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EXISTENCE, UNIQUENESS AND CONTINUITY RESULTS OF WEAK SOLUTIONS FOR NONLOCAL NONLINEAR PARABOLIC PROBLEMS

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Abstract. This paper is concerned with the study of a nonlocal nonlinear parabolic problem associated with the equation $u_t - M(\int_\Omega \phi u \,\mathrm{d}x)\mathrm{div}\,(A(x,t,u)\nabla u) = g(x,t,u)$ in $\Omega \times (0,T)$, where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geqslant 1)$, $T > 0$ is a positive number, $A(x,t,u)$ is an $n \times n$ matrix of variable coefficients depending on u and $M\colon \mathbb{R} \to \mathbb{R}$, $\phi\colon \Omega \to \mathbb{R}$, $g\colon \Omega\times(0,T)\times\mathbb{R} \to \mathbb{R}$ are given functions. We consider two different assumptions on g. The existence of a weak solution for this problem is proved using the Schauder fixed point theorem for each of these assumptions. Moreover, if $A(x,t,u) = a(x,t)$ depends only on the variable (x,t), we investigate two uniqueness theorems and give a continuity result depending on the initial data.

Keywords: nonlocal nonlinear parabolic problem; Schauder fixed point theorem; weak solution; existence; uniqueness

MSC 2020: 35D30, 35K55, 35Q92

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geqslant 1)$, T be a positive number, $Q = \Omega \times (0,T)$ and ϕ, $u_0 \in L^2(\Omega)$. Let $A\colon Q \times \mathbb{R} \to \mathbb{R}^{n\times n}$ be a vector function such that $(x,t) \mapsto A(x,t,s)$ is measurable for all $s \in \mathbb{R}$, $s \mapsto A(x,t,s)$ is continuous for a.e. $(x,t) \in Q$ and

$$\forall\, \xi \in \mathbb{R}^n\colon\ \lambda|\xi|^2 \leqslant A(x,t,s)\xi \cdot \xi, \tag{1.1}$$

$$\forall\, \xi \in \mathbb{R}^n\colon\ |A(x,t,s)\xi| \leqslant \Lambda|\xi|, \tag{1.2}$$

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for all $s \in \mathbb{R}$ and a.e. $(x,t) \in Q$, where λ and Λ are positive constants, and let $M\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying for some constants $m_1 \geqslant m_0 > 0$,

$$m_0 \leqslant M(s) \leqslant m_1 \quad \forall\, s \in \mathbb{R}. \tag{1.3}$$

On the other hand, let $g\colon Q \times \mathbb{R} \to \mathbb{R}$ be a function such that $(x,t) \mapsto g(x,t,s)$ is measurable for all $s \in \mathbb{R}$, $g(\cdot,\cdot,0) \in L^2(Q)$ and satisfies one of the two following assumptions:

$$\begin{gathered}\forall\, s_1, s_2 \in \mathbb{R},\ \text{a.e. } (x,t) \in Q\colon \\ |g(x,t,s_1) - g(x,t,s_2)| \leqslant \begin{cases} b_p(x,t)|s_1 - s_2|^p, & p \in (0,1); \\ b_1(x,t)|s_1 - s_2|, & p = 1,\end{cases}\end{gathered} \tag{1.4}$$

where $b_p \in L^{2/(1-p)}(Q)$ for $p \in (0,1)$, $b_1 \in L^\infty(Q)$, $C_\Omega(2T/(\lambda m_0))^{1/2}\|b_1\|_{L^\infty(Q)} < 1$ with C_Ω denoting a Poincaré constant for $H_0^1(\Omega)$, and for a.e. $(x,t) \in Q$, $s \mapsto g(x,t,s)$ is continuous,

$$\exists\, h \in L^2(Q) \quad \forall\, s \in \mathbb{R},\ \text{a.e. } (x,t) \in Q\colon\ |g(x,t,s)| \leqslant h(x,t). \tag{1.5}$$

We now consider the following weak formulation of nonlocal nonlinear parabolic problems:

$$\begin{cases} u \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega)), \quad u_t \in L^2(0,T;H^{-1}(\Omega)), \\ u(\cdot,0) = u_0 \quad \text{a.e. in } \Omega, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(u,\xi) + M(l(u)) \displaystyle\int_\Omega A(x,t,u)\nabla u \cdot \nabla\xi \,\mathrm{d}x = \int_\Omega g(x,t,u)\xi \,\mathrm{d}x \\ \hfill \text{in } \mathcal{D}'(0,T) \quad \forall\, \xi \in H_0^1(\Omega), \end{cases} \tag{1.6}$$

where

$$(u,\xi) = \int_\Omega u\xi \,\mathrm{d}x \quad \text{and} \quad l(u) = l(u)(t) = \int_\Omega \phi(x)u(x,t)\,\mathrm{d}x.$$

The study of nonlocal problems has attracted the attention of many authors and several results have been established. These types of problems (1.6) arise in a wide variety of applications in physics and population dynamics. For instance, the solution u can be used to describe the population density of bacteria in space and time. Also, this model can address some questions concerning the heat conduction. See [7], [8], [9], [10], [17] for more details.

In the homogenous case when $A(x,t,u)$ is the identity matrix, the authors in [8] and [3] established existence theorems for (1.6) by using the Galerkin Method, respectively, for the second term g depending only on $(x,t), g(x,t,u) = g(x,t) \in$

$L^2(0,T;H^{-1}(\Omega))$ and for $g(x,t,u)=k(x)-f(u)$ such that $k\in L^2(\Omega)$ and f is a continuously differentiable function satisfying $f(u)u\geqslant -\mu u^2-c_1$, $f'(u)\geqslant -\alpha$, where c_1, α are two positive constants, $0<\mu<m_0\lambda_1$ with $\lambda_1>0$ is the first eigenvalue of the operator $(-\Delta, H_0^1(\Omega))$. Moreover, the uniqueness and the asymptotic behavior of solutions and other results have been obtained under some additional conditions. By means of the Schauder fixed point theorem, the existence of weak solutions for (1.6) has been proved in [13] for a Lipschitz continuous function $g(x,t,u)=f(u)$ such that $f(0)=0$ and $f'(0)$ exists. Also, the uniqueness and the result on existence of periodic solution have been established. If g depends only on the variable (x,t), $g(x,t,u)=g(x,t)\in L^2(0,T;H^{-1}(\Omega))$ and $l\colon\ L^2(\Omega)\to\mathbb{R}$ is a continuous function, the authors in [10] gave an existence theorem for (1.6) and by imposing additional conditions, they investigated the uniqueness and asymptotic behavior of solutions. In [16], the authors considered the following nonlocal parabolic problem:

$$\begin{cases} u_t-a\Big(\displaystyle\int_\Omega |u|^\gamma\,\mathrm{d}x\Big)\Delta u=f(u) & \text{in } \Omega\times(0,T),\\ u=0 & \text{on } \partial\Omega\times(0,T),\\ u(\cdot,0)=u_0 & \text{in } \Omega,\end{cases}$$

where Ω is a sufficiently regular domain, $\gamma\in[1,\infty)$, $T\in(0,\infty]$, $u_0\in C^{2+\alpha}(\overline{\Omega})$, $\alpha\in(0,1)$, $f\in C^1(\mathbb{R})$ and $a\in C^1([0,\infty))$ with $\inf\limits_{t\in[0,\infty)} a(t)\geqslant a(0):=a_0>0$. By using the sub-supersolution method, they proved the existence, uniqueness and long-time behavior of positive solutions.

For the nonlocal problems in the stationary case, we refer to [1], [2], [4], [5], [9], [11], [15], [18], [19], [20] and the references therein in which various methods have been used for studying the existence and uniqueness of solutions and other questions.

In this paper, we suppose that (1.1)–(1.3) and one of (1.4) and (1.5) hold. We prove the existence of a solution for each problem (1.6) by applying the Schauder fixed point theorem. Moreover, if $A(x,t,u)=a(x,t)$ depends only on the variable (x,t), we investigate two uniqueness theorems and give a continuity result depending on the initial data as in [10], Lemma 5.1, assuming that M is a Lipschitz continuous function. Our first uniqueness theorem is concerned with the case $p=1$ in which the function g satisfies a generalized Lipschitz condition and under the assumption that g is decreasing with respect to u, we state and prove our second uniqueness theorem.

2. Existence of a solution

In this section, we prove the existence of a solution of (1.6) assuming that the function g satisfies one of assumptions (1.4) and (1.5).

Theorem 2.1. *If* (1.1)–(1.3) *and either* (1.4) *or* (1.5) *holds, problem* (1.6) *has a solution.*

P r o o f. We apply the Schauder fixed point theorem to show the existence of a solution of (1.6). For a fixed element v of $L^2(Q)$ and for a.e. $t \in (0,T)$, we define

$$\begin{aligned} A_v(u,\xi,t) &= M(l(v)) \int_\Omega A(x,t,v)\nabla u \cdot \nabla \xi \,\mathrm{d}x \quad \forall\, u,\xi \in H_0^1(\Omega), \\ f_v(x,t) &= g(x,t,v) \quad \text{a.e. } x \in \Omega. \end{aligned}$$

Observe that A_v is a bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$ which satisfies

$$t \mapsto A_v(u,\xi,t) \text{ is measurable } \forall\, u,\xi \in H_0^1(\Omega),$$

and by (1.1)–(1.3) and the Cauchy-Schwarz inequality, we have for a.e. $t \in (0,T)$,

$$\begin{aligned} |A_v(u,\xi,t)| &\leqslant \Lambda m_1 \|u\|_{H_0^1(\Omega)} \|\xi\|_{H_0^1(\Omega)} \quad \forall\, u,\xi \in H_0^1(\Omega), \\ A_v(u,u,t) + \lambda m_0 \|u\|_{L^2(\Omega)}^2 &\geqslant \lambda m_0 \|u\|_{H_0^1(\Omega)}^2 \quad \forall\, u \in H_0^1(\Omega). \end{aligned}$$

On the other hand, if one of assumptions (1.4) and (1.5) holds, the function f_v belongs to $L^2(Q)$. Thus, from [6], Theorem 11.7, we deduce that for each $v \in L^2(Q)$, there exists a unique solution of the following problem:

$$(2.1) \qquad \begin{cases} u \in L^2(0,T;H_0^1(\Omega)), \quad u_t \in L^2(0,T;H^{-1}(\Omega)), \\ u(\cdot,0) = u_0 \quad \text{a.e. in } \Omega, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(u,\xi) + M(l(v)) \displaystyle\int_\Omega A(x,t,v)\nabla u \cdot \nabla \xi \,\mathrm{d}x = \int_\Omega g(x,t,v)\xi \,\mathrm{d}x \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathcal{D}'(0,T) \quad \forall\, \xi \in H_0^1(\Omega). \end{cases}$$

Let us now define $F\colon L^2(Q) \to L^2(0,T;H_0^1(\Omega))$ by $F(v) = u$. If we choose $\xi = u$ as a test function in (2.1), we obtain

$$(2.2) \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2(\Omega)}^2 + M(l(v)) \int_\Omega A(x,t,v)\nabla u \cdot \nabla u \,\mathrm{d}x = \int_\Omega g(x,t,v)u \,\mathrm{d}x.$$

Integrating (2.2) from 0 to t, we get

$$(2.3) \qquad \begin{aligned} \frac{1}{2}\|u\|_{L^2(\Omega)}^2 &+ \int_0^t M(l(v)) \int_\Omega A(x,s,v)\nabla u \cdot \nabla u \,\mathrm{d}x \,\mathrm{d}s \\ &= \int_0^t \int_\Omega g(x,s,v)u \,\mathrm{d}x \,\mathrm{d}s + \frac{1}{2}\|u_0\|_{L^2(\Omega)}^2. \end{aligned}$$

Using (1.1), (1.3), (1.4), Hölder's inequality and Poincaré's inequality, we obtain from (2.3) for all $p \in (0,1)$,

$$\begin{aligned}
&\frac{1}{2}\|u\|^2_{L^2(\Omega)} + \lambda m_0 \int_0^t \|u\|^2_{H_0^1(\Omega)}\,\mathrm{d}s \\
&\leqslant \int_0^t \int_\Omega (b_p(x,s)|v|^p + |g(x,s,0)|)|u|\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} \\
&\leqslant (\|b_p\|_{L^{2/(1-p)}(Q)}\|v\|^p_{L^2(Q)} + \|g(\cdot,\cdot,0)\|_{L^2(Q)})\|u\|_{L^2(\Omega\times(0,t))} + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} \\
&\leqslant C_\Omega(\|b_p\|_{L^{2/(1-p)}(Q)}\|v\|^p_{L^2(Q)} + \|g(\cdot,\cdot,0)\|_{L^2(Q)})\Big(\int_0^t \|u\|^2_{H_0^1(\Omega)}\,\mathrm{d}s\Big)^{1/2} + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)},
\end{aligned}$$

which, by using Young's inequality, leads to

$$\begin{aligned}
(2.4)\quad &\frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{\lambda m_0}{2}\int_0^t \|u\|^2_{H_0^1(\Omega)}\,\mathrm{d}s \\
&\leqslant \frac{C_\Omega^2}{2\lambda m_0}(\|b_p\|_{L^{2/(1-p)}(Q)}\|v\|^p_{L^2(Q)} + \|g(\cdot,\cdot,0)\|_{L^2(Q)})^2 + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} \\
&\leqslant \frac{C_\Omega^2}{\lambda m_0}(\|b_p\|^2_{L^{2/(1-p)}(Q)}\|v\|^{2p}_{L^2(Q)} + \|g(\cdot,\cdot,0)\|^2_{L^2(Q)}) + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.
\end{aligned}$$

From (2.4), we derive the following estimates:

(2.5)

$$\begin{aligned}
&\|u\|_{L^2(0,T;H_0^1(\Omega))} \\
&\leqslant \frac{1}{(\lambda m_0)^{1/2}}\Big\{\frac{2C_\Omega^2}{\lambda m_0}(\|b_p\|^2_{L^{2/(1-p)}(Q)}\|v\|^{2p}_{L^2(Q)} + \|g(\cdot,\cdot,0)\|^2_{L^2(Q)}) + \|u_0\|^2_{L^2(\Omega)}\Big\}^{1/2}
\end{aligned}$$

and

(2.6)

$$\begin{aligned}
&\|u\|_{L^2(Q)} \\
&\leqslant T^{1/2}\Big\{\frac{2C_\Omega^2}{\lambda m_0}(\|b_p\|^2_{L^{2/(1-p)}(Q)}\|v\|^{2p}_{L^2(Q)} + \|g(\cdot,\cdot,0)\|^2_{L^2(Q)}) + \|u_0\|^2_{L^2(\Omega)}\Big\}^{1/2} \\
&\leqslant T^{1/2}\Big\{C_\Omega\Big(\frac{2}{\lambda m_0}\Big)^{1/2}(\|b_p\|_{L^{2/(1-p)}(Q)}\|v\|^p_{L^2(Q)} + \|g(\cdot,\cdot,0)\|_{L^2(Q)}) + \|u_0\|_{L^2(\Omega)}\Big\}.
\end{aligned}$$

Moreover, if $p = 1$, we arrive at

$$\begin{aligned}
(2.7)\quad &\|u\|_{L^2(0,T;H_0^1(\Omega))} \\
&\leqslant \frac{1}{(\lambda m_0)^{1/2}}\Big\{\frac{2C_\Omega^2}{\lambda m_0}(\|b_1\|^2_{L^\infty(Q)}\|v\|^2_{L^2(Q)} + \|g(\cdot,\cdot,0)\|^2_{L^2(Q)}) + \|u_0\|^2_{L^2(\Omega)}\Big\}^{1/2}
\end{aligned}$$

and

(2.8)

$$\|u\|_{L^2(Q)} \leqslant T^{1/2}\Big\{C_\Omega\Big(\frac{2}{\lambda m_0}\Big)^{1/2}(\|b_1\|_{L^\infty(Q)}\|v\|_{L^2(Q)} + \|g(\cdot,\cdot,0)\|_{L^2(Q)}) + \|u_0\|_{L^2(\Omega)}\Big\}.$$

By using (2.5)–(2.8) and the fact that $C_\Omega(2T/(\lambda m_0))^{1/2}\|b_1\|_{L^\infty(Q)} < 1$, we deduce that for all $p \in (0,1]$, there exist positive constants $R_1 = R_1(p)$, $R_2 = R_2(p)$ such that

$$\|v\|_{L^2(Q)} \leqslant R_1 \Rightarrow (\|u\|_{L^2(Q)} \leqslant R_1 \text{ and } \|u\|_{L^2(0,T;H_0^1(\Omega))} \leqslant R_2). \tag{2.9}$$

In particular, $F(\overline{B}(0,R_1)) \subset \overline{B}(0,R_1)$, where $\overline{B}(0,R_1)$ denotes the closed ball in $L^2(Q)$ of center 0 and radius R_1. Also, if g satisfies (1.5), we can find another positive constant R_3 independent of v such that F sends $L^2(Q)$ to $\overline{B}(0,R_3)$ and

$$\|u\|_{L^2(0,T;H_0^1(\Omega))} \leqslant R_3. \tag{2.10}$$

Now, we prove that $F\colon \overline{B}(0,R_4) \to \overline{B}(0,R_4)$ is continuous, where $R_4 = R_1$ or R_3. Let $(v_j)_{j\in\mathbb{N}}$ be a sequence in $\overline{B}(0,R_4)$ which converges to $v_\infty \in \overline{B}(0,R_4)$ and set $u_j = F(v_j)$, $u_\infty = F(v_\infty)$. There exists a subsequence j_k such that

$$v_{j_k} \to v_\infty \quad \text{strongly in } L^2(Q), \tag{2.11}$$

$$v_{j_k} \to v_\infty \quad \text{a.e. in } Q, \tag{2.12}$$

$$l(v_{j_k}) \to l(v_\infty) \quad \text{strongly in } L^2(0,T). \tag{2.13}$$

The last equation of (2.1) can be written in $L^2(0,T;H^{-1}(\Omega))$ as

$$\frac{\mathrm{d}u}{\mathrm{d}t} - M(l(v))\operatorname{div}(A(x,t,v)\nabla u) = g(x,t,v). \tag{2.14}$$

Writing (2.14) for $u = u_{j_k}$ and $v = v_{j_k}$, and multiplying the obtained equation by $\xi \in \mathcal{D}(0,T;H_0^1(\Omega))$, we get

$$\int_Q u_{j_k}\xi_t \,\mathrm{d}x\,\mathrm{d}t = \int_0^T M(l(v_{j_k}))\int_\Omega A(x,t,v_{j_k})\nabla u_{j_k}\cdot\nabla\xi \,\mathrm{d}x\,\mathrm{d}t - \int_Q g(x,t,v_{j_k})\xi \,\mathrm{d}x\,\mathrm{d}t. \tag{2.15}$$

Using (1.2), (1.3), the Cauchy-Schwarz inequality, Poincaré's inequality and either (1.4) with (2.9) or (1.5) with (2.10), we obtain from (2.15) for a constant c_2 independent of j_k and ξ,

$$\forall\, k \in \mathbb{N}\colon \left|\int_Q u_{j_k}\xi_t \,\mathrm{d}x\,\mathrm{d}t\right| \leqslant c_2\|\xi\|_{L^2(0,T;H_0^1(\Omega))},$$

which means that

$$\forall\, k \in \mathbb{N}\colon \|u_{j_k t}\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant c_2. \tag{2.16}$$

We notice that the balls of $E=\{v\in L^2(0,T;H_0^1(\Omega))/v_t\in L^2(0,T;H^{-1}(\Omega))\}$, which is a Hilbert space when equipped with the norm

$$\{\|v\|^2_{L^2(0,T;H_0^1(\Omega))}+\|v_t\|^2_{L^2(0,T;H^{-1}(\Omega))}\}^{1/2}$$

(see [6]), are relatively compact in $L^2(0,T;L^2(\Omega))=L^2(Q)$ (see [12]). Then, by (2.9), (2.10) and (2.16), the set $F(\overline{B}(0,R_4))$ is relatively compact in $\overline{B}(0,R_4)$. On the other hand, there exist a subsequence, which we still denote by j_k, and $\overline{u}\in L^2(0,T;H_0^1(\Omega))$ such that

$$u_{j_k}\rightharpoonup\overline{u}\quad\text{weakly in } L^2(0,T;H_0^1(\Omega)),\tag{2.17}$$

$$u_{j_kt}\rightharpoonup\overline{u}_t\quad\text{weakly in } L^2(0,T;H^{-1}(\Omega)),\tag{2.18}$$

$$u_{j_k}\to\overline{u}\quad\text{strongly in } L^2(Q).\tag{2.19}$$

For $\varphi\in\mathcal{D}(0,T)$ and $\xi\in H_0^1(\Omega)$ we have

$$\int_0^T\varphi(t)M(l(v_{j_k}))\int_\Omega A(x,t,v_{j_k})\nabla u_{j_k}\cdot\nabla\xi\,\mathrm{d}x\,\mathrm{d}t-\int_Q u_{j_k}\xi\varphi_t\,\mathrm{d}x\,\mathrm{d}t=\int_Q g(x,t,v_{j_k})\xi\varphi\,\mathrm{d}x\,\mathrm{d}t.\tag{2.20}$$

Observe that from (1.2), (1.3), (2.12), (2.13), (2.17), the continuity of M and $s\mapsto A(\cdot,\cdot,s)$, and the dominated convergence theorem, we have up to a subsequence

$$M(l(v_{j_k}))\nabla\xi\to M(l(v_\infty))\nabla\xi\quad\text{strongly in } (L^2(Q))^n,\tag{2.21}$$

$$A(\cdot,\cdot,v_{j_k})\nabla u_{j_k}\rightharpoonup A(\cdot,\cdot,v_\infty)\nabla\overline{u}\quad\text{weakly in } (L^2(Q))^n.\tag{2.22}$$

On the other hand, if (1.4) or (1.5) holds, then

$$\lim_{k\to\infty}\int_Q g(x,t,v_{j_k})\xi\,\mathrm{d}x\,\mathrm{d}t=\int_Q g(x,t,v_\infty)\xi\,\mathrm{d}x\,\mathrm{d}t.\tag{2.23}$$

Indeed, if (1.4) is satisfied, Hölder's inequality gives

$$\left|\int_Q(g(x,t,v_{j_k})-g(x,t,v_\infty))\xi\varphi\,\mathrm{d}x\,\mathrm{d}t\right|\leqslant\begin{cases}\|b_p\|_{L^{2/(1-p)}(Q)}\|v_{j_k}-v_\infty\|^p_{L^2(Q)}\|\xi\varphi\|_{L^2(Q)}, & p\in(0,1);\\ \|b_1\|_{L^\infty(Q)}\|v_{j_k}-v_\infty\|_{L^2(Q)}\|\xi\varphi\|_{L^2(Q)}, & p=1,\end{cases}$$

which leads to (2.23) by passing to the limit as $k\to\infty$ and using (2.11). Also, if g satisfies (1.5), it is sufficient to use (2.12), the continuity of $s\mapsto g(\cdot,\cdot,s)$ and the dominated convergence theorem to get (2.23). Then, passing to the limit in (2.20) as $k \to \infty$ and using (2.19) and (2.21)–(2.23), we obtain

$$\int_0^T \varphi(t) M(l(v_\infty)) \int_\Omega A(x,t,v_\infty) \nabla \overline{u} \cdot \nabla \xi \,\mathrm{d}x\,\mathrm{d}t - \int_Q \overline{u} \xi \varphi_t \,\mathrm{d}x\,\mathrm{d}t = \int_Q g(x,t,v_\infty) \xi \varphi \,\mathrm{d}x\,\mathrm{d}t,$$

which can be written as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\overline{u},\xi) + M(l(v_\infty)) &\int_\Omega A(x,t,v_\infty) \nabla \overline{u} \cdot \nabla \xi \,\mathrm{d}x \\ &= \int_\Omega g(x,t,v_\infty)\xi \,\mathrm{d}x \quad \text{in } \mathcal{D}'(0,T) \quad \forall\, \xi \in H_0^1(\Omega).\end{aligned}$$

Let us prove that $\overline{u}(\cdot,0) = u_0$ a.e. in Ω. For a.e. $t \in (0,T)$ and $\xi \in H_0^1(\Omega)$, we have

$$\int_0^t \langle u_{j_k t}, \xi\rangle \,\mathrm{d}s = (u_{j_k}(\cdot,t),\xi) - (u_{j_k}(\cdot,0),\xi) = (u_{j_k}(\cdot,t),\xi) - (u_0,\xi), \tag{2.24}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. We notice that from (2.19) we have up to a subsequence

$$u_{j_k} \to \overline{u} \quad \text{strongly in } L^2(\Omega), \text{ a.e. in } (0,T), \tag{2.25}$$

then, passing to the limit in (2.24) as $k \to \infty$, we obtain by using (2.18) and (2.25),

$$\int_0^t \langle \overline{u}_t, \xi\rangle \,\mathrm{d}s = (\overline{u}(\cdot,t),\xi) - (u_0,\xi),$$

which can be written as $(\overline{u}(\cdot,t),\xi) - (\overline{u}(\cdot,0),\xi) = (\overline{u}(\cdot,t),\xi) - (u_0,\xi)$. Therefore,

$$\overline{u}(\cdot,0) = u_0 \quad \text{a.e. in } \Omega.$$

Now, we see that $\overline{u}$ satisfies

$$\begin{cases} \overline{u} \in L^2(0,T;H_0^1(\Omega)), \quad \overline{u}_t \in L^2(0,T;H^{-1}(\Omega)), \\ \overline{u}(\cdot,0) = u_0 \quad \text{a.e. in } \Omega, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(\overline{u},\xi) + M(l(v_\infty)) \displaystyle\int_\Omega A(x,t,v_\infty)\nabla \overline{u} \cdot \nabla \xi \,\mathrm{d}x = \int_\Omega g(x,t,v_\infty)\xi \,\mathrm{d}x \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathcal{D}'(0,T) \quad \forall\, \xi \in H_0^1(\Omega), \end{cases}$$

and by uniqueness we have $\overline{u} = u_\infty$. In view of the above, we observe that every subsequence of $(u_j)_{j\in\mathbb{N}}$ has a sub-subsequence that converges to the same limit u_∞. So, the sequence $(F(v_j))_{j\in\mathbb{N}}$ converges to $F(v_\infty)$ in $\overline{B}(0,R_4)$, and then, $F\colon \overline{B}(0,R_4) \to \overline{B}(0,R_4)$ is continuous. Thus, by the Schauder fixed point theorem, there exists a fixed point u of F in E which is a solution of (1.6) since $E \subset C([0,T];L^2(\Omega))$. □

3. Uniqueness and continuity results of solutions

In this section, we assume that $A(x,t,u)=a(x,t)$ depends only on the variable (x,t). We investigate two uniqueness theorems of solutions for (1.6) and give a continuity result depending on the initial data assuming that M is a Lipschitz continuous function,

$$\exists\, m_2\geqslant 0\colon\ |M(s_1)-M(s_2)|\leqslant m_2|s_1-s_2|\quad \forall\, s_1,s_2\in\mathbb{R}. \tag{3.1}$$

In the following first uniqueness theorem, we are concerned with the case $p=1$ in which the function g satisfies a generalized Lipschitz condition.

Theorem 3.1. *Assume that* (1.1)–(1.3), (3.1) *hold and that g satisfies* (1.4) *with* $p=1$. *Then the solution of* (1.6) *is unique.*

Proof. Let u_1 and u_2 be two solutions of (1.6) corresponding to the same initial data $u_0\in L^2(\Omega)$. Then for all $\xi\in H_0^1(\Omega)$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(u_1,\xi)+M(l(u_1))\int_\Omega a(x,t)\nabla u_1\cdot\nabla\xi\,\mathrm{d}x&=\int_\Omega g(x,t,u_1)\xi\,\mathrm{d}x,\\ \frac{\mathrm{d}}{\mathrm{d}t}(u_2,\xi)+M(l(u_2))\int_\Omega a(x,t)\nabla u_2\cdot\nabla\xi\,\mathrm{d}x&=\int_\Omega g(x,t,u_2)\xi\,\mathrm{d}x\end{aligned}$$

in the distributional sense in $\mathcal{D}'(0,T)$. Choosing $\xi=u_1-u_2$ and subtracting the two equations from each other, we arrive at

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_1-u_2\|^2_{L^2(\Omega)}&+M(l(u_1))\int_\Omega a(x,t)\nabla u_1\cdot\nabla(u_1-u_2)\,\mathrm{d}x\\ &-M(l(u_2))\int_\Omega a(x,t)\nabla u_2\cdot\nabla(u_1-u_2)\,\mathrm{d}x\\ &=\int_\Omega(g(x,t,u_1)-g(x,t,u_2))(u_1-u_2)\,\mathrm{d}x,\end{aligned}$$

which can be written as

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_1-u_2\|^2_{L^2(\Omega)}&+M(l(u_1))\int_\Omega a(x,t)\nabla(u_1-u_2)\cdot\nabla(u_1-u_2)\,\mathrm{d}x\\ &=(M(l(u_2))-M(l(u_1)))\int_\Omega a(x,t)\nabla u_2\cdot\nabla(u_1-u_2)\,\mathrm{d}x\\ &\quad+\int_\Omega(g(x,t,u_1)-g(x,t,u_2))(u_1-u_2)\,\mathrm{d}x.\end{aligned}\tag{3.2}$$

Using (1.1)–(1.3), (1.4) for $p = 1$, (3.1) and Cauchy-Schwarz inequality, we obtain from (3.2),

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_1 - u_2\|^2_{L^2(\Omega)} + \lambda m_0 \int_\Omega |\nabla(u_1 - u_2)|^2 \,\mathrm{d}x \\
&\quad\leqslant m_2\Lambda|l(u_2) - l(u_1)| \int_\Omega |\nabla u_2||\nabla(u_1 - u_2)| \,\mathrm{d}x + \int_\Omega |b_1(x,t)||u_1 - u_2|^2 \,\mathrm{d}x \\
&\quad\leqslant m_2\Lambda\|\phi\|_{L^2(\Omega)}\|u_1 - u_2\|_{L^2(\Omega)}\|u_2\|_{H^1_0(\Omega)}\|u_1 - u_2\|_{H^1_0(\Omega)} \\
&\qquad + \|b_1\|_{L^\infty(Q)}\|u_1 - u_2\|^2_{L^2(\Omega)},
\end{aligned}$$

which, by using Young's inequality, leads to

$$\begin{aligned}
\text{(3.3)}\qquad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_1 - u_2\|^2_{L^2(\Omega)} + \lambda m_0\|u_1 - u_2\|^2_{H^1_0(\Omega)} \\
&\quad\leqslant \left\{\frac{(m_2\Lambda\|\phi\|_{L^2(\Omega)}\|u_2\|_{H^1_0(\Omega)})^2}{2\lambda m_0} + \|b_1\|_{L^\infty(Q)}\right\}\|u_1 - u_2\|^2_{L^2(\Omega)} \\
&\qquad + \frac{\lambda m_0}{2}\|u_1 - u_2\|^2_{H^1_0(\Omega)}.
\end{aligned}$$

From (3.3), we obtain for a function $\theta \in L^1(0,T)$,

$$\text{(3.4)}\qquad \frac{\mathrm{d}}{\mathrm{d}t}\|u_1 - u_2\|^2_{L^2(\Omega)} \leqslant \theta(t)\|u_1 - u_2\|^2_{L^2(\Omega)}.$$

Multiplying (3.4) by $\exp(-\int_0^t \theta(s)\,\mathrm{d}s)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(-\int_0^t \theta(s)\,\mathrm{d}s\right)\|u_1 - u_2\|^2_{L^2(\Omega)}\right) \leqslant 0.$$

Finally, integrating from 0 to t and taking into account that $u_1(\cdot,0) = u_2(\cdot,0)$ a.e. in Ω, we find $u_1 = u_2$ a.e. in Q. □

With the assumption that the function g is decreasing with respect to u, we state and prove our second uniqueness theorem.

Theorem 3.2. *Assume that* (1.1)–(1.3), (3.1) *and either* (1.4) (*with* $p \in (0,1)$) *or* (1.5) *hold. In addition, for a.e.* $(x,t) \in Q$,

(3.5) *the function* $s \mapsto g(x,t,s)$ *is decreasing on* $\mathbb{R}$.

Then the solution of (1.6) *is unique.*

P r o o f. Let u_1 and u_2 be two solutions of (1.6) corresponding to the same initial data $u_0 \in L^2(\Omega)$. Setting $w = u_1 - u_2$, we can see that the function w satisfies

$$(3.6)\qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(w,\xi) + M(l(u_1)) \displaystyle\int_\Omega a(x,t)\nabla w \cdot \nabla \xi \,\mathrm{d}x + \int_\Omega G(x,t)\xi \,\mathrm{d}x \\ \qquad = (M(l(u_2)) - M(l(u_1))) \displaystyle\int_\Omega a(x,t)\nabla u_2 \cdot \nabla \xi \,\mathrm{d}x + \int_\Omega w\xi \,\mathrm{d}x, \\ \qquad\qquad \text{in } \mathcal{D}'(0,T) \quad \forall\, \xi \in H_0^1(\Omega), \\ w(\cdot,0) = 0 \quad \text{a.e. in } \Omega, \end{cases}$$

where $G(x,t) = g(x,t,u_2) - g(x,t,u_1) + w$. Choosing $\xi = w$ as test function in the first equation of (3.6) and integrating from 0 to t, we obtain

(3.7)

$$\frac{1}{2}\|w\|^2_{L^2(\Omega)} + \int_0^t M(l(u_1)) \int_\Omega a(x,s)\nabla w \cdot \nabla w \,\mathrm{d}x \,\mathrm{d}s + \int_0^t \int_\Omega G(x,s) w \,\mathrm{d}x \,\mathrm{d}s$$
$$= \int_0^t (M(l(u_2)) - M(l(u_1))) \int_\Omega a(x,s)\nabla u_2 \cdot \nabla w \,\mathrm{d}x \,\mathrm{d}s + \int_0^t \|w\|^2_{L^2(\Omega)} \,\mathrm{d}s.$$

Using (1.1)–(1.3), (3.1), (3.5), the Cauchy-Schwarz inequality and Young's inequality, we get

$$(3.8)\qquad \int_0^t M(l(u_1)) \int_\Omega a(x,s)\nabla w \cdot \nabla w \,\mathrm{d}x \,\mathrm{d}s \geqslant m_0\lambda \int_0^t \|w\|^2_{H_0^1(\Omega)} \,\mathrm{d}s,$$

$$(3.9)\qquad \left| \int_0^t (M(l(u_2)) - M(l(u_1))) \int_\Omega a(x,s)\nabla u_2 \cdot \nabla w \,\mathrm{d}x \,\mathrm{d}s \right|$$
$$\leqslant m_2\Lambda \int_0^t \int_\Omega |\phi||w| \,\mathrm{d}x \int_\Omega |\nabla u_2||\nabla w| \,\mathrm{d}x \,\mathrm{d}s$$
$$\leqslant m_2\Lambda \|\phi\|_{L^2(\Omega)} \int_0^t \|w\|_{L^2(\Omega)} \|u_2\|_{H_0^1(\Omega)} \|w\|_{H_0^1(\Omega)} \,\mathrm{d}s$$
$$\leqslant \frac{m_0\lambda}{2} \int_0^t \|w\|^2_{H_0^1(\Omega)} \,\mathrm{d}s + \frac{(m_2\Lambda\|\phi\|_{L^2(\Omega)})^2}{2m_0\lambda} \int_0^t \|w\|^2_{L^2(\Omega)} \|u_2\|^2_{H_0^1(\Omega)} \,\mathrm{d}s$$

and

$$(3.10)\qquad \int_0^t \int_\Omega G(x,s) w \,\mathrm{d}x \,\mathrm{d}s = \int_0^t \int_\Omega (g(x,s,u_2) - g(x,s,u_1) + w) w \,\mathrm{d}x \,\mathrm{d}s \geqslant 0.$$

Combining (3.8)–(3.10), we obtain from (3.7),

$$\frac{1}{2}\|w\|^2_{L^2(\Omega)} + \frac{m_0\lambda}{2} \int_0^t \|w\|^2_{H_0^1(\Omega)} \,\mathrm{d}s$$
$$\leqslant \frac{(m_2\Lambda\|\phi\|_{L^2(\Omega)})^2}{2m_0\lambda} \int_0^t \|w\|^2_{L^2(\Omega)} \|u_2\|^2_{H_0^1(\Omega)} \,\mathrm{d}s + \int_0^t \|w\|^2_{L^2(\Omega)} \,\mathrm{d}s,$$

which leads to

$$\|w\|^2_{L^2(\Omega)} \leqslant \int_0^t \Big\{\frac{(m_2\Lambda\|\phi\|_{L^2(\Omega)})^2}{m_0\lambda}\|u_2\|^2_{H^1_0(\Omega)} + 2\Big\}\|w\|^2_{L^2(\Omega)}\,\mathrm{d}s.$$

From Lemma 3.1 of [14] we have $\|w\|_{L^2(\Omega)} = 0$ in $[0,T]$, which implies that $w = 0$ a.e. in Q, and then, $u_1 = u_2$ a.e. in Q. $\square$

Our continuity result is now presented.

Theorem 3.3. *Suppose either Theorem* 3.1 *or* 3.2 *holds. Then for all* $t \in [0,T]$, *the mapping*

$$\mathcal{T}\colon\ L^2(\Omega) \to L^2(\Omega), \quad u_0 \mapsto u(\cdot,t),$$

where u *is the unique solution of* (1.6) *corresponding to the initial data* u_0, *satisfies the following continuity result:*

$$u_0^j \rightharpoonup u_0 \text{ weakly in } L^2(\Omega) \Rightarrow \mathcal{T}(u_0^j) \to \mathcal{T}(u_0) \text{ strongly in } L^2(\Omega).$$

Particularly, $\mathcal{T}$ *is weakly continuous.*

P r o o f. Let $(u_0^j)_{j\in\mathbb{N}} \subset L^2(\Omega)$ be a sequence of initial data which converges to u_0 weakly in $L^2(\Omega)$, and let u^j, u be, respectively, the solutions of (1.6) corresponding to u_0^j, u_0, $j \in \mathbb{N}$. For all $j \in \mathbb{N}$ and all $p \in (0,1)$ we have

$$\|u^j\|_{L^2(Q)} \leqslant T^{1/2}\Big\{C_\Omega\Big(\frac{2}{\lambda m_0}\Big)^{1/2}(\|b_p\|_{L^{2/(1-p)}(Q)}\|u^j\|^p_{L^2(Q)} + \|g(\cdot,\cdot,0)\|_{L^2(Q)}) + \|u_0^j\|_{L^2(\Omega)}\Big\}, \tag{3.11}$$

$$\|u^j\|_{L^2(\Omega)} \leqslant \Big\{\frac{2C_\Omega^2}{\lambda m_0}(\|b_p\|^2_{L^{2/(1-p)}(Q)}\|u^j\|^{2p}_{L^2(Q)} + \|g(\cdot,\cdot,0)\|^2_{L^2(Q)}) + \|u_0^j\|^2_{L^2(\Omega)}\Big\}^{1/2} \tag{3.12}$$

and

$$\|u^j\|_{L^2(0,T;H^1_0(\Omega))} \leqslant \frac{1}{(\lambda m_0)^{1/2}}\Big\{\frac{2C_\Omega^2}{\lambda m_0}(\|b_p\|^2_{L^{2/(1-p)}(Q)}\|u^j\|^{2p}_{L^2(Q)} + \|g(\cdot,\cdot,0)\|^2_{L^2(Q)}) + \|u_0^j\|^2_{L^2(\Omega)}\Big\}^{1/2}, \tag{3.13}$$

and if $p = 1$,

$$\|u^j\|_{L^2(Q)} \leqslant T^{1/2}\Big\{C_\Omega\Big(\frac{2}{\lambda m_0}\Big)^{1/2}(\|b_1\|_{L^\infty(Q)}\|u^j\|_{L^2(Q)} + \|g(\cdot,\cdot,0)\|_{L^2(Q)}) + \|u_0^j\|_{L^2(\Omega)}\Big\}, \tag{3.14}$$

$$\|u^j\|_{L^2(\Omega)} \leqslant \Big\{\frac{2C_\Omega^2}{\lambda m_0}(\|b_1\|^2_{L^\infty(Q)}\|u^j\|^2_{L^2(Q)} + \|g(\cdot,\cdot,0)\|^2_{L^2(Q)}) + \|u_0^j\|^2_{L^2(\Omega)}\Big\}^{1/2} \tag{3.15}$$

and

$$\|u^j\|_{L^2(0,T;H_0^1(\Omega))} \leqslant \frac{1}{(\lambda m_0)^{1/2}} \Big\{ \frac{2C_\Omega^2}{\lambda m_0} (\|b_1\|^2_{L^\infty(Q)} \|u^j\|^2_{L^2(Q)} + \|g(\cdot,\cdot,0)\|^2_{L^2(Q)}) + \|u_0^j\|^2_{L^2(\Omega)} \Big\}^{1/2}. \tag{3.16}$$

On the other hand, for all $\xi \in \mathcal{D}(0,T;H_0^1(\Omega))$,

$$\int_Q u^j \xi_t \,\mathrm{d}x\,\mathrm{d}t = \int_0^T M(l(u^j)) \int_\Omega a(x,t)\nabla u^j \cdot \nabla \xi \,\mathrm{d}x\,\mathrm{d}t - \int_Q g(x,t,u^j)\xi \,\mathrm{d}x\,\mathrm{d}t. \tag{3.17}$$

Observe that the sequence $(u_0^j)_{j\in\mathbb{N}}$ is bounded in $L^2(\Omega)$ since it is a weakly convergent sequence in $L^2(\Omega)$. Then, by taking into account that $C_\Omega(2T/(\lambda m_0))^{1/2}\|b_1\|_{L^\infty(Q)}<1$ (in the case $p=1$), we deduce from (3.11) and (3.14) that $(u^j)_{j\in\mathbb{N}}$ is bounded in $L^2(Q)$. Hence, using (3.12), (3.13) and (3.15)–(3.17), we can find a constant $c_3 = c_3(p)$ independent of j such that for all $j \in \mathbb{N}$,

$$\|u^j\|_{L^\infty(0,T;L^2(\Omega))} \leqslant c_3, \tag{3.18}$$

$$\|u^j\|_{L^2(0,T;H_0^1(\Omega))} \leqslant c_3, \tag{3.19}$$

$$\|u_t^j\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant c_3. \tag{3.20}$$

Similarly, if g satisfies (1.5), we arrive at

$$\forall\, j \in \mathbb{N}\colon\ \|u^j\|^2_{L^2(\Omega)} + \lambda m_0 \int_0^t \|u^j\|^2_{H_0^1(\Omega)} \,\mathrm{d}s \leqslant \frac{(C_\Omega \|h\|_{L^2(Q)})^2}{\lambda m_0} + \|u_0^j\|^2_{L^2(\Omega)},$$

which leads us to saying that inequalities (3.18)–(3.20) are satisfied for another constant independent of j. Therefore, there exists a subsequence j_k and $\overline{u} \in E$ such that

$$u^{j_k} \to \overline{u} \quad \text{strongly in } L^2(Q), \tag{3.21}$$

$$u^{j_k} \to \overline{u} \quad \text{a.e. in } Q, \tag{3.22}$$

$$M(l(u^{j_k})) \to M(l(\overline{u})) \quad \text{strongly in } L^2(0,T), \tag{3.23}$$

$$u^{j_k}(\cdot,t) \to \overline{u}(\cdot,t) \quad \text{strongly in } L^2(\Omega),\ \text{a.e. } t \in (0,T), \tag{3.24}$$

$$u^{j_k} \rightharpoonup \overline{u} \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)), \tag{3.25}$$

$$u_t^{j_k} \rightharpoonup \overline{u}_t \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)), \tag{3.26}$$

$$u^{j_k} \rightharpoonup^* \overline{u} \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega)), \tag{3.27}$$

$$u_0^{j_k} \to u_0 \quad \text{weakly in } L^2(\Omega). \tag{3.28}$$

We have

$$\int_0^T \varphi(t) M(l(u^{j_k})) \int_\Omega a(x,t) \nabla u^{j_k} \cdot \nabla \xi \,\mathrm{d}x \,\mathrm{d}t - \int_Q u^{j_k} \xi \varphi_t \,\mathrm{d}x \,\mathrm{d}t = \int_Q g(x,t,u^{j_k}) \xi \varphi \,\mathrm{d}x \,\mathrm{d}t \quad \forall\, k \in \mathbb{N},\ \forall\, \varphi \in \mathcal{D}(0,T),\ \forall\, \xi \in H_0^1(\Omega). \tag{3.29}$$

We notice that if one of assumptions (1.4) and (1.5) holds, we obtain

$$\lim_{k\to\infty} \int_Q g(x,t,u^{j_k}) \xi \varphi \,\mathrm{d}x \,\mathrm{d}t = \int_Q g(x,t,\overline{u}) \xi \varphi \,\mathrm{d}x \,\mathrm{d}t \tag{3.30}$$

by using, respectively, (3.21) and (3.22), then, letting $k \to \infty$ in (3.29) we get by taking into account (1.2), (3.21), (3.23), (3.25) and (3.30),

$$\int_0^T \varphi(t) M(l(\overline{u})) \int_\Omega a(x,t) \nabla \overline{u} \cdot \nabla \xi \,\mathrm{d}x \,\mathrm{d}t - \int_Q \overline{u} \xi \varphi_t \,\mathrm{d}x \,\mathrm{d}t = \int_Q g(x,t,\overline{u}) \xi \varphi \,\mathrm{d}x \,\mathrm{d}t.$$

So, the limit $\overline{u}$ is a solution to

$$\begin{cases} \overline{u} \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega)), \quad \overline{u}_t \in L^2(0,T;H^{-1}(\Omega)), \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(\overline{u},\xi) + M(l(\overline{u})) \displaystyle\int_\Omega a(x,t) \nabla \overline{u} \cdot \nabla \xi \,\mathrm{d}x = \int_\Omega g(x,t,\overline{u}) \xi \,\mathrm{d}x \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathcal{D}'(0,T) \quad \forall\, \xi \in H_0^1(\Omega). \end{cases}$$

Moreover,

$$\forall\, \xi \in H_0^1(\Omega),\ \text{a.e. } t \in (0,T)\colon \int_0^t \langle u_t^{j_k}, \xi \rangle \,\mathrm{d}s = (u^{j_k}(\cdot,t),\xi) - (u_0^{j_k},\xi).$$

Passing to the limit as $k \to \infty$ and using (3.24), (3.26) and (3.28),

$$\int_0^t \langle \overline{u}_t, \xi \rangle \,\mathrm{d}s = (\overline{u}(\cdot,t),\xi) - (u_0,\xi),$$

which leads to

$$(\overline{u}(\cdot,t),\xi) - (\overline{u}(\cdot,0),\xi) = (\overline{u}(\cdot,t),\xi) - (u_0,\xi).$$

Hence, $\overline{u}(\cdot,0) = u_0$ a.e. in Ω. Therefore, due to the uniqueness of the limit, we have $\overline{u} = u$ a.e. in Q. Thus, by (3.27),

$$u^{j_k} \rightharpoonup^* u \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega)).$$

Particularly, if $\xi \in H_0^1(\Omega)$, it follows that

$$(u^{j_k}(\cdot,t),\xi) \rightharpoonup^* (u(\cdot,t),\xi) \quad \text{weakly star in } L^\infty(0,T). \tag{3.31}$$

Now, let t_1 and t_2 be elements of $[0,T]$ such that $t_2 > t_1$. We have

$$(3.32)\quad (u^{j_k}(\cdot,t_2),\xi) - (u^{j_k}(\cdot,t_1),\xi) = \int_{t_1}^{t_2} \langle u_t^{j_k},\xi\rangle\,\mathrm{d}t \leqslant \int_{t_1}^{t_2} \|u_t^{j_k}\|_{H^{-1}(\Omega)}\|\xi\|_{H_0^1(\Omega)}\,\mathrm{d}t$$
$$\leqslant (t_2-t_1)^{1/2}\|\xi\|_{H_0^1(\Omega)}\|u_t^{j_k}\|_{L^2(0,T;H^{-1}(\Omega))}$$
$$\leqslant c\|\xi\|_{H_0^1(\Omega)}(t_2-t_1)^{1/2}.$$

Due to (3.32), the sequence of functions $(u^{j_k}(\cdot,t),\xi)$ is equicontinuous, and then, it is relatively compact in $C([0,T])$. By (3.31) and the uniqueness of the limit, we have up to a subsequence

$$\forall\,\xi\in H_0^1(\Omega)\colon\ (u^{j_k}(\cdot,t),\xi)\to(u(\cdot,t),\xi)\quad\text{in } C([0,T]).$$

Thanks to the density of $H_0^1(\Omega)$ in $L^2(\Omega)$,

$$\forall\,\xi\in L^2(\Omega)\colon\ (u^{j_k}(\cdot,t),\xi)\to(u(\cdot,t),\xi)\quad\text{strongly in } \mathbb{R}\quad\forall\,t\in[0,T],$$

which means that

$$u^{j_k}(\cdot,t)\to u(\cdot,t)\quad\text{strongly in } L^2(\Omega)\quad\forall\,t\in[0,T].$$

Since every subsequence of $(u^j(\cdot,t))_{j\in\mathbb{N}}$ has a sub-subsequence that converges to the same limit $u(\cdot,t)$, we deduce that

$$\mathcal{T}(u_0^j)=u^j(\cdot,t)\to\mathcal{T}(u_0)=u(\cdot,t)\quad\text{strongly in } L^2(\Omega)\quad\forall\,t\in[0,T],$$

which completes the proof of the theorem. □

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