X, R_X) and (Y, R_Y) be two L-approximation spaces and $f\colon X \longrightarrow Y$ be a mapping. We define a subsethood degree $S\colon L^{X\times X} \times L^{X\times X} \longrightarrow L$ on $L^{X\times X}$ as follow:

$$\forall R_1, R_2 \in L^{X\times X},\ S(R_1, R_2) = \bigwedge_{(x,y)\in X\times X} (R_1(x,y) \to R_2(x,y)).$$

Then $\mathscr{D}(f) = S(R_X, R_Y \circ f^2)$, where $f^2 : X \times X \longrightarrow Y \times Y$ is defined by $f^2(x,y) = (f(x), f(y))$ for each $(x,y) \in X \times X$.

Proof. It is straightforward. □

In the following, we will present some characterizations of the compatible degree of mappings by using four kinds of cut sets.

Lemma 4.3. Let (X, R_X) and (Y, R_Y) be two L-approximation spaces and $f\colon X \longrightarrow Y$ be a mapping. Then for each $a \in L$,

$$a \le \mathscr{D}(f) \Longleftrightarrow \forall x, y \in X,\ R_X(x,y) \wedge a \le R_Y(f(x), f(y)).$$

Theorem 4.4. Let (X, R_X) and (Y, R_Y) be two L-approximation spaces and $f\colon X \longrightarrow Y$ be a mapping. Then

$$\mathscr{D}(f) = \bigvee\{a \in L \mid R_X(x,y) \wedge a \le R_Y(f(x), f(y)),\ \forall x, y \in X\}.$$

Proof. It follows immediately from Lemma 4.3. □

Theorem 4.5. Let (X, R_X) and (Y, R_Y) be two L-approximation spaces and $f\colon X \longrightarrow Y$ be a mapping. Then

(1) $\mathscr{D}(f) = \bigvee\{a \in L \mid \forall b \le a,\ f : (X, (R_X)_{[b]}) \to (Y, (R_Y)_{[b]})\ \text{is compatible}\}$.

(2) $\mathscr{D}(f) = \bigvee\{a \in L \mid \forall b \in P(L),\ a \not\le b,\ f : (X, (R_X)^{(b)}) \to (Y, (R_Y)^{(b)})\ \text{is compatible}\}$.

(3) $\mathscr{D}(f) = \bigvee\{a \in L \mid \forall b \notin \alpha(a),\ f : (X, (R_X)^{[b]}) \to (Y, (R_Y)^{[b]})\ \text{is compatible}\}$.

(4) $\mathscr{D}(f) = \bigvee\{a \in L \mid \forall b \in \beta(a),\ f : (X, (R_X)_{(b)}) \to (Y, (R_Y)_{(b)})\ \text{is compatible}\}$, if $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ for each $a, b \in L$.

Proof. It can be proven in a similar way as that Theorem 3.8. □

From Theorems 2.7, 2.8, 4.5 and Definition 4.1, we can obtain the following corollaries.

Corollary 4.6. Let (X, R_X) and (Y, R_Y) be two L-approximation spaces and $f\colon X \longrightarrow Y$ be a mapping. Then the following statements are equivalent:

(1) f is a compatible mapping from (X, R_X) to (Y, R_Y).

(2) f is a compatible mapping from $(X, (R_X)_{[a]})$ to $(Y, (R_Y)_{[a]})$ for each $a \in L$.

(3) f is a compatible mapping from $(X, (R_X)_{[a]})$ to $(Y, (R_Y)_{[a]})$ for each $a \in J(L)$.

(4) f is a compatible mapping from $(X, (R_X)^{[a]})$ to $(Y, (R_Y)^{[a]})$ for each $a \in L$.

(5) f is a compatible mapping from $(X, (R_X)^{[a]})$ to $(Y, (R_Y)^{[a]})$ for each $a \in P(L)$.

(6) f is a compatible mapping from $(X, (R_X)^{(a)})$ to $(Y, (R_Y)^{(a)})$ for each $a \in P(L)$.

Corollary 4.7. Let (X, R_X) and (Y, R_Y) be two L-approximation spaces and $A \in L^X$. If $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ for each $a, b \in L$, then the following statements are equivalent:

(1) f is a compatible mapping from (X, R_X) to (Y, R_Y).

(2) f is a compatible mapping from $(X, (R_X)_{(a)})$ to $(Y, (R_Y)_{(a)})$ for each $a \in \beta(1)$.

Proposition 4.8. Let (X, R_X), (Y, R_Y) and (Z, R_Z) be three L-approximation spaces, $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be two mappings. Then the following statements hold.

(1) $\mathscr{D}(g) \wedge \mathscr{D}(f) \le \mathscr{D}(g \circ f)$.

(2) $\mathscr{D}(g \circ f) \le \mathscr{D}(f)$ if g is generalized injective.

(3) $\mathscr{D}(g \circ f) \le \mathscr{D}(g)$ if f is surjective and generalized injective.

P r o o f. (1) By Definition 4.1 and Lemma 2.3, it follows that

$$\begin{aligned}\mathscr{D}(g)\wedge\mathscr{D}(f) &= \bigwedge_{u,w\in Y}(R_Y(u,w)\to R_Z(g(u),g(w)))\wedge\bigwedge_{x,y\in X}(R_X(x,y)\to R_Y(f(x),f(y)))\\ &\leq \bigwedge_{x,y\in X}(R_Y(f(x),f(y))\to R_Z(g(f(x)),g(f(y))))\wedge\\ &\quad \bigwedge_{x,y\in X}(R_X(x,y)\to R_Y(f(x),f(y)))\\ &= \bigwedge_{x,y\in X}((R_X(x,y)\to R_Y(f(x),f(y)))\wedge\\ &\qquad (R_Y(f(x),f(y))\to R_Z(g(f(x)),g(f(y)))))\\ &\leq \bigwedge_{x,y\in X}(R_X(x,y)\to R_Y(g\circ f(x),g\circ f(y)))\\ &= \mathscr{D}(g\circ f).\end{aligned}$$

(2) Since g is generalized injective, it follows from Definition 2.12 and Lemma 2.3 that

$$\begin{aligned}\mathscr{D}(g\circ f) &= \bigwedge_{x,y\in X}(R_X(x,y)\to R_Y(g\circ f(x),g\circ f(y)))\\ &= \bigwedge_{x,y\in X}(R_X(x,y)\to R_Z(g(f(x)),g(f(y))))\\ &\leq \bigwedge_{x,y\in X}(R_X(x,y)\to R_Y(f(x),f(y)))\\ &= \mathscr{D}(f).\end{aligned}$$

(3) Since f is generalized injective, it follows that

$$\begin{aligned}\mathscr{D}(g\circ f) &= \bigwedge_{x,y\in X}(R_X(x,y)\to R_Z(g(f(x)),g(f(y))))\\ &\leq \bigwedge_{x,y\in X}(R_X(f(x),f(y))\to R_Z(g(f(x)),g(f(y))))\\ &= \bigwedge_{u,w\in Y}(R_Y(u,w)\to R_Z(g(u),g(w)))\quad (\text{Since } f \text{ is surjective})\\ &= \mathscr{D}(g),\end{aligned}$$

as desired. □

When $\mathscr{D}(f)=1$, then f is a compatible mapping in the sense of Definition 2.11. By Proposition 4.8, we can obtain the following corollary.

Corollary 4.9. Let (X,R_X), (Y,R_Y) and (Z,R_Z) be three L-approximation spaces, $f:\ X\longrightarrow Y$ and g: $Y\longrightarrow Z$ be two mappings. Then the following statements hold:

(1) If f and g are compatible mappings, then $g \circ f$ is a compatible mapping.

(2) If g is generalized injective and $g \circ f$ is a compatible mapping, then f is a compatible mapping.

(3) If f is surjective and generalized injective and $g \circ f$ is a compatible mapping, then g is a compatible mapping.

4.2. Characterizations of the compatible degree of mappings in terms of the compatible degree of L-subsets

In this subsection, after exploring the relationship between compatible mappings and the image and preimage of compatible L-subsets, we will present a new characterization of the compatible degree of mappings in terms of the compatible degree of L-subsets, and give its another four characterizations by four kinds of cut sets of the compatible degree of L-subsets.

Theorem 4.10. Let (X, R_X) and (Y, R_Y) be two L-approximation spaces and $f\colon X \longrightarrow Y$ be a mapping. Then f is a compatible mapping from (X, R_X) to (Y, R_Y) if and only if $B \in L^{(Y,R_Y)}$ implies $f^{\leftarrow}(B) \in L^{(X,R_X)}$ for each $B \in L^Y$.

Proof. Necessity. Take each $B \in L^Y$ such that $B \in L^{(Y,R_Y)}$. Then for each $x, y \in X$, it follows from the definitions of compatible L-subsets and compatible mappings that

$$\begin{aligned} f^{\leftarrow}(B)(x) \wedge R_X(x,y) &= B(f(x)) \wedge R_X(x,y) \\ &\le B(f(x)) \wedge R_Y(f(x), f(y)) \\ &\le B(f(y)) \\ &= f^{\leftarrow}(B)(y). \end{aligned}$$

That is to say, $f^{\leftarrow}(B)$ is a compatible L-subset, i. e., $f^{\leftarrow}(B) \in L^{(X,R_X)}$.

Sufficiency. Take each $x, y \in X$. Then it follows from Example 2.10 that $R_{f(x)} \in L^{(Y,R_Y)}$, which implies $f^{\leftarrow}(R_{f(x)}) \in L^{(X,\sim_X)}$. Thus we have

$$f^{\leftarrow}(R_{f(x)})(x) \wedge R_X(x,y) \le f^{\leftarrow}(R_{f(x)})(y).$$

That is to say, $R_X(x,y) \le R_Y(f(x), f(y))$. This shows that f is a compatible mapping. □

Theorem 4.11. Let $\mathscr{C}_X$ and $\mathscr{C}_Y$ be two compatible degrees induced from (X, R_X) and (Y, R_Y), respectively. Then f is a compatible mapping from (X, R_X) to (Y, R_Y) if and only if $\mathscr{C}_Y(B) \le \mathscr{C}_X(f^{\leftarrow}(B))$ for each $B \in L^Y$.

Proof. Necessity. Take each $B \in L^Y$. Then it follows from the definition of compatible mappings that

$$\begin{aligned}\mathscr{C}_Y(B) &= \bigwedge_{y,w\in Y} (B(y)\wedge R_Y(y,w)\to B(w))\\ &\le \bigwedge_{x,z\in Y} B(f(x))\wedge R_Y(f(x),f(z))\to B(f(z)))\\ &\le \bigwedge_{x,z\in Y} B(f(x))\wedge R_X(x,z)\to B(f(z)))\\ &= \mathscr{C}_X(f^{\leftarrow}(B)).\end{aligned}$$

Sufficiency. Take each $B\in L^Y$ such that $B\in L^{(Y,R_Y)}$. Then $\mathscr{C}_Y(B)=1$, By $\mathscr{C}_Y(B)\le \mathscr{C}_X(f^{\leftarrow}(B))$, we have $\mathscr{C}_X(f^{\leftarrow}(B))=1$, which implies $f^{\leftarrow}(B)\in L^{(X,R_X)}$. By Theorem 4.10, it follows that f is a compatible mapping. □

In the following, we will give a lemma before introducing a new characterization of the compatible degree of mappings.

Lemma 4.12. Let (X,R_X) and (Y,R_Y) be two L-approximation spaces and $B\in L^Y$. Then the following statements are equivalent:

(1) $B\in(\mathscr{C}_Y)_{[a]}$ implies $f^{\leftarrow}(B)\in(\mathscr{C}_X)_{[a]}$ for each $a\in L$, $B\in L^Y$.

(2) f is a compatible mapping from $(X,(R_X)_{[a]})$ to $(Y,(R_Y)_{[a]})$ for each $a\in L$.

Proof. Take each $B\in L^Y$. Then we have

$B\in(\mathscr{C}_Y)_{[a]}$ implies $f^{\leftarrow}(B)\in(\mathscr{C}_X)_{[a]}$ for each $a\in L$, $B\in L^Y$
$\Longleftrightarrow a\le\mathscr{C}_Y(B)$ implies $a\le\mathscr{C}_X(f^{\leftarrow}(B))$ for each $a\in L$, $B\in L^Y$
$\Longleftrightarrow \mathscr{C}_Y(B)\le\mathscr{C}_X(f^{\leftarrow}(B))$ for each $B\in L^Y$ (By Lemma 2.2)
$\Longleftrightarrow f$ is a compatible mapping from (X,R_X) to (Y,R_Y) (By Theorem 4.11)
$\Longleftrightarrow f$ is a compatible mapping from $(X,(R_X)_{[a]})$ to $(Y,(R_Y)_{[a]})$ for each $a\in L$, (By Corollary 4.6)

as desired. □

Theorem 4.13. Let (X,R_X) and (Y,R_Y) be two L-approximation spaces and $f\colon X\longrightarrow Y$ be a mapping. Then

$$\mathscr{D}(f)=\bigwedge_{B\in L^Y}(\mathscr{C}_Y(B)\to\mathscr{C}_X(f^{\leftarrow}(B))).$$

Proof. Take each $a\in L$ such that $a\le\bigwedge_{B\in L^Y}(\mathscr{C}_Y(B)\to\mathscr{C}_X(f^{\leftarrow}(B)))$. Then it follows that $a\le\mathscr{C}_Y(B)\to\mathscr{C}_X(f^{\leftarrow}(B))$ for each $B\in L^Y$. By Lemma 2.3, we have $a\wedge\mathscr{C}_Y(B)\le\mathscr{C}_X(f^{\leftarrow}(B))$ for each $B\in L^Y$. For each $b\le a$, let $B\in(\mathscr{C}_Y)_{[b]}$, i.e., $b\le\mathscr{C}_Y(B)$. Then it follows that $b\le\mathscr{C}_Y(B)\wedge a\le\mathscr{C}_X(f^{\leftarrow}(B))$, which implies $f^{\leftarrow}(B)\in$

$(\mathscr{C}_X)_{[b]}$. This shows $B \in (\mathscr{C}_Y)_{[b]}$ implies $f^{\triangleleft}(B) \in (\mathscr{C}_X)_{[b]}$ for each $b \leq a$. By Lemma 4.12, we can obtain that f is a compatible mapping from $(X, (R_X)_{[b]})$ to $(Y, (R_Y)_{[b]})$ for each $b \leq a$. Applying (1) of Theorem 4.5 and Theorem 4.4, we have

$$\begin{aligned} a &\in \{c \in L \mid \forall b \leq c,\ f : (X, (R_X)_{[b]}) \to (Y, (R_Y)_{[b]}) \text{ is compatible}\} \\ &= \{c \in L \mid R_X(x, y) \wedge c \leq R_Y(f(x), f(y)),\ \forall x, y \in X\}. \end{aligned}$$

This means $R_X(x, y) \wedge a \leq R_Y(f(x), f(y))$, i.e, $a \leq R_X(x, y) \to R_Y(f(x), f(y))$ for each $x, y \in X$. That is to say,

$$a \leq \bigwedge_{x,y \in X} (R_X(x, y) \to R_Y(f(x), f(y))) = \mathscr{D}(f).$$

By the arbitrariness of a, we have

$$\bigwedge_{B \in L^Y} (\mathscr{C}_Y(B) \to \mathscr{C}_X(f^{\leftarrow}(B))) \leq \mathscr{D}(f).$$

Now, we will show that $\mathscr{D}(f) \leq \bigwedge_{B \in L^Y} (\mathscr{C}_Y(B) \to \mathscr{C}_X(f^{\leftarrow}(B)))$. It follows from (6), (7) and (8) of Lemma 2.3 that

$$\begin{aligned} &\bigwedge_{B \in L^Y} (\mathscr{C}_Y(B) \to \mathscr{C}_X(f^{\leftarrow}(B))) \\ &= \bigwedge_{B \in L^Y} \Big(\bigwedge_{y,w \in Y} (B(y) \wedge R_Y(y, w) \to B(w)) \to \Big(\bigwedge_{x,z \in X} f^{\leftarrow}(B)(x) \wedge R_X(x, z) \to f^{\leftarrow}(B)(z) \Big) \Big) \\ &\geq \bigwedge_{B \in L^Y} \Big(\bigwedge_{x,z \in X} ((B(f(x)) \wedge R_Y(f(x), f(z))) \to B(f(z))) \to \\ &\qquad \bigwedge_{x,z \in X} (f^{\leftarrow}(B)(x) \wedge R_X(x, z) \to f^{\leftarrow}(B)(z)) \Big) \\ &\geq \bigwedge_{B \in L^Y} \bigwedge_{x,z \in X} ((f^{\leftarrow}(B)(x) \wedge R_Y(f(x), f(z))) \to f^{\leftarrow}(B)(z)) \to \\ &\qquad ((f^{\leftarrow}(B)(x) \wedge R_X(x, z)) \to f^{\leftarrow}(B)(z)) \\ &\geq \bigwedge_{B \in L^Y} \bigwedge_{x,z \in X} (f^{\triangleleft}(B)(x) \wedge R_X(x, z)) \to (f^{\leftarrow}(B)(x) \wedge R_Y(f(x), f(z))) \\ &\geq \bigwedge_{x,z \in X} (R_X(x, z) \to R_Y(f(x), f(z))) \\ &= \mathscr{D}(f), \end{aligned}$$

as desired. □

In a manner of speaking, the above theorem provides a new definition of the compatible degree of mappings. Through the characterization approach with four kinds of cut sets, we can easily get the following conclusions, so we do not give the proof process.

Lemma 4.14. Let (X, R_X) and (Y, R_Y) be two L-approximation spaces and $f\colon X \longrightarrow Y$ be a mapping. Then for each $a \in L$,

$$a \leq \mathscr{D}(f) \Longleftrightarrow \forall B \in L^Y,\ a \wedge \mathscr{C}_Y(B) \leq \mathscr{C}_X(f^{\leftarrow}(B)).$$

Theorem 4.15. Let (X, R_X) and (Y, R_Y) be two L-approximation spaces and $f\colon X \longrightarrow Y$ be a mapping. Then

$$\mathscr{D}(f) = \bigvee\{a \in L \mid a \wedge \mathscr{C}_Y(B) \leq \mathscr{C}_X(f^{\leftarrow}(B)),\ \forall B \in L^Y\}.$$

Proof. It follows immediately from Lemma 4.14. □

Theorem 4.16. Let (X, R_X) and (Y, R_Y) be two L-approximation spaces and $f\colon X \longrightarrow Y$ be a mapping. Then

(1) $\mathscr{D}(f) = \bigvee\{a \in L \mid \forall b \leq a,\ B \in (\mathscr{C}_Y)_{[b]} \text{ implies } f^{\leftarrow}(B) \in (\mathscr{C}_X)_{[b]},\ \forall B \in L^Y\}$.

(2) $\mathscr{D}(f) = \bigvee\{a \in L \mid \forall b \in P(L),\ a \not\leq b,\ B \in (\mathscr{C}_Y)^{(b)} \text{ implies } f^{\leftarrow}(B) \in (\mathscr{C}_X)^{(b)},\ \forall B \in L^Y\}$.

(3) $\mathscr{D}(f) = \bigvee\{a \in L \mid \forall b \notin \alpha(a),\ B \in (\mathscr{C}_Y)^{[b]} \text{ implies } f^{\leftarrow}(B) \in (\mathscr{C}_X)^{[b]},\ \forall B \in L^Y\}$.

(4) $\mathscr{D}(f) = \bigvee\{a \in L \mid \forall b \in \beta(a),\ B \in (\mathscr{C}_Y)_{(b)} \text{ implies } f^{\leftarrow}(B) \in (\mathscr{C}_X)_{(b)},\ \forall B \in L^Y\}$, if $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ for each $a, b \in L$.

Proof. It follows immediately from Lemma 4.12 and Theorem 4.13. □

From Theorem 4.16, we can obtain the following corollaries.

Corollary 4.17. Let (X, R_X) and (Y, R_Y) be two L-approximation spaces and $B \in L^Y$. Then the following statements are equivalent:

(1) $B \in L^{(Y,R_Y)}$ implies $f^{\leftarrow}(B) \in L^{(X,R_X)}$.

(2) $B \in (\mathscr{C}_Y)_{[a]}$ implies $f^{\leftarrow}(B) \in (\mathscr{C}_X)_{[a]}$ for each $a \in L$.

(3) $B \in (\mathscr{C}_Y)_{[a]}$ implies $f^{\leftarrow}(B) \in (\mathscr{C}_X)_{[a]}$ for each $a \in J(L)$.

(4) $B \in (\mathscr{C}_Y)^{[a]}$ implies $f^{\leftarrow}(B) \in (\mathscr{C}_X)^{[a]}$ for each $a \in L$.

(5) $B \in (\mathscr{C}_Y)^{[a]}$ implies $f^{\leftarrow}(B) \in (\mathscr{C}_X)^{[a]}$ for each $a \in P(L)$.

(6) $B \in (\mathscr{C}_Y)^{(a)}$ implies $f^{\leftarrow}(B) \in (\mathscr{C}_X)^{(a)}$ for each $a \in P(L)$.

Corollary 4.18. Let (X, R_X) and (Y, R_Y) be two L-approximation spaces and $B \in L^Y$. If $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ for each $a, b \in L$, then the following statements are equivalent:

(1) $B \in L^{(Y,R_Y)}$ implies $f^{\leftarrow}(B) \in L^{(X,R_X)}$.

(2) $B \in (\mathscr{C}_Y)_{(a)}$ implies $f^{\leftarrow}(B) \in (\mathscr{C}_X)_{(a)}$ for each $a \in \beta(1)$.

5. COMPLETE L-SUBLATTICE INDUCED BY THE COMPATIBLE DEGREE OF L-SUBSETS

In this section, we will prove that the set of all compatible L-subsets on X is a complete sublattice of $(L^X, \leq)$, and introduce the notion of a complete L-sublattice. Then we will give some characterizations of a complete L-sublattice and prove that the compatible degree of L-subsets is a complete L-sublattice of $(L^X, \leq)$. Finally, we will discuss that the product of two complete L-sublattices is still a complete L-sublattice.

Theorem 5.1. Let (X, R) be an L-approximation space. Then $L^{(X,R)}$ is a complete sublattice of $(L^X, \leq)$.

Proof. It is obvious that $\underline{1}$ and $\underline{0}$ are in $L^{(X,R)}$.

Take each subfamily $\{A_i \mid i \in I\} \subseteq L^{(X,R)}$. Then it follows from Definition 2.4 that for each $x, y \in X$,

$$\begin{aligned}\Big(\bigwedge_{i\in I} A_i)(x) \wedge R(x,y) &\leq \bigwedge_{i\in I}(A_i(x) \wedge R(x,y)\Big)\\ &\leq \bigwedge_{i\in I} A_i(y)\end{aligned}$$

and

$$\begin{aligned}(\bigvee_{i\in I} A_i)(x) \wedge R(x,y) &\leq \bigvee_{i\in I}(A_i(x) \wedge R(x,y))\\ &\leq \bigvee_{i\in I} A_i(y).\end{aligned}$$

This means that $\bigwedge_{i\in I} A_i \in L^{(X,R)}$ and $\bigwedge_{i\in I} A_i \in L^{(X,R)}$. This shows that $L^{(X,R)}$ is a complete sublattice of L^X. □

Theorem 5.2. Let (X, R_X) and (Y, R_Y) be two L-approximation spaces and $f\colon X \longrightarrow Y$ be a mapping. Then $f^{\leftarrow}\colon L^{(Y,R_Y)} \longrightarrow L^{(X,R_X)}$ is a complete lattice homomorphism.

Proof. Take each subfamily $\{B_i \mid i \in I\} \subseteq L^{(Y,R_Y)}$. Then for each $x \in X$,

$$f^{\leftarrow}(\bigvee_{i\in I} B_i)(x) = \bigvee_{i\in I} B_i(f(x)) = \bigvee_{i\in I} f^{\leftarrow}(B_i)(x),$$

which implies $f^{\leftarrow}(\bigvee_{i\in I} B_i) = \bigvee_{i\in I} f^{\leftarrow}(B_i)$. And for each $x \in X$,

$$f^{\leftarrow}\Big(\bigwedge_{i\in I} B_i\Big)(x) = \bigwedge_{i\in I} B_i(f(x)) = \bigwedge_{i\in I} f^{\leftarrow}(B_i)(x),$$

which implies $f^{\leftarrow}\Big(\bigwedge_{i\in I} B_i\Big) = \bigwedge_{i\in I} f^{\leftarrow}(B_i)$. It shows that $f^{\leftarrow}$ is a complete lattice homomorphism. □

Definition 5.3. Let $(M,\leq)$ be a complete lattice. Then an L-subset $\mathscr{D}\colon M\longrightarrow L$ is called a complete L-sublattice of $(M,\leq)$, if it satisfies, for each $S\subseteq M$,

(1) $\bigwedge_{s\in S}\mathscr{D}(s)\leq\mathscr{D}(\bigvee S)$;

(2) $\bigwedge_{s\in S}\mathscr{D}(s)\leq\mathscr{D}\Big(\bigwedge S\Big)$.

Theorem 5.4. Let $\mathscr{D}\colon M\longrightarrow L$ be an L-subset on $(M,\leq)$. Then the following statements are equivalent:

(1) $\mathscr{D}$ is a complete L-sublattice of M.

(2) $\mathscr{D}_{[a]}$ is a complete sublattice of M for each $a\in L$.

(3) $\mathscr{D}^{[a]}$ is a complete sublattice of M for each $a\in L$.

P r o o f. (1) $\Longrightarrow$ (2) Take each $a\in L$ and $S\subseteq M$ such that $S\subseteq\mathscr{D}_{[a]}$, i. e., $a\leq\mathscr{D}(s)$ for each $s\in S$. Then it follows from Definition 5.3 that

$$a\leq\bigwedge_{s\in S}\mathscr{D}(s)\leq\mathscr{D}\Big(\bigvee S\Big)$$

and

$$a\leq\bigwedge_{s\in S}\mathscr{D}(s)\leq\mathscr{D}\Big(\bigwedge S\Big).$$

Thus, $\bigvee S\in\mathscr{D}_{[a]}$ and $\bigwedge S\in\mathscr{D}_{[a]}$. Thus $\mathscr{D}_{[a]}$ is a complete sublattice.

(2) $\Longrightarrow$ (1) Take each $S\subseteq M$ and $a\in L$ such that $a\leq\bigwedge_{s\in S}\mathscr{D}(s)$, i. e., $a\leq\mathscr{D}(s)$ for each $s\in S$. Then $s\in\mathscr{D}_{[a]}$ for each $s\in S$, i. e., $S\subseteq\mathscr{D}_{[a]}$. Since $\mathscr{D}_{[a]}$ is a complete sublattice, we have $\bigvee S\in\mathscr{D}_{[a]}$ and $\bigwedge S\in\mathscr{D}_{[a]}$, which implies $a\leq\mathscr{D}(\bigvee S)$ and $a\leq\mathscr{D}\Big(\bigwedge S\Big)$. By the arbitrariness of a, $\bigwedge_{s\in S}\mathscr{D}(s)\leq\mathscr{D}(\bigvee S)$ and $\bigwedge_{s\in S}\mathscr{D}(s)\leq\mathscr{D}\Big(\bigwedge S\Big)$. This shows that $\mathscr{D}$ is a complete L-sublattice of M.

(1) $\Longrightarrow$ (3) Take each $a\in L$ and $S\subseteq M$ such that $S\subseteq\mathscr{D}^{[a]}$, i.e., $a\notin\alpha(\mathscr{D}(s))$ for each $s\in S$. Then it follows that

$$a\notin\bigcup_{s\in S}\alpha(\mathscr{D}(s))=\alpha\Big(\bigwedge_{s\in S}\mathscr{D}(s)\Big).$$

Since $\bigwedge_{s\in S}\mathscr{D}(s)\leq\mathscr{D}(\bigvee S)$ and $\bigwedge_{s\in S}\mathscr{D}(s)\leq\mathscr{D}(\bigwedge S)$, we have

$$\alpha(\mathscr{D}(\bigvee S))\subseteq\alpha(\bigwedge_{s\in S}\mathscr{D}(s))$$

and

$$\alpha(\mathscr{D}(\bigwedge S))\subseteq\alpha(\bigwedge_{s\in S}\mathscr{D}(s)),$$

which implies $a\notin\alpha(\mathscr{D}(\bigvee S))$ and $a\notin\alpha(\mathscr{D}(\bigwedge S))$. This shows, $\bigvee S\in\mathscr{D}^{[a]}$ and $\bigwedge S\in\mathscr{D}^{[a]}$. Thus, $\mathscr{D}^{[a]}$ is a complete sublattice.

(3) $\Longrightarrow$ (1) Take each $S \subseteq M$ and $a \in L$ such that $a \notin \alpha\Big(\bigwedge_{s\in S}\mathscr{D}(s)\Big)$. Then it follows that $a \notin \bigcup_{s\in S}\alpha(\mathscr{D}(s))$, which implies $S \subseteq \mathscr{D}^{[a]}$. Since $\mathscr{D}^{[a]}$ is a complete sublattice of M, we have $\bigvee S \in \mathscr{D}^{[a]}$ and $\bigwedge S \in \mathscr{D}^{[a]}$, which implies $a \notin \alpha(\mathscr{D}(\bigvee S))$ and $a \notin \alpha(\mathscr{D}\Big(\bigwedge S\Big))$. By the arbitrariness of a, we can obtain $\bigwedge_{s\in S}\mathscr{D}(s) \le \mathscr{D}(\bigvee S)$ and $\bigwedge_{s\in S}\mathscr{D}(s) \le \mathscr{D}\Big(\bigwedge S\Big)$. This shows that $\mathscr{D}$ is a complete L-sublattice of M. $\square$

Actually, in Definition 3.1, we defined a mapping $\mathscr{C}\colon L^X \longrightarrow L$ which assigns each L-subset to some degree to which this L-subset becomes a compatible L-subset. In the following theorem, we will prove that the compatible degree of L-subset is a complete L-sublattice of L^X.

Theorem 5.5. Let (X, R) be an L-approximation space. Then the compatible degree $\mathscr{C}$ is a complete L-sublattice of $(L^X, \le)$.

Proof. Take each subfamily $\{A_i \mid i \in I\} \subseteq L^X$. Then it follows from Definition 3.1 that

$$\begin{aligned}\mathscr{C}(\bigvee_{i\in I} A_i) &= \bigwedge_{x,y\in X}(\bigvee_{i\in I} A_i(x)\wedge R(x,y)\to \bigvee_{i\in I} A_i(y))\\ &\ge \bigwedge_{x,y\in X}\bigwedge_{i\in I}(A_i(x)\wedge R(x,y)\to A_i(y))\\ &= \bigwedge_{i\in I}\mathscr{C}(A_i)\end{aligned}$$

and

$$\begin{aligned}\mathscr{C}\Big(\bigwedge_{i\in I} A_i\Big) &= \bigwedge_{x,y\in X}\Big(\bigwedge_{i\in I} A_i(x)\wedge R(x,y)\to \bigwedge_{i\in I} A_i(y)\Big)\\ &\ge \bigwedge_{x,y\in X}\bigwedge_{i\in I}(A_i(x)\wedge R(x,y)\to A_i(y))\\ &= \bigwedge_{i\in I}\mathscr{C}(A_i).\end{aligned}$$

This shows that $\mathscr{C}$ is a complete L-sublattice of L^X. $\square$

Let X and Y be two nonempty sets and (X, R_X) and (Y, R_Y) be two L-approximation spaces. Then L^X and L^Y are two complete lattices in a pointwise way. By Theorem 5.1, we have $L^{(X,R_X)}$ (resp. $L^{(Y,R_Y)}$) is complete sublattice of L^X (resp. L^Y). Now we will prove the product of $L^{(X,R_X)}$ and $L^{(Y,R_Y)}$ is still a complete sublattice of $L^X \times L^Y$. Define $\mathscr{F} = L^X \times L^Y$ by $\mathscr{F} = \{(A, B) \mid A \in L^X,\ \ B \in L^Y\}$, and a partial order $\le$ on $\mathscr{F}$ by

$$\begin{aligned}(A_1, B_1) \le (A_2, B_2) &\Longleftrightarrow A_1 \le A_2 \text{ and } B_1 \le B_2\\ &\Longleftrightarrow A_1(x) \le A_2(x) \text{ and } B_1(y) \le B_2(y) \text{ for each } x \in X,\ y \in Y.\end{aligned}$$

Then $\mathscr{F}$ is a complete lattice, where

$$\bigvee_{i\in I}(A_i,B_i)=\Big(\bigvee_{i\in I}A_i,\bigvee_{i\in I}B_i\Big);$$
$$\bigwedge_{i\in I}(A_i,B_i)=\Big(\bigwedge_{i\in I}A_i,\bigwedge_{i\in I}B_i\Big).$$

Theorem 5.6. Let (X,R_X) and (Y,R_Y) be two L-approximation spaces, $\mathscr{F}=L^X\times L^Y$ and $\mathscr{F}_C=L^{(X,R_X)}\times L^{(Y,R_Y)}$. Then $\mathscr{F}_C$ is a complete sublattice of $(\mathscr{F},\leq)$.

Proof. Take each subfamily $\{(A_i,B_i)\mid i\in I\}\subseteq\mathscr{F}_C\subseteq\mathscr{F}$. Then $\bigvee_{i\in I}(A_i,B_i)=(\bigvee_{i\in I}A_i,\bigvee_{i\in I}B_i)$. By Theorem 5.1, we have $\bigvee_{i\in I}A_i\in L^{(X,\sim_X)}$ and $\bigvee_{i\in I}B_i\in L^{(Y,\sim_Y)}$. Thus $(\bigvee_{i\in I}A_i,\bigvee_{i\in I}B_i)\in\mathscr{F}_C$, i.e., $\bigvee_{i\in I}(A_i,B_i)\in\mathscr{F}_C$. Similarly, we can obtain $\bigwedge_{i\in I}(A_i,B_i)\in\mathscr{F}_C$. Hence, $\mathscr{F}_C$ is a complete sublattice of $\mathscr{F}$. □

In the following, we will prove that the product of two compatible degrees of L-subsets is still a complete L-sublattice.

Let $(M,\leq_M)$ and $(N,\leq_N)$ be two complete lattices. We define a partial order $\leq$ on $M\times N$ by

$$(a_1,b_1)\leq(a_2,b_2)\Longleftrightarrow a_1\leq_M a_2 \text{ and } b_1\leq_N b_2.$$

Definition 5.7. Let $(M,\leq_M)$ and $(N,\leq_N)$ be two complete lattices, $\mathscr{D}\colon M\longrightarrow L$ and $\mathscr{E}\colon N\longrightarrow L$. Then $\mathscr{D}\times\mathscr{E}\colon M\times N\longrightarrow L$ defined by

$$\mathscr{D}\times\mathscr{E}(x,y)=\mathscr{D}(x)\wedge\mathscr{E}(y),\quad\forall(x,y)\in M\times N$$

is called the product of $\mathscr{D}$ and $\mathscr{E}$.

Theorem 5.8. Let $(M,\leq_M)$ and $(N,\leq_N)$ be two complete lattices, $\mathscr{D}\colon M\longrightarrow L$ be a complete L-sublattice of M and $\mathscr{E}\colon N\longrightarrow L$ be a complete L-sublattice of N. Then $\mathscr{D}\times\mathscr{E}$ is a complete L-sublattice of $(M\times N,\leq)$.

Proof. Take each subfamily $\{(S_i,T_i)\mid i\in I\}\subseteq M\times N$. Then it follows that

$$\begin{aligned}\mathscr{D}\times\mathscr{E}\Big(\bigvee_{i\in I}(S_i,T_i)\Big)&=\mathscr{D}\times\mathscr{E}\Big(\bigvee_{i\in I}S_i,\bigvee_{i\in I}T_i\Big)\\&=\mathscr{D}(\bigvee_{i\in I}S_i)\wedge\mathscr{E}\Big(\bigvee_{i\in I}T_i\Big)\\&\geq\bigwedge_{i\in I}\mathscr{C}(S_i)\wedge\bigwedge_{i\in I}\mathscr{E}(T_i)\\&=\bigwedge_{i\in I}\mathscr{C}\times\mathscr{E}(S_i,T_i)\end{aligned}$$

and

$$\begin{aligned}\mathscr{C}\times\mathscr{E}\Big(\bigwedge_{i\in I}(S_i,T_i)\Big) &= \mathscr{C}\times\mathscr{E}\Big(\bigwedge_{i\in I}S_i,\bigwedge_{i\in I}T_i\Big)\\ &= \mathscr{C}\Big(\bigwedge_{i\in I}S_i\Big)\wedge\mathscr{E}\Big(\bigwedge_{i\in I}T_i\Big)\\ &\geq \bigwedge_{i\in I}\mathscr{C}(S_i)\wedge\bigwedge_{i\in I}\mathscr{E}(T_i)\\ &= \bigwedge_{i\in I}\mathscr{C}\times\mathscr{E}(S_i,T_i).\end{aligned}$$

By Definition 5.3, we have $\mathscr{C}\times\mathscr{E}$ is a complete L-sublattice of $M\times N$. □

Corollary 5.9. Let $(X,\sim_X)$ and $(Y,\sim_Y)$ be two L-approximation spaces, $\mathscr{C}_X\colon L^X\longrightarrow L$ and $\mathscr{C}_Y\colon L^Y\longrightarrow L$ be two compatible degrees. Then $\mathscr{C}_X\times\mathscr{C}_Y$ is the compatible degree of $\mathscr{F}$, i. e., a complete L-sublattice of $(\mathscr{F},\leq)$.

Proof. It follows immediately from Theorems 5.5 and 5.8. □

6. CONCLUSIONS

In this paper, we introduced the notions of the compatible degrees of L-subsets and mappings on a completely distributive lattice L, which is further fuzzification of compatible mappings and compatible L-subsets, respectively. We gave their characterizations by four kinds of cut sets. Then we explored the relationships between compatible mappings and compatible L-subsets, the relationships between the compatible degree of L-subsets and that of mappings. Inspired by the notion of the compatible degree of L-subsets, we defined a complete L-sublattice and did some simple research. In the future, we will further study complete L-sublattices, and look for more examples of existing structures that are complete L-sublattices.

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