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Mathematica Bohemica, Vol. 150 (2025), No. 1, 139–159

Persistent URL: http://dml.cz/dmlcz/152880

DYNAMIC BEHAVIOR OF VECTOR SOLUTIONS OF A CLASS OF 2-D NEUTRAL DIFFERENTIAL SYSTEMS

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Received October 19, 2023. Published online July 18, 2024.
Communicated by Josef Diblík

Abstract. This work deals with the analysis pertaining some dynamic behavior of vector solutions of first order two-dimensional neutral delay differential systems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u(t)+pu(t-\tau) \\ v(t)+pv(t-\tau) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u(t-\alpha) \\ v(t-\beta) \end{bmatrix}.$$

The effort has been made to study

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x(t)-p(t)h_1(x(t-\tau)) \\ y(t)-p(t)h_2(y(t-\tau)) \end{bmatrix} + \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}\begin{bmatrix} f_1(x(t-\alpha)) \\ f_2(y(t-\beta)) \end{bmatrix} = 0,$$

where $p, a, b, c, d, h_1, h_2, f_1, f_2 \in C(\mathbb{R},\mathbb{R})$; $\alpha, \beta, \tau \in \mathbb{R}^+$. We verify our results with the examples.

Keywords: oscillation; nonoscillation; nonlinear system of neutral differential equations; asymptotically stable; Banach's fixed point theorem

MSC 2020: 34K40, 34C10, 34A34

1. Introduction

Suppose A and B are two countries having their common boundaries engaged in so called sensitive border issues. Let the expenditure on arms by the country A and B be $x(t)$ and $y(t)$, respectively. If the rate of change of expenditure by country A is $\mathrm{d}x/\mathrm{d}t$, then it depends upon the expenditure on arms by B as well as its own arm expenditure. It may be possible that it contains a term depending on mutual hostility or mutual goodwill, which is independent of expenditures. Similarly for the

DOI: 10.21136/MB.2024.0156-23

country B. Therefore, a model citing the arm race problem could be of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ax + by + r,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = cx - dy + s,$$

where $a, b, c, d \in \mathbb{R}$. Moreover, r and s are less than zero in case of mutual goodwill and greater than zero in case of mutual hostility. So, we have the vector differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -a & b \\ c & -d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} r \\ s \end{bmatrix} \tag{1.1}$$

which is known as Richardson's armed race model.

Let $x(t)$ be the matured male population size and $y(t)$ be the matured female population size of a region at time t. Assume that before a certain period of time τ they are immature and the size could be $x(t-\tau)$ and $y(t-\tau)$, respectively. After a period τ, the population size depends on $x(t)$, $x(t-\tau)$ and $y(t)$, $y(t-\tau)$. Therefore, a mathematical description of the change in population size can be treated as

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) + p(t)x(t-\tau)] = a(t)x(t-\alpha) + b(t)y(t-\beta),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[y(t) + p(t)y(t-\tau)] = c(t)x(t-\alpha) + d(t)y(t-\beta),$$

where $p(t), a(t), b(t), c(t), d(t) \in \mathbb{C}(\mathbb{R}, \mathbb{R})$ and $\alpha, \beta, \tau \in \mathbb{R}^+$. In the above system, if the growth of population depends on the rate of emigration and immigration of people, then the required population model could be of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) + p(t)x(t-\tau)] = a(t)x(t-\alpha) + b(t)y(t-\beta) + r(t),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[y(t) + p(t)y(t-\tau)] = c(t)x(t-\alpha) + d(t)y(t-\beta) + s(t),$$

where $r, s \in \mathbb{C}(\mathbb{R}, \mathbb{R})$. Therefore, the representation of the population model is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x(t) + p(t)x(t-\tau) \\ y(t) + p(t)y(t-\tau) \end{bmatrix} = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}\begin{bmatrix} x(t-\alpha) \\ y(t-\beta) \end{bmatrix} + \begin{bmatrix} r(t) \\ s(t) \end{bmatrix}. \tag{1.2}$$

In the above two mathematical models, (1.1) is a special case of (1.2) and both are worth studying. In this work, we emphasize to study the dynamic behavior of vector solutions of (1.2).

Consider a 2-dimensional first order neutral delay differential systems with constant coefficients of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u(t)+pu(t-\tau) \\ v(t)+pv(t-\tau) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u(t-\alpha) \\ v(t-\beta) \end{bmatrix}, \tag{1.3}$$

where $a,b,c,d,p \in \mathbb{R}$ and $\tau,\alpha,\beta \in \mathbb{R}^{+}$. It is easy to see that the system (1.3) can be reduced to a second order delay differential equation with constant coefficients of the form

$$\begin{aligned} v''(t) &+ 2pv''(t-\tau) + p^2v''(t-2\tau) - dv'(t-\beta) - av'(t-\alpha) \\ &- pdv'(t-\tau-\beta) - pav'(t-\tau-\alpha) + (ad-bc)v(t-\alpha-\beta) = 0. \end{aligned}$$

We note that the ongoing equation has the characteristic equation

$$f(\zeta) = \zeta^2(1+p\mathrm{e}^{-\tau\zeta})^2 - \zeta(1+p\mathrm{e}^{-\tau\zeta})(a\mathrm{e}^{-\alpha\zeta} + d\mathrm{e}^{-\beta\zeta}) + (ad-bc)\mathrm{e}^{-(\alpha+\beta)\zeta} = 0 \tag{1.4}$$

provided $b \neq 0$. On the other hand, the system (1.3) can also be written as

$$\begin{aligned} u''(t) &+ 2pu''(t-\tau) + p^2u''(t-2\tau) - du'(t-\beta) - au'(t-\alpha) \\ &- pdu'(t-\tau-\beta) - pau'(t-\tau-\alpha) + (ad-bc)u(t-\alpha-\beta) = 0 \end{aligned}$$

with the same characteristic equation (1.4) provided $c \neq 0$. It is easy to prove the following result.

Proposition 1.1. *Let $bc \neq 0$. If the characteristic equation* (1.4) *has no real roots, then the system* (1.3) *is oscillatory.*

In [5] and [6], Naito and Opluštil have established the oscillation and nonoscillation criteria for two-dimensional nonlinear differential systems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 0 & g(t) \\ -p(t) & 0 \end{bmatrix}\begin{bmatrix} |u|^{1/\alpha}\operatorname{sgn} u \\ |v|^{1/\alpha}\operatorname{sgn} v \end{bmatrix}$$

and, in a similar fashion, Mihalíková (see [4]) has obtained oscillation results for the neutral differential system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1(t) - px_1(t-\tau) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & a_1(t) \\ a_2(t) & 0 \end{bmatrix}\begin{bmatrix} f_2(x_1(g_1(t))) \\ f_1(x_2(g_2(t))) \end{bmatrix}.$$

Note that the above works deal with the special cases of the systems like (1.1), (1.2) and (1.3). However, Grigorian's work in [2] presents an interesting application of the

Riccati technique for study of oscillation properties of a two-dimensional differential system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\varphi(t)\\ \psi(t)\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}\varphi(t)\\ \psi(t)\end{bmatrix}$$

which is in closed form. This observation motivates us to consider the closed form of two-dimensional neutral delay differential systems and to investigate their dynamic behavior. For more on delay differential systems, we refer the reader to the monographs [1] and [3] and the references cited therein.

The objective of this work is to discuss the oscillatory, nonoscillatory and asymptotic stability of two-dimensional linear neutral delay differential systems with constant coefficients of the form (1.3), and by applying the results for linear differential system, we are interested in linearized oscillation theory for the nonlinear two-dimensional differential system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x(t)-p(t)h_1(x(t-\tau))\\ y(t)-p(t)h_2(y(t-\tau))\end{bmatrix}+\begin{bmatrix}a(t) & b(t)\\ c(t) & d(t)\end{bmatrix}\begin{bmatrix}f_1(x(t-\alpha))\\ f_2(y(t-\beta))\end{bmatrix}=0. \tag{1.5}$$

Definition 1.1. By the solution of (1.3) (or (1.5)), we mean a continuous vector function $X(t)=[u(t),v(t)]^\top$ with $u,v\in C^1(\mathbb{R},\mathbb{R})$ which satisfies (1.3) (or (1.5)) for $t\in(-\varrho,\infty)$, where $\varrho=\max\{\tau,\alpha,\beta\}$. A nontrivial vector solution of (1.3) (or (1.5)) is said to be oscillatory if all its components are oscillatory; otherwise, the system is called nonoscillatory.

2. Differential systems with constant coefficients

This section deals with necessary and sufficient conditions for oscillation of all vector solutions of the system (1.3) by means of its characteristic equation (1.4).

Theorem 2.1. *Assume that $ad-bc>0$. Then the system* (1.3) *is oscillatory if and only if $f(\zeta)$ has no real roots in* $(0,\infty)$.

Proof. Suppose that (1.3) is oscillatory. Now, $f(0)=ad-bc>0$ and

$$\begin{aligned}\lim_{\zeta\to\infty}f(\zeta)&=\lim_{\zeta\to\infty}(\zeta^2(1+p\mathrm{e}^{-\tau\zeta})^2-\zeta(1+p\mathrm{e}^{-\tau\zeta})(a\mathrm{e}^{-\alpha\zeta}+d\mathrm{e}^{-\beta\zeta})\\&\qquad+(ad-bc)\mathrm{e}^{-(\alpha+\beta)\zeta})\\&=\infty\end{aligned}$$

implies that $f(\zeta)$ has no real roots in $(0,\infty)$.

Conversely, assume that $f(\zeta)$ has no real roots in $(0,\infty)$. Then $f(\zeta)$ has complex roots. Let $n+\mathrm{i}m$ and $n-\mathrm{i}m$ be the complex roots of $f(\zeta)$. Therefore, we have the components like

$$u(t) = c_1\mathrm{e}^{(n+\mathrm{i}m)t} = c_1\mathrm{e}^{nt}(\cos mt + \mathrm{i}\sin mt)$$

and

$$v(t) = c_2\mathrm{e}^{(n-\mathrm{i}m)t} = c_2\mathrm{e}^{nt}(\cos mt - \mathrm{i}\sin mt).$$

Hence, there is an oscillatory vector solution of the system (1.3) and it is of the form

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} c_1\mathrm{e}^{nt} & \mathrm{i}c_1\mathrm{e}^{nt} \\ c_2\mathrm{e}^{nt} & -\mathrm{i}c_2\mathrm{e}^{nt} \end{bmatrix} \begin{bmatrix} \cos mt \\ \sin mt \end{bmatrix}.$$

By taking all complex roots of (1.3), we get oscillatory vector solutions. □

Theorem 2.2. *If $ad-bc \leqslant 0$, then the system* (1.3) *has a nonoscillatory vector solution in* $(-\infty,\infty)$.

Proof. If $ad-bc=0$, then $\zeta=0$ is a real root of $f(\zeta)$. When $ad-bc<0$, we have $f(0)=ad-bc<0$ and $\lim\limits_{\zeta\to\infty} f(\zeta)=\infty$ shows that $f(\zeta)=0$ has a real root lying in $(-\infty,\infty)$. Hence, the system (1.3) is nonoscillatory in $(-\infty,\infty)$. □

Theorem 2.3. *In* $[0,\infty)$, *the system* (1.3) *is oscillatory if and only if* $ad-bc>0$.

Theorem 2.4. *If $p=-1$, $ad-bc>0$, $a<0$, $d<0$, then the system* (1.3) *is oscillatory if and only if $f(\zeta)$ has no real roots in* $(-\infty,\infty)$.

Theorem 2.5. *Let $p \geqslant 0$, $ad-bc<0$, $a<0$, $d<0$, $\alpha,\beta>\tau$. Then the system* (1.3) *is oscillatory if and only if $f(\zeta)$ has no real roots in* $(-\infty,0)$.

Proof. Assume that the system (1.3) is oscillatory. We have $f(0)=ad-bc<0$. Clearly,

$$f(\zeta) = \zeta^2(1+p\mathrm{e}^{-\tau\zeta})^2\Big(1-\frac{a\mathrm{e}^{-\alpha\zeta}+d\mathrm{e}^{-\beta\zeta}}{\zeta(1+p\mathrm{e}^{-\tau\zeta})}\Big) + (ad-bc)\mathrm{e}^{-(\alpha+\beta)\tau}$$

and

$$\begin{aligned}\lim_{\zeta\to-\infty}\frac{a\mathrm{e}^{-\alpha\zeta}+d\mathrm{e}^{-\beta\zeta}}{\zeta(1+p\mathrm{e}^{-\tau\zeta})} &= \lim_{\zeta\to-\infty}\frac{-a\alpha\mathrm{e}^{-\alpha\zeta}-d\beta\mathrm{e}^{-\beta\zeta}}{1+p\mathrm{e}^{-\tau\zeta}-\tau p\zeta\mathrm{e}^{-\tau\zeta}} \\ &= \lim_{\zeta\to-\infty}\frac{-aa_1\alpha^2\mathrm{e}^{-a_1\zeta}}{\tau^2 p} + \lim_{\zeta\to-\infty}\frac{-da_2\beta^2\mathrm{e}^{-a_2\zeta}}{\tau^2 p} = \infty,\end{aligned}$$

where $-\alpha+\tau=-a_1<0$ and $-\beta+\tau=-a_2<0$ implies that $\lim\limits_{\zeta\to-\infty} f(\zeta)=-\infty$, that is, $f(\zeta)$ has no real roots in $(-\infty,0)$. The converse part follows from Theorem 2.1. □

Example 2.1. Consider a two-dimensional neutral delay differential system of the form

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u(t)+3u(t-\frac{1}{2}\pi) \\ v(t)+3v(t-\frac{1}{2}\pi) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} u(t-\frac{3}{2}\pi) \\ v(t-\frac{7}{2}\pi) \end{bmatrix} \tag{2.1}
$$

which satisfies all the conditions of Theorem 2.1. In particular, $[u(t), v(t)]^\top = [\sin t, \cos t]^\top$ is one of oscillatory vector solutions of the system (2.1).

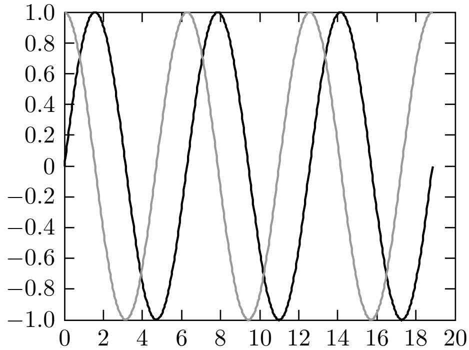

Figure 1.

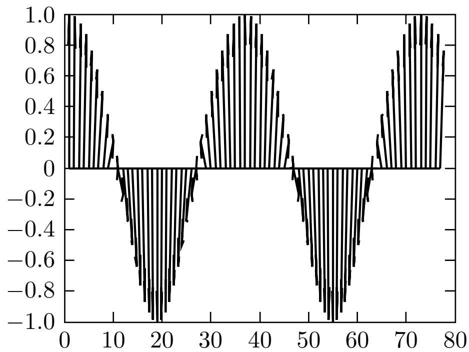

Figure 2.

In Figure 1, the solution satisfies component-wise oscillation, which is explained in the proof of the theorem. However, Figure 2 deals with the oscillatory vector directly.

Theorem 2.6. *Let* $p \geqslant 0$, $a, d \geqslant 0$, $\alpha, \beta > \tau$, $ad - bc < 0$, *then the system* (1.3) *is oscillatory if and only if*

$$
g(\zeta) = \frac{f(\zeta)}{\zeta^2(1 + p\mathrm{e}^{-\tau\zeta})^2}
$$

has no real roots in $(0, \zeta_1) \cup (\zeta_1, \infty)$, *where* ζ_1 *is a root of* $g(\zeta)$.

P r o o f. We note that $g(0) = -\infty$ and

$$\lim_{\zeta\to\infty} g(\zeta) = \lim_{\zeta\to\infty} \Big(1 + \frac{(ad-bc)\mathrm{e}^{-(\alpha+\beta)\zeta}}{\zeta^2(1+p\mathrm{e}^{-\tau\zeta})^2} - \frac{(a\mathrm{e}^{-\alpha\zeta} + d\mathrm{e}^{-\beta\zeta})}{\zeta(1+p\mathrm{e}^{-\tau\zeta})}\Big) = 1.$$

Indeed,

$$\begin{aligned} g'(\zeta) &= \frac{(ad-bc)\mathrm{e}^{-(\alpha+\beta)\zeta}}{\zeta^4(1+p\mathrm{e}^{-\tau\zeta})^4}(-\zeta^2(1+p\mathrm{e}^{-\tau\zeta})^2(\alpha+\beta) \\ &\quad - 2\zeta(1+p\mathrm{e}^{-\tau\zeta})^2 + 2p\tau\mathrm{e}^{-\tau\zeta}\zeta^2(1+p\mathrm{e}^{-\tau\zeta})) \\ &\quad - \frac{1}{\zeta^2(1+p\mathrm{e}^{-\tau\zeta})^2}(-\zeta(1+p\mathrm{e}^{-\tau\zeta})(a\alpha\mathrm{e}^{-\alpha\zeta} + d\beta\mathrm{e}^{-\beta\zeta}) \\ &\quad - (a\mathrm{e}^{-\alpha\zeta} + d\mathrm{e}^{-\beta\zeta})(1 + p\mathrm{e}^{-\tau\zeta} - \zeta\tau p\mathrm{e}^{-\tau\zeta})) \\ &= \frac{(ad-bc)\mathrm{e}^{-(\alpha+\beta)\zeta}}{\zeta^3(1+p\mathrm{e}^{-\tau\zeta})^3}(-2(1+p\mathrm{e}^{-\tau\zeta}) - \zeta(\alpha+\beta+p\mathrm{e}^{-\tau\zeta}(\alpha+\beta-2\tau))) \\ &\quad - \frac{1}{\zeta^2(1+p\mathrm{e}^{-\tau\zeta})^2}(-(1+p\mathrm{e}^{-\tau\zeta})(a\mathrm{e}^{-\alpha\zeta} + d\mathrm{e}^{-\beta\zeta}) - \zeta(a\alpha\mathrm{e}^{-\alpha\zeta} + d\beta\mathrm{e}^{-\beta\zeta}) \\ &\quad - \zeta p\mathrm{e}^{-\tau\zeta}(a\mathrm{e}^{-\alpha\zeta}(\alpha-\tau) + d\mathrm{e}^{-\beta\zeta}(\beta-\tau))) \\ &> 0 \end{aligned}$$

for all $\zeta > 0$, which implies $g(\zeta)$ is an increasing function on $(0,\infty)$ and hence $g(\zeta)$ may cut the real axis once at $\zeta = \zeta_1$. Therefore, $g(\zeta)$ has no real root in $(0,\zeta_1)\cup(\zeta_1,\infty)$ and the statement of the theorem follows. $\square$

Theorem 2.7. *Let $p \leqslant -1$, $a, d \geqslant 0$, $\alpha, \beta > \tau$, $ad - bc < 0$. Then the system* (1.3) *is oscillatory if and only if $g(\zeta)$ has no real roots in* $(\log(-p)/\tau, \zeta_2)\cup(\zeta_2,\infty)$, *where ζ_2 is a root of $g(\zeta)$.*

P r o o f. Here, we observe that $g(\log(-p)/\tau) = -\infty$ and $\lim_{\zeta\to\infty} g(\zeta) = 1$. The rest of the proof follows from Theorem 2.6. $\square$

Corollary 2.1. *Let $-1 < p < 0$, $a, d < 0$ and $ad - bc \geqslant 0$; then the system* (1.3) *is oscillatory if and only if $g(\zeta)$ has no real roots in* $(-\infty, \log(-p)/\tau)$.

R e m a r k 2.1. We have seen in Corollary 2.1 that $g(\zeta)$ has no real roots in $(-\infty, \log(-p)/\tau)$ and so also $f(\zeta)$. That is, the roots have negative real parts in $(-\infty, \log(-p)/\tau)$. Therefore, the components take the form

$$u(t) = \mathrm{e}^{(x_1+\mathrm{i}x_2)t} = \mathrm{e}^{x_1 t}(\cos x_2 t + \mathrm{i}\sin x_2 t)$$

and

$$v(t) = \mathrm{e}^{(x_1'+\mathrm{i}x_2')t} = \mathrm{e}^{x_1' t}(\cos x_2' t + \mathrm{i}\sin x_2' t).$$

Consequently, $\lim_{t\to\infty} Z(t) = \lim_{t\to\infty} [u(t), v(t)]^\top = 0$. Hence, the vector solution of (1.3) asymptotically converges to zero.

Corollary 2.2. *Assume that Corollary* 2.1 *holds true. Then the system* (1.3) *is asymptotically stable in* $(-\infty, \log(-p)/\tau)$.

P r o o f. Let $\zeta_j = \alpha_j + \mathrm{i}\beta_j$, $j = 1, 2, 3, \ldots$ Since the characteristic equation (1.4) of (1.3) is the same for both equations in $u(t)$ and $v(t)$, then the corresponding components of the system (1.3), taking multiplicity into account, can be of the form

$$u(t) = t^p \exp(\zeta_j t) = v(t), \tag{2.2}$$

where p is a nonnegative integer. Let $\theta > 0$ be a number such that $-\theta > \max \mathrm{Re}(\zeta_j)$. Then, we have

$$\alpha_j + \theta < 0. \tag{2.3}$$

We claim that $|u_j(t)\exp(\theta t)| \to 0$ as $t \to \infty$ and $|v_j(t)\exp(\theta t)| \to 0$ as $t \to \infty$. Indeed,

$$\begin{aligned} |u_j(t)\exp(\theta t)| &= |t^p \exp(\zeta_j t)\exp(\theta t)| = |t^p \exp(\alpha_j t)\exp(\theta t)| \\ &= |t^p \exp((\alpha_j + \theta)t)| \to 0 \quad \text{as } t \to \infty \end{aligned}$$

due to (2.3). So, we can find constants $k_j > 0$ for $j = 1, 2, 3, \ldots$ such that

$$|u_j(t)\exp(\theta t)| \leqslant k_j \quad \text{and} \quad |v_j(t)\exp(\theta t)| \leqslant k_j,$$

that is,

$$|u_j(t)| \leqslant k_j \exp(-\theta t) \tag{2.4}$$

and

$$|v_j(t)| \leqslant k_j \exp(-\theta t). \tag{2.5}$$

We know that the general solution of the system (1.3), by means of components, has the form

$$u(t) = \sum_{j=1}^{\infty} c_j u_j(t) \quad \text{and} \quad v(t) = \sum_{j=1}^{\infty} {c_j}' v_j(t),$$

where $c_1, c_2, \ldots$ and $c'_1, c'_2, \ldots$ are arbitrary constants. From (2.4) and (2.5), we get

$$|u(t)| \leqslant \sum_{j=1}^{\infty} |c_j| |u_j(t)| \leqslant \max_j |c_j| \sum_{j=1}^{\infty} |u_j(t)| \leqslant \max_j |c_j| \sum_{j=1}^{\infty} k_j \exp(-\theta t) = k \exp(-\theta t),$$

where $k = \max\limits_j |c_j| \sum\limits_{j=1}^{\infty} k_j$ and

$$|v(t)| \leqslant \sum_{j=1}^{\infty} |c_j'| |v_j(t)| \leqslant \max_j |c_j'| \sum_{j=1}^{\infty} |v_j(t)| \leqslant \max_j |c_j| \sum_{j=1}^{\infty} k_j \exp(-\theta t) = k' \exp(-\theta t),$$

where $k' = \max\limits_j |c_j'| \sum\limits_{j=1}^{\infty} k_j$. Consequently,

$$\lim_{t\to\infty} |u(t)| = 0 \quad \text{and} \quad \lim_{t\to\infty} |v(t)| = 0$$

and hence $\|Z(t)\| \to 0$ as $t \to \infty$. Therefore, the zero solution of the system (1.3) is asymptotically stable. $\square$

Corollary 2.3. *If* $p \geqslant 0$, $ad - bc < 0$, $a < 0$, $d < 0$, $\alpha, \beta > \tau$, *then the system* (1.3) *is asymptotically stable in* $(-\infty, 0)$.

Proof. The proof of the corollary follows from Theorem 2.5 and Corollary 2.2. Hence, the details are omitted. $\square$

Example 2.2. Consider a two-dimensional neutral delay differential system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} u(t) - \mathrm{e}^{-2\pi} u(t - 2\pi) \\ v(t) - \mathrm{e}^{-2\pi} v(t - 2\pi) \end{bmatrix} = \begin{bmatrix} -2\mathrm{e}^{-3\pi/2} & 2 \\ 3 & -3\mathrm{e}^{3\pi/2} \end{bmatrix} \begin{bmatrix} u(t - 4\pi) \\ v(t - \frac{5\pi}{2}) \end{bmatrix}, \tag{2.6}$$

which satisfies all the conditions of Corollary 2.2. In particular, $[u(t), v(t)]^\top = [\mathrm{e}^{-t} \sin t, \mathrm{e}^{-t} \cos t]^\top$ is one such asymptotically stable vector solution of the system (2.6).

In Figure 3, the solution converges asymptotically to zero, which is explained in the proof of the theorem. However, Figure 4 deals with the vector that directly converges asymptotically to zero.

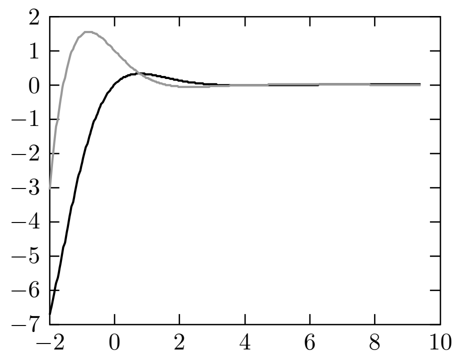

Figure 3.

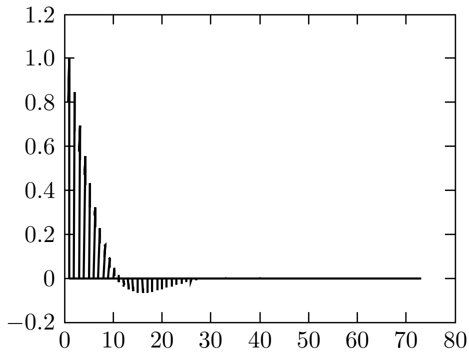

Figure 4.

3. Linearized oscillation for differential systems

This section deals with some applications of constant coefficient results to two-dimensional nonlinear neutral differential systems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u(t)-p(t)h_1(u(t-\tau)) \\ v(t)-p(t)h_2(v(t-\tau)) \end{bmatrix} + \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}\begin{bmatrix} f_1(u(t-\alpha)) \\ f_2(v(t-\beta)) \end{bmatrix} = 0 \tag{3.1}$$

through linearized oscillation technique. For this, we need the following hypotheses in the sequel:

(H_1) $\limsup\limits_{t\to\infty} p(t) = p_0 \in [1,\infty)$;

(H_2) $\lim\limits_{t\to\infty} a(t) = a_0 \in (0,\infty)$, $\lim\limits_{t\to\infty} b(t) = b_0 \in (0,\infty)$, $\lim\limits_{t\to\infty} c(t) = c_0 \in (0,\infty)$, $\lim\limits_{t\to\infty} d(t) = d_0 \in (0,\infty)$;

(H_3) $h_1(s)/s \geqslant 1$ for $s \neq 0$ and $\lim\limits_{|s|\to\infty} h_1(s)/s = 1$, $h_2(s)/s \geqslant 1$ for $s \neq 0$ and $\lim\limits_{|s|\to\infty} h_2(s)/s = 1$;

(H_4) $sf_1(s) > 0$ for $s \neq 0$, $|f_1(s)| \geqslant f_{10} > 0$ for $|s| \geqslant \zeta_0 > 0$, $\lim\limits_{|s|\to\infty} f_1(s)/s = 1$, $sf_2(s) > 0$ for $s \neq 0$, $|f_2(s)| \geqslant f_{20} > 0$ for $|s| \geqslant \zeta_0 > 0$, $\lim\limits_{|s|\to\infty} f_2(s)/s = 1$.

We consider the limiting system of (3.1) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \nu_1(t)-p_0\nu_1(t-\tau) \\ \nu_2(t)-p_0\nu_2(t-\tau) \end{bmatrix} + \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix}\begin{bmatrix} \nu_1(t-\alpha) \\ \nu_2(t-\beta) \end{bmatrix} = 0, \tag{3.2}$$

whose characteristic equation is given by

$$\begin{aligned} E(\zeta) = \zeta^2(1-p_0\mathrm{e}^{-\tau\zeta})^2 &+ \zeta(1-p_0\mathrm{e}^{-\tau\zeta})(a_0\mathrm{e}^{-\alpha\zeta}+d_0\mathrm{e}^{-\beta\zeta}) \\ &+ (a_0d_0-b_0c_0)\mathrm{e}^{-(\alpha+\beta)\zeta} \\ = 0.& \end{aligned} \tag{3.3}$$

By Theorem 2.1, (3.2) is oscillatory if and only if (3.3) has no real roots in $(0,\infty)$ provided $a_0d_0-b_0c_0 \geqslant 0$ and $b_0c_0 \neq 0$.

Lemma 3.1. *Let $(a_0+d_0)>(b_0+c_0)$. Assume that every vector solution of the system (3.2) oscillates. Then there exists an $\varepsilon_0>0$ such that for each ε with $0\leqslant\varepsilon\leqslant\varepsilon_0$, every vector solution of*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\nu_1(t)-(p_0+2\varepsilon)\nu_1(t-\tau)\\ \nu_2(t)-(p_0+2\varepsilon)\nu_2(t-\tau)\end{bmatrix}+\begin{bmatrix}(a_0-\varepsilon) & (b_0-\varepsilon)\\ (c_0-\varepsilon) & (d_0-\varepsilon)\end{bmatrix}\begin{bmatrix}\nu_1(t-\alpha)\\ \nu_2(t-\beta)\end{bmatrix}=0 \tag{3.4}$$

oscillates.

P r o o f. Since (3.2) is oscillatory, then (3.3) has no real roots in $(0,\infty)$ when $a_0d_0\geqslant b_0c_0$. Let $0<\varepsilon^*<\min\{a_0,b_0,c_0,d_0\}$. For $\zeta\in\mathbb{R}$, set

$$\begin{aligned}M^*(\zeta)=\varepsilon^*(&4\zeta^2\mathrm{e}^{-\tau\zeta}(1-p_0\mathrm{e}^{-\tau\zeta})+2\zeta\mathrm{e}^{-\tau\zeta}(a_0\mathrm{e}^{-\alpha\zeta}+d_0\mathrm{e}^{-\beta\zeta})\\ &+\zeta(\mathrm{e}^{-\alpha\zeta}+\mathrm{e}^{-\beta\zeta})(1-p_0\mathrm{e}^{-\tau\zeta})+\mathrm{e}^{-(\alpha+\beta)\zeta}((a_0+d_0)-(b_0+c_0))).\end{aligned}$$

Therefore,

$$\begin{aligned}E(\zeta)-M^*(\zeta)=\;&\zeta^2(1-p_0\mathrm{e}^{-\tau\zeta})(1-(p_0+4\varepsilon^*)\mathrm{e}^{-\tau\zeta})\\ &+\mathrm{e}^{-(\alpha+\beta)\zeta}(a_0d_0-b_0c_0-\varepsilon^*((a_0+d_0)-(b_0+c_0)))\\ &-2\varepsilon^*\zeta\mathrm{e}^{-\tau\zeta}(a_0\mathrm{e}^{-\alpha\zeta}+d_0\mathrm{e}^{-\beta\zeta})\\ &+\zeta(1-p_0\mathrm{e}^{-\tau\zeta})(\mathrm{e}^{-\alpha\zeta}(a_0-\varepsilon^*)+\mathrm{e}^{-\beta\zeta}(d_0-\varepsilon^*))\end{aligned}$$

implies that $E(\zeta)-M^*(\zeta)\to\infty$ as $\zeta\to\infty$. Hence, it is possible to find a $\zeta_0>0$ such that $\zeta\geqslant\zeta_0$ implies that $E(\zeta)-M^*(\zeta)>\frac12m$, where $m=\inf\limits_{\zeta\geqslant0}E(\zeta)>0$. Let

$$\varepsilon_0=\min\Big\{\varepsilon^*,\frac{\frac12m}{(2\zeta_0+1)(2\zeta_0+a_0+d_0)}\Big\}$$

and $0<\varepsilon\leqslant\varepsilon_0$. Now, the characteristic equation of (3.4) is given by

$$\begin{aligned}M(\zeta)=\;&\zeta^2(1-(p_0+2\varepsilon)\mathrm{e}^{-\tau\zeta})^2\\ &+\zeta(1-(p_0+2\varepsilon)\mathrm{e}^{-\tau\zeta})((a_0-\varepsilon)\mathrm{e}^{-\alpha\zeta}+(d_0-\varepsilon)\mathrm{e}^{-\beta\zeta})\\ &+((a_0-\varepsilon)(d_0-\varepsilon)-(b_0-\varepsilon)(c_0-\varepsilon))\mathrm{e}^{-(\alpha+\beta)\zeta}.\end{aligned} \tag{3.5}$$

To complete the proof of the lemma, we are required to show that (3.5) has no real roots on $(0,\infty)$. If $\zeta\geqslant\zeta_0$, then

$$\begin{aligned}M(\zeta)=E(\zeta)-\varepsilon(&4\zeta^2\mathrm{e}^{-\tau\zeta}(1-(p_0+\varepsilon)\mathrm{e}^{-\tau\zeta})+2\zeta\mathrm{e}^{-\tau\zeta}(a_0\mathrm{e}^{-\alpha\zeta}+d_0\mathrm{e}^{-\beta\zeta})\\ &+\zeta(\mathrm{e}^{-\alpha\zeta}+\mathrm{e}^{-\beta\zeta})(1-(p_0+2\varepsilon)\mathrm{e}^{-\tau\zeta})+\mathrm{e}^{-(\alpha+\beta)\zeta}((a_0+d_0)-(b_0+c_0)))\\ \geqslant E(\zeta)-\varepsilon^*(&4\zeta^2\mathrm{e}^{-\tau\zeta}(1-p_0\mathrm{e}^{-\tau\zeta})+2\zeta\mathrm{e}^{-\tau\zeta}(a_0\mathrm{e}^{-\alpha\zeta}+d_0\mathrm{e}^{-\beta\zeta})\\ &+\zeta(\mathrm{e}^{-\alpha\zeta}+\mathrm{e}^{-\beta\zeta})(1-p_0\mathrm{e}^{-\tau\zeta})+\mathrm{e}^{-(\alpha+\beta)\zeta}((a_0+d_0)-(b_0+c_0)))\\ \geqslant E(\zeta)-M^*(&\zeta)>\frac{m}{2}.\end{aligned}$$

If $\zeta \in (0, \zeta_0)$, then

$$\begin{aligned}
M(\zeta) &= E(\zeta) - \varepsilon(4\zeta^2 \mathrm{e}^{-\tau\zeta}(1 - (p_0 + \varepsilon)\mathrm{e}^{-\tau\zeta}) + 2\zeta \mathrm{e}^{-\tau\zeta}(a_0 \mathrm{e}^{-\alpha\zeta} + d_0 \mathrm{e}^{-\beta\zeta}) \\
&\quad + \zeta(\mathrm{e}^{-\alpha\zeta} + \mathrm{e}^{-\beta\zeta})(1 - (p_0 + 2\varepsilon)\mathrm{e}^{-\tau\zeta}) + \mathrm{e}^{-(\alpha+\beta)\zeta}((a_0 + d_0) - (b_0 + c_0))) \\
&\geqslant E(\zeta) - \varepsilon(4\zeta^2 \mathrm{e}^{-\tau\zeta} + 2\zeta \mathrm{e}^{-\tau\zeta}(a_0 \mathrm{e}^{-\alpha\zeta} + d_0 \mathrm{e}^{-\beta\zeta}) + \zeta(\mathrm{e}^{-\alpha\zeta} + \mathrm{e}^{-\beta\zeta}) \\
&\quad + \mathrm{e}^{-(\alpha+\beta)\zeta}((a_0 + d_0) - (b_0 + c_0))) \\
&\geqslant E(\zeta) - \varepsilon(4\zeta^2 + 2\zeta(a_0 + d_0) + 2\zeta + (a_0 + d_0)) \\
&\geqslant E(\zeta) - \varepsilon(2\zeta(2\zeta + 1) + (2\zeta + 1)(a_0 + d_0)) \\
&\geqslant E(\zeta) - \varepsilon_0((2\zeta_0 + 1)(2\zeta_0 + a_0 + d_0)) \geqslant \frac{m}{2}.
\end{aligned}$$

Hence, $M(\zeta)$ has no real roots in $(0, \infty)$. This completes the proof. $\square$

Proceeding as in Lemma 3.1, we can prove the following result.

Lemma 3.2. *Let* $(a_0 + d_0) > (b_0 + c_0)$. *Assume that every vector solution of the system* (3.2) *is oscillatory. Then there exists an* $\varepsilon_0 > 0$ *such that for each* ε *with* $0 \leqslant \varepsilon \leqslant \varepsilon_0$, *every solution of*

$$\begin{aligned}
\text{(3.6)} \qquad & \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \nu_1(t) - (p_0 + 2\varepsilon)\nu_1(t - \tau) \\ \nu_2(t) - (p_0 + 2\varepsilon)\nu_2(t - \tau) \end{bmatrix} \\
& \quad + \begin{bmatrix} (a_0 - \varepsilon) & (b_0 - \varepsilon)(1 - \varepsilon_1) \\ (c_0 - \varepsilon)(1 - \varepsilon_1) & (d_0 - \varepsilon) \end{bmatrix} \begin{bmatrix} \nu_1(t - \alpha) \\ \nu_2(t - \beta) \end{bmatrix} = 0
\end{aligned}$$

oscillates, where $0 < \varepsilon_1 < 1$.

Theorem 3.1. *Let* (H_1)–(H_4) *hold and* $(a_0 + d_0) > (b_0 + c_0)$. *If* (3.2) *is oscillatory, then every vector solution of* (3.1) *oscillates.*

P r o o f. Let $[u(t), v(t)]^\top$ be a nonoscillatory vector solution of (3.1). We have the following possibilities:

Case 1: $u(t) > 0$, $u(t - \tau) > 0$, $u(t - \alpha) > 0$ and $v(t) > 0$, $v(t - \tau) > 0$, $v(t - \beta) > 0$ for $t \geqslant t_1$. Setting

$$\text{(3.7)} \qquad \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} u(t) - p(t)h_1(u(t - \tau)) \\ v(t) - p(t)h_2(v(t - \tau)) \end{bmatrix}$$

for $t \geqslant t_2 > t_1 + \sigma$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix} \begin{bmatrix} f_1(u(t - \alpha)) \\ f_2(v(t - \beta)) \end{bmatrix} = 0$$

and it is equivalent to

$$z_1'(t) = -a(t)f_1(u(t-\alpha)) - b(t)f_2(v(t-\beta)) < 0, \tag{3.8}$$

$$z_2'(t) = -c(t)f_1(u(t-\alpha)) - d(t)f_2(v(t-\beta)) < 0. \tag{3.9}$$

Consequently, $z_1(t)$ and $z_2(t)$ are nonincreasing functions for $t \geqslant t_2$. Assume that $z_1(t), z_2(t) > 0$ for $t \geqslant t_3$. Integrating (3.8) and (3.9), we find

$$\int_{t_3}^{t} z_1'(s)\,\mathrm{d}s = \int_{t_3}^{t} (-a(s)f_1(u(s-\alpha)) - b(s)f_2(v(s-\beta)))\,\mathrm{d}s,$$
$$\int_{t_3}^{t} z_2'(s)\,\mathrm{d}s = \int_{t_3}^{t} (-c(s)f_1(u(s-\alpha)) - d(s)f_2(v(s-\beta)))\,\mathrm{d}s,$$

that is,

$$\int_{t_3}^{t} (a(s)f_1(u(s-\alpha)) + b(s)f_2(v(s-\beta)))\,\mathrm{d}s = z_1(t_3) - z_1(t) < z_1(t_3), \tag{3.10}$$

$$\int_{t_3}^{t} (c(s)f_1(u(s-\alpha)) + d(s)f_2(v(s-\beta)))\,\mathrm{d}s = z_2(t_3) - z_2(t) < z_2(t_3). \tag{3.11}$$

By (H_2) and (H_4), we can rewrite (3.10) and (3.11) as

$$\int_{t_3}^{t} (a(s)f_{10} + b(s)f_{20})\,\mathrm{d}s < z_1(t_3), \quad \int_{t_3}^{t} (c(s)f_{10} + d(s)f_{20})\,\mathrm{d}s < z_2(t_3),$$

which contradicts the fact, that

$$\int_{t_3}^{\infty} (a(s)f_{10} + b(s)f_{20})\,\mathrm{d}s = \infty \quad \text{and} \quad \int_{t_3}^{\infty} (c(s)f_{10} + d(s)f_{20})\,\mathrm{d}s = \infty.$$

So, $z_1(t), z_2(t) \leqslant 0$ for $t \geqslant t_3$. If we don't agree that $\lim\limits_{t\to\infty} z_1(t) = -\infty$ and $\lim\limits_{t\to\infty} z_2(t) = -\infty$, then we let

$$\lim_{t\to\infty} z_1(t) = \mu_1 \quad \text{and} \quad \lim_{t\to\infty} z_2(t) = \mu_2, \quad \mu_1, \mu_2 \in (-\infty, 0].$$

Again, on integration of (3.8) and (3.9), we find contradictions. Therefore, $\lim\limits_{t\to\infty} z_1(t) = -\infty$ and $\lim\limits_{t\to\infty} z_2(t) = -\infty$.

Next, we claim that $\liminf\limits_{t\to\infty} u(t) = \infty = \liminf\limits_{t\to\infty} v(t)$. If not, we suppose that $\liminf\limits_{t\to\infty} u(t) = \eta_1 \in (0,\infty)$ and $\liminf\limits_{t\to\infty} v(t) = \eta_2 \in (0,\infty)$. Hence, there exists a sequence $\{t_m\}$ such that $t_m \to \infty$ as $m \to \infty$ and $u(t_m) \to \eta_1$, $v(t_m) \to \eta_2$ as $m \to \infty$. From (3.7), we get

$$-z_1(t_m+\tau) < p(t_m+\tau)h_1(u(t_m)), \quad -z_2(t_m+\tau) < p(t_m+\tau)h_2(v(t_m))$$

for $t_m \geqslant t_4$. Taking the lim sup as $m \to \infty$, we have

$$\infty \leqslant \limsup_{m\to\infty}(p(t_m+\tau)h_1(u(t_m))) \leqslant \limsup_{m\to\infty} p(t_m+\tau) \lim_{m\to\infty} h_1(u(t_m)) \leqslant p_0 h_1(\eta_1) < \infty$$

and

$$\infty \leqslant \limsup_{m\to\infty}(p(t_m+\tau)h_2(v(t_m))) \leqslant p_0 h_2(\eta_2) < \infty,$$

which are contradiction. So, our claim holds and hence $\lim_{t\to\infty} u(t) = \infty$ and $\lim_{t\to\infty} v(t) = \infty$. If we set

$$\begin{aligned} P_1(t) &= \frac{p(t)h_1(u(t-\tau))}{u(t-\tau)}, \quad P_2(t) = \frac{p(t)h_2(v(t-\tau))}{v(t-\tau)}, \\ A(t) &= \frac{a(t)f_1(u(t-\alpha))}{u(t-\alpha)}, \quad B(t) = \frac{b(t)f_2(v(t-\beta))}{v(t-\beta)}, \\ C(t) &= \frac{c(t)f_1(u(t-\alpha))}{u(t-\alpha)}, \quad D(t) = \frac{d(t)f_2(v(t-\beta))}{v(t-\beta)} \end{aligned}$$

for $t \geqslant t_4$, then (3.1) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \begin{bmatrix} u(t-\alpha) \\ v(t-\beta) \end{bmatrix} = 0. \tag{3.12}$$

Using (H_1) and (H_3), we get

$$\begin{aligned} \limsup_{t\to\infty} P_1(t) &\leqslant \limsup_{t\to\infty} p(t) \lim_{t\to\infty} \frac{h_1(u(t-\tau))}{u(t-\tau)} = p_0, \\ \limsup_{t\to\infty} P_2(t) &\leqslant \limsup_{t\to\infty} p(t) \lim_{t\to\infty} \frac{h_2(v(t-\tau))}{v(t-\tau)} = p_0. \end{aligned}$$

Also, it is easy to see that

$$\limsup_{t\to\infty} P_1(t) \geqslant \limsup_{t\to\infty} p(t) = p_0, \quad \limsup_{t\to\infty} P_2(t) \geqslant \limsup_{t\to\infty} p(t) = p_0.$$

Therefore,

$$\limsup_{t\to\infty} P_1(t) = p_0, \quad \limsup_{t\to\infty} P_2(t) = p_0.$$

Integrating (3.12) from t_4 to t for $t > t_4$, we get

$$\begin{aligned} z_1(t) + \int_{t_4}^{t} (A(s)u(s-\alpha) + B(s)v(s-\beta))\,\mathrm{d}s = z_1(t_4) < 0, \\ z_2(t) + \int_{t_4}^{t} (C(s)u(s-\alpha) + D(s)v(s-\beta))\,\mathrm{d}s = z_2(t_4) < 0, \end{aligned}$$

that is,

$$u(t)-P_1(t)u(t-\tau)+\int_{t_4}^{t}(A(s)u(s-\alpha)+B(s)v(s-\beta))\,\mathrm{d}s<0,$$
$$v(t)-P_2(t)v(t-\tau)+\int_{t_4}^{t}(C(s)u(s-\alpha)+D(s)v(s-\beta))\,\mathrm{d}s<0,$$

which are equivalent to

$$u(t)>\frac{1}{P_1(t+\tau)}\Big(u(t+\tau)+\int_{t_4}^{t+\tau}(A(s)u(s-\alpha)+B(s)v(s-\beta))\,\mathrm{d}s\Big), \tag{3.13}$$
$$v(t)>\frac{1}{P_2(t+\tau)}\Big(v(t+\tau)+\int_{t_4}^{t+\tau}(C(s)u(s-\alpha)+D(s)v(s-\beta))\,\mathrm{d}s\Big). \tag{3.14}$$

Let $0<\varepsilon<\min\{a_0,b_0,c_0,d_0\}$ be given. We can find positive constants T_1, T_2, T_3, T_4, T_5, T_6 such that $P_1(t)<p_0+\varepsilon$ for $t\geqslant T_1$, $P_2(t)<p_0+\varepsilon$ for $t\geqslant T_2$, $A(t)>a_0-\varepsilon$ for $t\geqslant T_3$, $B(t)>b_0-\varepsilon$ for $t\geqslant T_4$, $C(t)>c_0-\varepsilon$ for $t\geqslant T_5$, $D(t)>d_0-\varepsilon$ for $t\geqslant T_6$. Let $T_0=\max\{t_4,T_1,T_2,T_3,T_4,T_5,T_6\}$ and choose $1<\gamma<1+\varepsilon/(p_0+\varepsilon)$. Then, $P_1(t),P_2(t)<(p_0+2\varepsilon)/\gamma$ for $t\geqslant T_0$. Hence, (3.13) and (3.14) reduce to

$$u(t)>\frac{\gamma}{p_0+2\varepsilon}\Big(u(t+\tau)+(a_0-\varepsilon)\int_{T_0}^{t+\tau}u(s-\alpha)\,\mathrm{d}s+(b_0-\varepsilon)\int_{T_0}^{t+\tau}v(s-\beta)\,\mathrm{d}s\Big), \tag{3.15}$$

$$v(t)>\frac{\gamma}{p_0+2\varepsilon}\Big(v(t+\tau)+(c_0-\varepsilon)\int_{T_0}^{t+\tau}u(s-\alpha)\,\mathrm{d}s+(d_0-\varepsilon)\int_{T_0}^{t+\tau}v(s-\beta)\,\mathrm{d}s\Big). \tag{3.16}$$

Let $X=BC([T_0-\varrho,\infty),\mathbb{R}^2)$ be the Banach space of all real valued bounded continuous functions on $I=[T_0-\varrho,\infty)$ defined by

$$X=\Big\{Y\colon\ I\to\mathbb{R}^2,\ \|Y\|=\sup_{t\in I}|Y|<\infty\Big\}.$$

For $Y=[x(t),y(t)]^{\top}$, we put

$$B=\{Y\in X\colon\ 0\leqslant x(t),\ y(t)\leqslant 1,\ t\geqslant T_0-\varrho \text{ but } Y(t)\neq 0 \text{ on any subinterval of } I\}.$$

Indeed, B is a closed, bounded and convex subset of X. Define $T\colon B\to X$ as $(TY)(t)=(TY)(T_0)$, $t\in[T_0-\varrho,T_0]$,

$$(TY)(t)=\begin{bmatrix}\dfrac{1}{(p_0+2\varepsilon)u(t)}\Big(u(t+\tau)x(t+\tau)+(a_0-\varepsilon)\displaystyle\int_{T_0}^{t+\tau}u(s-\alpha)x(s-\alpha)\,\mathrm{d}s\\ \qquad +(b_0-\varepsilon)\displaystyle\int_{T_0}^{t+\tau}v(s-\beta)y(s-\beta)\,\mathrm{d}s\Big)\\ \dfrac{1}{(p_0+2\varepsilon)v(t)}\Big(v(t+\tau)y(t+\tau)+(c_0-\varepsilon)\displaystyle\int_{T_0}^{t+\tau}u(s-\alpha)x(s-\alpha)\,\mathrm{d}s\\ \qquad +(d_0-\varepsilon)\displaystyle\int_{T_0}^{t+\tau}v(s-\beta)y(s-\beta)\,\mathrm{d}s\Big)\end{bmatrix}$$

for $t \geqslant T_0$. We notice that $(TY)(t) > 0$ for $t \geqslant T_0 - \varrho$,

$$\begin{aligned}
&(Tx)(t)\\
&\quad\leqslant \frac{1}{(p_0+2\varepsilon)u(t)}\left(u(t+\tau)+(a_0-\varepsilon)\int_{T_0}^{t+\tau}u(s-\alpha)\,\mathrm{d}s+(b_0-\varepsilon)\int_{T_0}^{t+\tau}v(s-\beta)\,\mathrm{d}s\right)\\
&\quad\leqslant \frac{1}{\gamma} < 1
\end{aligned}$$

and

$$\begin{aligned}
&(Ty)(t)\\
&\quad\leqslant \frac{1}{(p_0+2\varepsilon)v(t)}\left(v(t+\tau)+(c_0-\varepsilon)\int_{T_0}^{t+\tau}u(s-\alpha)\,\mathrm{d}s+(d_0-\varepsilon)\int_{T_0}^{t+\tau}v(s-\beta)\,\mathrm{d}s\right)\\
&\quad\leqslant \frac{1}{\gamma} < 1
\end{aligned}$$

implies that $T\colon\ B \to B$. For $x_1, x_2, y_1, y_2 \in B$,

$$\begin{aligned}
&|(Tx_1)(t)-(Tx_2)(t)|\\
&\qquad = \begin{cases} |(Tx_1)(T_0)-(Tx_2)(T_0)|, \quad t\in[T_0-\varrho,T_0],\\ \dfrac{1}{(p_0+2\varepsilon)u(t)}\Bigg|\Bigg(u(t+\tau)(x_1(t+\tau)-x_2(t+\tau))\\ \qquad +(a_0-\varepsilon)\displaystyle\int_{T_0}^{t+\tau}u(s-\alpha)(x_1(s-\alpha)-x_2(s-\alpha))\,\mathrm{d}s\Bigg)\Bigg|, \quad t\geqslant T_0, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&|(Ty_1)(t)-(Ty_2)(t)|\\
&\qquad = \begin{cases} |(Ty_1)(T_0)-(Ty_2)(T_0)|, \quad t\in[T_0-\varrho,T_0],\\ \dfrac{1}{(p_0+2\varepsilon)v(t)}\Bigg|\Bigg(v(t+\tau)(y_1(t+\tau)-y_2(t+\tau))\\ \qquad +(d_0-\varepsilon)\displaystyle\int_{T_0}^{t+\tau}v(s-\beta)(y_1(s-\beta)-y_2(s-\beta))\,\mathrm{d}s\Bigg)\Bigg|, \quad t\geqslant T_0. \end{cases}
\end{aligned}$$

Therefore,

$$\begin{aligned}
|(Tx_1)(t)-(Tx_2)(t)| &\leqslant \frac{\|x_1-x_2\|}{(p_0+2\varepsilon)u(t)}\left(u(t+\tau)+(a_0-\varepsilon)\int_{T_0}^{t+\tau}u(s-\alpha)\,\mathrm{d}s\right)\\
&\leqslant \frac{\|x_1-x_2\|}{\gamma}
\end{aligned}$$

and

$$\begin{aligned}
|(Ty_1)(t)-(Ty_2)(t)| &\leqslant \frac{\|y_1-y_2\|}{(p_0+2\varepsilon)v(t)}\left(v(t+\tau)+(d_0-\varepsilon)\int_{T_0}^{t+\tau}v(s-\beta)\,\mathrm{d}s\right)\\
&\leqslant \frac{\|y_1-y_2\|}{\gamma}
\end{aligned}$$

show that T is a contraction mapping. Hence, by Banach's contraction principle, T has a unique fixed point Y such that $(TY)(t) = Y(t)$, that is, $Y(t) = Y(T_0)$, $t \in [T_0 - \varrho, T_0]$ and

$$Y(t) = \begin{bmatrix} \dfrac{1}{(p_0+2\varepsilon)u(t)}\Big(u(t+\tau)x(t+\tau) + (a_0-\varepsilon)\displaystyle\int_{T_0}^{t+\tau} u(s-\alpha)x(s-\alpha)\,\mathrm{d}s \\ \qquad + (b_0-\varepsilon)\displaystyle\int_{T_0}^{t+\tau} v(s-\beta)y(s-\beta)\,\mathrm{d}s\Big) \\ \dfrac{1}{(p_0+2\varepsilon)v(t)}\Big(v(t+\tau)y(t+\tau) + (c_0-\varepsilon)\displaystyle\int_{T_0}^{t+\tau} u(s-\alpha)x(s-\alpha)\,\mathrm{d}s \\ \qquad + (d_0-\varepsilon)\displaystyle\int_{T_0}^{t+\tau} v(s-\beta)y(s-\beta)\,\mathrm{d}s\Big) \end{bmatrix}$$

for $t \geqslant T_0$. Setting

$$\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} x(t)u(t) \\ y(t)v(t) \end{bmatrix}, \quad t \geqslant T_0 - \varrho,$$

we find

$$\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{(p_0+2\varepsilon)}\Big(w_1(t+\tau) + (a_0-\varepsilon)\displaystyle\int_{T_0}^{t+\tau} w_1(s-\alpha)\,\mathrm{d}s \\ \qquad + (b_0-\varepsilon)\displaystyle\int_{T_0}^{t+\tau} w_2(s-\beta)\,\mathrm{d}s\Big) \\ \dfrac{1}{(p_0+2\varepsilon)}\Big(w_2(t+\tau) + (c_0-\varepsilon)\displaystyle\int_{T_0}^{t+\tau} w_1(s-\alpha)\,\mathrm{d}s \\ \qquad + (d_0-\varepsilon)\displaystyle\int_{T_0}^{t+\tau} w_2(s-\beta)\,\mathrm{d}s\Big) \end{bmatrix},$$

which is a nonoscillatory vector solution of (3.4), a contradiction to Lemma 3.1.

Case 2: $u(t) < 0$, $u(t-\tau) < 0$, $u(t-\alpha) < 0$ and $v(t) < 0$, $v(t-\tau) < 0$, $v(t-\beta) < 0$ for $t \geqslant t_1$. The proof is similar as in Case 1.

Case 3: $u(t) > 0$, $u(t-\tau) > 0$, $u(t-\alpha) > 0$ and $v(t) < 0$, $v(t-\tau) < 0$, $v(t-\beta) < 0$ for $t \geqslant t_1$. In this case we rewrite (3.1) as

(3.17)
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u(t) - p(t)h_1(u(t-\tau)) - \displaystyle\int_t^{\infty} b(s)f_2(v(s-\beta))\,\mathrm{d}s \\ v(t) - p(t)h_2(v(t-\tau)) - \displaystyle\int_t^{\infty} c(s)f_1(u(s-\alpha))\,\mathrm{d}s \end{bmatrix} = \begin{bmatrix} -a(t)f_1(u(t-\alpha)) \\ -d(t)f_2(v(t-\beta)) \end{bmatrix}.$$

By putting $r(t) = -v(t)$ in (3.17), we get

(3.18)
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u(t) - p(t)h_1(u(t-\tau)) + \displaystyle\int_t^{\infty} b(s)f_2(r(s-\beta))\,\mathrm{d}s \\ r(t) - p(t)h_2(r(t-\tau)) + \displaystyle\int_t^{\infty} c(s)f_1(u(s-\alpha))\,\mathrm{d}s \end{bmatrix} = \begin{bmatrix} -a(t)f_1(u(t-\alpha)) \\ -d(t)f_2(r(t-\beta)) \end{bmatrix}.$$

If

$$(3.19)\qquad \begin{bmatrix} z_1^*(t) \\ z_2^*(t) \end{bmatrix} = \begin{bmatrix} u(t) - p(t)h_1(u(t-\tau)) + \int_t^\infty b(s)f_2(r(s-\beta))\,\mathrm{d}s \\ r(t) - p(t)h_2(r(t-\tau)) + \int_t^\infty c(s)f_1(u(s-\alpha))\,\mathrm{d}s \end{bmatrix},$$

then we obtain

$$(3.20)\qquad z_1^{*\prime}(t) = -a(t)f_1(u(t-\alpha)) < 0,$$

$$(3.21)\qquad z_2^{*\prime}(t) = -d(t)f_2(r(t-\beta)) < 0.$$

Therefore, $z_1^*(t)$ and $z_2^*(t)$ are monotonic nonincreasing functions for $t \geqslant t_2$. Proceeding as in Case 1, we obtain $\lim\limits_{t\to\infty} z_1^*(t) = -\infty$ and $\lim\limits_{t\to\infty} z_2^*(t) = -\infty$ upon the ultimate choice of $z_1^*(t)$, $z_2^*(t) < 0$ for $t \geqslant t_3$. Now, we set

$$P_1(t) = \frac{p(t)h_1(u(t-\tau))}{u(t-\tau)}, \quad P_2(t) = \frac{p(t)h_2(r(t-\tau))}{r(t-\tau)},$$
$$A(t) = \frac{a(t)f_1(u(t-\alpha))}{u(t-\alpha)}, \quad D(t) = \frac{d(t)f_2(r(t-\beta))}{r(t-\beta)},$$
$$P_1^*(t) = \frac{\int_t^\infty b(s)f_2(r(s-\beta))\,\mathrm{d}s}{\int_t^\infty r(s-\beta)\,\mathrm{d}s}, \quad P_2^*(t) = \frac{\int_t^\infty c(s)f_1(u(s-\alpha))\,\mathrm{d}s}{\int_t^\infty u(s-\alpha)\,\mathrm{d}s}$$

for $t \geqslant t_4$. So, (3.1) becomes

$$(3.22)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1^*(t) \\ z_2^*(t) \end{bmatrix} + \begin{bmatrix} A(t)u(t-\alpha) \\ D(t)r(t-\beta) \end{bmatrix} = 0.$$

Using (H_1) and (H_3), we get

$$\limsup_{t\to\infty} P_1(t) \leqslant \limsup_{t\to\infty} p(t) \lim_{t\to\infty} \frac{h_1(u(t-\tau))}{u(t-\tau)} = p_0,$$
$$\limsup_{t\to\infty} P_2(t) \leqslant \limsup_{t\to\infty} p(t) \lim_{t\to\infty} \frac{h_2(r(t-\tau))}{r(t-\tau)} = p_0.$$

Therefore, we can find $t_4 > t_3$ such that

$$P_1(t), P_2(t) < \frac{(p_0 + 2\varepsilon)}{\gamma}$$

for $t \geqslant t_4$. Integrating (3.22) from t_5 to t ($> t_5 > t_4$), we obtain

$$\int_{t_5}^t z_1^{*\prime}(t) + \int_{t_5}^t A(s)u(s-\alpha)\,\mathrm{d}s = 0, \quad \int_{t_5}^t z_2^{*\prime}(t) + \int_{t_5}^t D(s)r(s-\beta)\,\mathrm{d}s = 0,$$

that is,

$$z_1^*(t) + \int_{t_5}^t A(s)u(s-\alpha)\,\mathrm{d}s < 0, \quad z_2^*(t) + \int_{t_5}^t D(s)r(s-\beta)\,\mathrm{d}s < 0.$$

Substituting for the values of $z_1^*(t)$ and $z_2^*(t)$, we get

$$\begin{aligned} &u(t) - P_1(t)u(t-\tau) + \int_t^\infty b(s)f_2(r(s-\beta))\,\mathrm{d}s + \int_{t_5}^t A(s)u(s-\alpha)\,\mathrm{d}s < 0,\\ &r(t) - P_2(t)r(t-\tau) + \int_t^\infty c(s)f_1(u(s-\alpha))\,\mathrm{d}s + \int_{t_5}^t D(s)r(s-\beta)\,\mathrm{d}s < 0, \end{aligned}$$

and equivalently

$$\text{(3.23)}\quad u(t) > \frac{1}{P_1(t+\tau)}\Big(u(t+\tau)+P_1^*(t+\tau)\int_{t+\tau}^\infty r(s-\beta)\,\mathrm{d}s+\int_{t_5}^{t+\tau} A(s)u(s-\alpha)\,\mathrm{d}s\Big),$$

$$\text{(3.24)}\quad r(t) > \frac{1}{P_2(t+\tau)}\Big(r(t+\tau)+P_2^*(t+\tau)\int_{t+\tau}^\infty u(s-\alpha)\,\mathrm{d}s+\int_{t_5}^{t+\tau} D(s)r(s-\beta)\,\mathrm{d}s\Big)$$

for $t \geqslant t_5$. Let $0 < \varepsilon < \min\{a_0, b_0, c_0, d_0\}$. We can find positive constants T_1, T_2, T_3, T_4, T_5, T_6 such that $P_1(t) < p_0+\varepsilon$ for $t \geqslant T_1$, $P_2(t) < p_0+\varepsilon$ for $t \geqslant T_2$, $A(t) > a_0-\varepsilon$ for $t \geqslant T_3$, $b(t) > b_0-\varepsilon$ for $t \geqslant T_4$, $c(t) > c_0-\varepsilon$ for $t \geqslant T_5$, $D(t) > d_0-\varepsilon$ for $t \geqslant T_6$. Clearly,

$$\begin{aligned} P_1^*(t) &= \frac{\int_t^\infty b(s)f_2(r(s-\beta))\,\mathrm{d}s}{\int_t^\infty r(s-\beta)\,\mathrm{d}s} \geqslant (b_0-\varepsilon)\frac{\int_t^\infty f_2(r(s-\beta))\,\mathrm{d}s}{\int_t^\infty r(s-\beta)\,\mathrm{d}s},\\ P_2^*(t) &= \frac{\int_t^\infty c(s)f_1(u(s-\alpha))\,\mathrm{d}s}{\int_t^\infty u(s-\alpha)\,\mathrm{d}s} \geqslant (c_0-\varepsilon)\frac{\int_t^\infty f_1(u(s-\alpha))\,\mathrm{d}s}{\int_t^\infty u(s-\alpha)\,\mathrm{d}s}. \end{aligned}$$

For $0 < \varepsilon_1 < 1$, there exists $T_7 > 0$ such that

$$\begin{aligned} P_1^*(t) &\geqslant (b_0-\varepsilon)(1-\varepsilon_1)\frac{\int_t^\infty r(s-\beta)\,\mathrm{d}s}{\int_t^\infty r(s-\beta)\,\mathrm{d}s},\\ P_2^*(t) &\geqslant (c_0-\varepsilon)(1-\varepsilon_1)\frac{\int_t^\infty u(s-\alpha)\,\mathrm{d}s}{\int_t^\infty u(s-\alpha)\,\mathrm{d}s} \end{aligned}$$

for $t \geqslant T_7$. Let $T_0 = \max\{t_5, T_1, T_2, T_3, T_4, T_5, T_6, T_7\}$. Therefore, (3.23) and (3.24) reduce to

(3.25)

$$u(t) > \frac{\gamma}{p_0+2\varepsilon}\Big(u(t+\tau)+(a_0-\varepsilon)\int_{T_0}^{t+\tau} u(s-\alpha)\,\mathrm{d}s+(b_0-\varepsilon)(1-\varepsilon_1)\int_{t+\tau}^\infty r(s-\beta)\,\mathrm{d}s\Big)$$

and

(3.26)

$$r(t) > \frac{\gamma}{p_0+2\varepsilon}\Big(r(t+\tau)+(d_0-\varepsilon)\int_{T_0}^{t+\tau} r(s-\beta)\,\mathrm{d}s+(c_0-\varepsilon)(1-\varepsilon_1)\int_{t+\tau}^\infty u(s-\alpha)\,\mathrm{d}s\Big).$$

Similar to Case 1, we define $T_1\colon B \to X$ as $(T_1Y)(t) = (T_1Y)(T_0)$, $t \in [T_0 - \varrho, T_0]$,

$$(T_1Y)(t) = \begin{bmatrix} \dfrac{1}{(p_0+2\varepsilon)u(t)}\Big(u(t+\tau)x(t+\tau) + (a_0-\varepsilon)\displaystyle\int_{T_0}^{t+\tau} u(s-\alpha)x(s-\alpha)\,\mathrm{d}s \\ \qquad +(b_0-\varepsilon)(1-\varepsilon_1)\displaystyle\int_{t+\tau}^{\infty} r(s-\beta)y(s-\beta)\,\mathrm{d}s\Big) \\ \dfrac{1}{(p_0+2\varepsilon)r(t)}\Big(r(t+\tau)y(t+\tau) + (d_0-\varepsilon)\displaystyle\int_{T_0}^{t+\tau} r(s-\beta)y(s-\beta)\,\mathrm{d}s \\ \qquad +(c_0-\varepsilon)(1-\varepsilon_1)\displaystyle\int_{t+\tau}^{\infty} u(s-\alpha)x(s-\alpha)\,\mathrm{d}s\Big) \end{bmatrix}$$

for $t \geqslant T_0$. The rest of the proof follows from Case 1.

Case 4: $u(t) < 0$, $u(t-\tau) < 0$, $u(t-\alpha) < 0$ and $v(t) > 0$, $v(t-\tau) > 0$, $v(t-\beta) > 0$ for $t \geqslant t_1$. The proof is similar as Case 3. This completes the proof of the theorem. $\square$

E x a m p l e 3.1. Consider a two-dimensional first order neutral delay differential system of the form

(3.27)
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u(t) - (1+1/t)h_1(u(t-2\pi)) \\ v(t) - (1+1/t)h_2(v(t-2\pi)) \end{bmatrix} + \begin{bmatrix} (2+1/t) & (2+1/t) \\ (7+1/t) & (7+1/t) \end{bmatrix}\begin{bmatrix} f_1(u(t-\frac{5}{2}\pi)) \\ f_2(v(t-4\pi)) \end{bmatrix} = 0,$$

where

$$h_1(u) = u(1+\mathrm{e}^{-|u|}), \quad f_1(u) = u(1+\mathrm{e}^{-|u|}),$$
$$h_2(v) = v(1+\mathrm{e}^{-|v|}), \quad f_2(v) = v(1+\mathrm{e}^{-|v|}).$$

Here,

$$\lim_{t\to\infty} p(t) = 1 \in [1,\infty), \quad \lim_{t\to\infty} a(t) = 2 \in (0,\infty), \quad \lim_{t\to\infty} b(t) = 2 \in (0,\infty),$$
$$\lim_{t\to\infty} c(t) = 7 \in (0,\infty), \quad \lim_{t\to\infty} d(t) = 7 \in (0,\infty).$$

Hence, the limiting system of (3.27) can be put in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u(t) - u(t-2\pi) \\ v(t) - v(t-2\pi) \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 7 & 7 \end{bmatrix}\begin{bmatrix} u(t-\frac{5}{2}\pi) \\ v(t-4\pi) \end{bmatrix} = 0 \tag{3.28}$$

which satisfies all conditions of Theorem 3.1. Since the limiting system is oscillatory, then so also is (3.27). In particular, $[u(t), v(t)]^\top = [\sin t, \cos t]^\top$ is one of such oscillatory vector solutions of the system (3.28).

R e m a r k 3.1. It would be interesting to establish linearized oscillation theory for the system (3.1) when $-\infty < p(t) < 1$ for all t.

A c k n o w l e d g e m e n t s. The authors are thankful to the referees for their valuable suggestions and comments for necessary corrections of this paper.

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