Mathematica Bohemica

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Mathematica Bohemica, Vol. 149 (2024), No. 4, 569–583

Persistent URL: http://dml.cz/dmlcz/152680

ON THE HYPER-ORDER OF ANALYTIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS NEAR A FINITE SINGULAR POINT

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Received May 9, 2023. Published online March 21, 2024.
Communicated by Grigore Sălăgean

Abstract. We study the hyper-order of analytic solutions of linear differential equations with analytic coefficients having the same order near a finite singular point. We improve previous results given by S. Cherief and S. Hamouda (2021). We also consider the nonhomogeneous linear differential equations.

Keywords: linear differential equation; hyper-order; a finite singular point; Nevanlinna theory

MSC 2020: 34M10, 30D35

1. Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distrubition theory of meromorphic function in the complex plane $\mathbb{C}$ and in the unit disc $D = \{z \in \mathbb{C}\colon |z| < 1\}$ (see [2], [3], [8], [10]). We denote the order of growth of a meromorphic function f in $\mathbb{C}$ by $\sigma(f)$.

Recently the authors in [4], [6], [7] have investigated the growth of solutions of linear differential equations near a finite singular point. They studied the order and the hyper-order of analytic solutions of different types of linear differential equations with analytic coefficients near a finite singular point by using an adapted definitions and properties of Nevanlinna theory. In this paper, we continue this investigation near a finite singular point.

First, we recall the appropriate definitions. Set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and suppose that f is meromorphic in $\overline{\mathbb{C}} \setminus \{z_0\}$, where $z_0 \in \mathbb{C}$. Define the counting function near z_0 by

$$N_{z_0}(r,f) = -\int_{\infty}^{r} \frac{n(t,f) - n(\infty,f)}{t}\,\mathrm{d}t - n(\infty,f)\log r,$$

DOI: 10.21136/MB.2024.0075-23

where $n(t,f)$ counts the number of poles of f in the region $\{z\in\mathbb{C}\colon\ t\leqslant|z_0-z|\}\cup\{\infty\}$, each pole according to its multiplicity, and the proximity function by

$$m_{z_0}(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(z_0-re^{i\varphi})|\,d\varphi.$$

The characteristic function of f is defined in the usual manner by

$$T_{z_0}(r,f)=m_{z_0}(r,f)+N_{z_0}(r,f).$$

In addition, the order of a meromorphic function f near z_0 is defined by

$$\sigma_T(f,z_0)=\limsup_{r\to0}\frac{\log^+T_{z_0}(r,f)}{-\log r}.$$

For an analytic function f in $\overline{\mathbb{C}}\setminus\{z_0\}$, we have also the definition

$$\sigma_M(f,z_0)=\limsup_{r\to0}\frac{\log^+\log^+M_{z_0}(r,f)}{-\log r},$$

where $M_{z_0}(r,f)=\max\limits_{|z_0-z|=r}|f(z)|$.

When the order is infinite, we introduce the notion of a hyper-order near z_0 that is defined as

$$\sigma_{2,T}(f,z_0)=\limsup_{r\to0}\frac{\log^+\log^+T_{z_0}(r,f)}{-\log r},$$
$$\sigma_{2,M}(f,z_0)=\limsup_{r\to0}\frac{\log^+\log^+\log^+M_{z_0}(r,f)}{-\log r}.$$

Remark 1.1 ([4]). It is shown in [6] that if f is a non-constant meromorphic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ and $g(\omega)=f(z_0-1/\omega)$, then $g(\omega)$ is meromorphic in $\mathbb{C}$ and we have

$$T(R,g)=T_{z_0}\Big(\frac{1}{R},f\Big),$$

where $R>0$ and so $\sigma_T(f,z_0)=\sigma(g)$. Also, if f is analytic in $\overline{\mathbb{C}}\setminus\{z_0\}$, then $g(\omega)$ is entire and thus $\sigma_T(f,z_0)=\sigma_M(f,z_0)$ and $\sigma_{2,T}(f,z_0)=\sigma_{2,M}(f,z_0)$. Then we can use the notations $\sigma(f,z_0)$ and $\sigma_2(f,z_0)$ without any ambiguity.

Many authors [1], [2], [3], [9], [10] have studied the linear differential equation

$$f''+A(z)e^{az}f'+B(z)e^{bz}f=0, \tag{1.1}$$

where $A(z)$ and $B(z)$ are entire functions and a,b are complex numbers. Kwon in [9] proved that if $ab\neq0$ and $\arg a\neq\arg b$ or $a=cb$ with $0<c<1$, then every solution $f\not\equiv0$ of the equation (1.1) is of infinite order.

Recently, Fettouch and Hamouda (see [6]) proved the following result.

Theorem 1.1 ([6]). *Let z_0 a, b be complex constants, such that* $\arg a \neq \arg b$ *or* $a = cb$ *with* $0 < c < 1$ *and* $n \in \mathbb{N} \setminus \{0\}$. *Let* $A(z), B(z) \not\equiv 0$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *with* $\max\{\sigma(A, z_0), \sigma(B, z_0)\} < n$. *Then every solution* $f \not\equiv 0$ *of the differential equation*

$$f'' + A(z)\exp\Big\{\frac{a}{(z_0 - z)^n}\Big\}f' + B(z)\exp\Big\{\frac{b}{(z_0 - z)^n}\Big\}f = 0$$

satisfies $\sigma(f, z_0) = \infty$ *and* $\sigma_2(f, z_0) = n$.

In [4], Cherief and Hamouda have extended Theorem 1.1 to higher order linear differential equations and proved the following two results.

Theorem 1.2 ([4]). *Let* $n \in \mathbb{N} \setminus \{0\}$, $k \geqslant 2$ *be an integer and* $A_j(z)$ $(j = 0, \ldots, k-1)$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$, *such that* $\sigma(A_j, z_0) < n$ *and* $A_0(z) \not\equiv 0$. *If* a_j $(j = 0, \ldots, k-1)$ *are distinct complex numbers, then every solution* $f \not\equiv 0$ *of the differential equation*

$$(1.2)\quad f^{(k)} + A_{k-1}(z)\exp\Big\{\frac{a_{k-1}}{(z_0 - z)^n}\Big\}f^{(k-1)} + \ldots + A_0(z)\exp\Big\{\frac{a_0}{(z_0 - z)^n}\Big\}f = 0$$

that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$, *satisfies* $\sigma(f, z_0) = \infty$.

Theorem 1.3 ([4]). *Let* $n \in \mathbb{N} \setminus \{0\}$, $k \geqslant 2$ *be an integer and* $A_j(z)$ $(j = 0, \ldots, k-1)$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$, *such that* $\sigma(A_j, z_0) < n$ *and* $A_0(z) \not\equiv 0$. *Let* a_j $(j = 0, \ldots, k-1)$ *be complex constants. Suppose that there exist nonzero numbers* a_s *and* a_l, *such that* $0 < s < l \leqslant k-1$, $a_s = |a_s|e^{i\theta_s}$, $a_l = |a_l|e^{i\theta_l}$, $\theta_s, \theta_l \in [0, 2\pi)$, $\theta_s \neq \theta_l$. *Let* $A_s A_l \not\equiv 0$ *and for* $j \neq s, l$, a_j *satisfy either* $a_j = d_j a_s$ $(0 < d_j < 1)$ *or* $a_j = d_j a_l$ $(0 < d_j < 1)$. *Then every solution* $f \not\equiv 0$ *of the equation* (1.2) *that is analytic in* $\overline{\mathbb{C}} \setminus \{z_0\}$, *satisfies* $\sigma(f, z_0) = \infty$.

In this paper, we continue to consider the above theorems and investigate the hyper-order of analytic solutions of the equation (1.2). We also consider the nonhomogeneous equation. We prove the following results.

Theorem 1.4. *Let* $n \in \mathbb{N} \setminus \{0\}$, $k \geqslant 2$ *be an integer and* $A_j(z)$, a_j $(j = 0, \ldots, k-1)$ *satisfy the additional hypotheses of Theorem* 1.2. *Then every solution* f *of the equation* (1.2) *that is analytic in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *satisfies* $\sigma_2(f, z_0) = n$, *where* z_0 *is an essential singular point for* f.

E x a m p l e 1.1. Consider the differential equation

$$f''' + \frac{3}{z}\Big(2+\frac{1}{z}\Big)f'' - \frac{1}{z^4}\exp\Big\{\frac{2}{z}\Big\}f' - \frac{2}{z^4}\Big(3+\frac{3}{z}+\frac{1}{z^2}\Big)\exp\Big\{\frac{1}{z}\Big\}f = 0. \tag{1.3}$$

Obviously, the conditions of Theorem 1.4 are satisfied. Hence every solution f of the equation (1.3) that is analytic in $\overline{\mathbb{C}}\setminus\{0\}$ satisfies $\sigma_2(f,0)=1$, where 0 is an essential singular point for f.

Let us remark that the function $f(z)=\exp\{\exp(1/z)\}$ is a solution of the equation (1.3) that is analytic in $\overline{\mathbb{C}}\setminus\{0\}$ with $\sigma_2(f,0)=1$.

Theorem 1.5. *Let $n\in\mathbb{N}\setminus\{0\}$, $k\geqslant 2$ be an integer and $A_j(z)$, a_j $(j=0,\dots,k-1)$ satisfy the additional hypotheses of Theorem* 1.3. *Then every solution f of the equation* (1.2) *that is analytic in $\overline{\mathbb{C}}\setminus\{z_0\}$, satisfies $\sigma_2(f,z_0)=n$, where z_0 is an essential singular point for f.*

Theorem 1.6. *Let $n\in\mathbb{N}\setminus\{0\}$, $k\geqslant 2$ be an integer and $A_j(z)$, a_j $(j=0,\dots,k-1)$ satisfy the hypotheses of Theorem* 1.3 *or those of Theorem* 1.4. *Let $F\not\equiv 0$ be analytic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ of order $\sigma=\sigma(F,z_0)<n$. Then every solution f of the equation*

$$f^{(k)} + A_{k-1}(z)\exp\Big\{\frac{a_{k-1}}{(z_0-z)^n}\Big\}f^{(k-1)} + \ldots + A_0(z)\exp\Big\{\frac{a_0}{(z_0-z)^n}\Big\}f = F \tag{1.4}$$

that is analytic in $\overline{\mathbb{C}}\setminus\{z_0\}$ satisfies $\sigma(f,z_0)=\infty$ and $\sigma_2(f,z_0)=n$, where z_0 is an essential singular point for f, with at most one exceptional analytic solution f_0 of finite order in $\overline{\mathbb{C}}\setminus\{z_0\}$.

2. Preliminary lemmas

Lemma 2.1 ([6])**.** *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}}\setminus\{z_0\}$. Let $\alpha>0$ be a given real constant and $j\in\mathbb{N}$. Then there exists a set $E_1\subset(0,1)$ of finite logarithmic measure, that is $\int_0^1\chi_{E_1}(t)\,\mathrm{d}t/t<\infty$, and a constant $A>0$, that depends on α and j, such that for all $r=|z-z_0|$ satisfying $r\notin E_1$, we have*

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| \leqslant A\Big[\frac{1}{r^2}T_{z_0}(\alpha r,f)\log T_{z_0}(\alpha r,f)\Big]^j,$$

where χ_{E_1} is the characteristic function of the set E_1.

Lemma 2.2 ([10]). *Let g be a transcendental entire function, let $0 < \eta_1 < \frac{1}{4}$ and ω_R be a point such that $|\omega_R| = R$ and $|g(\omega_R)| > M(R,g)V(R)^{-1/4+\eta_1}$ holds. Then there exists a set $F_1 \subset (1,\infty)$ of finite logarithmic measure, that is $\int_1^\infty \chi_{F_1}(t)\,\mathrm{d}t/t < \infty$, such that*

$$\frac{g^{(j)}(\omega_R)}{g(\omega_R)} = \Big(\frac{V(R)}{\omega_R}\Big)^j (1+o(1)), \quad j \in \mathbb{N}$$

holds as $R \to \infty$ and $R \notin F_1$, where $V(R)$ is the central index of g and $M(R,g) = \max\limits_{|\omega|=R} |g(\omega)|$.

Remark 2.1 ([7]). Let f be a non-constant analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$. Then the function $g(\omega) = f(z_0 - 1/\omega)$ is entire in $\mathbb{C}$ and $V_{z_0}(r) = V(R)$, where $R = 1/r$, $R > 0$, $V(R)$ is the central index of g in $\mathbb{C}$ and $V_{z_0}(r)$ is the central index of f near the singular point z_0.

By using Lemma 2.2, Remark 2.1 and similar arguments as in the proof of Theorem 8 in [7], we can obtain the following lemma.

Lemma 2.3. *Let f be a non-constant analytic function in $\overline{\mathbb{C}}\setminus\{z_0\}$. Let $0 < \eta_1 < \frac{1}{4}$ and z_r be a point such that $|z_0 - z_r| = r$ and $|f(z_r)| > M_{z_0}(r,f)V_{z_0}(r)^{-1/4+\eta_1}$ holds. Then there exists a set $E_2 \subset (0,1)$ of finite logarithmic measure, such that*

$$\frac{f^{(j)}(z_r)}{f(z_r)} = \Big(\frac{V_{z_0}(r)}{z_0 - z_r}\Big)^j (1+o(1)), \quad j \in \mathbb{N}$$

holds as $r \to 0$, $r \notin E_2$, where $V_{z_0}(r)$ is the central index of f near a singular point z_0 and $M_{z_0}(r,f) = \max\limits_{|z_0-z|=r} |f(z)|$.

Lemma 2.4. *Let f be a non-constant analytic function in $\overline{\mathbb{C}}\setminus\{z_0\}$. For $|z_0 - z| = r$ sufficiently small, let $z_r = z_0 - r\mathrm{e}^{\mathrm{i}\theta_r}$ be a point satisfying $|f(z_r)| = \max\limits_{|z_0-z|=r} |f(z)|$. Then there exist a constant $\delta_r > 0$ and a set $E_3 \subset (0,1)$ of finite logarithmic measure, such that for all z satisfying $|z_0 - z| = r \notin E_3$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have*

$$\frac{f^{(j)}(z)}{f(z)} = \Big(\frac{V_{z_0}(z)}{z_0 - z}\Big)^j (1+o(1)), \quad j \in \mathbb{N},$$

where $V_{z_0}(z)$ is the central index of f near a singular point z_0.

P r o o f. If $z_r = z_0 - re^{i\theta_r}$ is a point satisfying $|f(z_r)| = M_{z_0}(r,f)$, since $|f(z)|$ is continuous in $|z_0 - z| = r$, then there exists a constant δ_r (> 0), such that for all z satisfying $|z_0 - z| = r$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$||f(z)| - |f(z_r)|| < \varepsilon,$$

that is

$$|f(z)| > \frac{1}{2}|f(z_r)| = \frac{1}{2}M_{z_0}(r,f) > M_{z_0}(r,f)V_{z_0}(r)^{-1/4+\eta_1}.$$

By Lemma 2.3,

$$\frac{f^{(j)}(z)}{f(z)} = \Big(\frac{V_{z_0}(z)}{z_0 - z}\Big)^j(1 + o(1)), \quad j \in \mathbb{N}$$

holds for all z satisfying $|z_0 - z| = r \notin E_2$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$.

□

Lemma 2.5. *Let f be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$. For $|z_0 - z| = r$, let $z_r = z_0 - re^{i\theta_r}$ be a point satisfying $|f(z_r)| = \max\limits_{|z_0 - z| = r}|f(z)|$. Then there exist a constant $\delta_r > 0$ and a set $E_4 \subset (0,1)$ of finite logarithmic measure, such that for all z satisfying $|z_0 - z| = r \notin E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have*

$$\Big|\frac{f(z)}{f^{(j)}(z)}\Big| \leqslant 2r^j, \quad j \in \mathbb{N},$$

where z_0 is an essential singular point for f.

P r o o f. Let $z_r = z_0 - re^{i\theta_r}$ be a point satisfying $|f(z_r)| = \max\limits_{|z_0 - z| = r}|f(z)|$. Then by Lemma 2.4 there exist a constant $\delta_r > 0$ and a set $E_3 \subset (0,1)$ of finite logarithmic measure, such that for all z satisfying $|z_0 - z| = r \notin E_3$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \Big(\frac{V_{z_0}(z)}{z_0 - z}\Big)^j(1 + o(1)), \quad j \in \mathbb{N}. \tag{2.1}$$

Since $g(\omega) = f(z_0 - 1/\omega)$ is a transcendental entire function, it follows that $V(R) \to \infty$ as $R \to \infty$. On the other hand, $V(R) = V_{z_0}(r)$ $(R = 1/r)$. Hence $V_{z_0}(r) \to \infty$ as $r \to 0$. Then by (2.1), for all z satisfying $|z_0 - z| = r \notin E_3$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| \geqslant \frac{1}{2}r^{-j},$$

that is,

$$\Big|\frac{f(z)}{f^{(j)}(z)}\Big| \leqslant 2r^j, \quad j \in \mathbb{N}.$$

□

Lemma 2.6 ([6]). *Let $A(z)$ be an analytic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ with $\sigma(A,z_0)<n$ $(n\in\mathbb{N}\setminus\{0\})$. Set $g(z)=A(z)\exp\{a/(z_0-z)^n\}$, where $a=\alpha+\mathrm{i}\beta\neq 0$ is a complex number, $z_0-z=r\mathrm{e}^{\mathrm{i}\varphi}$, $\delta_a(\varphi)=\alpha\cos(n\varphi)+\beta\sin(n\varphi)$, and $H=\{\varphi\in[0,2\pi)\colon \delta_a(\varphi)=0\}$ (obviously, H is a finite set). Then for any given $\varepsilon>0$ and for any $\varphi\in[0,2\pi)\setminus H$, there exists $r_0>0$, such that for $0<r<r_0$, we have*

(i) *if $\delta_a(\varphi)>0$, then*

$$\exp\Big\{(1-\varepsilon)\delta_a(\varphi)\frac{1}{r^n}\Big\}\leqslant|g(z)|\leqslant\exp\Big\{(1+\varepsilon)\delta_a(\varphi)\frac{1}{r^n}\Big\}, \tag{2.2}$$

(ii) *if $\delta_a(\varphi)<0$, then*

$$\exp\Big\{(1+\varepsilon)\delta_a(\varphi)\frac{1}{r^n}\Big\}\leqslant|g(z)|\leqslant\exp\Big\{(1-\varepsilon)\delta_a(\varphi)\frac{1}{r^n}\Big\}. \tag{2.3}$$

Lemma 2.7 ([4]). *Let $k\geqslant 2$ be an integer and $A_j(z)$ $(j=0,\dots,k-1)$ be analytic functions in $\overline{\mathbb{C}}\setminus\{z_0\}$, such that $\sigma(A_j,z_0)\leqslant\alpha<\infty$. If f is a solution of the equation*

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\ldots+A_1(z)f+A_0(z)f=0 \tag{2.4}$$

that is analytic in $\overline{\mathbb{C}}\setminus\{z_0\}$, then $\sigma_2(f,z_0)\leqslant\alpha$.

Lemma 2.8 ([7]). *Let f be a non-constant analytic function in $\overline{\mathbb{C}}\setminus\{z_0\}$. Then there exists a set $E_5\subset(0,1)$ of finite logarithmic measure, such that*

$$\frac{f^{(j)}(z_r)}{f(z_r)}=(1+o(1))\Big(\frac{V_{z_0}(z)}{z_0-z_r}\Big)^j,\quad j\in\mathbb{N}$$

holds as $r\to 0$, $r\notin E_5$, where z_r is a point on the circle $|z_0-z|=r$ that satisfies $|f(z_r)|=M_{z_0}(r,f)=\max\limits_{|z_0-z|=r}|f(z)|$.

Lemma 2.9 ([5]). *Let f be a non-constant analytic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ of infinite order with the hyper-order $\sigma_2(f,z_0)=\sigma$ and $V_{z_0}(r)$ be the central index of f. Then*

$$\limsup_{r\to 0}\frac{\log^+\log^+V_{z_0}(r)}{-\log r}=\sigma.$$

Lemma 2.10. *Let $k\geqslant 2$ be an integer, $A_j(z)$ $(j=0,\dots,k-1)$ and $F\not\equiv 0$ be analytic functions in $\overline{\mathbb{C}}\setminus\{z_0\}$, such that $\max\{\sigma(A_j,z_0),\sigma(F,z_0)\}\leqslant\alpha<\infty$. If f is an infinite order solution of the equation*

$$f^{(k)}+A_{k-1}(z)f^{k-1}+\ldots+A_1(z)f'+A_0(z)f=F \tag{2.5}$$

that is analytic in $\overline{\mathbb{C}}\setminus\{z_0\}$, then $\sigma_2(f,z_0)\leqslant\alpha$.

Proof. Assume that f is an infinite analytic solution in $\overline{\mathbb{C}} \setminus \{z_0\}$ of the equation (2.5). By (2.5), we have

$$\left|\frac{f^{(k)}}{f}\right| \leqslant |A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f}\right| + \ldots + |A_1(z)|\left|\frac{f'}{f}\right| + \left|\frac{F}{f}\right| + |A_0(z)|. \tag{2.6}$$

By Lemma 2.8, there exists a set $E_5 \subset (0,1)$ of finite logarithmic measure, such that for all $j = 0, 1, \ldots, k$, we have

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1 + o(1))\Big(\frac{V_{z_0}(z_r)}{z_0 - z_r}\Big)^j \tag{2.7}$$

as $r \to 0$, $r \notin E_5$, where z_r is a point on the circle $|z_0 - z| = r$ that satisfies $|f(z_r)| = M_{z_0}(r, f) = \max\limits_{|z_0 - z| = r} |f(z)|$. For any given $\varepsilon > 0$, there exists $r_0 > 0$, such that for all $0 < r = |z_0 - z| < r_0$ we have

$$\Big|A_j(z)\Big| \leqslant \exp\Big\{\frac{1}{r^{\alpha+\varepsilon}}\Big\}, \quad j = 0, 1, \ldots, k-1 \tag{2.8}$$

and

$$\Big|F(z)\Big| \leqslant \exp\Big\{\frac{1}{r^{\alpha+\varepsilon}}\Big\}. \tag{2.9}$$

Since $M_{z_0}(r, f) \geqslant 1$ as $r \to 0$, it follows from (2.9) that

$$\frac{|F(z)|}{M_{z_0}(r, f)} \leqslant \exp\Big\{\frac{1}{r^{\alpha+\varepsilon}}\Big\} \quad \text{as } r \to 0. \tag{2.10}$$

By substituting (2.7), (2.8) and (2.10) into (2.6), we obtain

$$\Big(\frac{V_{z_0}(r)}{r}\Big)^k |1 + o(1)| \leqslant (k+1)\Big(\frac{V_{z_0}(r)}{r}\Big)^{k-1} |1 + o(1)| \exp\Big\{\frac{1}{r^{\alpha+\varepsilon}}\Big\} \tag{2.11}$$

for all $|z_0 - z_r| = r \notin E_5$, $r \to 0$ and $|f(z_r)| = M_{z_0}(r, f)$. By (2.11) and Lemma 2.9, we get

$$\sigma_2(f, z_0) \leqslant \alpha.$$

$\square$

3. Proof of theorems

Proof of Theorem 1.4. Assume that f is an analytic solution of (1.2) in $\overline{\mathbb{C}} \setminus \{z_0\}$, where z_0 is an essential singular point for f. By Lemma 2.1, there exist a set $E_1 \subset (0,1)$ of finite logarithmic measure and a constant $\lambda > 0$, such that for all $r = |z_0 - z|$ satisfying $r \notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant \lambda\Big[\frac{1}{r} T_{z_0}(\alpha r, f)\Big]^{2j}, \quad j = 1, \ldots, k. \tag{3.1}$$

For each sufficiently small $|z_0 - z| = r$, let $z_r = z_0 - re^{i\theta_r}$ be a point satisfying $|f(z_r)| = \max\limits_{|z_0 - z| = r} |f(z)|$. By Lemma 2.5, there exist a constant $\delta_r > 0$ and a set $E_4 \subset (0,1)$ of finite logarithmic measure such that for all z satisfying $|z_0 - z| = r \notin E_4$, $r \to 0$, and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\left|\frac{f(z)}{f^{(j)}(z)}\right| \leqslant 2r^j, \quad j = 1, \ldots, k. \tag{3.2}$$

Set $a_j = \alpha_j + i\beta_j$, $\delta_{a_j}(\theta) = \alpha_j \cos(n\theta) + \beta_j \sin(n\theta)$, $z_0 - z = re^{i\theta}$,

$$H_1 = \bigcup_{j=0}^{k-1} \{\theta \in [0, 2\pi) \colon \delta_{a_j}(\theta) = 0\},$$

$$H_2 = \bigcup_{0 \leqslant i < j \leqslant k-1} \{\theta \in [0, 2\pi) \colon \delta_{a_j - a_i}(\theta) = 0\}.$$

Since a_j are distinct complex numbers, then there exists only one $s \in \{0, \ldots, k-1\}$, such that for any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have

$$\delta_1 = \delta_{a_s}(\theta) = \max\{\delta_{a_j}(\theta) \colon j = 0, \ldots, k-1\}.$$

We have $\delta_1 > 0$ or $\delta_1 < 0$.

Case 1. $\delta_1 > 0$. Set $\delta_2 = \max\{\delta_{a_j}(\theta) \colon j \neq s\}$. Then $\delta_2 < \delta_1$.

Subcase 1.1. If $\delta_2 > 0$ then $0 < \delta_2 < \delta_1$. Thus by Lemma 2.6, for any given ε $(0 < 2\varepsilon < (\delta_1 - \delta_2)/(\delta_1 + \delta_2))$, for all z satisfying $|z_0 - z| = r$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have

$$\left|A_s(z) \exp\left\{\frac{a_s}{(z_0 - z)^n}\right\}\right| \geqslant \exp\left\{(1 - \varepsilon)\frac{\delta_1}{r^n}\right\} \tag{3.3}$$

and

$$\left|A_j(z) \exp\left\{\frac{a_j}{(z_0 - z)^n}\right\}\right| \leqslant \exp\left\{(1 + \varepsilon)\frac{\delta_2}{r^n}\right\}, \quad j \neq s. \tag{3.4}$$

By (1.2), it follows that

$$\begin{aligned} -A_s(z) \exp\left\{\frac{a_s}{(z_0 - z)^n}\right\} &= \frac{f^{(k)}}{f^{(s)}} + \sum_{j=s+1}^{k-1} A_j(z) \exp\left\{\frac{a_j}{(z_0 - z)^n}\right\} \frac{f^{(j)}}{f^{(s)}} \\ &\quad + \sum_{j=0}^{s-1} A_j(z) \exp\left\{\frac{a_j}{(z_0 - z)^n}\right\} \frac{f^{(j)}}{f} \frac{f}{f^{(s)}}. \end{aligned} \tag{3.5}$$

Substituting (3.1)–(3.4) into (3.5), for all z satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we obtain

$$\exp\Big\{(1-\varepsilon)\frac{\delta_1}{r^n}\Big\} \leqslant M_1 r^s \exp\Big\{(1+\varepsilon)\frac{\delta_2}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.6}$$

where $M_1 > 0$ is a constant. Hence by (3.6), we obtain $\sigma_2(f, z_0) \geqslant n$. On the other hand, by Lemma 2.7, we have $\sigma_2(f, z_0) = n$.

Subcase 1.2. Let $\delta_2 < 0$. By Lemma 2.6, for any given ε $(0 < 2\varepsilon < 1)$, for all z satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have (3.3) and

$$\Big|A_j(z)\exp\Big\{\frac{a_j}{(z_0-z)^n}\Big\}\Big| \leqslant \exp\Big\{(1-\varepsilon)\frac{\delta_2}{r^n}\Big\} < 1, \quad j \neq s. \tag{3.7}$$

Substituting (3.1)–(3.3), (3.7) into (3.5), for all z satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we obtain

$$\exp\Big\{(1-\varepsilon)\frac{\delta_1}{r^n}\Big\} \leqslant M_2 r^s \Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.8}$$

where $M_2 > 0$ is a constant. Hence by (3.8), we obtain $\sigma_2(f, z_0) \geqslant n$. On the other hand, by Lemma 2.7, we have $\sigma_2(f, z_0) = n$.

Case 2. Let $\delta_1 < 0$. By Lemma 2.6, for any given ε $(0 < 2\varepsilon < 1)$, for all z satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have

$$\Big|A_j(z)\exp\Big\{\frac{a_j}{(z_0-z)^n}\Big\}\Big| \leqslant \exp\Big\{(1-\varepsilon)\frac{\delta_1}{r^n}\Big\} < 1, \quad j = 0, \ldots, k-1. \tag{3.9}$$

By (1.2), we get

$$-1 = \sum_{j=1}^{k-1} A_j(z)\exp\Big\{\frac{a_j}{(z_0-z)^n}\Big\}\frac{f^{(j)}}{f}\frac{f}{f^{(k)}} + A_0(z)\exp\Big\{\frac{a_0}{(z_0-z)^n}\Big\}\frac{f}{f^{(k)}}. \tag{3.10}$$

Substituting (3.1)–(3.3), (3.9) into (3.10), for all z satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we obtain

$$1 \leqslant M_3 r^k \exp\Big\{(1+\varepsilon)\frac{\delta_1}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.11}$$

where $M_3 > 0$ is a constant. Hence by (3.11), we obtain $\sigma_2(f, z_0) \geqslant n$. On the other hand, by Lemma 2.7, we have $\sigma_2(f, z_0) = n$. $\square$

P r o o f of Theorem 1.5. Assume that f is an analytic solution of (1.2) in $\overline{\mathbb{C}}\setminus\{z_0\}$, where z_0 is an essential singular point for f. By Lemma 2.1, there exist a set $E_1\subset(0,1)$ of finite logarithmic measure and a constant $\lambda>0$, such that for all $r=|z_0-z|$ satisfying $r\notin E_1$, we have (3.1).

For each sufficiently small $|z_0-z|=r$, let $z_r=z_0-re^{\mathrm{i}\theta_r}$ be a point satisfying $|f(z_r)|=\max\limits_{|z_0-z|=r}|f(z)|$. By Lemma 2.5, there exist a constant $\delta_r>0$ and a set $E_4\subset(0,1)$ of finite logarithmic measure such that for all z satisfying $|z_0-z|=r\notin E_4$ and $\arg(z_0-z)=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]$, we have (3.2).

Set

$$H_3=\{\theta\in[0,2\pi)\colon\ \delta_{a_s}(\theta)=0 \text{ or } \delta_{a_l}(\theta)=0\}$$

and

$$H_4=\{\theta\in[0,2\pi)\colon\ \delta_{a_s}(\theta)=\delta_{a_l}(\theta)\}.$$

For any given $\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus(H_3\cup H_4)$, we have $\delta_{a_s}(\theta)\neq0$, $\delta_{a_l}(\theta)\neq0$ and $\delta_{a_s}(\theta)>\delta_{a_l}(\theta)$ or $\delta_{a_s}(\theta)<\delta_{a_l}(\theta)$.

Set $c_1=\delta_{a_s}(\theta)$ and $c_2=\delta_{a_l}(\theta)$.

Case 1. $c_1>c_2$. Here we also divide our proof in three subcases.

Subcase 1.1. $c_1>c_2>0$. Set $c_3=\max\{\delta_{a_j}(\theta)\colon\ j\neq s\}$. Then $0<c_3<c_1$. Thus by Lemma 2.6, for any given ε $(0<2\varepsilon<(c_1-c_3)/(c_1+c_3))$, for all z satisfying $|z_0-z|=r\notin E_1\cup E_4$, $r\to0$ and $\arg(z_0-z)=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus(H_3\cup H_4)$, we have

$$\Bigl|A_s(z)\exp\Bigl\{\frac{a_s}{(z_0-z)^n}\Bigr\}\Bigr|\geqslant\exp\Bigl\{(1-\varepsilon)\frac{c_1}{r^n}\Bigr\}\tag{3.12}$$

and

$$\Bigl|A_j(z)\exp\Bigl\{\frac{a_j}{(z_0-z)^n}\Bigr\}\Bigr|\leqslant\exp\Bigl\{(1+\varepsilon)\frac{c_3}{r^n}\Bigr\},\quad j\neq s.\tag{3.13}$$

Substituting (3.1), (3.2), (3.12), (3.13) into (3.5), for all z satisfying $|z_0-z|=r\notin E_1\cup E_4$, $r\to0$ and $\arg(z_0-z)=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus(H_1\cup H_2)$, we obtain

$$\exp\Bigl\{(1-\varepsilon)\frac{c_1}{r^n}\Bigr\}\leqslant M_4r^s\exp\Bigl\{(1+\varepsilon)\frac{c_3}{r^n}\Bigr\}\Bigl[\frac{T_{z_0}(\alpha r,f)}{r}\Bigr]^{2k},\tag{3.14}$$

where $M_4>0$ is a constant. Hence by (3.14), we obtain $\sigma_2(f,z_0)\geqslant n$. On the other hand, by Lemma 2.7, we have $\sigma_2(f,z_0)=n$.

Subcase 1.2. $c_1>0>c_2$. Set $\gamma_1=\max\{d_j\colon\ j\neq s,l\}$. Thus, by Lemma 2.6, for any given ε $(0<2\varepsilon<(1-\gamma_1)/(1+\gamma_1))$, for all z satisfying $|z_0-z|=r\notin E_1\cup E_4$, $r\to0$ and $\arg(z_0-z)=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus(H_3\cup H_4)$, we have

$$\Bigl|A_j(z)\exp\Bigl\{\frac{a_j}{(z_0-z)^n}\Bigr\}\Bigr|\leqslant\exp\Bigl\{(1+\varepsilon)\frac{\gamma_1c_1}{r^n}\Bigr\},\quad j\neq s.\tag{3.15}$$

Substituting (3.1), (3.2), (3.12), (3.15) into (3.5), for all z satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we obtain

$$\exp\Big\{(1-\varepsilon)\frac{c_1}{r^n}\Big\} \leqslant M_5 r^s \exp\Big\{(1+\varepsilon)\frac{\gamma_1 c_1}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.16}$$

where $M_5 > 0$ is a constant. Hence by (3.16), we obtain $\sigma_2(f, z_0) \geqslant n$. On the other hand, by Lemma 2.7, we have $\sigma_2(f, z_0) = n$.

Subcase 1.3. $0 > c_1 > c_2$. Set $\gamma_2 = \min\{d_j\colon\ j \neq s, l\}$. By Lemma 2.6, for any given ε $(0 < 2\varepsilon < 1)$, for all z satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_3 \cup H_4)$, we have

$$\Big|A_s(z)\exp\Big\{\frac{a_s}{(z_0 - z)^n}\Big\}\Big| \leqslant \exp\Big\{(1-\varepsilon)\frac{c_1}{r^n}\Big\} \tag{3.17}$$

and

$$\Big|A_j(z)\exp\Big\{\frac{a_j}{(z_0 - z)^n}\Big\}\Big| \leqslant \exp\Big\{(1+\varepsilon)\frac{\gamma_2 c_1}{r^n}\Big\}, \quad j \neq s. \tag{3.18}$$

Substituting (3.1), (3.2), (3.17), (3.18) into (3.10), for all z satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we obtain

$$1 \leqslant M_6 r^k \exp\Big\{(1+\varepsilon)\frac{\gamma_2 c_1}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.19}$$

where $M_6 > 0$ is a constant. Hence by (3.19), we obtain $\sigma_2(f, z_0) \geqslant n$. On the other hand, by Lemma 2.7, we have $\sigma_2(f, z_0) = n$.

Case 2. $c_1 < c_2$. Using the same reasoning as in Case 1, we can also obtain $\sigma_0(f, z_0) = n$. $\square$

P r o o f of Theorem 1.6. First we show that (1.4) can possess at most one exceptional analytic solutionf_0 in $\overline{\mathbb{C}} \setminus \{z_0\}$ of finite order. In fact, if f^* is another analytic solution of finite order of the equation (1.4), then $f_0 - f^*$ is an analytic solution in $\overline{\mathbb{C}}\setminus\{z_0\}$ of finite order of the corresponding homogeneous equation of (1.4). This contradicts Theorem 1.4 and Theorem 1.5.

We assume that f is an infinite order analytic solution in $\overline{\mathbb{C}} \setminus \{z_0\}$ of the equation (1.4), where z_0 is an essential singular point for f. By Lemma 2.10, it follows that $\sigma_2(f, z_0) \leqslant n$.

Now we prove that $\sigma_2(f, z_0) \geqslant n$. By Lemma 2.1, there exist a set $E_1 \subset (0, 1)$ of finite logarithmic measure and a constant $\lambda > 0$, such that for all z satisfying $|z_0 - z| = r \notin E_1$, we have (3.1). For each sufficiently small $|z_0 - z| = r$, let $z_r =$

$z_0 - re^{\mathrm{i}\theta_r}$ be a point satisfying $|f(z_r)| = \max_{|z_0-z|=r} |f(z)|$. By Lemma 2.5, there exist a constant $\delta_r > 0$ and a set $E_4 \subset (0,1)$ of finite logarithmic measure such that for all z satisfying $|z_0 - z| = r \notin E_4$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have (3.2). Since $|f(z)|$ is continous in $|z_0 - z| = r$, then there exists a constant $\lambda_r > 0$ such that for all z satisfying $|z_0 - z| = r$ sufficiently small and $\arg(z_0 - z) = \theta \in [\theta_r - \lambda_r, \theta_r + \lambda_r]$, we have

$$\frac{1}{2}|f(z_r)| < |f(z)| < \frac{3}{2}|f(z_r)|. \tag{3.20}$$

On the other hand, for any given ε $(0 < 2\varepsilon < n - \sigma)$, there exists $r_0 > 0$, such that for all $0 < r = |z_0 - z| < r_0$, we have

$$|F(z)| \leqslant \exp\Big\{\frac{1}{r^{\sigma+\varepsilon}}\Big\}. \tag{3.21}$$

Since $M_{z_0}(r, f) \geqslant 1$ as $r \to 0$, it follows from (3.20) and (3.21) that

$$\Big|\frac{F(z)}{f(z)}\Big| \leqslant 2\exp\Big\{\frac{1}{r^{\sigma+\varepsilon}}\Big\} \quad \text{as } r \to 0. \tag{3.22}$$

Set $\gamma = \min\{\delta_r, \lambda_r\}$.

(i) Suppose that $a_j (j = 0, \dots, k-1)$ satisfy the hypotheses of Theorem 1.4. Since a_j are distinct complex numbers, then there exists only $s \in \{0, \dots, k-1\}$ such that for any given $\theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_1 \cup H_2)$, where H_1 and H_2 are definied above, we have

$$\delta_1 = \delta_{a_s}(\theta) = \max\{\delta_{a_j}(\theta)\colon\ j = 0, \dots, k-1\}.$$

We have $\delta_1 > 0$ or $\delta_1 < 0$.

Case 1. $\delta_1 > 0$. Set $\delta_2 = \max\{\delta_{a_j}(\theta)\colon\ j \neq s\}$. Then $\delta_2 < \delta_1$.

Subcase 1.1. $\delta_2 > 0$. From (3.1)–(3.4), (3.22) and (1.4), for all z satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_1 \cup H_2)$, we obtain

$$\exp\Big\{(1-\varepsilon)\frac{\delta_1}{r^n}\Big\} \leqslant B_1 r^s \exp\Big\{\frac{1}{r^{\sigma+\varepsilon}}\Big\} \exp\Big\{(1+\varepsilon)\frac{\delta_2}{r^n}\Big\} \Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.23}$$

where $B_1 > 0$ is a constant. From (3.23), we get $\sigma_2(f, z_0) \geqslant n$. This and the fact that $\sigma_2(f, z_0) \leqslant n$ yield $\sigma_2(f, z_0) = n$.

Subcase 1.2. $\delta_2 < 0$. From (3.1)–(3.3), (3.7), (3.22) and (1.4), for all z satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_1 \cup H_2)$, we obtain

$$\exp\Big\{(1-\varepsilon)\frac{\delta_1}{r^n}\Big\} \leqslant B_2 r^s \exp\Big\{\frac{1}{r^{\sigma+\varepsilon}}\Big\} \Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.24}$$

where $B_2 > 0$ is a constant. From (3.24), we get $\sigma_2(f, z_0) \geqslant n$. This and the fact that $\sigma_2(f, z_0) \leqslant n$ yield $\sigma_2(f, z_0) = n$.

Case 2. $\delta_1 < 0$. From (3.1), (3.2), (3.9), (3.22) and (1.4), for all z satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_1 \cup H_2)$, we have

$$1 \leqslant B_3 r^k \exp\Big\{\frac{1}{r^{\sigma+\varepsilon}} v\Big\} \exp\Big\{(1+\varepsilon)\frac{\delta_1}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.25}$$

where $B_3 > 0$ is a constant. From (3.25), we get $\sigma_2(f, z_0) \geqslant n$. This and the fact that $\sigma_2(f, z_0) \leqslant n$ yield $\sigma_2(f, z_0) = n$.

(ii) Suppose that a_j $(j = 0, \ldots, k-1)$ satisfy the hypotheses of Theorem 1.5. For any given $\theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_3 \cup H_4)$, where H_3 and H_4 are defined above, we have $\delta_{a_s}(\theta) \neq 0$, $\delta_{a_l}(\theta) \neq 0$ and $\delta_{a_s}(\theta) > \delta_{a_l}(\theta)$ or $\delta_{a_s}(\theta) < \delta_{a_l}(\theta)$.

Set $c_1 = \delta_{a_s}(\theta)$ and $c_2 = \delta_{a_l}(\theta)$.

Case 1. $c_1 > c_2$. Here we also divide our proof in three subcases.

Subcase 1.1 $c_1 > c_2 > 0$. From (3.1), (3.2), (3.12), (3.13), (3.22) and (1.4), for all z satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_3 \cup H_4)$, we obtain

$$\exp\Big\{(1-\varepsilon)\frac{c_1}{r^n}\Big\} \leqslant B_4 r^s \exp\Big\{\frac{1}{r^{\sigma+\varepsilon}}\Big\} \exp\Big\{(1+\varepsilon)\frac{c_3}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.26}$$

where $B_4 > 0$ is a constant. Hence by (3.26), we get $\sigma_2(f, z_0) \geqslant n$. This and the fact that $\sigma_2(f, z_0) \leqslant n$ yield $\sigma_2(f, z_0) = n$.

Subcase 1.2. $c_1 > 0 > c_2$. From (3.1), (3.2), (3.12), (3.15), (3.22) and (1.4), for all z satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_3 \cup H_4)$, we obtain

$$\exp\Big\{(1-\varepsilon)\frac{c_1}{r^n}\Big\} \leqslant B_5 r^s \exp\Big\{\frac{1}{r^{\sigma+\varepsilon}}\Big\} \exp\Big\{(1+\varepsilon)\frac{\gamma_1 c_3}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.27}$$

where $B_5 > 0$ is a constant. From (3.27), we get $\sigma_2(f, z_0) \geqslant n$. This and the fact that $\sigma_2(f, z_0) \leqslant n$ yield $\sigma_2(f, z_0) = n$.

Subcase 1.3. $0 > c_1 > c_2$. From (3.1), (3.2), (3.17), (3.18), (3.22) and (1.4), for all z satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_3 \cup H_4)$, we obtain

$$1 \leqslant B_6 r^k \exp\Big\{\frac{1}{r^{\sigma+\varepsilon}}\Big\} \exp\Big\{(1+\varepsilon)\frac{\gamma_2 c_1}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.28}$$

where $B_6 > 0$ is a constant. From (3.28), we get $\sigma_2(f, z_0) \geqslant n$. This and the fact that $\sigma_2(f, z_0) \leqslant n$ yield $\sigma_2(f, z_0) = n$. $\square$

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