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A GENERALIZATION OF REFLEXIVE RINGS

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Abstract. We introduce a class of rings which is a generalization of reflexive rings and J-reversible rings. Let R be a ring with identity and $J(R)$ denote the Jacobson radical of R. A ring R is called J-reflexive if for any $a, b \in R$, $aRb = 0$ implies $bRa \subseteq J(R)$. We give some characterizations of a J-reflexive ring. We prove that some results of reflexive rings can be extended to J-reflexive rings for this general setting. We conclude some relations between J-reflexive rings and some related rings. We investigate some extensions of a ring which satisfies the J-reflexive property and we show that the J-reflexive property is Morita invariant.

Keywords: reflexive ring; reversible ring; J-reflexive ring; J-reversible ring; ring extension

MSC 2020: 13C99, 16D80, 16U80

1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. We write $M_n(R)$ for the ring of all $n \times n$ matrices and $T_n(R)$ for the ring of all $n \times n$ upper triangular matrices over a ring R. Also we write $R[x]$, $R[[x]]$, $N(R)$, $U(R)$ and $J(R)$ for the polynomial ring, the power series ring over a ring R, the set of all nilpotent elements, the set of all invertible elements and the Jacobson radical of a ring R, respectively. The ring of integers is denoted by $\mathbb{Z}$.

In [6], Mason introduced the reflexive property for ideals. Let R be a ring (without identity) and I an ideal of R. Then I is called *reflexive*, if $aRb \subseteq I$ for $a, b \in R$ implies $bRa \subseteq I$. It is clear that every semiprime ideal is reflexive. Also, the ring R is called *reflexive*, if 0 is a reflexive ideal (i.e., $aRb = 0$ implies $bRa = 0$ for $a, b \in R$). In [4], Kwak and Lee studied reflexive rings. They investigated the reflexive property of rings related to matrix rings and polynomial rings. According to Cohn (see [2]), a ring R is said to be *reversible* if for any a, $b \in R$, $ab = 0$ implies $ba = 0$. It is clear

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that every reversible ring is reflexive. Recently, as a generalization of reversible ring, the so-called J-reversible ring has been studied in [1]. A ring R is called *J-reversible*, if $ab = 0$ implies that $ba \in J(R)$ for $a, b \in R$. As an application it is shown that every J-clean ring is directly finite. Motivated by these studies, we introduce a class of rings which generalize J-reversible rings and reflexive rings. A ring R is called *J-reflexive*, if $bRa \subseteq J(R)$ whenever $aRb = 0$ for $a, b \in R$.

We summarize the contents of this paper. In Section 2, we study main properties of J-reflexive rings. We give some characterizations of J-reflexive rings. We prove that every J-reversible ring is J-reflexive and we supply an example (Example 2.4) to show that the converse is not true in general. Moreover, we see that if R is a Baer ring, then J-reversible rings are J-reflexive. It is clear that reflexive rings are J-reflexive. Example 2.6 shows that J-reflexive rings need not be reflexive.

We give a necessary and sufficient condition for a quotient ring to be J-reflexive. Also we conclude some results which clarify relations between J-reflexive rings and some class of rings. With our finding, we prove that every uniquely clean ring is J-reflexive and quasi-duo rings are J-reflexive. Moreover, we show that the converse is not true in general.

Having the Morita invariant property is very important for a class of rings. A ring-theoretic property $\mathcal{P}$ is *Morita invariant* if and only if whenever a ring R satisfies $\mathcal{P}$ so does eRe, for any idempotent e and $M_n(R)$ with $n > 1$. There are a lot of studies on the Morita invariant property of rings. In Section 3, we prove that the J-reflexive property is Morita invariant. Furthermore, we study the J-reflexive property in several kinds of ring extensions (Dorroh extension, upper triangular matrix ring, Laurent polynomial ring, trivial extension etc.).

2. J-reflexive rings

In this section we define the J-reflexive property of a ring. We investigate some properties of J-reflexive rings and exert relations between J-reflexive rings and some related rings.

Definition 2.1. A ring R is called *J-reflexive*, if $aRb = 0$ implies that $bRa \subseteq J(R)$ for $a, b \in R$.

For a nonempty subset X of a ring R, the set $r_R(X) = \{a \in R\colon\ Xa = 0\}$ is called the *right annihilator of X in R* and the set $l_R(X) = \{b \in R\colon\ bX = 0\}$ is called the *left annihilator of X in R*.

Now we give our main characterization for J-reflexive rings.

Theorem 2.2. *The following statements are equivalent for a ring R.*

(1) *R is J-reflexive.*

(2) *For all $a \in R$, $r_R(aR)Ra \subseteq J(R)$ and $aRl_R(Ra) \subseteq J(R)$.*

(3) *$IRK = 0$ implies $KRI \subseteq J(R)$ for every nonempty subset I, K of R.*

(4) *$\langle a\rangle\langle b\rangle = 0$ implies $\langle b\rangle\langle a\rangle \subseteq J(R)$ for any $a, b \in R$.*

(5) *$IK = 0$ implies $KI \subseteq J(R)$ for every right (left) ideal I, K of R.*

(6) *$IK = 0$ implies $KI \subseteq J(R)$ for every ideal I, K of R.*

P r o o f. (1) $\Rightarrow$ (2): Let $b \in r_R(aR)$. Then $aRb = 0$ for $a, b \in R$. Since R is J-reflexive, $bRa \subseteq J(R)$. So we have $r_R(Ra)Ra \subseteq J(R)$. Similarly, one can show that $aRl_R(Ra) \subseteq J(R)$.

(2) $\Rightarrow$ (1): Assume that $aRb = 0$ for $a, b \in R$. Then, $b \in r_R(aR)$. By (2) we have $bRa \subseteq J(R)$. So R is a J-reflexive ring.

(3) $\Rightarrow$ (4) $\Rightarrow$ (5)$\Rightarrow$ (6): It is clear.

(6) $\Rightarrow$ (1): Let $aRb = 0$ for $a, b \in R$. Then $RaRRbR = 0$. By hypothesis, $RbRRaR \subseteq J(R)$. As $bRa \subseteq RbRRaR$, we have $bRa \subseteq J(R)$.

(1) $\Rightarrow$ (3): Assume that $IRK = 0$ for nonempty subsets I, K of R. Then for any $a \in I$ and $b \in K$, $aRb = 0$. As R is J-reflexive, $bRa \subseteq J(R)$. This implies that $KRI \subseteq J(R)$. $\square$

Examples of J-reflexive rings are abundant. All reduced rings, symmetric rings, reversible rings and reflexive rings are J-reflexive. In the sequel, we show that every J-reversible ring, uniquely clean ring and every right (left) quasi-duo ring is J-reflexive.

Proposition 2.3. *Every J-reversible ring is J-reflexive.*

P r o o f. Let R be a J-reversible ring and $aRb = 0$ for some $a, b \in R$. Then $ab = 0$ and $abr = 0$ for any $r \in R$. As R is J-reversible, $bra \in J(R)$. Hence, $bRa \subseteq J(R)$. $\square$

The converse of Proposition 2.3 is not true in general as the following example shows.

E x a m p l e 2.4. Consider the ring $R = M_2(\mathbb{Z})$. It can be easily shown that R is a J-reflexive ring. Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$. Although $AB = 0$, $BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin J(R)$. So R is not J-reversible.

Recall that a ring R is called *Baer* if the right (left) annihilator of every nonempty subset of R is generated by an idempotent (see [3] for details). We show that the converse of Proposition 2.3 is true for Baer rings.

Theorem 2.5. *Let R be a Baer ring. Then the following statements are equivalent.*

(1) *R is a J-reversible ring.*

(2) *R is a J-reflexive ring.*

Proof. (1) $\Rightarrow$ (2): It is clear by Proposition 2.3.

(2) $\Rightarrow$ (1): Let $ab = 0$ for $a, b \in R$. Then $abR = 0$ and so $a \in l_R(bR)$. As R is a Baer ring, there exists an idempotent $e \in R$ such that $l_R(bR) = eR$. Then we have $eRbR = 0$. Since R is J-reflexive, $bReR \subseteq J(R)$ and so $ba \in J(R)$, as desired. □

Though reflexive rings are J-reflexive, J-reflexive rings are not reflexive as the following example shows.

Example 2.6. Let R be a commutative ring. Consider the ring

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in R \right\}.$$

By [1], Propoosition 3.7, S is J-reversible and by Proposition 2.3, it is J-reflexive. For $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in S$, $ASB = 0$ but $BSA \neq 0$. Thus, S is not a reflexive ring.

The following result can be easily obtained by the definition of J-reflexive rings.

Corollary 2.7. *The following statements are equivalent for a ring R.*

(1) *If $R/J(R)$ is reflexive, then R is J-reflexive.*

(2) *If $R/J(R)$ is commutative, then R is J-reflexive.*

An element a in a ring R is called *uniquely clean* if $a = e + u$ where $e^2 = e \in R$ and $u \in U(R)$ and this representation is unique. A ring R is called a *uniquely clean ring* if every element of R is uniquely clean (see [7]).

Corollary 2.8. *Every uniquely clean ring is J-reflexive.*

Proof. Assume that R is uniquely clean. Then $R/J(R)$ is Boolean by [7], Theorem 20. Hence, R is J-reflexive by Corollary 2.7. □

The converse of Corollary 2.8 is not true in general as the following example shows.

Example 2.9. For a commutative ring R, consider the ring $M_2(R)$. Since $M_2(R)$ is not an abelian ring, $M_2(R)$ is not a uniquely clean ring. Also, it can be easily shown that $M_2(R)$ is a J-reflexive ring by Theorem 3.1.

Proposition 2.10. *Let R be a ring. If $N(R) \subseteq J(R)$, then R is J-reflexive.*

P r o o f. Assume that $aRb = 0$ for $a, b \in R$. Then for any $r \in R$, $arb = 0$ and so $ab = 0$. Hence, $(bra)^2 = brabra = 0$ for all $r \in R$. So $bra \in N(R)$. By hypothesis we have $bra \in J(R)$, as asserted. $\square$

A ring R is called *right (left) quasi-duo* if every right (left) maximal ideal of R is an ideal (see [5]).

Corollary 2.11. *Every right (left) quasi-duo ring is J-reflexive.*

P r o o f. It is clear by [8], Lemma 2.3. $\square$

We now give a necessary and sufficient condition for a quotient ring to be J-reflexive.

Theorem 2.12. *Let R be a ring and I a nilpotent ideal of R. Then R is J-reflexive if and only if R/I is J-reflexive.*

P r o o f. Let $R/I = \overline{R}$, $a + I = \overline{a} \in \overline{R}$ and $\overline{aRb} = \overline{0}$ for $\overline{a}, \overline{b} \in \overline{R}$. So $aRb \subseteq I$. As I is nilpotent there exists $k \in \mathbb{Z}^+$ such that $(RaRbR)^k = 0$. $(RbRaR)^k \subseteq J(R)$, since R is J-reflexive. Thus $RbRaR \subseteq J(R)$ as a Jacobson radical is semiprime. Hence $\overline{RbRaR} \subseteq J(R)/I = J(\overline{R})$. So $\overline{bRa} \subseteq J(\overline{R})$.

Conversely, assume that $aRb = 0$ for $a, b \in R$. Then $\overline{aRb} = \overline{0}$. So $aRb \subseteq I$ and $RaRbR \subseteq I$. Therefore there exists $k \in \mathbb{Z}^+$ such that $I^k = 0$, and so $(RaRbR)^k = RaRbRaRbR \ldots RaRbR = 0$. Hence, $\overline{(RaRbR)^k} = \overline{0}$. Since R/I is J-reflexive, $\overline{(RbRaRbRaR \ldots RbRaR)} = \overline{(RbRaR)}^k \subseteq J(\overline{R})$. As the Jacobson radical is a semiprime ideal, we have $RbRaR \subseteq J(\overline{R})$. Thus, $\overline{bRa} \subseteq J(\overline{R})$. Hence, for all $\overline{r} \in \overline{R}$, we have $\overline{1} - (\overline{bra})\overline{x} \in U(\overline{R})$ for some $\overline{x} \in J(\overline{R})$. Then, there exists $\overline{s} \in \overline{R}$ such that $(\overline{1} - (\overline{bra})\overline{x})\overline{s} = \overline{1}$. Hence, $1 - (1 - bra)xs \in I$. As every nilpotent ideal is nil, $1-(brax)s \in U(R)$. This implies that $bra \in J(R)$ and so $bRa \subseteq J(R)$, as desired. $\square$

Corollary 2.13. *Let R be a ring. Then the following statements hold.*

(1) *If $J(R)$ is a nilpotent ideal, then R is J-reflexive if and only if $R/J(R)$ is J-reflexive.*
(2) *If R is an Artinian ring, then R is J-reflexive if and only if $R/J(R)$ is J-reflexive.*

P r o o f. (1) It is clear by Theorem 2.12.
(2) Since the Jacobson radical of an Artinian ring is nilpotent, it is clear by (1). $\square$

Proposition 2.14. *Let R be a ring and I an ideal of R with $I \subseteq J(R)$. If R/I is J-reflexive, then R is J-reflexive.*

P r o o f. Let $\overline{R} = R/I$ and $\overline{a} = a + I \in R/I$. Assume that $aRb = 0$ for $a, b \in R$. So $\overline{aRb} = \overline{0}$. Since $\overline{R}$ is J-reflexive, $\overline{bRa} \subseteq J(\overline{R})$ and $\overline{bra} \in J(\overline{R})$ for any $\overline{r} \in \overline{R}$. Thus, for all $\overline{x} \in \overline{R}$ we have $\overline{1} - (\overline{bra})\overline{x} \in U(\overline{R})$. Then, there exists $\overline{s} \in \overline{R}$ such that $(\overline{1} - (\overline{bra})\overline{x})\overline{s} = \overline{1}$. Hence, $1 - (1 - brax)s \in I$. As I is contained in $J(R)$, $(1 - brax)s \in U(R)$. This implies that $bra \in J(R)$ and so $bRa \subseteq J(R)$, as desired. □

Proposition 2.15. *Let R be a ring and I a reflexive ideal of R. Then R/I is J-reflexive.*

P r o o f. Let $\overline{R} = R/I$ and $\overline{a} = a + I \in R/I$. Suppose that $\overline{aRb} = \overline{0}$ for $\overline{a}, \overline{b} \in \overline{R}$. Then $aRb \subseteq I$. Since I is a reflexive ideal, we have $bRa \subseteq I$. Hence, $\overline{bRa} = \overline{0} \in J(\overline{R})$. □

Theorem 2.16. *Every subdirect product of a J-reflexive ring is J-reflexive.*

P r o o f. Let R be a ring, I, K ideals of R and R a subdirect product of R/I and R/K. Assume that R/I and R/K are J-reflexive. Let $aRb = 0$ for $a, b \in R$. Then $\overline{aRb} = \overline{0}$ in R/I and R/K. Since R/I and R/K are J-reflexive, $\overline{bRa} \subseteq J(R/I)$ and $\overline{bRa} \subseteq J(R/K)$. Then for each $x \in R$ we have $\overline{1 - brax} \in U(R/I)$ and $\overline{1 - brax} \in U(R/K)$. Hence, there exist $y \in R/I$ and $z \in R/K$ such that $\overline{(1 - brax)y} = \overline{1} \in R/I$ and $\overline{(1 - brax)z} = \overline{1} \in R/K$. So $1 - (1 - brax)y \in I$ and $1 - (1 - brax)z \in K$. If we multiply the last two elements, we have $(1 - (1 - brax)y)(1 - (1 - brax)z) \in IK \subseteq I \cap K = 0$. Thus, $1 - (1 - brax)t = 0$ and $(1 - brax)t = 1$. This implies that $bRa \subseteq J(R)$. □

Corollary 2.17. *Let I and K be ideals of a ring R. If R/I and R/K are J-reflexive, then $R/I \cap K$ is J-reflexive.*

P r o o f. Let $\alpha\colon R/(I \cap K) \Rightarrow R/I$ and $\beta\colon R/(I \cap K) \Rightarrow R/K$ where

$$\alpha(r + (I \cap K)) = r + I \quad \text{and} \quad \beta(r + (I \cap K)) = r + K.$$

It can be shown that α and β are surjective ring homomorphisms and $\ker \alpha \cap \ker \beta = 0$. Hence $R/(I \cap K)$ is a subdirect product of R/I and R/K. Therefore, $R/(I \cap K)$ by Theorem 2.16. □

Corollary 2.18. *Let R be a ring and I, K ideals of R. If R/I and R/K are J-reflexive, then R/IK is J-reflexive.*

P r o o f. Assume that R/I and R/K are J-reflexive. Since

$$R/I \cap K \cong (R/IK)/(I \cap K/IK)$$

and $(I \cap K/IK)^2 = 0$, we complete the proof by Theorem 2.12. □

3. Extensions of J-reflexive rings

In this section we show that several extensions (Dorroh extension, upper triangular matrix ring, Laurent polynomial ring, trivial extension etc.) of a J-reflexive ring are J-reflexive. In particular, it is proved that the J-reflexive condition is Morita invariant.

Two rings R and S are said to be *Morita equivalent* if the categories of all right R-modules and all right S-modules are equivalent. Properties shared between equivalent rings are called *Morita invariant properties.* $\mathcal{P}$ is Morita invariant if and only if whenever a ring R satisfies $\mathcal{P}$, then so does eRe for every idempotent e and so does every matrix ring $M_n(R)$ for every positive integer n.

Next result shows that the property of J-reflexivity is Morita invariant.

Theorem 3.1. *Let R be a ring. Then we have the following statements.*

(1) *If R is J-reflexive, then eRe is J-reflexive for all idempotents $e \in R$.*

(2) *R is a J-reflexive ring if and only if $M_n(R)$ is J-reflexive for any positive integer n.*

Proof. (1) Assume that R is a J-reflexive ring. Let $a, b \in eRe$ with $aeReb = 0$. As R is J-reflexive, $ebRae = ebRea \subseteq J(eRe) = eJ(R)e$. This implies that eRe is a J-reflexive ring.

(2) Assume that $M_n(R)$ is a J-reflexive ring. It is clear that R is J-reflexive by (1). Conversely, suppose that R is J-reflexive and I, K are ideals of $M_n(R)$ such that $IK = 0$. Then, there exist ideals I_1, K_1 of R such that $I = M_n(I_1)$ and $K = M_n(K_1)$. So $0 = IK = M_n(I_1)M_n(K_1) = M_n(I_1K_1)$. Thus, $I_1K_1 = 0$. Since R is J-reflexive, $K_1I_1 \subseteq J(R)$. This implies that $KI = M_n(K_1)M_n(I_1) = M_n(K_1I_1) \subseteq J(M_n(R)) = M_n(J(R))$. This completes the proof. $\square$

Corollary 3.2. *Let M be finitely generated projective modules over a J-reflexive ring R. Then $\mathrm{End}_R(M)$ is J-reflexive.*

Proof. It is obvious by Theorem 3.1. $\square$

Proposition 3.3. *The following statements are equivalent for a ring R.*

(1) *R is J-reflexive.*

(2) $M = \left\{ \begin{pmatrix} r & x_{12} & \dots & x_{1n} \\ 0 & r & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & r \end{pmatrix} : r \in R,\, x_{ij} \in R \right\}$ *is J-reflexive.*

P r o o f. (1) $\Leftrightarrow$ (2): Take

$$I = \begin{pmatrix} 0 & x_{12} & \dots & x_{1n} \\ 0 & 0 & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix}.$$

The proof is clear by Theorem 2.12. $\square$

Recall that the *trivial extension* of R by an R-module M is the ring denoted by $R \propto M$ whose underlying additive group is $R \oplus M$ with multiplication given by $(r, m)(r', m') = (rr', rm' + mr')$. The ring $R \propto M$ is isomorphic to $S = \left\{\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \colon x \in R,\, y \in M\right\}$ under the usual matrix operations.

Proposition 3.4. *The following statements are equivalent for a ring R.*

(1) *The trivial extension $R \propto R$ of the ring R is J-reflexive.*

(2) *R is a J-reflexive ring.*

P r o o f. (1) $\Rightarrow$ (2): Assume that $R \propto R$ is J-reflexive. Let $aRb = 0$ for $a, b \in R$. Then, for $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \in R \propto R$, we have $A(R \propto R)B = \begin{pmatrix} aRb & aRb \\ 0 & aRb \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. As $R \propto R$ is J-reflexive, $B(R \propto R)A \subseteq J(R \propto R)$. Hence, $bRa \subseteq J(R)$.

(2) $\Rightarrow$ (1): Suppose that R is J-reflexive. Let $A(R \propto R)B = 0$ for $A = \begin{pmatrix} a & x \\ 0 & a \end{pmatrix}$, $B = \begin{pmatrix} b & y \\ 0 & b \end{pmatrix} \in R \propto R$. Then for any $M = \begin{pmatrix} s & t \\ 0 & s \end{pmatrix} \in R \propto R$, we have $AMB = \begin{pmatrix} asb & asy + atb + xsb \\ 0 & asb \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Since R is J-reflexive and $aRb = 0$, we conclude that $bsa \in J(R)$ for any $s \in R$. Note that $J(R \propto R) = \begin{pmatrix} J(R) & R \\ 0 & J(R) \end{pmatrix}$. Hence, $B(R \propto R)A \subseteq J(R \propto R)$, as asserted. $\square$

Proposition 3.5. *Let $\{R_i\}_{i \in \mathcal{I}}$ be an indexed set of the ring R_i. Then R_i is J-reflexive for all $i \in \mathcal{I}$ if and only if $\prod\limits_{i \in \mathcal{I}} R_i$ is J-reflexive.*

P r o o f. ($\Rightarrow$): Let $\prod\limits_{i \in \mathcal{I}} M_i K_i = 0$ for ideals $\prod\limits_{i \in \mathcal{I}} M_i$, $\prod\limits_{i \in \mathcal{I}} K_i$ of $\prod\limits_{i \in \mathcal{I}} R_i$. Then $\prod\limits_{i \in \mathcal{I}} M_i K_i = 0$. Therefore, $M_i K_i = 0$ for all $i \in \mathcal{I}$. Since R_i is J-reflexive, $K_i M_i \subseteq J(R_i)$ for all $i \in \mathcal{I}$. So $\prod\limits_{i \in \mathcal{I}} K_i \prod\limits_{i \in \mathcal{I}} M_i = \prod\limits_{i \in \mathcal{I}} K_i M_i \subseteq J\Big(\prod\limits_{i \in \mathcal{I}} R_i\Big) = \prod\limits_{i \in \mathcal{I}} J(R_i)$.

($\Leftarrow$): Assume that $M_\varphi K_\varphi = 0$ for ideals M_φ, K_φ of R_φ. Choose $M = (M_\varphi)_{\varphi \in \mathcal{I}}$ and $K = (K_\varphi)_{\varphi \in \mathcal{I}}$ as only φ components are a nonzero ideal. So M and K are ideals of $\prod\limits_{i \in \mathcal{I}} R_i$. Also we have $MK = 0$. As $\prod\limits_{i \in \mathcal{I}} R_i$ is J-reflexive, $KM \subseteq J\Big(\prod\limits_{i \in \mathcal{I}} R_i\Big)$. Thus, $K_\varphi M_\varphi \subseteq J(R_\varphi)$. $\square$

Proposition 3.6. *The following statements are equivalent for a ring R.*

(1) *R is a J-reflexive ring.*
(2) *$T_n(R)$ is J-reflexive for any $n \in \mathbb{Z}^+$.*

Proof. (1) $\Rightarrow$ (2): For $n = 1$ it is clear. Consider the ring $T_2(R)$. Choose the ideal $I = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$. It is clear that $I^2 = 0$. So $T_2(R)/I \cong R \times R$. By Proposition 3.5, $T_2(R)/I$ is J-reflexive. Hence $T_2(R)$ is J-reflexive by Theorem 2.12. By induction, $T_n(R)$ is J-reflexive for any $n \in \mathbb{Z}^+$.

(2) $\Rightarrow$ (1): It is evident from Theorem 3.1(1). $\square$

Proposition 3.7. *Let R be a ring and $e^2 = e \in R$ be central. Then, R is a J-reflexive ring if and only if eR and $(1 - e)R$ are J-reflexive.*

Proof. The necessity is obvious by Theorem 3.1. For the sufficiency suppose that eR and $(1-e)R$ are J-reflexive for a central idempotent $e \in R$. It is well-known that $R \cong eR \times (1 - e)R$. By Proposition 3.5, R is J-reflexive. $\square$

For an algebra R over a commutative ring S, the *Dorroh extension* $I(R; S)$ of R by S is the additive abelian group $I(R; S) = R \oplus S$ with multiplication $(r, v)(s, w) = (rs, rw + vs + vw)$.

Proposition 3.8. *Let R be a ring and $M = I(R; S)$ a Dorroh extension of R by a commutative ring S. Assume that for all $s \in S$ there exists $s' \in S$ such that $s + s' + ss' = 0$. Then the following statements are equivalent.*

(1) *R is J-reflexive.*
(2) *M is J-reflexive.*

Proof. (1) $\Rightarrow$ (2): Let $(a_1, b_1)M(a_2, b_2) = (0, 0)$ for $(a_1, b_1), (a_2, b_2) \in M$. So for any $(x, y) \in M$, we have $(a_1, b_1)(x, y)(a_2, b_2) = (0, 0)$. Then

$$(a_1xa_2, a_1xb_2 + a_1ya_2 + b_1xa_2 + b_1ya_2 + a_1yb_2 + b_1xb_2 + b_1yb_2) = (0, 0).$$

Hence, $a_1xa_2 = 0$ and $a_1xb_2+a_1ya_2+b_1xa_2+b_1ya_2+a_1yb_2+b_1xb_2+b_1yb_2 = 0$. As R is J-reflexive, $a_2xa_1 \in J(R)$ for any $x \in R$. Thus, $(a_2, b_2)(x, y)(a_1, b_1) = (a_2xa_1, *)$. By hypothesis, $(0, S) \subseteq J(M)$. It can be easy to show that $(a_2xa_1, 0) \in J(M)$ for every $x \in R$. Therefore, $(a_2, b_2)S(a_1, b_1) \subseteq J(M)$.

(2) $\Rightarrow$ (1): Let $aRb = 0$ for $a, b \in R$. Then $(a, 0)M(b, 0) = (0, 0)$. Since M is J-reflexive, $(b, 0)M(a, 0) \subseteq J(M)$. By hypothesis, $(0, S) \subseteq J(M)$. This implies that $(bRa, 0) \subseteq J(S)$. Hence, $bRa \subseteq J(R)$. $\square$

If R is a ring and $f\colon R \to R$ is a ring homomorphism, let $R[[x, f]]$ denote the *ring of skew formal power series over* R; that is all formal power series in x with coefficients from R with multiplication defined by $xr = f(r)x$ for all $r \in R$. Note that $J(R[[x, f]]) = J(R) + \langle x \rangle$. Since $R[[x, f]] \cong I(R; \langle x \rangle)$ where $\langle x \rangle$ is the ideal generated by x, we have the following result.

Corollary 3.9. *Let R be a ring and $f\colon R \to R$ a ring homomorphism. Then the following statements are equivalent.*
(1) *R is a J-reflexive ring.*
(2) *$R[[x, f]]$ is J-reflexive.*

If f is taken as $f = 1_R\colon R \to R$ (i.e., $1_R(r) = r$ for all $r \in R$), we have that $R[[x]] = R[[x, 1_R]]$ is the ring of formal power series over R.

Corollary 3.10. *The following statements are equivalent for a ring R.*
(1) *R is a J-reflexive ring.*
(2) *$R[[x]]$ is J-reflexive.*

Let R be a ring and $u \in R$. Recall that u is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, a *left regular* element can be defined. An element is *regular* if it is both left and right regular.

Proposition 3.11. *Let R be a ring and M multiplicatively closed subset of R consisting of central regular elements. Then the following statements are equivalent.*
(1) *R is J-reflexive.*
(2) *$S = M^{-1}R = \{a/b\colon a \in R, b \in M\}$ is J-reflexive.*

Proof. (1) $\Rightarrow$ (2): Let $aSb = 0$ for $a, b \in S$. So there exist $a_1, b_1 \in R$ and $u^{-1}, v^{-1} \in M$ such that $a = a_1u^{-1}$ and $b = b_1v^{-1}$. Then $0 = aSb = a_1u^{-1}Sb_1v^{-1} = a_1Sbv^{-1}$. Hence for any $rs^{-1} \in S$ we have $a_1rs^{-1}bv^{-1}$. Thus, $a_1rb_1 = 0$ for each $r \in R$. As R is J-reflexive, $b_1ra_1 \in J(R)$. This implies that $b_1v^{-1}rs^{-1}a_1u^{-1} \in J(R)$. As $J(R) \subseteq J(S)$, $aSb \subseteq J(S)$.

(2) $\Rightarrow$ (1): Let $aRb = 0$ for $a, b \in R$ and $u, v \in M$. So we have $auRbv = 0$. Then for any $m \in M$ and $r \in R$, $aurmbv = 0$. Since S is J-reflexive, $bvrmau \in J(S)$. If we multiply $bvrmau$ with inverses of u, m, v, then we have $bra \in J(R)$ for any $r \in R$. This completes the proof. □

The following result is a direct consequence of Proposition 3.11.

Corollary 3.12. *Let R be a ring. Then the following statements are equivalent.*
(1) *$R[x]$ is J-reflexive.*
(2) *$R[x, x^{-1}]$ is J-reflexive.*

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