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ON SOLUTIONS OF A CERTAIN NONLINEAR DIFFERENTIAL-DIFFERENCE FUNCTIONAL EQUATION

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Abstract. We investigate all the possible finite order entire solutions of the Fermat-type differential-difference functional equation $(Af(z))^2 + R^2(z)(Bf^{(m)}(z+c) + Cf^{(n)}(z))^2 = Q(z)$, where $m, n \in \mathbb{N}$, $A, B, C \in \mathbb{C} \setminus \{0\}$ and $R(z)$, $Q(z)$ are nonzero polynomials. The results significantly improve some earlier findings, especially the results due to A. Banerjee and T. Biswas (2021). We also show that the equation does not have any non-entire meromorphic solution. We provide some examples to support the results.

Keywords: functional equation; differential-difference equation; Fermat-type equation; Nevanlinna theory

MSC 2020: 39B32, 34M05, 30D35

1. Introduction, definitions and results

Let $\mathscr{M}(\mathbb{C})$ (or $\mathscr{E}(\mathbb{C})$) denote the set of all meromorphic (or entire, respectively) functions on $\mathbb{C}$. We denote by $\mathscr{M}_T(\mathbb{C})$, $\mathscr{M}^{<\infty}(\mathbb{C})$ and $\mathscr{M}_T^{<\infty}(\mathbb{C})$ (or $\mathscr{E}_T(\mathbb{C})$, $\mathscr{E}^{<\infty}(\mathbb{C})$, $\mathscr{E}_T^{<\infty}(\mathbb{C})$) the set of all transcendental meromorphic functions, finite order meromorphic functions and finite order transcendental meromorphic functions (or transcendental entire functions, finite order entire functions and finite order transcendental entire functions, respectively) on $\mathbb{C}$. The Nevanlinna value distribution theory of meromorphic functions has been extensively applied to resolve growth, value distribution (see [7], [9], [27]), and solvability of meromorphic solutions of linear and nonlinear differential equations (see [8], [11], [24], [26]). Let f be a non-constant meromorphic function in the complex plane. We assume that the reader is familiar with the standard notations and results such as the proximity function $m(r, f)$,

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counting function $N(r,f)$, characteristic function $T(r,f)$, the first and second main theorems, lemma on the logarithmic derivatives etc. of the Nevanlinna theory (see [7], [9], [27]). Recall that a meromorphic function α is said to be a small function of f, if $T(r,\alpha)=S(r,f)$, where $S(r,f)$ is used to denote any quantity that satisfies $S(r,f)=o(T(r,f))$ as $r\to\infty$, possibly outside of a set of r of finite logarithmic measure. We denote by $\mathscr{S}(f)$ the set of all small functions of f. We denote the order of $f\in\mathscr{M}(\mathbb{C})$ by $\varrho(f)$ such that

$$\varrho(f)=\limsup_{r\to\infty}\frac{\log^+T(r,f)}{\log r},\quad \text{where } \log^+(x)=\max\{\log x,0\}.$$

Next we recall Hadamard's factorization theorem.

Let $f(z)$ be a meromorphic function with $\varrho(f)<\infty$. Let $P_0(z)$ and $P_\infty(z)$ be the canonical products formed with the zeros and poles of $f(z)$ in $\mathbb{C}\setminus\{0\}$, respectively. Let c_mz^m with $c_m\neq 0$ be the first non-vanishing term in the Laurent series of $f(z)$ near 0. Then there exists a polynomial $Q(z)$ with $\deg(Q)\leqslant\varrho(f)$ such that $f(z)=z^m\mathrm{e}^{Q(z)}P_0(z)/P_\infty(z)$.

The next definition is necessary for understanding this paper.

Definition A. Given a meromorphic function $f(z)$, $f(z+c)$ is called a shift of f and $\Delta_c(f)=f(z+c)-f(z)$ is called a difference operator of f, where $c\in\mathbb{C}\setminus\{0\}$.

A difference polynomial (or a differential-difference polynomial) in f is a finite sum of the difference products of f and its shifts (or of the products of f, the derivatives of f and of their shifts, respectively) with all the coefficients of these monomials being the small functions of f.

We now consider the Fermat-type functional equation

$$f^n(z)+g^n(z)=1,\quad \text{where } n\in\mathbb{N}. \tag{1.1}$$

We summarize the classical results for solutions of the equation (1.1) on $\mathbb{C}$ in the following propositions.

Proposition A.

(i) *The functional equation* (1.1) *with* $n=2$ *has the non-constant entire solutions* $f(z)=\cos(\eta(z))$ *and* $g(z)=\sin(\eta(z))$, *where* $\eta(z)$ *is any entire function. No other solutions exist (see* [6], [4]*).*

(ii) *For* $n\geqslant 3$, *there are no non-constant entire solutions of* (1.1) *on* $\mathbb{C}$ *(see* [5], [6], *and* [20]*).*

Proposition B.

(i) *The functional equation* (1.1) *with* $n = 2$ *has the non-constant meromorphic solutions* $f = 2\omega/(1+\omega^2)$ *and* $g = (1-\omega^2)/(1+\omega^2)$, *where* ω *is an arbitrary meromorphic function on* $\mathbb{C}$ *(see* [6]*).*

(ii) *The functional equation* (1.1) *with* $n = 3$ *has the non-constant meromorphic solutions* $f = (1 + \frac{1}{\sqrt{3}}\wp'(h))/(2\wp(h))$, $g = (1 - \frac{1}{\sqrt{3}}\wp'(h))/(2\wp(h))$, *where* $\wp(z)$ *denotes the Weierstrass elliptic* $\wp$*-function with periods* ω_1 *and* ω_2 *which is defined as*

$$\wp(z;\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{\mu,\nu;\mu^2+\nu^2\neq 0}\Big\{\frac{1}{(z+\mu\omega_1+\nu\omega_2)^2} - \frac{1}{(\mu\omega_1+\nu\omega_2)^2}\Big\},$$

even and satisfying, after appropriately choosen ω_1 *and* ω_2, $(\wp')^2 = 4\wp^3 - 1$ *(see* [1], [5]*).*

(iii) *For* $n \geqslant 4$, *there are no non-constant meromorphic solutions of (1.1) on* $\mathbb{C}$ *(see* [5], [6]*).*

In 1970, Yang (see [23]) investigated the solutions of the functional equation

$$a(z)f^n(z) + b(z)g^m(z) = 1, \quad \text{where } a(z) \in \mathscr{S}(f), b(z) \in \mathscr{S}(g) \text{ and } m, n \in \mathbb{N}, \tag{1.2}$$

and obtained the following result.

Theorem A. *Let* $m, n \in \mathbb{N}$ *satisfy* $1/m + 1/n < 1$. *Then there are no non-constant entire functions* $f(z)$ *and* $g(z)$ *satisfying* (1.2).

Let $L(f) = \sum\limits_{k=0}^{n} b_k f^{(k)}$ be a linear differential polynomial in f, where $n \in \mathbb{N}$, a, $b_0, b_1, \ldots, b_{n-1}$ are polynomials and $b_n \in \mathbb{C} \setminus \{0\}$. In 2004, Yang and Li (see [26]) obtained that the solution of the Fermat-type equation

$$f^2 + (L(f))^2 = a, \tag{1.3}$$

must have the form $f(z) = \frac{1}{2}(P(z)\mathrm{e}^{R(z)} + Q(z)\mathrm{e}^{-R(z)})$, where P, Q and R are polynomials with $PQ = a$.

Some meromorphic solutions of the functional equation (1.2) are found in [21], [28]. The study of finding the solutions of the functional equation (1.2) has taken on a new dimension with the replacement of $g(z)$ by the finite order difference function $f(z+c)$. The results due to Liu (see [12]) is the gateway in this direction. For the entire solution, Liu (see [12]) obtained the following results.

Theorem B ([12]). *Let f be a non-constant finite order entire solution of the nonlinear difference equation*

$$f^2(z) + f^2(z+c) = a^2(z), \tag{1.4}$$

then $f(z) = \frac{1}{2}(h_1(z) + h_2(z))$, where $h_1(z+c) = \mathrm{i}h_1(z)$ and $h_2(z+c) = -\mathrm{i}h_2(z)$, $h_1(z)h_2(z) = a^2(z)$, where $a(z)$ is a non-vanishing small function of $f(z)$ with period c.

Theorem C ([12]). *There is no non-constant finite order entire solution of the nonlinear difference equation*

$$f^2(z) + (\Delta_c f)^2 = a^2,$$

where a is a nonzero constant.

In 2012, Liu et al. (see [14]) obtained some results in this direction. For a result in [14], they considered the functional difference equation

$$f^n(z) + f^m(z+c) = 1 \tag{1.5}$$

and proved that if $n \geqslant m$, then there are no non-constant polynomial solutions of (1.5). Clearly from Theorem A, if $n > m > 1$ or $n = m > 2$, then there is no non-constant entire solution of (1.5). Moreover, if $n > m$, then there is no transcendental entire solution of finite order (see [18], Theorem 1.4). Some related results can also be found in [25]. When $m = n = 1$, it is clear that $f(z)$ is a periodic function with period $2c$. Thus, the general entire solution is $f(z) = \frac{1}{2} + \mathrm{e}^{\pi z\mathrm{i}/c}h(z)$, where $h(z)$ is any periodic entire function with period c. Hence $h(z)$ can be written as $h(z) = g(\mathrm{e}^{2\pi\mathrm{i}z/c})$, where $g(z)$ is an entire function in $\mathbb{C}\setminus\{0\}$. If $h(z)$ is a polynomial in $\mathrm{e}^{2\pi\mathrm{i}z/c}$ or $\mathrm{e}^{-2\pi\mathrm{i}z/c}$, then there are many solutions with $\varrho(f) = 1$.

Remark A. Clearly the case $m = n = 2$ can be treated when the functions with finite order in (1.5) have some special relationship, i.e., when $m = n = 2$, the problem is still open. This was the starting point of a new era for studying the solutions, mainly the entire solutions of the functional equation like (1.5). As a result, successively several results have been found (see [3], [10], [12]–[14], [19], [21], [22], [26], [28]).

In 2013, Liu and Yang (see [16]) considered the Fermat-type equation

$$f^2(z) + P^2(z)(\Delta_c(f))^2 = Q(z), \tag{1.6}$$

where $P(z)$ ($\not\equiv 0$), $Q(z)$ ($\not\equiv 0$) are polynomials, and obtained the following result.

Theorem D ([16]). *There is no transcendental entire solution with finite order of* (1.6).

In 2015, Liu and Dong (see [15]) considered the Fermat-type equation

$$C^2f^2(z)+(Af(z+c)+Bf(z))^2=1 \tag{1.7}$$

and obtained the following result.

Theorem E ([15]). *Let $A,B\in\mathbb{C}$. Then there are $f\in\mathscr{E}_T^{<\infty}$ satisfying the nonlinear difference equation* (1.7) *with $A^2=B^2+C^2$.*

Also, Liu and Dong in [15] discussed that the necessary condition of existence of $f\in\mathscr{E}_T^{<\infty}$ satisfying the Fermat-type equation $f^2(z)+(Af^{(m)}(z+c)+Bf^{(n)}(z))^2=1$ is: $m,n\in\mathbb{N}$ are odd and $m+n$ is even.

In 2021, Banerjee and Biswas (see [2]) considered the nonlinear delay-differential functional equations

$$f^2(z)+R^2(z)(\Delta_c(f^{(k)}))^2=Q(z) \tag{1.8}$$

and

$$f^2(z)+R^2(z)(Af^{(m)}(z+c)+Bf^{(n)}(z))^2=1, \tag{1.9}$$

where $m,n\in\mathbb{N}$, $A,B\in\mathbb{C}\setminus\{0\}$, $R(z),Q(z)$ are nonzero polynomials, and obtained the following results.

Theorem F. *Those $f\in\mathscr{E}_T^{<\infty}$ satisfying the functional equation* (1.8) *must have the form $f(z)=\frac{1}{2}(Q_1(z)\mathrm{e}^{az+b}+Q_2(z)\mathrm{e}^{-az-b})$, where $a\,(\neq 0),b\in\mathbb{C}$ such that $Q_1(z)Q_2(z)=Q(z)$, with $Q_1(z)$ and $Q_2(z)$ being nonzero polynomials. Moreover, one of the following conclusions holds:*

(I) *if $\mathrm{e}^{ac}\neq 1$, then k must be odd, $Q(z)$ and $R(z)$ reduces to constants satisfying $2\mathrm{i}Ra^k+1=0$;*

(II) *if $\mathrm{e}^{ac}=1$, then $\deg(R(z))=1$, none of $Q_1(z),Q_2(z)$ are constants with $R(z)=Q_1(z)/P(Q_1(z))=Q_2(z)/((-1)^{k-l+1}P(Q_2(z)))$, where $P(x)=\mathrm{i}\sum\limits_{l=0}^{k}\binom{k}{l}a^{k-l}\times(x^{(l)}(z+c)-x^{(l)}(z))$.*

Theorem G. *If $f\in\mathscr{E}_T^{<\infty}$ satisfies the functional equation* (1.9), *then $R(z)\equiv R\in\mathbb{C}\setminus\{0\}$ and f must have the form $f(z)=\frac{1}{2}(\mathrm{e}^{az+b}+\mathrm{e}^{-az-b})$ with the following possibilities:*

(I) *when m,n are even, then $a^{m-n}\neq\pm B/A$, $R^2=(a^{2m}A^2-a^{2n}B^2)^{-1}$ and $\mathrm{e}^{ac}=(-a^nB\pm\sqrt{(a^nB)^2-(a^mA)^2})/(a^mA)\notin\{\pm1,-((a^mA)/(a^nB))^{\pm1}\}$;*

(II) *when m,n are odd, then $\mathrm{e}^{ac}=\pm1$, $a^{m-n}\neq\pm B/A$ and $R=-\mathrm{i}(a^nB\pm a^mA)^{-1}$;*

(III) *when m is even, n is odd, then* $\mathrm{e}^{ac} = \pm\mathrm{i}$, $a^{m-n} \neq \pm\mathrm{i}B/A$ *and* $R = -\mathrm{i}(a^n B \pm \mathrm{i}a^m A)^{-1}$;

(IV) *when m is odd, n is even, then* $a^{m-n} \neq \pm\mathrm{i}B/A$, $R^2 = -(a^{2n}B^2 + a^{2m}A^2)^{-1}$ *and* $\mathrm{e}^{ac} = (-a^n B \pm \sqrt{(a^n B)^2 + (a^m A)^2})/(a^m A) \notin \{\pm 1, (a^m A)/(a^n B), -(a^n B)/(a^m A)\}$.

Motivated by the above results, especially [2], [12], [16], in this paper we consider the delay-differential functional equation

$$(Af(z))^2 + R^2(z)(Bf^{(m)}(z+c) + Cf^{(n)}(z))^2 = Q(z) \tag{1.10}$$

to investigate all the possible meromorphic and finite order entire solutions, where $m, n \in \mathbb{N}$, $A, B, C \in \mathbb{C} \setminus \{0\}$ and $R(z)$, $Q(z)$ are nonzero polynomials.

2. Main results on entire solutions

When $m = n$ in (1.10), we obtain the following results.

Theorem 2.1. *Let $f \in \mathscr{E}_T^{<\infty}$ satisfy* (1.10) *with $B = \pm C$. Then $f(z)$ is of the form*

$$f(z) = \frac{Q_1(z)\mathrm{e}^{az+b} + Q_2(z)\mathrm{e}^{-az-b}}{2A}, \tag{2.1}$$

where $a\,(\neq 0), b \in \mathbb{C}$ and $Q_1(z)Q_2(z) = Q(z)$ with $Q_1(z)$ and $Q_2(z)$ being nonzero polynomials. Moreover, the solutions exist for any of the following possibilities:

(I) *if m is odd and* $\deg(R(z)) > 0$, *then* $\deg(R(z)) = 1$, *both $Q_1(z)$, $Q_2(z)$ are non-constant polynomials and* $\mathrm{e}^{ac} = -C/B$. *Also*

$$\begin{aligned} R(z) &= \frac{AQ_1(z)}{\pm\mathrm{i}B\sum_{j=0}^{m}\binom{m}{j}a^{m-j}\{Q_1^{(j)}(z+c) - Q_1^{(j)}(z)\}} \\ &= \frac{-AQ_2(z)}{\pm\mathrm{i}B\sum_{j=0}^{m}\binom{m}{j}(-a)^{m-j}\{Q_2^{(j)}(z+c) - Q_2^{(j)}(z)\}}; \end{aligned}$$

(II) *if m is odd and* $\deg(R(z)) = 0$, *then* $\mathrm{e}^{ac} = C/B$, $A - \mathrm{i}Ra^m(B\mathrm{e}^{ac} + C) = 0$ *and* $Q_1, Q_2 \in \mathbb{C}$.

Theorem 2.2. *Let $f \in \mathscr{E}_T^{<\infty}$ satisfy* (1.10) *with $B \neq \pm C$. Then $f(z)$ is of the form* (2.1). *Moreover, the solutions exist for any of the following possibilities:*

(I) *if m is even, then* $\deg(R(z)) = 0$, $\mathrm{e}^{ac} = (-C \pm \sqrt{C^2 - B^2})/B$, $R = A/(\pm\mathrm{i}a^m \times \sqrt{C^2 - B^2})$ *and* $Q_1, Q_2 \in \mathbb{C}$;

(II) *if m is odd and* $\deg(R(z)) = 0$, *then* $A - 2\mathrm{i}CRa^m \neq 0$, $\mathrm{e}^{ac} = \pm 1$, $R = A/(\mathrm{i}a^m(C \pm B))$ *and* $Q_1, Q_2 \in \mathbb{C}$.

When $m \neq n$ in (1.10), we obtain the following results.

Theorem 2.3. *Let $f \in \mathscr{E}_T^{<\infty}$ satisfy (1.10) with $B = \pm C$. Then $f(z)$ is of the form (2.1). Moreover, the solutions exist for any of the following possibilities:*

(I) *if m, n are both even and $a^{m-n} \neq \pm 1$, then $\deg(R(z)) = 0$, $\mathrm{e}^{ac} = (-Ca^n \pm B\sqrt{a^{2n} - a^{2m}})/(Ba^m)$, $R = (\mp \mathrm{i}A)/(B\sqrt{a^{2n} - a^{2m}})$ and $Q_1, Q_2 \in \mathbb{C}$;*

(II) *if m, n are both odd and also*

(II$_1$) *if $a^{m-n} \neq \pm 1$ and $\deg(R(z)) = 0$, then $A - 2\mathrm{i}CRa^n \neq 0$, $\mathrm{e}^{ac} = \pm 1$, $R = (-\mathrm{i}A)/(\pm Ba^m + Ca^n)$ and $Q_1, Q_2 \in \mathbb{C}$;*

(II$_2$) *if $a^{m-n} = \pm 1$ and $\deg(R(z)) > 0$, then $\deg(R(z)) = 1$, $\mathrm{e}^{ac} = \pm 1$ and both $Q_1(z)$, $Q_2(z)$ are non-constant polynomials. Also when $\mathrm{e}^{ac} = \pm 1$, then*

$$\begin{aligned} R(z) &= \frac{-\mathrm{i}AQ_1(z)}{\pm B\sum_{j=0}^{m}\binom{m}{j}a^{m-j}Q_1^{(j)}(z+c) + C\sum_{j=0}^{n}\binom{n}{j}a^{n-j}Q_1^{(j)}(z)} \\ &= \frac{\mathrm{i}AQ_2(z)}{\pm B\sum_{j=0}^{m}\binom{m}{j}(-a)^{m-j}Q_2^{(j)}(z+c) + C\sum_{j=0}^{n}\binom{n}{j}(-a)^{n-j}Q_2^{(j)}(z)} \end{aligned}$$

and if $B = C$ (or $B = -C$), then $a^{m-n} = \mp 1$ (or $a^{m-n} = \pm 1$, respectively);

(II$_3$) *if $a^{m-n} = \pm 1$ and $\deg(R(z)) = 0$, then $A - 2\mathrm{i}CRa^n = 0$, $\mathrm{e}^{ac} = \pm 1$, $R = (-\mathrm{i}A)/(\pm Ba^m + Ca^n)$ and $Q_1, Q_2 \in \mathbb{C}$;*

(III) *if m is even and n is odd and also*

(III$_1$) *if $a^{m-n} \neq \pm \mathrm{i}$ and $\deg(R(z)) = 0$, then $A - 2\mathrm{i}CRa^n \neq 0$, $\mathrm{e}^{ac} = \pm \mathrm{i}$, $R = (-\mathrm{i}A)/(\pm \mathrm{i}Ba^m + Ca^n)$ and $Q_1, Q_2 \in \mathbb{C}$;*

(III$_2$) *if $a^{m-n} = \pm \mathrm{i}$ and $\deg(R(z)) > 0$, then $\deg(R(z)) = 1$, $\mathrm{e}^{ac} = \pm \mathrm{i}$ and both $Q_1(z)$, $Q_2(z)$ are non-constant polynomials. Also when $\mathrm{e}^{ac} = \pm \mathrm{i}$, then*

$$\begin{aligned} R(z) &= \frac{-\mathrm{i}AQ_1(z)}{\pm \mathrm{i}B\sum_{j=0}^{m}\binom{m}{j}a^{m-j}Q_1^{(j)}(z+c) + C\sum_{j=0}^{n}\binom{n}{j}a^{n-j}Q_1^{(j)}(z)} \\ &= \frac{\mathrm{i}AQ_2(z)}{\pm \mathrm{i}B\sum_{j=0}^{m}\binom{m}{j}(-a)^{m-j}Q_2^{(j)}(z+c) + C\sum_{j=0}^{n}\binom{n}{j}(-a)^{n-j}Q_2^{(j)}(z)} \end{aligned}$$

and if $B = C$ (or $B = -C$), then $a^{m-n} = \pm \mathrm{i}$ (or $a^{m-n} = \mp \mathrm{i}$, respectively);

(III$_3$) *if $a^{m-n} = \pm \mathrm{i}$ and $\deg(R(z)) = 0$, then $A - 2\mathrm{i}CRa^n = 0$, $\mathrm{e}^{ac} = \pm \mathrm{i}$, $R = -\mathrm{i}A/(\pm \mathrm{i}Ba^m + Ca^n)$ and $Q_1, Q_2 \in \mathbb{C}$;*

(IV) *if m is odd, n is even and $a^{m-n} \neq \pm \mathrm{i}$, then $\deg(R(z)) = 0$, $\mathrm{e}^{ac} = (-Ca^n \pm B\sqrt{a^{2n} + a^{2m}})/(Ba^m)$, $R = \mp \mathrm{i}A/(B\sqrt{a^{2n} + a^{2m}})$ and $Q_1, Q_2 \in \mathbb{C}$.*

Theorem 2.4. *Let $f \in \mathscr{E}_T^{<\infty}$ satisfy* (1.10) *with $B \neq \pm C$. Then $f(z)$ is of the form* (2.1). *Moreover, the solutions exist for any of the following possibilities:*

(I) *if m, n are both even and $a^{m-n} \neq \pm C/B$, then $\deg(R(z)) = 0$, $\mathrm{e}^{ac} = (-Ca^n \pm \sqrt{C^2a^{2n} - B^2a^{2m}})/(Ba^m)$, $R = \mp \mathrm{i}A/(\sqrt{C^2a^{2n} - B^2a^{2m}})$ and $Q_1, Q_2 \in \mathbb{C}$;*

(II) *if m, n are both odd and also*

(II$_1$) *if $a^{m-n} \neq \pm C/B$ and $\deg(R(z)) = 0$, then $A - 2\mathrm{i}CRa^n \neq 0$, $\mathrm{e}^{ac} = \pm 1$, $R = -\mathrm{i}A/(\pm Ba^m + Ca^n)$ and $Q_1, Q_2 \in \mathbb{C}$;*

(II$_2$) *if $a^{m-n} = \pm C/B$ and $\deg(R(z)) > 0$, then $\deg(R(z)) = 1$, $\mathrm{e}^{ac} = \pm 1$ and both $Q_1(z)$, $Q_2(z)$ are non-constant polynomials. Also when $\mathrm{e}^{ac} = \pm 1$, then*

$$R(z) = \frac{-\mathrm{i}AQ_1(z)}{\pm B\sum_{j=0}^{m}\binom{m}{j}a^{m-j}Q_1^{(j)}(z+c) + C\sum_{j=0}^{n}\binom{n}{j}a^{n-j}Q_1^{(j)}(z)}$$
$$= \frac{\mathrm{i}AQ_2(z)}{\pm B\sum_{j=0}^{m}\binom{m}{j}(-a)^{m-j}Q_2^{(j)}(z+c) + C\sum_{j=0}^{n}\binom{n}{j}(-a)^{n-j}Q_2^{(j)}(z)}$$

and $a^{m-n} = \mp C/B$;

(II$_3$) *if $a^{m-n} = \pm C/B$ and $\deg(R(z)) = 0$, then $A - 2\mathrm{i}CRa^n = 0$, $\mathrm{e}^{ac} = \pm 1$, $R = -\mathrm{i}A/(\pm Ba^m + Ca^n)$ and $Q_1, Q_2 \in \mathbb{C}$;*

(III) *if m is even and n is odd and also*

(III$_1$) *if $a^{m-n} \neq \pm \mathrm{i}C/B$ and $\deg(R(z)) = 0$, then $A - 2\mathrm{i}CRa^n \neq 0$ and $\mathrm{e}^{ac} = \pm\mathrm{i}$, $R = -\mathrm{i}A/(\pm\mathrm{i}Ba^m + Ca^n)$ and $Q_1, Q_2 \in \mathbb{C}$;*

(III$_2$) *if $a^{m-n} = \pm\mathrm{i}C/B$ and $\deg(R(z)) > 0$, then $\deg(R(z)) = 1$, $\mathrm{e}^{ac} = \pm i$ and both $Q_1(z)$, $Q_2(z)$ are non-constant polynomials. Also when $\mathrm{e}^{ac} = \pm\mathrm{i}$, then*

$$R(z) = \frac{-\mathrm{i}AQ_1(z)}{\pm\mathrm{i}B\sum_{j=0}^{m}\binom{m}{j}a^{m-j}Q_1^{(j)}(z+c) + C\sum_{j=0}^{n}\binom{n}{j}a^{n-j}Q_1^{(j)}(z)}$$
$$= \frac{\mathrm{i}AQ_2(z)}{\pm\mathrm{i}B\sum_{j=0}^{m}\binom{m}{j}(-a)^{m-j}Q_2^{(j)}(z+c) + C\sum_{j=0}^{n}\binom{n}{j}(-a)^{n-j}Q_2^{(j)}(z)}$$

and $a^{m-n} = \pm\mathrm{i}C/B$;

(III$_3$) *if $a^{m-n} = \pm\mathrm{i}C/B$ and $\deg(R(z)) = 0$, then $A - 2\mathrm{i}CRa^n = 0$, $\mathrm{e}^{ac} = \pm\mathrm{i}$, $R = -\mathrm{i}A/(\pm\mathrm{i}Ba^m + Ca^n)$ and $Q_1, Q_2 \in \mathbb{C}$;*

(IV) *if m is odd, n is even and $a^{m-n} \neq \pm\mathrm{i}C/B$, then $\deg(R(z)) = 0$, $\mathrm{e}^{ac} = (-Ca^n \pm \sqrt{C^2a^{2n} + B^2a^{2m}})/(Ba^m)$, $R = \mp\mathrm{i}A/\sqrt{C^2a^{2n} + B^2a^{2m}}$ and $Q_1, Q_2 \in \mathbb{C}$.*

Remark 2.1. In particular, $A = B = 1$, $C = -1$ and $m = n = k \in \mathbb{N}$ in (1.10). Then (1.10) becomes (1.8). Also if we replace A, B, C by 1, A, B, respectively, and $Q(z) \equiv 1$, then (1.10) becomes (1.9). Clearly then the conclusions of Theorem 2.1

arise as the conclusions of Theorem F and the conclusions (I), (II_1), (III_1), (IV) of Theorem 2.4 arise as the conclusions of Theorem G. In this sense, Theorems 2.1, 2.4 are improvements of Theorem F and Theorem G.

Clearly, Theorems 2.1 and 2.2 improve Theorem F whereas Theorems 2.3 and 2.4 significantly improve Theorem G.

Remark 2.2. The key tools in the proof of the main results are (i) the Hadamard's factorization theorem and (ii) the core part of value distribution theory, namely the lemma given in the Lemma section.

The following examples related to Theorem 2.1 are reasonable.

Example 2.1. We consider the entire function $f(z)=\frac{1}{2\sqrt{3}}(z\mathrm{e}^{3z+7}+z\mathrm{e}^{-3z-7})$. Clearly it satisfies $(\sqrt{3}f(z))^2+\frac{1}{196}\pi^{-2}3z^2(7f'(z+c)-7f'(z))^2=z^2$. Note that $A=\sqrt{3}$, $B=7$, $C=-7$, $a=3$, $c=\frac{2}{3}\pi\mathrm{i}$ and $R(z)=-\frac{\sqrt{3}}{14}z\pi^{-1}$ in (1.10).

Example 2.2. We consider the entire function $f(z)=\frac{1}{2\sqrt{5}}(z+3)(\mathrm{e}^{5z+8}+\mathrm{e}^{-5z-8})$. Clearly it satisfies $(\sqrt{5}f(z))^2+\frac{1}{45}\pi^{-2}(z+3)^2(3f'(z+c)+3f'(z))^2=(z+3)^2$. Note that $A=\sqrt{5}$, $B=C=3$, $a=5$, $c=\pi\mathrm{i}$ and $R(z)=-\frac{1}{3\sqrt{5}}\pi^{-1}(z+3)$ in (1.10).

Example 2.3. We consider the entire function $f(z)=\frac{1}{\mathrm{i}2\sqrt{2}}(5\mathrm{e}^{(2\mathrm{i}z/3)+5}+\sqrt{7}\mathrm{e}^{-2\mathrm{i}z/3-5})$. Clearly it satisfies $(\mathrm{i}\sqrt{2}f(z))^2-\frac{9}{128}(4f'(z+c)+4f'(z))^2=5\sqrt{7}$ with $A=\mathrm{i}\sqrt{2}$, $B=C=4$, $a=\frac{2}{3}\mathrm{i}$, $c=3\pi$ and $R=-\frac{3}{8\sqrt{2}}\mathrm{i}$ in (1.10).

Example 2.4. Clearly $f(z)=\frac{1}{\mathrm{i}2\sqrt{2}}(5\mathrm{e}^{7z+1}+\sqrt{7}\mathrm{e}^{-7z-1})$ satisfies

$$(\pi f(z))^2-\frac{\pi^2}{392}(\sqrt{2}f'(z+c)-\sqrt{2}f'(z))^2=5\sqrt{7}$$

also with $A=\pi$, $B=\sqrt{2}$, $C=-\sqrt{2}$, $a=7$, $c=\pi\mathrm{i}$ and $R=\frac{1}{14\sqrt{2}}\pi\mathrm{i}$ in (1.10).

The following example related to Theorem 2.2 is reasonable.

Example 2.5. We consider the entire function $f(z)=\frac{1}{10}(\sqrt{5}\mathrm{e}^{az+b}+7\mathrm{e}^{-az-b})$. Clearly it satisfies $(5f(z))^2-\frac{25}{6}a^{-4}(\sqrt{2}f''(z+c)+2\sqrt{2}f''(z))^2=7\sqrt{5}$. Note that $A=\pi$, $B=\sqrt{2}$, $C=2\sqrt{2}$, $\mathrm{e}^{ac}=\sqrt{3}-2$ and $R=\frac{5}{\mathrm{i}\sqrt{6}}a^{-2}$ in (1.10).

The following example related to Theorem 2.3 is reasonable.

Example 2.6. We consider the entire function $f(z)=\frac{1}{2\sqrt{5}}(\pi\mathrm{e}^{az+b}+7\mathrm{e}^{-az-b})$. Clearly it satisfies $(\sqrt{5}f(z))^2-\frac{5}{9}a^{-4}(1-a^4)^{-1}(3f^{(4)}(z+c)+3f^{(2)}(z))^2=7\pi$. Note that $A=\sqrt{5}$, $B=C=3$, $\mathrm{e}^{ac}=(-1+\sqrt{1-a^4})/a^2$, $a^4\neq 1$ and $R=-\mathrm{i}\sqrt{5}/(3a^2\sqrt{1-a^4})$ in (1.10).

The following examples related to Theorem 2.4 are reasonable.

E x a m p l e 2.7. We consider the entire function $f(z)=\frac{1}{2}(\sqrt{2}\mathrm{e}^{az+b}+5\mathrm{e}^{-az-b})/\pi$. Clearly it satisfies $(\pi f(z))^2-5a^{-4}(25a^4-9)^{-1}(3f''(z+c)+5f^{iv}(z))^2=5\sqrt{2}$. Note that $A=\pi$, $B=3$, $C=5$, $\mathrm{e}^{ac}=\frac{1}{3}(-5a^2+\sqrt{25a^4-9})$, $a^2\neq\pm\frac{3}{5}$ and $R=-\mathrm{i}\pi a^{-2}/\sqrt{25a^4-9}$ in (1.10).

E x a m p l e 2.8. We consider the entire function $f(z)=\frac{1}{12}(2\mathrm{e}^{az+b}+\pi\mathrm{e}^{-az-b})$. Clearly it satisfies $(6f(z))^2-36a^{-4}(7f'''(z+c)+3f''(z))^2/(9+49a^2)=2\pi$. Note that $A=6$, $B=7$, $C=3$, $\mathrm{e}^{ac}=(-3-\sqrt{9+49a^2})/(7a)$, $a\neq\pm\mathrm{i}\frac{3}{7}$ and $R=6\mathrm{i}a^{-2}/\sqrt{9+49a^2}$ in (1.10).

3. Lemma section

We apply the following lemma in the proof of the main results of this paper.

Lemma 3.1 ([27], Corollary on page 77). *Suppose $f_j(z)$ $(j=1,2,\ldots,n+1)$ and g_k $(k=1,2,\ldots,n)$ $(n\geqslant 1)$ are entire functions satisfying the following conditions:*

(i) $\sum\limits_{j=1}^{n} f_j(z)\mathrm{e}^{g_j(z)}\equiv f_{n+1}(z)$.

(ii) *The order of $f_j(z)$ is less than the order of $\mathrm{e}^{g_k(z)}$ for $1\leqslant j\leqslant n+1$, $1\leqslant k\leqslant n$ and, furthermore, the order of $f_j(z)$ is less than the order of $\mathrm{e}^{g_h(z)-g_k(z)}$ for $n\geqslant 2$ and $1\leqslant j\leqslant n+1$, $1\leqslant h<k\leqslant n$.*

Then $f_j(z)\equiv 0$ for $1\leqslant j\leqslant n+1$.

4. Proof of the main results on entire solutions

P r o o f of Theorem 2.1. Let $f\in\mathscr{E}_T^{<\infty}$ satisfy (1.10). Then (1.10) can be expressed as

$$\begin{aligned}(4.1)\qquad &(Af(z)+\mathrm{i}R(z)(Bf^{(m)}(z+c)+Cf^{(m)}(z)))\\ &\quad\times(Af(z)-\mathrm{i}R(z)(Bf^{(m)}(z+c)+Cf^{(m)}(z)))=Q(z).\end{aligned}$$

Clearly, $Af(z)\pm\mathrm{i}R(z)(Bf^{(m)}(z+c)+Cf^{(m)}(z))\in\mathscr{E}_T^{<\infty}$ for $f\in\mathscr{E}_T^{<\infty}$ and has finitely many zeros in view of (4.1). So by the Hadamard factorization theorem, we have

$$Af(z)+\mathrm{i}R(z)(Bf^{(m)}(z+c)+Cf^{(m)}(z))=Q_1(z)\mathrm{e}^{P(z)}$$

and

$$Af(z)-\mathrm{i}R(z)(Bf^{(m)}(z+c)+Cf^{(m)}(z))=Q_2(z)\mathrm{e}^{-P(z)},$$

where $Q_1(z)$, $Q_2(z)$ are nonzero polynomials such that $Q_1(z)Q_2(z) = Q(z)$ and $P(z)$ is a non-constant polynomial. Thus, we have

$$f(z) = \frac{Q_1(z)\mathrm{e}^{P(z)} + Q_2(z)\mathrm{e}^{-P(z)}}{2A} \tag{4.2}$$

and

$$Bf^{(m)}(z+c) + Cf^{(m)}(z) = \frac{Q_1(z)\mathrm{e}^{P(z)} - Q_2(z)\mathrm{e}^{-P(z)}}{2\mathrm{i}R(z)}.$$

Differentiating on both sides of the first equation in (4.2), we deduce that

$$f^{(m)}(z) = \frac{p_1(z)\mathrm{e}^{P(z)} + p_2(z)\mathrm{e}^{-P(z)}}{2A}, \tag{4.3}$$

where

$$\begin{aligned} p_1(z) = {} & Q_1(z)((P'(z))^m + M_{1,m}(P'(z), P''(z), \ldots, P^{(m)}(z))) \\ & + Q_1'(z)M_{2,m-1}(P'(z), P''(z), \ldots, P^{(m-1)}(z)) \\ & + \ldots + Q_1^{(m-1)}(z)M_{m,1}(P'(z)) + Q_1^{(m)}(z), \end{aligned} \tag{4.4}$$

$$\begin{aligned} p_2(z) = {} & Q_2(z)((-1)^m(P'(z))^m + N_{1,m}(P'(z), P''(z), \ldots, P^{(m)}(z))) \\ & + (-1)^{m-1}Q_2'(z)N_{2,m-1}(P'(z), P''(z), \ldots, P^{(m-1)}(z)) \\ & + \ldots + Q_2^{(m-1)}(z)N_{m,1}(P'(z)) + Q_2^{(m)}(z), \end{aligned} \tag{4.5}$$

and $M_{j,m-j+1}$ (or $N_{j,m-j+1}$, respectively) are differential polynomials of $P'(z)$ of degree $m-1$ for $j = 1, 2$ and of degree $m-j+1$ for $j = 3, 4, \ldots, m$. From (4.2) and (4.3), we have

$$\begin{aligned} & \Big(Bp_1(z+c)\mathrm{e}^{\Delta_c P(z)} + Cp_1(z) - \frac{AQ_1(z)}{\mathrm{i}R(z)}\Big)\mathrm{e}^{P(z)} \\ & \quad + \Big(Bp_2(z+c)\mathrm{e}^{-\Delta_c P(z)} + Cp_2(z) + \frac{AQ_2(z)}{\mathrm{i}R(z)}\Big)\mathrm{e}^{-P(z)} \equiv 0. \end{aligned} \tag{4.6}$$

Clearly, $T(r, \mathrm{e}^{\Delta_c P(z)}) = S(r, \mathrm{e}^{P(z)})$. Hence by Lemma 3.1, we get from (4.6) that

$$Bp_1(z+c)\mathrm{e}^{\Delta_c P(z)} + Cp_1(z) - \frac{AQ_1(z)}{\mathrm{i}R(z)} \equiv 0 \tag{4.7}$$

and

$$Bp_2(z+c)\mathrm{e}^{-\Delta_c P(z)} + Cp_2(z) + \frac{AQ_2(z)}{\mathrm{i}R(z)} \equiv 0.$$

We claim that $\deg(P(z)) = 1$. If not, let $\deg(P(z)) \geqslant 2 \Rightarrow \deg(\Delta_c P(z)) \geqslant 1$. By Lemma 3.1, we have $p_1(z+c) \equiv 0$ and $p_2(z+c) \equiv 0$. So from (4.3), we see that f is a polynomial, which is not possible. Hence, $\deg(P(z)) = 1 \Rightarrow P(z) = az + b$, where $a\,(\neq 0), b \in \mathbb{C}$. From (4.2), we get

$$f(z) = \frac{Q_1(z)\mathrm{e}^{az+b} + Q_2(z)\mathrm{e}^{-az-b}}{2A}.$$

Also from (4.7), we deduce that

(4.8)

$$Bp_1(z+c)\mathrm{e}^{ac}+Cp_1(z)-\frac{AQ_1(z)}{\mathrm{i}R(z)}\equiv 0 \quad\text{and}\quad Bp_2(z+c)\mathrm{e}^{-ac}+Cp_2(z)+\frac{AQ_2(z)}{\mathrm{i}R(z)}\equiv 0,$$

where $p_1(z)=\sum\limits_{j=0}^{m}\binom{m}{j}a^{m-j}Q_1^{(j)}(z)$, $p_2(z)=\sum\limits_{j=0}^{m}\binom{m}{j}(-a)^{m-j}Q_2^{(j)}(z)$. Combining two equations in (4.8), we deduce that

$$\begin{aligned}(4.9)\qquad & B^2R^2(z)(p_1(z+c)p_2(z+c)-p_1(z)p_2(z))\\
&\equiv A^2Q(z)-\mathrm{i}ACR(z)(p_1(z)Q_2(z)-p_2(z)Q_1(z))\\
&\Rightarrow B^2R^2(z)\Big((-1)^m a^{2m}(Q(z+c)-Q(z))\\
&\quad+(-1)^{m-1}a^{2m-1}\binom{m}{1}(Q_1(z+c)Q_2'(z+c)-Q_1(z)Q_2'(z))\\
&\quad+(-1)^{m}a^{2m-1}\binom{m}{1}(Q_1'(z+c)Q_2(z+c)-Q_1'(z)Q_2(z))\\
&\quad+\ldots+(Q_1^{(m)}(z+c)Q_2^{(m)}(z+c)-Q_1^{(m)}(z)Q_2^{(m)}(z))\Big)\\
&\equiv A^2Q(z)-\mathrm{i}ACR(z)\Big((a^m-(-a)^m)Q(z)\\
&\quad+\binom{m}{1}(a^{m-1}Q_1'(z)Q_2(z)-(-a)^{m-1}Q_1(z)Q_2'(z))\\
&\quad+\ldots+(Q_1^{(m)}(z)Q_2(z)-Q_1(z)Q_2^{(m)}(z))\Big).\end{aligned}$$

Now, we investigate the following cases.

Case 1. Let $m\in\mathbb{N}$ be even. If $\deg(R(z))>0$, then by considering the degrees of $R(z)$, $Q_1(z)$ and $Q_2(z)$ from (4.9), we see that $2\deg(R(z))+\deg(Q(z))-1=\deg(R(z))+\deg(Q(z))-1\Rightarrow\deg(R(z))=0$, which is a contradiction.

Now if $\deg(R(z))=0$ and $m\in\mathbb{N}$ is even, then by considering the degrees of $Q_1(z)$ and $Q_2(z)$ from (4.9), again we see that $\deg(Q(z))-1=\deg(Q(z))$, which is absurd. Therefore, the solution of (1.10) does not exist in this case.

Case 2. Now let $m\in\mathbb{N}$ be odd. If $\deg(R(z))=0$ and $A-2\mathrm{i}CRa^m\neq 0$, then by considering the degrees of $Q_1(z)$ and $Q_2(z)$ from (4.9), again we see that $\deg(Q(z))-1=\deg(Q(z))$, which is absurd. Hence, $\deg(R(z))=0$ with odd $m\in\mathbb{N}$ and $A-2\mathrm{i}CRa^m=0$ are possible. From (4.8), we have

$$\begin{aligned}(4.10)\quad \mathrm{i}R\Big(&a^m(B\mathrm{e}^{ac}Q_1(z+c)+CQ_1(z))+a^{m-1}\binom{m}{1}(B\mathrm{e}^{ac}Q_1'(z+c)+CQ_1'(z))\\
&+\ldots+(B\mathrm{e}^{ac}Q_1^{(m)}(z+c)+CQ_1^{(m)}(z))\Big)\equiv AQ_1(z)\end{aligned}$$

and

(4.11)
$$\mathrm{i}R\biggl(-a^m(B\mathrm{e}^{-ac}Q_2(z+c)+CQ_2(z))+a^{m-1}\binom{m}{1}(B\mathrm{e}^{-ac}Q_2'(z+c)+CQ_2'(z))\\ +\ldots+(B\mathrm{e}^{-ac}Q_2^{(m)}(z+c)+CQ_2^{(m)}(z))\biggr)\equiv -AQ_2(z).$$

Comparing the highest power of z on both sides of (4.10) and (4.11), we get

$$\mathrm{i}Ra^m(B\mathrm{e}^{ac}+C)=A \quad\text{and}\quad \mathrm{i}Ra^m(B\mathrm{e}^{-ac}+C)=A\Rightarrow \mathrm{e}^{ac}=\frac{C}{B} \text{ and } \mathrm{e}^{-ac}=\frac{C}{B},$$

respectively. We now have the following cases to consider.

Sub-case 2.1. If $B=C$, then $\mathrm{e}^{ac}=1$, i.e., $ac=2l\pi\mathrm{i}$ ($l\in\mathbb{Z}\setminus\{0\}$). We claim that $\deg(Q_1(z))=0$. If not, let $\deg(Q_1(z))\geqslant 1$ and let $Q_1(z)\equiv\sum_{j=0}^{q_1}a_jz^j$, where $a_j\in\mathbb{C}$ with $a_{q_1}\neq 0$ and $q_1\geqslant 1$ ($0\leqslant j\leqslant q_1$).

Comparing the coefficient of z^{q_1-1} on both sides of (4.10), we get $m=-l\pi\mathrm{i}$, which leads to a contradiction with $m\in\mathbb{N}$, m odd. Similarly from (4.11) we get $\deg(Q_2(z))=0$.

Sub-case 2.2. If $B=-C$, then $\mathrm{e}^{ac}=-1$, i.e., $ac=(2l+1)\pi\mathrm{i}$ ($l\in\mathbb{Z}$). Similarly as in Case 1, we get $\deg(Q_1(z))=0$ and $\deg(Q_2(z))=0$.

Case 3. Let $m\in\mathbb{N}$ be odd. If $\deg(R(z))>0$, then by considering the degrees of $R(z)$, $Q_1(z)$ and $Q_2(z)$ from (4.9), we see that $2\deg(R(z))+\deg(Q(z))-1=\deg(R(z))+\deg(Q(z))\Rightarrow\deg(R(z))=1$. Let $R(z)=R_1z+R_2$, where $R_1\,(\neq 0)$, $R_2\in\mathbb{C}$. Clearly from (4.9), we get that $Q(z)$ is a non-constant polynomial. Now the following cases arise.

Sub-case 3.1. When $m\in\mathbb{N}$ is odd, $\deg(R(z))=1$, $Q_1(z)\equiv Q_1\in\mathbb{C}\setminus\{0\}$ and $Q_2(z)=\sum_{j=0}^{q_2}b_jz^j$, where $b_j\in\mathbb{C}$ with $b_{q_2}\neq 0$ ($0\leqslant j\leqslant q_2$). From (4.8), we get $\mathrm{i}a^m(B\mathrm{e}^{ac}+C)R_1z+\mathrm{i}a^m(B\mathrm{e}^{ac}+C)R_2-A\equiv 0\Rightarrow B\mathrm{e}^{ac}+C=0$ and $\mathrm{i}a^m(B\mathrm{e}^{ac}+C)R_2-A=0\Rightarrow A=0$, a contradiction arises.

Sub-case 3.2. When $m\in\mathbb{N}$ is odd, $\deg(R(z))=1$, $Q_2(z)\equiv Q_2\in\mathbb{C}\setminus\{0\}$ and $Q_1(z)=\sum_{j=0}^{q_1}a_jz^j$, where $a_j\in\mathbb{C}$ with $a_{q_1}\neq 0$ ($0\leqslant j\leqslant q_1$). From (4.8), we get $-\mathrm{i}a^m(B\mathrm{e}^{-ac}+C)R_1z-\mathrm{i}a^m(B\mathrm{e}^{-ac}+C)R_2+A\equiv 0\Rightarrow B\mathrm{e}^{-ac}+C=0$ and $-\mathrm{i}a^m(B\mathrm{e}^{-ac}+C)R_2+A=0\Rightarrow A=0$, a contradiction arises.

Sub-case 3.3. When $m\in\mathbb{N}$ is odd, $\deg(R(z))=1$, and let $Q_1(z)\equiv\sum_{i=0}^{q_1}a_jz^j$, $Q_2(z)\equiv\sum_{k=0}^{q_2}b_kz^k$, where $a_j,b_k\in\mathbb{C}$ with $a_{q_1}\neq 0$, $b_{q_2}\neq 0$ ($0\leqslant j\leqslant q_1$, $0\leqslant k\leqslant q_2$).

From (4.8) we get

$$\begin{aligned}\text{(4.12)}\qquad \mathrm{i}(R_1 z+R_2)\Big(&a^m(B\mathrm{e}^{ac}Q_1(z+c)+CQ_1(z))\\ &+a^{m-1}\binom{m}{1}(B\mathrm{e}^{ac}Q_1'(z+c)+CQ_1'(z))\\ &+\ldots+(B\mathrm{e}^{ac}Q_1^{(m)}(z+c)+CQ_1^{(m)}(z))\Big)\equiv AQ_1(z).\end{aligned}$$

Comparing the coefficient of highest power of z on both sides of (4.12), we get $\mathrm{i}R_1 a^m(B\mathrm{e}^{ac}+C)=0 \Rightarrow B\mathrm{e}^{ac}+C=0 \Rightarrow \mathrm{e}^{ac}=-C/B$. When $\mathrm{e}^{ac}=1$, then from (4.12) we get

$$\begin{aligned}R(z)&=\frac{AQ_1(z)}{\mathrm{i}B\sum_{j=0}^{m}\binom{m}{j}a^{m-j}(Q_1^{(j)}(z+c)-Q_1^{(j)}(z))}\\ &=\frac{-AQ_2(z)}{\mathrm{i}B\sum_{j=0}^{m}\binom{m}{j}(-a)^{m-j}(Q_2^{(j)}(z+c)-Q_2^{(j)}(z))}.\end{aligned}$$

If $\mathrm{e}^{ac}=-1$, then from (4.12), we get

$$\begin{aligned}R(z)&=\frac{-AQ_1(z)}{\mathrm{i}B\sum_{j=0}^{m}\binom{m}{j}a^{m-j}(Q_1^{(j)}(z+c)-Q_1^{(j)}(z))}\\ &=\frac{AQ_2(z)}{\mathrm{i}B\sum_{j=0}^{m}\binom{m}{j}(-a)^{m-j}(Q_2^{(j)}(z+c)-Q_2^{(j)}(z))}.\end{aligned}$$

This completes the proof. □

P r o o f of Theorem 2.2. Proceeding similarly to the proof of Theorem 2.1, we get the required conclusions. For this reason, we omit the details. □

P r o o f of Theorem 2.3. Let $f\in\mathscr{E}_T^{<\infty}$ satisfy (1.10). By similar arguments as in the proof of Theorem 2.1, we have

$$\text{(4.13)}\qquad f(z)=\frac{Q_1(z)\mathrm{e}^{P(z)}+Q_2(z)\mathrm{e}^{-P(z)}}{2A}$$

and

$$Bf^{(m)}(z+c)+Cf^{(n)}(z)=\frac{Q_1(z)\mathrm{e}^{P(z)}-Q_2(z)\mathrm{e}^{-P(z)}}{2\mathrm{i}R(z)},$$

where $Q_1(z)$, $Q_2(z)$ are nonzero polynomials such that $Q_1(z)Q_2(z)=Q(z)$ and $P(z)$ is a non-constant polynomial. Differentiating on the both sides of first equation in (4.13), we deduce that

(4.14)

$$f^{(m)}(z)=\frac{p_1(z)\mathrm{e}^{P(z)}+p_2(z)\mathrm{e}^{-P(z)}}{2A}\quad\text{and}\quad f^{(n)}(z)=\frac{q_1(z)\mathrm{e}^{P(z)}+q_2(z)\mathrm{e}^{-P(z)}}{2A},$$

where $p_1(z)$ and $p_2(z)$ are given in (4.4) and (4.5), respectively, and

$$\begin{aligned}(4.15)\qquad q_1(z) = {} & Q_1(z)((P'(z))^n + M_{1,n}(P'(z), P''(z), \ldots, P^{(n)}(z))) \\ & + Q_1'(z)M_{2,n-1}(P'(z), P''(z), \ldots, P^{(n-1)}(z)) \\ & + \ldots + Q_1^{(n-1)}(z)M_{n,1}(P'(z)) + Q_1^{(n)}(z)\end{aligned}$$

and

$$\begin{aligned}(4.16)\qquad q_2(z) = {} & Q_2(z)((-1)^n(P'(z))^n + N_{1,n}(P'(z), P''(z), \ldots, P^{(n)}(z))) \\ & + (-1)^{n-1}Q_2'(z)N_{2,n-1}(P'(z), P''(z), \ldots, P^{(n-1)}(z)) \\ & + \ldots + Q_2^{(n-1)}(z)N_{n,1}(P'(z)) + Q_2^{(n)}(z),\end{aligned}$$

where $M_{j,\mu-j+1}$ (or $N_{j,\mu-j+1}$) are differential polynomials of $P'(z)$ of degree $\mu-1$ for $j=1,2$ and of degree $\mu-j+1$ for $j=3,4,\ldots,\mu$ and $\mu\in\{m,n\}$. From (4.13) and (4.14), we have

$$\begin{aligned}(4.17)\qquad & \Big(Bp_1(z+c)\mathrm{e}^{\Delta_c P(z)} + Cq_1(z) - \frac{AQ_1(z)}{\mathrm{i}R(z)}\Big)\mathrm{e}^{P(z)} \\ & + \Big(Bp_2(z+c)\mathrm{e}^{-\Delta_c P(z)} + Cq_2(z) + \frac{AQ_2(z)}{\mathrm{i}R(z)}\Big)\mathrm{e}^{-P(z)} \equiv 0.\end{aligned}$$

Clearly, $T(r, \mathrm{e}^{\Delta_c P(z)}) = S(r, \mathrm{e}^{P(z)})$. Hence by Lemma 3.1, we get from (4.17) that

$$(4.18)\qquad Bp_1(z+c)\mathrm{e}^{\Delta_c P(z)} + Cq_1(z) - \frac{AQ_1(z)}{\mathrm{i}R(z)} \equiv 0$$

and

$$Bp_2(z+c)\mathrm{e}^{-\Delta_c P(z)} + Cq_2(z) + \frac{AQ_2(z)}{\mathrm{i}R(z)} \equiv 0.$$

We claim that $\deg(P(z)) = 1$. If not, let $\deg(P(z)) \geqslant 2 \Rightarrow \deg(\Delta_c P(z)) \geqslant 1$. By Lemma 3.1, we have $p_1(z+c)\equiv 0$ and $p_2(z+c)\equiv 0$. So from (4.14), we see that f is a polynomial, which is not possible. Hence, $\deg(P(z)) = 1 \Rightarrow P(z) = az+b$, where $a(\neq 0), b\in\mathbb{C}$. From (4.13), we get

$$f(z) = \frac{Q_1(z)\mathrm{e}^{az+b} + Q_2(z)\mathrm{e}^{-az-b}}{2A}.$$

Also from (4.18), we deduce that

$$(4.19)\qquad Bp_1(z+c)\mathrm{e}^{ac} + Cq_1(z) - \frac{AQ_1(z)}{\mathrm{i}R(z)} \equiv 0 \quad\text{and}\quad Bp_2(z+c)\mathrm{e}^{-ac} + Cq_2(z) + \frac{AQ_2(z)}{\mathrm{i}R(z)} \equiv 0,$$

where $p_1(z) = \sum\limits_{j=0}^{m}\binom{m}{j}a^{m-j}Q_1^{(j)}(z)$, $p_2(z) = \sum\limits_{j=0}^{m}\binom{m}{j}(-a)^{m-j}Q_2^{(j)}(z)$, $q_1(z) = \sum\limits_{j=0}^{n}\binom{n}{j}a^{n-j}Q_1^{(j)}(z)$ and $q_2(z) = \sum\limits_{j=0}^{n}\binom{n}{j}(-a)^{n-j}Q_2^{(j)}(z)$. Combining the two equations in (4.19), we deduce that

$$
\begin{aligned}
(4.20)\quad & R^2(z)(B^2p_1(z+c)p_2(z+c)-C^2q_1(z)q_2(z)) \\
&\equiv A^2Q(z)-\mathrm{i}ACR(z)(q_1(z)Q_2(z)-q_2(z)Q_1(z)) \\
&\Rightarrow B^2R^2(z)\sum_{j=0}^{m}\binom{m}{j}a^{m-j}Q_1^{(j)}(z+c)\sum_{j=0}^{m}\binom{m}{j}(-a)^{m-j}Q_2^{(j)}(z+c) \\
&\quad -C^2R^2(z)\sum_{j=0}^{n}\binom{n}{j}a^{n-j}Q_1^{(j)}(z)\sum_{j=0}^{n}\binom{n}{j}(-a)^{n-j}Q_2^{(j)}(z) \\
&\equiv A^2Q(z)-\mathrm{i}ACR(z)Q_2(z)\sum_{j=0}^{n}\binom{n}{j}a^{n-j}Q_1^{(j)}(z) \\
&\quad +\mathrm{i}ACR(z)Q_1(z)\sum_{j=0}^{n}\binom{n}{j}(-a)^{n-j}Q_2^{(j)}(z) \\
&\Rightarrow B^2R^2(z)\Bigg(((-1)^m a^{2m}Q(z+c)-(-1)^n a^{2n}Q(z)) \\
&\quad +\Bigg((-1)^{m-1}a^{2m-1}\binom{m}{1}Q_1(z+c)Q_2'(z+c) \\
&\quad -(-1)^{n-1}a^{2n-1}\binom{n}{1}Q_1(z)Q_2'(z)\Bigg) \\
&\quad +\Bigg((-1)^{m}a^{2m-1}\binom{m}{1}Q_1'(z+c)Q_2(z+c) \\
&\quad -(-1)^{n}a^{2n-1}\binom{n}{1}Q_1'(z)Q_2(z)\Bigg)+\dots\Bigg) \\
&\equiv A^2Q(z)-\mathrm{i}ACR(z)\Bigg((a^n-(-a)^n)Q(z) \\
&\quad +\binom{n}{1}(a^{n-1}Q_1'(z)Q_2(z)-(-a)^{n-1}Q_1(z)Q_2'(z)) \\
&\quad +\dots+(Q_1^{(n)}(z)Q_2(z)-Q_1(z)Q_2^{(n)}(z))\Bigg).
\end{aligned}
$$

Now we consider the following cases.

Case 1. Let $m,n\in\mathbb{N}$ be both even. Then the following cases arise.

Sub-case 1.1. Let $a^{2m}-a^{2n}=0\Rightarrow a^{m-n}=\pm1$ for $a\in\mathbb{C}\setminus\{0\}$. If $\deg(R(z))>0$, then by considering the degrees of $R(z)$, $Q_1(z)$ and $Q_2(z)$ from (4.20), we see that $2\deg(R(z))+\deg(Q(z))-1=\deg(R(z))+\deg(Q(z))-1\Rightarrow\deg(R(z))=0$, which is a contradiction. If $\deg(R(z))=0$, then by considering the degrees of $Q_1(z)$ and $Q_2(z)$ from (4.20), we see that $\deg(Q(z))-1=\deg(Q(z))$, which is absurd. So, in this case, the solution of (1.10) does not exist.

Sub-case 1.2. Let $a^{2m} - a^{2n} \neq 0 \Rightarrow a^{m-n} \neq \pm 1$ for $a \in \mathbb{C} \setminus \{0\}$. We claim that $\deg(R(z)) = 0$. If not, let $\deg(R(z)) > 0$, then by considering the degrees of $R(z)$, $Q_1(z)$ and $Q_2(z)$ from (4.20), we see that $2\deg(R(z)) + \deg(Q(z)) = \deg(R(z)) + \deg(Q(z)) - 1 \Rightarrow \deg(R(z)) = -1$, which is absurd. Hence, $\deg(R(z)) = 0$ for $a^{m-n} \neq \pm 1$. From (4.19), we get

$$
\begin{aligned}
(4.21)\ &(Ba^m \mathrm{e}^{ac} Q_1(z+c) + Ca^n Q_1(z)) \\
&+ \left(B\binom{m}{1} a^{m-1} \mathrm{e}^{ac} Q_1'(z+c) + C\binom{n}{1} a^{n-1} Q_1'(z)\right) + \ldots \equiv \frac{A}{\mathrm{i}R} Q_1(z), \\
(4.22)\ &(Ba^m \mathrm{e}^{-ac} Q_2(z+c) + Ca^n Q_2(z)) \\
&+ \left(-B\binom{m}{1} a^{m-1} \mathrm{e}^{-ac} Q_2'(z+c) - C\binom{n}{1} a^{m-1} Q_2'(z)\right) + \ldots \equiv -\frac{A}{\mathrm{i}R} Q_2(z).
\end{aligned}
$$

Comparing the coefficient of highest power of z on both sides of (4.21) and (4.22), we get $Ba^m \mathrm{e}^{ac} + Ca^n = A/(\mathrm{i}R)$ and $Ba^m \mathrm{e}^{-ac} + Ca^n = -A/(\mathrm{i}R) \Rightarrow \mathrm{e}^{ac} = (-Ca^n \pm B\sqrt{a^{2n} - a^{2m}})/(Ba^m) \Rightarrow R = \mp \mathrm{i}A/(B\sqrt{a^{2n} - a^{2m}})$, respectively. Now the following cases arise.

Sub-case 1.2.1. When $\mathrm{e}^{ac} = (-Ca^n + B\sqrt{a^{2n} - a^{2m}})/(Ba^m)$, we claim that $\deg(Q_1(z)) = 0$. If not, let $Q_1(z) \equiv \sum\limits_{j=0}^{q_1} a_j z^j$, where $a_j \in \mathbb{C}$ with $q_1 \geqslant 1$ and $a_{q_1} \neq 0$ $(0 \leqslant j \leqslant q_1)$. Comparing the coefficient of z^{q_1} and $z^{q_1 - 1}$ on both sides of (4.21), we get

$$
(4.23) \qquad (a^m B \mathrm{e}^{ac} + Ca^n) \equiv \frac{A}{\mathrm{i}R}
$$

and

$$
\begin{aligned}
(4.24)\quad &(a^m B \mathrm{e}^{ac} (q_1 c a_{q_1} + a_{q_1 - 1}) + a^n C a_{q_1 - 1}) \\
&\quad + (m a^{m-1} B \mathrm{e}^{ac} q_1 a_{q_1} + n a^{n-1} C q_1 a_{q_1}) = \frac{A}{\mathrm{i}R} a_{q_1 - 1} \\
&\Rightarrow ac(-C + B\sqrt{1 - a^{2m-2n}}) + (m(-C + B\sqrt{1 - a^{2m-2n}}) + Cn) = 0 \\
&\Rightarrow m - n = \frac{ac(-C + B\sqrt{1 - a^{2m-2n}}) + Bm\sqrt{1 - a^{2m-2n}}}{C},
\end{aligned}
$$

respectively. Since $B = \pm C$, from (4.24) we deduce that $m - n = ac(-1 \pm \sqrt{1 - a^{2m-2n}}) \pm m\sqrt{1 - a^{2m-2n}}$, which is not possible with even $m, n \in \mathbb{N}$, $a \in \mathbb{C} \setminus \{0\}$ and $a^{2m-2n} \neq 1$. Hence, $\deg(Q_1(z)) = 0$. Similarly from (4.22), we can deduce that $\deg(Q_2(z)) = 0$.

Sub-case 1.2.2. When $\mathrm{e}^{ac} = (-Ca^n - B\sqrt{a^{2n} - a^{2m}})/(Ba^m)$, then using similar arguments as in Sub-case 1.2.1 we get $\deg(Q_1(z)) = 0$ and $\deg(Q_2(z)) = 0$.

Case 2. When $m, n \in \mathbb{N}$ are both odd. Then the following cases arise.

Sub-case 2.1. Let $a^{2m} - a^{2n} \neq 0 \Rightarrow a^{m-n} \neq \pm 1$ for $a \in \mathbb{C} \setminus \{0\}$. If $\deg(R(z)) > 0$, then by considering the degrees of $R(z)$, $Q_1(z)$ and $Q_2(z)$ from (4.20), we see that $2\deg(R(z)) + \deg(Q(z)) = \deg(R(z)) + \deg(Q(z)) \Rightarrow \deg(R(z)) = 0$, which is a contradiction. If $\deg(R(z)) = 0$ and $A - 2\mathrm{i}CRa^n = 0$, then by considering the degrees of $Q_1(z)$ and $Q_2(z)$ from (4.20), we see that $\deg(Q(z)) = \deg(Q(z)) - 1$, which is not possible. Hence, $\deg(R(z)) = 0$ with $A - 2\mathrm{i}CRa^n \neq 0$ is possible. From (4.19), we get

$$\begin{aligned}(4.25)\quad &(Ba^m \mathrm{e}^{ac} Q_1(z+c) + Ca^n Q_1(z)) \\ &\quad + \left(B\binom{m}{1} a^{m-1} \mathrm{e}^{ac} Q_1'(z+c) + C\tbinom{n}{1} a^{n-1} Q_1'(z)\right) + \ldots \equiv \frac{A}{\mathrm{i}R} Q_1(z), \\ (4.26)\quad &(-Ba^m \mathrm{e}^{-ac} Q_2(z+c) - Ca^n Q_2(z)) \\ &\quad + \left(B\binom{m}{1} a^{m-1} \mathrm{e}^{-ac} Q_2'(z+c) + C\binom{n}{1} a^{m-1} Q_2'(z)\right) + \ldots \equiv -\frac{AQ_2(z)}{\mathrm{i}R}.\end{aligned}$$

Comparing the coefficient of highest power of z on both sides of (4.25) and (4.26), we get $Ba^m \mathrm{e}^{ac} + Ca^n = A/(\mathrm{i}R)$ and $Ba^m \mathrm{e}^{-ac} + Ca^n = A/(\mathrm{i}R) \Rightarrow \mathrm{e}^{ac} = \pm 1 \Rightarrow R = -\mathrm{i}A/(\pm Ba^m + Ca^n)$, respectively. Now the following cases arise.

Sub-case 2.1.1. When $a^{m-n} \neq \pm 1$ and $\mathrm{e}^{ac} = 1$, we claim that $\deg(Q_1(z)) = 0$. If not, let $Q_1(z) \equiv \sum_{j=0}^{q_1} a_j z^j$, where $a_j \in \mathbb{C}$ with $q_1 \geqslant 1$ and $a_{q_1} \neq 0$ $(0 \leqslant j \leqslant q_1)$. Comparing the coefficient of z^{q_1} and $z^{q_1 - 1}$ on both sides of (4.25), we get

$$a^m B + Ca^n \equiv \frac{A}{\mathrm{i}R}$$

and

$$\begin{aligned}(4.27)\quad &(a^m B(q_1 c a_{q_1} + a_{q_1-1}) + a^n C a_{q_1-1}) \\ &\qquad + (ma^{m-1} B q_1 a_{q_1} + na^{n-1} C q_1 a_{q_1}) = \frac{A}{\mathrm{i}R} a_{q_1-1} \\ &\Rightarrow Bca^m + (Bma^{m-1} + Cna^{n-1}) = 0 \Rightarrow m = -2l\pi\mathrm{i} - \frac{C}{B} na^{n-m},\end{aligned}$$

respectively. Since $B = \pm C$, then from (4.27) we get $m = -2l\pi\mathrm{i} \mp na^{n-m}$, which is not possible with odd $m, n \in \mathbb{N}$, $l \in \mathbb{Z} \setminus \{0\}$ and $a^{m-n} \neq \pm 1$. So $\deg(Q_1(z)) = 0$. Similarly from (4.26), we get $\deg(Q_2(z)) = 0$.

Sub-case 2.1.2. When $a^{m-n} \neq \pm 1$ and $\mathrm{e}^{ac} = -1$, then using similar arguments as in Sub-case 2.1.1, we get $\deg(Q_1(z)) = 0$ and $\deg(Q_2(z)) = 0$.

Sub-case 2.2. Let $a^{2m} - a^{2n} = 0 \Rightarrow a^{m-n} = \pm 1$ for $a \in \mathbb{C} \setminus \{0\}$. If $\deg(R(z)) > 0$, then by considering the degrees of $R(z)$, $Q_1(z)$ and $Q_2(z)$ from (4.20), we see that

$2\deg(R(z))+\deg(Q(z))-1=\deg(R(z))+\deg(Q(z))\Rightarrow\deg(R(z))=1$. Let $R(z)\equiv R_1z+R_2$, where $R_1(\neq 0), R_2\in\mathbb{C}$. From (4.20), it is clear that $Q(z)$ is a non-consant polynomial. Let $Q_1(z)\equiv Q_1\in\mathbb{C}\setminus\{0\}$ and $Q(z)\equiv Q_1Q_2(z)$. Then from (4.19) we get $\mathrm{i}R_1(Ba^m\mathrm{e}^{ac}+Ca^n)z+\mathrm{i}R_2(Ba^m\mathrm{e}^{ac}+Ca^n)-A\equiv 0\Rightarrow Ba^m\mathrm{e}^{ac}+Ca^n=0$ and $\mathrm{i}R_2(Ba^m\mathrm{e}^{ac}+Ca^n)-A=0\Rightarrow A=0$, which is not possible. Again if $Q_2(z)\equiv Q_2\in\mathbb{C}\setminus\{0\}$ and $Q(z)\equiv Q_2Q_1(z)$, then similarly we get a contradiction. Hence both $Q_1(z)$, $Q_2(z)$ are non-constant polynomials. Let $Q_1(z)\equiv\sum\limits_{j=0}^{q_1}a_jz^j$, $Q_2(z)\equiv\sum\limits_{k=0}^{q_2}b_kz^k$, where $a_j,b_k\in\mathbb{C}$ with $a_{q_1}\neq 0$, $b_{q_2}\neq 0$ $(0\leqslant j\leqslant q_1,\ 0\leqslant k\leqslant q_2)$. From (4.19), we get

$$\begin{aligned}(4.28)\quad \mathrm{i}(R_1z+R_2)\Bigg(&(B\mathrm{e}^{ac}a^mQ_1(z+c)+Ca^nQ_1(z))\\&+\left(B\mathrm{e}^{ac}a^{m-1}\binom{m}{1}Q_1'(z+c)+Ca^{n-1}\binom{n}{1}Q_1'(z)\right)+\ldots\Bigg)\equiv AQ_1(z).\end{aligned}$$

Comparing the coefficient of z^{q_1+1} on both sides of (4.28), we get

$$\mathrm{i}R_1(B\mathrm{e}^{ac}a^m+Ca^n)=0\Rightarrow B\mathrm{e}^{ac}a^m+Ca^n=0\Rightarrow\mathrm{e}^{ac}=-\frac{Ca^n}{Ba^m}. \tag{4.29}$$

Note that $B=\pm C$ and $a^{m-n}=\pm 1$. From (4.29), we get $\mathrm{e}^{ac}=\pm 1\Rightarrow Ba^m=\mp Ca^n$. Also from (4.19), we get

(4.30)
$$\begin{aligned}\mathrm{i}(R_1z+R_2)\Bigg(&-(B\mathrm{e}^{-ac}a^mQ_2(z+c)+Ca^nQ_2(z))\\&+\left(B\mathrm{e}^{-ac}a^{m-1}\binom{m}{1}Q_2'(z+c)+Ca^{n-1}\binom{n}{1}Q_2'(z)\right)+\ldots\Bigg)\equiv -AQ_2(z).\end{aligned}$$

If $\mathrm{e}^{ac}=\pm 1$, then from (4.28) and (4.30) we get

$$\begin{aligned}R(z)&=\frac{-\mathrm{i}AQ_1(z)}{\pm B\sum_{j=0}^{m}\binom{m}{j}a^{m-j}Q_1^{(j)}(z+c)+C\sum_{j=0}^{n}\binom{n}{j}a^{n-j}Q_1^{(j)}(z)}\\&=\frac{\mathrm{i}AQ_2(z)}{\pm B\sum_{j=0}^{m}\binom{m}{j}(-a)^{m-j}Q_2^{(j)}(z+c)+C\sum_{j=0}^{n}\binom{n}{j}(-a)^{n-j}Q_2^{(j)}(z)}.\end{aligned}$$

Also $\mathrm{e}^{ac}=\pm 1$ and $B=C$ gives $a^{m-n}=\mp 1$. Similarly $\mathrm{e}^{ac}=\pm 1$ and $B=-C$ gives $a^{m-n}=\pm 1$.

Sub-case 2.3. If $a^{2m}-a^{2n}=0\Rightarrow a^{m-n}=\pm 1$ for $a\in\mathbb{C}\setminus\{0\}$ with $\deg(R(z))=0$ and $A-2\mathrm{i}CRa^n\neq 0$, then by considering the degrees of $Q_1(z)$ and $Q_2(z)$ from (4.20), we see that $\deg(Q(z))-1=\deg(Q(z))$, which is absurd. Hence, $\deg(R(z))=0$ with

$A - 2\mathrm{i}CRa^n = 0$ is possible. From (4.19), we get

$$\text{(4.31)}\quad \mathrm{i}R\bigg((Ba^m\mathrm{e}^{ac}Q_1(z+c)+Ca^nQ_1(z)) + \left(B\binom{m}{1}a^{m-1}\mathrm{e}^{ac}Q_1'(z+c)+C\binom{n}{1}a^{n-1}Q_1'(z)\right)+\ldots\bigg) \equiv AQ_1(z),$$

$$\text{(4.32)}\quad \mathrm{i}R\bigg((-Ba^m\mathrm{e}^{-ac}Q_2(z+c)-Ca^nQ_2(z)) + \left(B\binom{m}{1}a^{m-1}\mathrm{e}^{-ac}Q_2'(z+c)+C\binom{n}{1}a^{m-1}Q_2'(z)\right)+\ldots\bigg) \equiv -AQ_2(z).$$

Comparing the highest power of z on both sides of (4.31) and (4.32), we get

$$\mathrm{i}R(Ba^m\mathrm{e}^{ac}+Ca^n) = A \quad \text{and} \quad \mathrm{i}R(Ba^m\mathrm{e}^{-ac}+Ca^n) = A \Rightarrow \mathrm{e}^{ac} = \pm 1,$$

respectively. Now we deduce that $\mathrm{e}^{ac} = \pm 1 \Rightarrow R = -\mathrm{i}A/(\pm Ba^m + Ca^n)$. Then the following cases arise.

Sub-case 2.3.1. When $ac = 2l\pi\mathrm{i}$ ($l \in \mathbb{Z}\setminus\{0\}$), $a^{m-n} = \pm 1$ and $\mathrm{i}R(Ba^m+Ca^n) = A$, we claim that $\deg(Q_1(z)) = 0$. If not, let $Q_1(z) = \sum_{j=0}^{q_1} a_jz^j$, where $a_j \in \mathbb{C}$ with $a_{q_1} \neq 0$ and $q_1 \geqslant 1$ ($0 \leqslant j \leqslant q_1$). Comparing the coefficient of z^{q_1} and z^{q_1-1} on both sides of (4.31), we get

$$\mathrm{i}R(Ba^m + Ca^n) = A$$

and

(4.33)

$$\begin{aligned}\mathrm{i}R(Ba^m(cq_1a_{q_1}+a_{q_1-1}) + Ca^na_{q_1-1} + Bma^{m-1}q_1a_{q_1} + Cna^{n-1}q_1a_{q_1}) = Aa_{q_1-1}\\ \Rightarrow Bca^m + Bma^{m-1} + Cna^{n-1} = 0.\end{aligned}$$

When $a^m = a^n$ and $B = \pm C$, then from (4.33) we get $m \pm n = -2l\pi\mathrm{i}$, which is not possible, since $m, n \in \mathbb{N}$ are both odd and $l \in \mathbb{Z}\setminus\{0\}$ is arbitrary.

When $a^m = -a^n$ and $B = \pm C$, then from (4.33) we get $m \mp n = -2l\pi\mathrm{i}$, which is not possible, since $m, n \in \mathbb{N}$ are both odd and $l \in \mathbb{Z}\setminus\{0\}$ is arbitrary. Hence $\deg(Q_1(z)) = 0$. Similarly from (4.32), we also get $\deg(Q_2(z)) = 0$.

Sub-case 2.3.2. When $ac=(2l+1)\pi\mathrm{i}(l\in\mathbb{Z})$, $a^{m-n}=\pm 1$ and $\mathrm{i}R\times(-Ba^m+Ca^n) = A$, similarly as in Sub-case 2.3.1, we get $\deg(Q_1(z)) = 0$ and $\deg(Q_2(z)) = 0$.

Case 3. When $m, n \in \mathbb{N}$ with m even and n odd, then the following cases arise.

Sub-case 3.1. Let $a^{2m} + a^{2n} \neq 0 \Rightarrow a^{m-n} \neq \pm\mathrm{i}$ for $a \in \mathbb{C}\setminus\{0\}$. If $\deg(R(z)) > 0$, then by considering the degrees of $R(z)$, $Q_1(z)$ and $Q_2(z)$ from (4.20), we see that $2\deg(R(z)) + \deg(Q(z)) = \deg(R(z)) + \deg(Q(z)) \Rightarrow \deg(R(z)) = 0$, which is a contradiction. If $\deg(R(z)) = 0$ with $A - 2\mathrm{i}CRa^n = 0$, then by considering the degrees of $Q_1(z)$ and $Q_2(z)$ from (4.20), we see that $\deg(Q(z)) = \deg(Q(z)) - 1$, which is not possible. Hence, $\deg(R(z)) = 0$ with $A - 2\mathrm{i}CRa^n \neq 0$ is possible. From (4.19), we get

$$\begin{aligned}(4.34)\quad &(Ba^m\mathrm{e}^{ac}Q_1(z+c) + Ca^nQ_1(z))\\ &\quad + \left(B\binom{m}{1}a^{m-1}\mathrm{e}^{ac}Q_1'(z+c) + C\binom{n}{1}a^{n-1}Q_1'(z)\right) + \ldots \equiv \frac{A}{\mathrm{i}R}Q_1(z),\\ (4.35)\quad &(Ba^m\mathrm{e}^{-ac}Q_2(z+c) - Ca^nQ_2(z))\\ &\quad + \left(-B\binom{m}{1}a^{m-1}\mathrm{e}^{-ac}Q_2'(z+c) + C\binom{n}{1}a^{m-1}Q_2'(z)\right) + \ldots \equiv \frac{-A}{\mathrm{i}R}Q_2(z).\end{aligned}$$

Comparing the coefficient of highest power of z on both sides of (4.34) and (4.35), we get $Ba^m\mathrm{e}^{ac} + Ca^n = A/(\mathrm{i}R)$ and $Ba^m\mathrm{e}^{-ac} - Ca^n = -A/(\mathrm{i}R) \Rightarrow \mathrm{e}^{ac} = \pm\mathrm{i} \Rightarrow R = -\mathrm{i}A/(\pm\mathrm{i}Ba^m + Ca^n)$, respectively. Now the following cases arise.

Sub-case 3.1.1. When $\mathrm{e}^{ac} = \mathrm{i} \Rightarrow ac = (4l+1)\frac{1}{2}\pi\mathrm{i}$ $(l \in \mathbb{Z})$, $A - 2\mathrm{i}CRa^n \neq 0$ and $R = -\mathrm{i}A/(\mathrm{i}Ba^m + Ca^n)$, we claim that $\deg(Q_1(z)) = 0$. If not, let $Q_1(z) = \sum\limits_{j=0}^{q_1} a_jz^j$, where $a_j \in \mathbb{C}$ with $a_{q_1} \neq 0$ and $q_1 \geqslant 1$ $(0 \leqslant j \leqslant q_1)$. Comparing the coefficient of z^{q_1} and z^{q_1-1} on both sides of (4.34), we get

$$\mathrm{i}Ba^m + Ca^n = \frac{A}{\mathrm{i}R}$$

and

(4.36)

$$\begin{aligned}&\mathrm{i}Ba^m(cq_1a_{q_1} + a_{q_1-1}) + Ca^na_{q_1-1} + \mathrm{i}Bma^{m-1}q_1a_{q_1} + Cna^{n-1}q_1a_{q_1} = \frac{A}{\mathrm{i}R}a_{q_1-1}\\ &\quad\Rightarrow \mathrm{i}Bca^m + \mathrm{i}Bma^{m-1} + Cna^{n-1} = 0 \Rightarrow m = -(4l+1)\frac{\pi\mathrm{i}}{2} + \frac{C}{B}\mathrm{i}na^{n-m}.\end{aligned}$$

Since $B = \pm C$, from (4.36) we deduce that $m = -(4l+1)\frac{1}{2}\pi\mathrm{i} \pm \mathrm{i}na^{n-m}$, which is not possible with both $m, n \in \mathbb{N}$ odd, $l \in \mathbb{Z}$ and $a^{m-n} \neq \pm\mathrm{i}$. Hence, $\deg(Q_1(z)) = 0$. Similarly from (4.35), we get $\deg(Q_2(z)) = 0$.

Sub-case 3.1.2. Let $\mathrm{e}^{ac} = -\mathrm{i} \Rightarrow ac = (4l-1)\frac{1}{2}\pi\mathrm{i}$ $(l \in \mathbb{Z})$, $a^{m-n} \neq \pm\mathrm{i}$ and $R = \mathrm{i}A/(\mathrm{i}Ba^m - Ca^n)$. Using similar arguments as in Sub-case 3.1.1 we get $\deg(Q_1(z)) = 0$ and $\deg(Q_2(z)) = 0$.

Sub-case 3.2. Let $a^{2m} + a^{2n} = 0 \Rightarrow a^{m-n} = \pm\mathrm{i}$ for $a \in \mathbb{C} \setminus \{0\}$ and $\deg(R(z)) > 0$. By considering the degrees of $R(z)$, $Q_1(z)$ and $Q_2(z)$ from (4.20), we see that $2\deg(R(z)) + \deg(Q(z)) - 1 = \deg(R(z)) + \deg(Q(z)) \Rightarrow \deg(R(z)) = 1$. Let $R(z) = R_1z + R_2$, where $R_1\,(\neq 0), R_2 \in \mathbb{C}$. Then from (4.20), we deduce that $Q(z)$ is a non-constant polynomial. If $Q_1(z) \equiv Q_1 \in \mathbb{C} \setminus \{0\}$ and $Q(z) = Q_1Q_2(z)$, then from (4.19) we get $\mathrm{i}R_1(Ba^m\mathrm{e}^{ac} + Ca^n)z + \mathrm{i}R_2(Ba^m\mathrm{e}^{ac} + Ca^n) - A \equiv 0 \Rightarrow$

$Ba^m e^{ac} + Ca^n = 0$ and $\mathrm{i}R_2(Ba^m e^{ac} + Ca^n) - A = 0 \Rightarrow A = 0$, which is not possible. Again, if $Q_2(z) \equiv Q_2 \in \mathbb{C}\setminus\{0\}$, then similarly we get a contradiction. So both $Q_1(z)$, and $Q_2(z)$ are non-constant polynomials. Let $Q_1(z) \equiv \sum_{i=0}^{q_1} a_j z^j$ and $Q_2(z) \equiv \sum_{k=0}^{q_2} b_k z^k$, where $a_j, b_k \in \mathbb{C}$ with $a_{q_1} \neq 0$, $b_{q_2} \neq 0$ $(0 \leqslant j \leqslant q_1,\ 0 \leqslant k \leqslant q_2)$. From (4.19), we get

$$\begin{aligned}\mathrm{i}(R_1 z + R_2)\Bigg(&(B\mathrm{e}^{ac}a^m Q_1(z+c) + Ca^n Q_1(z)) \\ &+ \left(B\mathrm{e}^{ac}a^{m-1}\binom{m}{1}Q_1'(z+c) + Ca^{n-1}\tbinom{n}{1}Q_1'(z)\right) + \dots\Bigg) \equiv AQ_1(z).\end{aligned} \tag{4.37}$$

Comparing the coefficient of z^{q_1+1} on both sides of (4.37), we get

$$\mathrm{i}R_1(B\mathrm{e}^{ac}a^m + Ca^n)a_{q_1} = 0 \Rightarrow B\mathrm{e}^{ac}a^m + Ca^n = 0 \Rightarrow \mathrm{e}^{ac} = -\frac{Ca^n}{Ba^m}. \tag{4.38}$$

Note that $B = \pm C$ and $a^{m-n} = \pm\mathrm{i}$. So from (4.38), we get $\mathrm{e}^{ac} = \pm\mathrm{i} \Rightarrow \mathrm{i}Ba^m = \mp Ca^n$. Similarly from (4.19), we get

$$\begin{aligned}&\mathrm{i}(R_1 z + R_2)\Bigg((B\mathrm{e}^{-ac}a^m Q_2(z+c) - Ca^n Q_2(z)) \\ &\quad+ \left(-B\mathrm{e}^{-ac}a^{m-1}\binom{m}{1}Q_2'(z+c) + Ca^{n-1}\binom{n}{1}Q_2'(z)\right) + \dots\Bigg) \equiv -AQ_2(z).\end{aligned} \tag{4.39}$$

If $\mathrm{e}^{ac} = \pm\mathrm{i}$, then from (4.37) and (4.39) we get

$$\begin{aligned}R(z) &= \frac{-\mathrm{i}AQ_1(z)}{\pm\mathrm{i}B\sum_{j=0}^{m}\binom{m}{j}a^{m-j}Q_1^{(j)}(z+c) + C\sum_{j=0}^{n}\binom{n}{j}a^{n-j}Q_1^{(j)}(z)} \\ &= \frac{\mathrm{i}AQ_2(z)}{\pm\mathrm{i}B\sum_{j=0}^{m}\binom{m}{j}(-a)^{m-j}Q_2^{(j)}(z+c) + C\sum_{j=0}^{n}\binom{n}{j}(-a)^{n-j}Q_2^{(j)}(z)}.\end{aligned}$$

Also $\mathrm{e}^{ac} = \pm\mathrm{i}$ and $B = C$ gives $a^{m-n} = \pm\mathrm{i}$. Similarly $\mathrm{e}^{ac} = \pm\mathrm{i}$ and $B = -C$ gives $a^{m-n} = \mp\mathrm{i}$.

Sub-case 3.3. Let $a^{2m} + a^{2n} = 0 \Rightarrow a^{m-n} = \pm\mathrm{i}$ for $a \in \mathbb{C}\setminus\{0\}$ with $\deg(R(z)) = 0$ and $A - 2\mathrm{i}CRa^n \neq 0$. By considering the degrees of $Q_1(z)$ and $Q_2(z)$ from (4.20), we see that $\deg(Q(z)) - 1 = \deg(Q(z))$, which is absurd. Hence, $\deg(R(z)) = 0$ with $A - 2\mathrm{i}CRa^n = 0$ is possible. From (4.19), we deduce that

$$\begin{aligned}\mathrm{i}R\Bigg(&(B\mathrm{e}^{ac}a^m Q_1(z+c) + Ca^n Q_1(z)) \\ &+ \left(B\mathrm{e}^{ac}a^{m-1}\binom{m}{1}Q_1'(z+c) + Ca^{n-1}\tbinom{n}{1}Q_1'(z)\right) + \dots\Bigg) \equiv AQ_1(z),\end{aligned} \tag{4.40}$$

$$(4.41)\quad \mathrm{i}R\bigg((B\mathrm{e}^{-ac}a^mQ_2(z+c)-Ca^nQ_2(z)) + \bigg(-B\mathrm{e}^{-ac}a^{m-1}\binom{m}{1}Q_2'(z+c)+Ca^{n-1}\tbinom{n}{1}Q_2'(z)\bigg)+\ldots\bigg)\equiv -AQ_2(z).$$

Comparing the coefficient of highest power of z on both sides of (4.40) and (4.41), we get $\mathrm{i}R(Ba^m\mathrm{e}^{ac}+Ca^n)=A$, and $\mathrm{i}R(Ba^m\mathrm{e}^{-ac}-Ca^n)=-A \Rightarrow \mathrm{e}^{ac}=\pm\mathrm{i}\Rightarrow \mathrm{i}R(\pm\mathrm{i}Ba^m+Ca^n)=A$, respectively.

Sub-case 3.3.1. When $\mathrm{e}^{ac}=\mathrm{i}\Rightarrow ac=(4l+1)\frac{1}{2}\pi\mathrm{i}(l\in\mathbb{Z})$, $a^{m-n}=\pm\mathrm{i}$ and $\mathrm{i}R(\mathrm{i}Ba^m+Ca^n)=A$, we claim that $\deg(Q_1(z))=0$. If not, let $Q_1(z)=\sum\limits_{j=0}^{q_1}a_jz^j$, where $a_j\in\mathbb{C}$ $(0\leqslant j\leqslant q_1)$ with $a_{q_1}\neq 0$ and $q_1\geqslant 1$. Comparing the coefficient of z^{q_1} and z^{q_1-1} on both sides of (4.40), we get

$$\mathrm{i}R(\mathrm{i}Ba^m+Ca^n)=A$$

and

$$(4.42)\quad \begin{aligned}&\mathrm{i}R(\mathrm{i}Ba^m(cq_1a_{q_1}+a_{q_1-1})+Ca^na_{q_1-1}\\&\qquad+\mathrm{i}Bma^{m-1}q_1a_{q_1}+Cna^{n-1}q_1a_{q_1})=Aa_{q_1-1}\\&\Rightarrow \mathrm{i}Bca^m+\mathrm{i}Bma^{m-1}+Cna^{n-1}=0.\end{aligned}$$

Note that $a^m=\mathrm{i}a^n$ and $B=\pm C$. From (4.42), we get $-m\pm n=(4l+1)\frac{1}{2}\pi\mathrm{i}$, which is not possible, since $l\in\mathbb{Z}$, $m,n\in\mathbb{N}$ with m even and n odd.

Also note that $a^m=-\mathrm{i}a^n$ and $B=\pm C$. From (4.42), we get $m\pm n=-(4l+1)\frac{1}{2}\pi\mathrm{i}$, which is not possible, since $l\in\mathbb{Z}$, $m,n\in\mathbb{N}$ with m even and n odd. Hence $\deg(Q_1(z))=0$. Similarly from (4.41) we can deduce that $\deg(Q_2(z))=0$.

Sub-case 3.3.2. Let $\mathrm{e}^{ac}=-\mathrm{i}\Rightarrow ac=(4l-1)\frac{1}{2}\pi\mathrm{i}$ $(l\in\mathbb{Z})$, $a^{m-n}=\pm\mathrm{i}$ and $\mathrm{i}R(-\mathrm{i}Ba^m+Ca^n)=A$. Using similar arguments as in Sub-case 3.3.1 we get $\deg(Q_1(z))=0$ and $\deg(Q_2(z))=0$.

Case 4. When $m,n\in\mathbb{N}$ with m odd and n even, then the following cases arise.

Sub-case 4.1. Let $a^{2m}+a^{2n}=0$ for $a\in\mathbb{C}\setminus\{0\}$. If $\deg(R(z))>0$, then by considering the degrees of $R(z)$, $Q_1(z)$ and $Q_2(z)$ from (4.20), we see that $2\deg(R(z))+\deg(Q(z))-1=\deg(R(z))+\deg(Q(z))-1\Rightarrow\deg(R(z))=0$, which is a contradiction. If $\deg(R(z))=0$, then by considering the degrees of $Q_1(z)$ and $Q_2(z)$ from (4.20), we see that $\deg(Q(z))-1=\deg(Q(z))$, which is absurd. So, in this case, the solution of (1.10) is not possible. If $a^{2m}+a^{2n}\neq 0\Rightarrow a^{m-n}\neq\pm\mathrm{i}$ for $a\in\mathbb{C}\setminus\{0\}$, then we claim that $\deg(R(z))=0$. If not, let $\deg(R(z))>0$, then by considering the degrees of $R(z)$, $Q_1(z)$ and $Q_2(z)$ from (4.20), we see that

$2\deg(R(z)) + \deg(Q(z)) = \deg(R(z)) + \deg(Q(z)) - 1 \Rightarrow \deg(R(z)) = -1$, which is absurd. Hence $\deg(R(z)) = 0$. From (4.19), we get that

$$\begin{aligned}
(4.43)\quad & (Ba^m e^{ac} Q_1(z+c) + Ca^n Q_1(z)) \\
&\quad + \left(B\binom{m}{1} a^{m-1} e^{ac} Q_1'(z+c) + C\binom{n}{1} a^{n-1} Q_1'(z)\right) + \ldots \equiv \frac{AQ_1(z)}{\mathrm{i}R}, \\
(4.44)\quad & (Ba^m e^{-ac} Q_2(z+c) - Ca^n Q_2(z)) \\
&\quad + \left(-B\binom{m}{1} a^{m-1} e^{-ac} Q_2'(z+c) + C\binom{n}{1} a^{m-1} Q_2'(z)\right) + \ldots \equiv \frac{AQ_2(z)}{\mathrm{i}R}.
\end{aligned}$$

Comparing the coefficient of highest power of z on both sides of (4.43) and (4.44), we get $Ba^m e^{ac} + Ca^n = A/(\mathrm{i}R)$ and $-Ba^m e^{-ac} + Ca^n = -A/(\mathrm{i}R) \Rightarrow e^{ac} = (-Ca^n \pm B\sqrt{a^{2n} + a^{2m}})/(Ba^m) \Rightarrow R = \mp \mathrm{i}A/(B\sqrt{a^{2n} + a^{2m}})$, respectively. Now the following cases arise.

Sub-case 4.1.1. When $a^{m-n} \neq \pm\mathrm{i}$ and $e^{ac} = (-Ca^n + B\sqrt{a^{2n} + a^{2m}})/(Ba^m)$, we claim that $\deg(Q_1(z)) = 0$. If not, let $Q_1(z) = \sum\limits_{j=0}^{q_1} a_j z^j$, where $a_j \in \mathbb{C}$ $(0 \leqslant j \leqslant q_1)$ with $a_{q_1} \neq 0$ and $q_1 \geqslant 1$. Comparing the coefficient of z^{q_1} and z^{q_1-1} on both sides of (4.43), we get

$$Ba^m e^{ac} + Ca^n = \frac{A}{\mathrm{i}R}$$

and

$$\begin{aligned}
(4.45)\quad & Ba^m e^{ac}(cq_1 a_{q_1} + a_{q_1-1}) + Ca^n a_{q_1-1} \\
&\quad + Bma^{m-1} e^{ac} q_1 a_{q_1} + Cna^{n-1} q_1 a_{q_1} = \frac{A}{\mathrm{i}R} a_{q_1-1} \\
&\Rightarrow Be^{ac} ca^m + Be^{ac} ma^{m-1} + Cna^{n-1} = 0 \\
&\Rightarrow m - n = \frac{ac(-C + B\sqrt{1 + a^{2m-2n}}) + Bm\sqrt{1 + a^{2m-2n}}}{C}.
\end{aligned}$$

Note that $B = \pm C$. Then from (4.45), we get $m - n = ac(-1 \pm \sqrt{1 + a^{2m-2n}}) \pm m\sqrt{1 + a^{2m-2n}}$, which is a contradiction, since $m, n \in \mathbb{N}$ with m odd and n even, $a \in \mathbb{C} \setminus \{0\}$ and $a^{2m-2n} \neq -1$. Hence, $\deg(Q_1(z)) = 0$. Similarly from (4.44), we get $\deg(Q_2(z)) = 0$.

Sub-case 4.1.2. When $a^{m-n} \neq \pm\mathrm{i}$, $e^{ac} = (-Ca^n - B\sqrt{a^{2n} + a^{2m}})/(Ba^m)$ and $m, n \in \mathbb{N}$ with m odd and n even, then using similar arguments as in Sub-case 4.1.1 we get $\deg(Q_1(z)) = 0$ and $\deg(Q_2(z)) = 0$. This completes the proof. □

P r o o f of Theorem 2.4. Proceeding similarly to the proof of Theorem 2.3, we get the required conclusions. Therefore, we omit the details. □

5. The results on meromorphic solutions

During the last decade, there have appeared great contributions for the finite order entire solutions on Fermat-type delay functional equations. But, as far as we know, there is only one contribution due to Liu and Yang (see [17]) on the meromorphic solutions in this direction. Actually, the authors of [17] considered the difference equations

$$f^2(z) + f^2(z+c) = 1 \tag{5.1}$$

and

$$f^2(z) + f^2(qz) = 1 \tag{5.2}$$

and obtained the results as follows.

Proposition E.

(i) *The meromorphic solutions of* (5.1) *must satisfy* $f(z) = \frac{1}{2}(h(z) + 1/h(z))$, *where* $h(z)$ *is a meromorphic function satisfying one of the following two cases:*
   (a) $h(z+c) = -\mathrm{i}h(z)$;
   (b) $h(z+c)h(z) = \mathrm{i}$.

(ii) *The meromorphic solutions of* (5.2) *must satisfy* $f(z) = \frac{1}{2}(h(z) + 1/h(z))$, *where* $h(z)$ *is a meromorphic function satisfying one of the following two cases:*
   (a) $h(qz) = -\mathrm{i}h(z)$;
   (b) $h(qz)h(z) = \mathrm{i}$.

The authors fortified the conditions obtained by exhibiting some examples. For meromorphic solutions of (1.10), we get the following result.

Theorem 5.1. *If f is a non-entire meromorphic function of any order satisfying* (1.10), *then the functional equation has no solution.*

6. Proof of the main result on meromorphic solutions

Proof of Theorem 5.1. From (1.10), we easily see that $f(z)$ and $(Bf^{(m)}(z+c) + Cf^{(n)}(z))$ share ∞ CM. If $f(z) = (Q_1(z)h(z) + Q_2(z)/h(z))/(2A)$ is a meromorphic solution of (1.10), then $Bf^{(m)}(z+c)+Cf^{(n)}(z) = (Q_1(z)h(z) - Q_2(z)/h(z))/(2\mathrm{i}R(z))$, where $h(z)$ is a non-entire meromorphic function and $Q(z) = Q_1(z)Q_2(z)$. Since $f(z)$ and $(Bf^{(m)}(z+c) + Cf^{(n)}(z))$ share ∞ CM, $R(z)$ reduces to a nonzero constant, say $R \in \mathbb{C} \setminus \{0\}$. We now easily write the given equation (1.10) as

$$BRf^{(m)}(z+c) = -(CRf^{(n)}(z) \mp \mathrm{i}\sqrt{A^2f^2(z) - Q(z)}). \tag{6.1}$$

Since $f^{(n)}(z)$ contributes all the poles of the RHS of (6.1), $f^{(m)}(z+c)$ and $f^{(n)}(z)$ share ∞ CM. Thus the poles of $(Bf^{(m)}(z+c)+Cf^{(n)}(z))$ coincide with the poles of $f^{(m)}(z+c)$ or $f^{(n)}(z)$. In view of (1.10), we conclude that $f(z)$ and $f^{(n)}(z)$ share ∞ CM. But this is possible only when $f(z)$ is an entire function. Thus the conclusion follows. $\square$

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