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A NEW RESULT ON STABILITY ANALYSIS AND H_∞ DYNAMIC OUTPUT FEEDBACK CONTROLLER FOR SYSTEMS WITH TIME-VARYING DELAYS

El khaloufi Ghizlane, Chaibi Noreddine, Boumhidi Ismail and El jimi Driss

The stability and stabilization of systems with time-varying delays and external disturbances are the subject of this study. To circumvent the limitation of the Bessel-Legendre inequality, which cannot treat a time-varying delay system because the resulting limit contains reciprocal convexity, the generalized free-matrix-based integral inequality is used to generate less conservative stability criteria. Improved stabilization requirements are proposed in the form of linear matrix inequalities by developing a new augmented Lyapuno–Krasovskii function. To achieve resolved controller gains, a method for designing a H_∞ dynamic output feedback controller based on linear matrix inequalities is then provided. Finally, three examples are used to validate the advantages of the approach over existing methods.

Keywords: stability, stabilization, free-matrix-based integral inequality, linear matrix inequality, H_∞ dynamic output feedback controller

Classification: 93Dxx, 93B52

1. INTRODUCTION

Delays are omnipresent in physical systems, such as biological systems, electrical systems, networked control systems and industrial automation systems. The presence of a delay in a system degrades performance or can even lead to system instability in a closed loop. The main focus of research into the stability of delay systems is delay-dependent stability and stabilization, [7, 9, 14, 18, 26]. Stability analysis is performed using the Lyapunov–Krasovskii (LK) theorem. The key elements of this strategy are choosing an appropriate LK functional and obtaining a tighter limit for its time derivative. The integral term in the derivative of the LK functional is treated by the free weighting matrix approach [28] and the model transformation method [3] to reduce the conservatism of the derived stability condition. The above works use several integral inequalities to estimate quadratic integral terms. These include Jensen's inequality [2], the Bessel–Legendre inequality (BLI) [16, 20] and the Wirtinger-based integral inequality [19]. For constant delay systems, the BLI has the potential to provide an analytical solution [21], but it has a shortcoming in the application to time-varying delay systems as the resultant bound

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contains a reciprocal convexity [17, 33], equiring a new type of function combination. For time-varying delay systems, the affine BLI was proposed in [9, 22]. However, this affine version cannot cover all the vectors in the Lyapunov–Krasovskii functional, and the derived conditions remain conservative, leaving room for improvement.

Delay-dependent stability requirements are no longer applicable and unattainable if a variable-delay system becomes unstable. Consequently, the creation of a stabilization controller capable of solving instability problems becomes crucial [1, 5, 10]. The same difficulties that occur when determining stability criteria also occur when determining stabilization criteria. In addition, a congruence transformation must be used to translate bilinear matrix inequalities into Linear Matrix Inequality (LMI) in order to derive the negative definite condition in terms of LMI. The advantage of the congruence transformation is that, while retaining its sign definition, it eliminates the bilinear elements of the inequalities [34].

The most relevant literature has assumed that state feedback control can be easily achieved and that the state of the dynamic system is completely quantifiable [24, 25]. In the real world, however, state information will inevitably not always be completely accessible. In these circumstances, there are two main approaches to controller analysis and design: the first is to create an observer-based controller that reconstructs the state of the dynamic system [7, 23]. Creating an Dynamic Output Feedback Controller (DOFC) is the second approach, in which feedback control is performed using the measured output signals of the dynamic systems [11, 13, 27]. When system state is unavailable due to the difficulty of obtaining complete state information, the output feedback control strategy plays a critical role in practical control applications and implementations [8, 6].

In this study, we propose a new stability and stabilization criterion for linear systems with time-varying delays and external perturbations. First, a new augmented LK function is constructed and its derivative is evaluated using the generalized free-matrix-based Integral Inequality [31], permitting the treatment of time-varying delay systems without recourse to the reciprocal convexity lemma. This approach overcomes the drawback of the Bessel–Legendre inequality [15, 22], and provides less conservative stability criteria. We then present a control design technique for linear systems with time-varying delay, where we focus on solving a specific nonlinearity problem, which then produces DOFC gains. Finally, we present new LMI conditions for robust stability evaluation according to H_∞ performance criteria and for DOFC design. Standard numerical packages are used to solve the derived delay-dependent stabilization criteria. The superiority of the proposed stabilization criterion over current criteria is demonstrated, and numerical examples are used to confirm the effectiveness of the proposed approach. The contributions of this paper are then summarized as follows:

1. Less conservative stability criteria are obtained by introducing new state-related terms into LK function and using generalized free-matrix-based integral inequality to evaluate its derivative.

2. A new H_∞ dynamic output feedback control algorithm for time-varying delay systems is implemented, guaranteeing robust closed-loop system stability.

3. The proposed method improves the upper delay limits compared with current results for different lower delay limits.

4. Numerical examples are presented to demonstrate the validity and interest of the proposed method.

The article is structured as follows. Some introductory lemmas used to construct stabilization criteria are provided in section 2. Closed-loop system stability analysis is described in section 3, where the dynamic output feedback controller is designed for linear systems with time-varying delays and external disturbances. Two simulation examples are performed and comparative results are presented in section 4. Finally, section 5 presents conclusions.

In addition to the standard notations used throughout the work, $diag(M, N)$ denotes the matrix $\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$ for any matrices M, N. Moreover, we define $He(M) = M + M^T$ for every square matrix M. The set $\mathbf{S}^n_+$ denotes the set of symmetric positive definite matrices. The exponents T and -1 represent the transpose and inverse of a matrix, respectively. $\left(\begin{smallmatrix} a \\ b \end{smallmatrix} \right)$ represents the binomial coefficients $\frac{a!}{b!(a-b)!}$.

2. MODEL DESCRIPTION AND PRELIMINARIES

Consider a linear system with time-varying delays:

$$\begin{cases} \dot{\xi}(t) = A\xi(t) + A_d\xi(t - d(t)) + B(u(t) + w(t)), \quad t \geq 0, \\ y(t) = C\xi(t) + w(t), \\ \xi(t) = \varrho(t), \quad -d_2 \leq t \leq 0, \end{cases} \tag{1}$$

where $A, A_d, B, C \in \mathbf{R}^{n\times n}$ are the system matrices, $\xi(t) \in \mathbf{R}^{n_\xi}, u(t) \in \mathbf{R}^{n_u}, y(t) \in \mathbf{R}^{n_y}$ are, respectively, the system state, the control input, the measured output, $w(t) \in \mathbf{R}^{n\times w}$ is the disturbance input, which belongs to $L_2[0, \infty)$ and $\varrho(t)$ is the initial condition. The time-varying delay $d(t)$ is continuous and satisfies,

$$0 \leq d_1 \leq d(t) \leq d_2, \quad d_{12} \triangleq d_2 - d_1. \tag{2}$$

We assume that the system states (1) are not accessible, in order to implement a complete state feedback control law. Therefore, to solve this problem, we use a DOFC described as follows:

$$\begin{cases} \dot{\xi}_c(t) = A_c\xi_c(t) + B_c y(t), \\ u(t) = C_c\xi_c(t) + D_c y(t), \end{cases} \tag{3}$$

where $\xi_c(t) \in \mathbf{R}^n$ is the controller state, and A_c, B_C, C_c, D_c are controller gains with appropriate dimensions to be determined.

The system (1) with controller (3) is provided by,

$$\begin{cases} \dot{\hat{\xi}}(t) = \hat{A}\hat{\xi}(t) + \hat{A}_d\hat{\xi}(t - d(t)) + \hat{B}w(t), \\ \hat{y}(t) = D\hat{\xi}(t), \end{cases} \tag{4}$$

where $\hat{\xi}(t) = \mathrm{col}\{\xi(t), \xi_c(t)\}$ and $\hat{A} = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}$, $\hat{A}_d = \begin{bmatrix} A_d & 0 \\ 0 & 0 \end{bmatrix}$, $\hat{B} = \begin{bmatrix} BD_c + B \\ B_c \end{bmatrix}$ and $D = \begin{bmatrix} I & 0 \end{bmatrix}$.

For a prescribed scalar $\delta > 0$, we define the performance index

$$\mathcal{J}(\hat{y}(t), w(t)) = \int_0^t \hat{y}^T(s)\hat{y}(s)\,\mathrm{d}s - \delta^2 \int_0^t w^T(s)w(s)\,\mathrm{d}s. \tag{5}$$

The objective of this paper is to establish new LMI conditions that ensure the asymptotic stability of system (1) and to design a H_∞ DOFC (3), such that, for any $d(t)$ satisfying (2),

1. The closed-loop system (4) of system (1) is asymptotically stable under the condition $w(t) = 0, \forall t \geqslant 0$.

2. The H_∞ performance

$$\|\hat{y}(t)\|_2 < \delta \|w(t)\|_2 \tag{6}$$

of the closed-loop system (4) is guaranteed for all nonzero $w(t) \in L_2[0, \infty)$ and a prescribed $\delta > 0$ under the condition $\hat{\xi}(t) = 0, \forall t \in [-d_2, 0]$.

Before presenting our main results, we introduce the following lemmas, which are essential for the derivation of the main results.

Lemma 2.1. (Xu and Lam [29] The following inequality holds for all $\alpha, \beta \in \mathbf{R}^n$ and $Q \in \mathbf{R}_+^{n\times n}$,

$$-2\alpha^T\beta \leq \alpha^T Q\alpha + \beta^T Q^{-1}. \tag{7}$$

Lemma 2.2. (Seuret and Gouaisbaut [21]) Let ξ be a continuous and differentiable function: $[a, b] \to \mathbf{R}^n$ and $N \in \mathbf{N}$. Any matrix $Z \in \mathbf{R}_+^{n\times n}$ satisfies the following inequality,

$$-\int_a^b \dot{\xi}^T(s)Z\dot{\xi}(s)\,\mathrm{d}s \leq -\frac{1}{b-a}\Upsilon_N^T \left[\sum_{k=0}^{N}(2k+1)\pi_N^T(k)Z\pi_N(k)\right]\Upsilon_N, \tag{8}$$

where

$$\Upsilon_N = \begin{cases} \begin{bmatrix} \xi^T(b) & \xi^T(a) \end{bmatrix}^T, & N = 0, \\ \begin{bmatrix} \xi^T(b) & \xi^T(a) & \frac{1}{b-a}\Sigma_0^T & \cdots \frac{1}{b-a}\Sigma_{N-1}^T \end{bmatrix}^T, & N > 1, \end{cases} \tag{9}$$

$$\pi_N(k) = \begin{cases} \begin{bmatrix} I & -I \end{bmatrix}, & N = 0, \\ \begin{bmatrix} I & (-1)^{k+1}I & \sigma_{Nk}^0 I \cdots \sigma_{Nk}^{N-1} I \end{bmatrix}, & N \geq 1, \end{cases} \tag{10}$$

$$\sigma_{Nk}^j = \begin{cases} (2j+1)\left((-1)^{k+j} - 1\right), & j \leq k, \\ 0, & j > k, \end{cases} \tag{11}$$

$$\mathrm{F}_k(s) = (-1)^k \sum_{i=0}^{k}\left[(-1)^i \begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k+i \\ i \end{pmatrix}\right]\left(\frac{s-a}{b-a}\right)^i, \tag{12}$$

$$\Sigma_k = \int_a^b \mathrm{F}_k(s)\xi(s)\,\mathrm{d}s. \tag{13}$$

Lemma 2.3. (Zeng et al. [31]) Let ξ be a continuous, differentiable function: $[a,b] \to R^n$ and $N \in \mathbf{N}, \psi \in \mathbf{R}^m$. This inequality holds for all matrices $Z \in \mathbf{R}_+^{n\times n}$, $M \in \mathbf{R}^{(N+1)n\times m}$,

$$-\int_a^b \dot{\xi}^{\mathrm{T}}(s) Z\dot{\xi}(s)\,\mathrm{d}s \le 2\Upsilon_N^T \Gamma_N^T M\psi + (b-a)\psi^T M^T \tilde{Z} M\psi, \tag{14}$$

where

$$\Gamma_N = \begin{bmatrix} \pi_N^T(0) & \pi_N^T(1)\cdots\pi_N^T(N) \end{bmatrix}^T, \tag{15}$$

$$\tilde{Z} = \mathrm{diag}\left\{\frac{1}{Z}, \frac{1}{3Z}, \ldots, \frac{1}{(2N+1)Z}\right\}, \tag{16}$$

Υ_N is defined according to Lemma 2.2 .

P r o o f. According to Lemma 2.2

$$\begin{aligned} -\int_a^b \dot{\xi}^T(s) Z\dot{\xi}(s)\mathrm{d}s &\le -\frac{1}{b-a}\Upsilon_N^T \left[\sum_{k=0}^{N}(2k+1)\pi_N^T(k) Z\pi_N(k)\right]\Upsilon_N \\ &\le -\frac{1}{b-a}\Upsilon_N^T \left[\pi_N^T(0), \pi_N^T(1), \ldots, \pi_N^T(N)\right] \\ &\quad \mathrm{diag}\left\{Z, 3Z, \ldots (2N+1)Z\right\}\left[\pi_N(0), \pi_N(1), \ldots, \pi_N(N)\right]\Upsilon_N \\ &\le -\frac{1}{b-a}\Upsilon_N^T \Gamma_N^T \hat{Z} \Gamma_N \Upsilon_N, \end{aligned}$$

we apply Lemma 2.1

$$-\frac{1}{b-a}\Upsilon_N^T \Gamma_N^T \hat{Z}\Gamma_N \Upsilon_N \le \frac{1}{b-a}\left(2\alpha^T\beta + \beta^T \hat{Z}^{-1}\beta\right), \tag{17}$$

let

$$\alpha = \Gamma_N \Upsilon_N, \tag{18}$$

$$\beta = (b-a)M\xi, \tag{19}$$

$$\tilde{Z} = \hat{Z}^{-1}. \tag{20}$$

By substituting (18), (19) and (20) into (17), inequality (14) in Lemma 2.3 is obtained. The proof is complete. □

Remark 2.4. The disadvantage of the BLI (8), which presents a reciprocal convexity that is difficult to handle in the resulting limit for determining stability conditions for time-varying delay systems, is resolved by Lemma 2.3, which allows us to transform this reciprocal convexity and using the convexity property, we obtain conditions that are fairly easy to handle. Note that many current inequalities can be seen as particular cases of inequality (14). Examples include Jensen's inequality [2], the Wirtinger-based integral inequality [19], the affine BLI [15] and the inequality based on free matrices [30].

3. STABILITY ANALYSIS AND CONTROL SYNTHESIS

In this section, the stability and stabilization of a linear system with a time-varying delay and an external disturbance are investigated using a generalized free-matrix-based Integral Inequality. This section is divided into two parts. The first presents the complete LK function proposed for the system with a time-varying delay and an external disturbance. The second aims at obtaining LMI conditions for synthesizing controller gains. The following notations will be used in this section to simplify the presentation:

$$\begin{aligned} \hat{e}_i &= \begin{bmatrix} 0_{n\times(i-1)n} & I_n & 0_{n\times(16-i)n} \end{bmatrix}, i = 1, \ldots, 16, \\ g &= \hat{A}\hat{e}_1 + \hat{A}_d\hat{e}_3 - \hat{e}_{15} + \hat{B}\hat{e}_{16}, \\ \bar{E} &= \hat{e}_{15} + \hat{e}_1 + \hat{e}_3. \end{aligned} \tag{21}$$

and

$$\begin{aligned} \varpi_1(t) &= \begin{bmatrix} \hat{\xi}^T(t) & \hat{\xi}^T(t-d_1) & \hat{\xi}^T(t-d(t)) & \hat{\xi}^T(t-d_2) \end{bmatrix}^T, \\ \varpi_2(t) &= \frac{1}{d_1}\left[\int_{-d_1}^{0} \hat{\xi}_t^T(s)\,\mathrm{d}s \quad \int_{-d_1}^{0} F_1(s)\hat{\xi}_t^T(s)\,\mathrm{d}s\right]^T, \\ \varpi_3(t) &= \frac{1}{d(t)-d_1}\left[\int_{-d(t)}^{-d_1} \hat{\xi}_t^T(s)\,\mathrm{d}s \quad \int_{-d(t)}^{-d_1} F_2(s)\hat{\xi}_t^T(s)\mathrm{d}s\right]^T, \\ \varpi_4(t) &= \frac{1}{d_2-d(t)}\left[\int_{-d_2}^{-d(t)} \hat{\xi}_t^T(s)\,\mathrm{d}s \quad \int_{-d_2}^{-d(t)} F_3(s)\hat{\xi}_t^T(s)\mathrm{d}s\right]^T, \\ \varpi_5(t) &= (d(t)-d_1)\,\varpi_3(t), \quad \varpi_6(t) = (d_2-d(t))\,\varpi_4(t), \\ \varpi_7(t) &= \begin{bmatrix} \int_{-d_2}^{-d_1} \hat{\xi}_t^T(s)\,\mathrm{d}s & d_{12}\int_{-d_2}^{-d_1} F_4(s)\hat{\xi}_t^T(s)\,\mathrm{d}s \end{bmatrix}^T, \\ \varpi_8(t) &= \begin{bmatrix} \dot{\hat{\xi}}(t) & w(t) \end{bmatrix}^T, \end{aligned} \tag{22}$$

and the functions F_k $(k = 1, \ldots, 4)$, presented in Lemma 2.2, are given by,

$$\begin{aligned} F_1(s) &= 2\frac{s+d_1}{d_1} - 1, & F_2(s) &= 2\frac{s+d(t)}{d(t)-d_1} - 1, \\ F_3(s) &= 2\frac{s+d_2}{d_2-d(t)} - 1, & F_4(s) &= 2\frac{s+d_2}{d_{12}} - 1. \end{aligned} \tag{23}$$

3.1. Stability analysis

The stability criterion for the system (4) is presented as follows,

Theorem 3.1. If there exist non-negative scalars d_1, d_2 satisfying the conditions in (2), constant $\delta > 0$, the matrices $P \in \mathbf{S}_{+}^{5n}$, $Q_1, Q_2, Z_1, Z_2 \in \mathbf{S}_{+}^{n}$, $S_1, S_2 \in \mathbf{R}^{16n\times 2n}$, $N_1, N_2 \in \mathbf{R}^{16n\times 3n}$ and matrices $V, \hat{A}, \hat{A}_d, \hat{B}$ such that LMIs (24) and (25) hold

$$\begin{bmatrix} \Xi(d_1) & d_{21}N_2 \\ d_{21}N_2^T & -\hat{Q}_2 \end{bmatrix} < 0, \tag{24}$$

$$\begin{bmatrix} \Xi(d_2) & d_{21}N_1 \\ d_{21}N_1^T & -\hat{Q}_2 \end{bmatrix} < 0, \tag{25}$$

for all $\theta \in \mathbf{R}$,

$$\Xi(\theta) = \Phi(\theta) + 2\bar{E}^T V g, \tag{26}$$

and

$$\begin{aligned}
\Phi(\theta) =& \operatorname{He}\left(L_1^T(\theta) P L_0 + S_1 g_1(\theta) + S_2 g_2(\theta)\right) + \hat{Z} + \hat{e}_{15}^T \left(d_1^2 Q_1 + d_{21}^2 Q_2\right) \hat{e}_{15} \\
& - \mathcal{K}_0^T \hat{Q}_1 \mathcal{K}_0 + d_{21} \operatorname{He}\left(N_1 \mathcal{K}_1 + N_2 \mathcal{K}_2\right) + \hat{e}_1^T \hat{e}_1 - \delta^2 \hat{e}_{16}^T \hat{e}_{16}, \\
\hat{Z} =& \operatorname{diag}\left(Z_1, -Z_1 + Z_2, 0_{n\times n}, -Z_2, 0_{12n\times 12n}\right), \\
\hat{Q}_i =& \operatorname{diag}\left(Q_i, 3Q_i, 5Q_i\right), \quad i = 1, 2, \\
\tilde{Q}_i =& \hat{Q}_i^{-1}, \\
L_0 =& \begin{bmatrix} \hat{e}_{15}^T & \hat{e}_1^T - \hat{e}_2^T & \hat{e}_1^T + \hat{e}_2^T - 2\hat{e}_5^T & \hat{e}_2^T - \hat{e}_4^T & \hat{L}_0^T \end{bmatrix}^T, \\
\hat{L}_0 =& d_{12}\left(\hat{e}_2 + \hat{e}_4\right) - 2\left(\hat{e}_{11} + \hat{e}_{13}\right), \\
\hat{L}_1(\theta) =& \left(d_2 - \theta\right)\left(\hat{e}_{11} + \hat{e}_{14}\right) + \left(\theta - d_1\right)\left(\hat{e}_{12} - \hat{e}_{13}\right), \\
L_1(\theta) =& \begin{bmatrix} \hat{e}_1^T & d_1\hat{e}_5^T & d_1\hat{e}_6^T & \hat{e}_{11}^T + \hat{e}_{13}^T & \hat{L}_1^T(\theta) \end{bmatrix}^T, \\
\mathcal{K}_0 =& \begin{bmatrix} \hat{e}_1^T - \hat{e}_3^T & \hat{e}_1^T + \hat{e}_3^T - 2\hat{e}_5^T & \hat{e}_1^T - \hat{e}_3^T - 6\hat{e}_6^T \end{bmatrix}^T, \\
\mathcal{K}_1 =& \begin{bmatrix} \hat{e}_1^T - \hat{e}_3^T & \hat{e}_1^T + \hat{e}_3^T - 2\hat{e}_7^T & \hat{e}_1^T - \hat{e}_3^T + 6\hat{e}_7^T - 12\hat{e}_8^T \end{bmatrix}^T, \\
\mathcal{K}_2 =& \begin{bmatrix} \hat{e}_3^T - \hat{e}_4^T & \hat{e}_3^T + \hat{e}_4^T - 2\hat{e}_9^T & \hat{e}_3^T - \hat{e}_4^T + 6\hat{e}_9^T - 12\hat{e}_{10}^T \end{bmatrix}^T, \\
g_1(\theta) =& \left(\theta - d_1\right)\begin{bmatrix} \hat{e}_7 \\ \hat{e}_8 \end{bmatrix} - \begin{bmatrix} \hat{e}_{11} \\ \hat{e}_{12} \end{bmatrix}, \\
g_2(\theta) =& \left(d_2 - \theta\right)\begin{bmatrix} \hat{e}_9 \\ \hat{e}_{10} \end{bmatrix} - \begin{bmatrix} \hat{e}_{13} \\ \hat{e}_{14} \end{bmatrix}.
\end{aligned} \tag{27}$$

Then, For any disturbance $w(t) \in L_2[0, \infty)$, the system (4) is asymptotically stable while ensuring an H_∞ performance index δ.

P r o o f. Consider a Lyapunov functional given by

$$V\left(\hat{\xi}_t, \dot{\hat{\xi}}_t\right) = V_1\left(\hat{\xi}_t\right) + V_2\left(\hat{\xi}_t\right) + V_3\left(\hat{\xi}_t, \dot{\hat{\xi}}_t\right), \tag{28}$$

where

$$V_1\left(\hat{\xi}_t\right) = \bar{\xi}^T(t)P\bar{\xi}(t), \tag{29}$$

$$V_2\left(\hat{\xi}_t\right) = \int_{t-d_1}^{t} \hat{\xi}^T(s)Z_1\hat{\xi}(s)\mathrm{d}s + \int_{t-d_2}^{t-d_1} \hat{\xi}^T(s)Z_2\hat{\xi}(s)\,\mathrm{d}s, \tag{30}$$

$$\begin{aligned} V_3\left(\hat{\xi}_t,\dot{\hat{\xi}}_t\right) =& d_1\int_{-d_1}^{0}\int_{t+\theta}^{t}\dot{\hat{\xi}}^T(s)Q_1\dot{\hat{\xi}}(s)\,\mathrm{d}s\mathrm{d}\theta \\ &+ d_{12}\int_{-d_2}^{-d_1}\int_{t+\theta}^{t}\dot{\hat{\xi}}^T(s)Q_2\dot{\hat{\xi}}(s)\,\mathrm{d}s\mathrm{d}\theta, \end{aligned} \tag{31}$$

and where $d_{21} = d_2 - d_1$ and $\bar{\xi} = \mathrm{col}\left\{\hat{\xi}_1(t), d_1\varpi_2(t), \varpi_7(t)\right\}$.
The derivative of $\dot{V}_1(\hat{\xi}_t)$ satisfies the relation,

$$\dot{V}_1(\hat{\xi}_t) = 2\bar{\xi}(t)^T P\dot{\bar{\xi}}(t). \tag{32}$$

The augmented vector (33) is used to represent $\bar{\xi}(t)$ and $\dot{\bar{\xi}}(t)$,

$$\psi(t) = \mathrm{col}\left\{\varpi_1(t),\varpi_2(t),\varpi_3(t),\varpi_4(t),\varpi_5(t),\varpi_6(t),\varpi_8(t)\right\}, \tag{33}$$

where $\varpi_i(t)$, for $i = 1, ..., 8$, are given in (22),

$$d_1\varpi_2(t) = d_1\left[\begin{array}{cc}\hat{e}_5^T & \hat{e}_6^T\end{array}\right]\psi, \tag{34}$$

$$\begin{aligned} \varpi_7(t) &= \left[\begin{array}{cc}\int_{-d_2}^{-d_1}\hat{\xi}_t^T(s)\,\mathrm{d}s & d_{21}\int_{-d_2}^{-d_1}F_4(s)\hat{\xi}_t^T(s)\,\mathrm{d}s\end{array}\right]^T \\ &= \left[\begin{array}{cc}\int_{-d_2}^{-d_1}\hat{\xi}_t^T(s)\,\mathrm{d}s & d_{21}(\int_{-d_2}^{-d}F_4(s)\hat{\xi}_t^T(s)\,\mathrm{d}s + \int_{-d}^{-d_1}F_4(s)\hat{\xi}_t^T(s)\,\mathrm{d}s)\end{array}\right]^T, \end{aligned} \tag{35}$$

from equation (23), we have

$$\begin{aligned} d_{21}F_4(s) =& (d-d_1)F_2(s) + (d_2-d) \\ =& (d_2-d)F_3(s) + (d-d_1), \end{aligned} \tag{36}$$

then

$$\begin{aligned} d_{21}\int_{-d_2}^{-d}F_4(s)\hat{\xi}_t^T(s)\,\mathrm{d}s &= (d_2-d)\int_{-d_2}^{-d}F_3(s)\hat{\xi}_t^T(s) - (d-d_1)\int_{-d_2}^{-d}\hat{\xi}_t^T(s), \\ d_{21}\int_{-d}^{-d_1}F_4(s)\hat{\xi}_t^T(s)\,\mathrm{d}s &= (d-d_1)\int_{-d}^{-d_1}F_2(s)\hat{\xi}_t^T(s) - (d_2-d)\int_{-d}^{-d_1}\hat{\xi}_t^T(s), \end{aligned} \tag{37}$$

by reintroducing (37) into (35), we obtain that

$$\varpi_7(t) = \left[\begin{array}{cc}\hat{e}_{11}^T + \hat{e}_{13}^T & (d-d_1)(\hat{e}_{12}^T - \hat{e}_{13}^T) + (d_2-d)(\hat{e}_{11}^T + \hat{e}_{14}^T)\end{array}\right]^T\psi, \tag{38}$$

$$\varpi_7(t) = \begin{bmatrix} \hat{e}_{11}^T + \hat{e}_{13}^T & \hat{L}_1^T(d) \end{bmatrix}^T \psi,$$

according to equation (34) and (38), we obtain

$$\begin{aligned} \bar{\xi}(t) &= \begin{bmatrix} \hat{e}_1^T & d_1\hat{e}_5^T & d_1\hat{e}_6^T & \hat{e}_{11}^T + \hat{e}_{13}^T & \hat{L}_1^T(d) \end{bmatrix}^T \psi, \\ \bar{\xi}(t) &= L_1(d), \end{aligned} \tag{39}$$

we now calculate the derivative of $\bar{\xi}(t)$,

$$\begin{aligned} \dot{\xi}_1(t) &= \hat{e}_{15}\psi, \\ d_1\dot{\varpi}_2(t) &= \begin{bmatrix} \hat{e}_1^T - \hat{e}_2^T & \hat{e}_1^T + \hat{e}_2^T - 2\hat{e}_5^T \end{bmatrix}^T \psi, \\ \dot{\varpi}_7(t) &= \begin{bmatrix} \hat{e}_2^T - \hat{e}_4^T & d_{21}(\hat{e}_2^T + \hat{e}_4^T) - 2(\hat{e}_{11}^T + \hat{e}_{13}^T) \end{bmatrix}^T \psi, \\ &= \begin{bmatrix} \hat{e}_2^T - \hat{e}_4^T & \hat{L}_0^T \end{bmatrix}^T \psi, \end{aligned} \tag{40}$$

then

$$\begin{aligned} \dot{\bar{\xi}}(t) &= \begin{bmatrix} \hat{e}_{15}^T & \hat{e}_1^T - \hat{e}_2^T & \hat{e}_1^T + \hat{e}_2^T - 2\hat{e}_5^T & \hat{e}_2^T - \hat{e}_4^T & \hat{L}_0^T \end{bmatrix}^T \psi, \\ \dot{\bar{\xi}}(t) &= L_0\psi, \end{aligned} \tag{41}$$

substituting (39) and (41) into (32) gives

$$\begin{aligned} \dot{V}_1(\hat{\xi}_t) &= 2(L_1(d)\psi)^T P L_0 \psi(t) \\ &= \psi^T \operatorname{He}\left(L_1(d)^T P L_0\right) \psi(t). \end{aligned} \tag{42}$$

Additionally, we have $\varpi_5(t) = (d - d_1)\varpi_3(t)$ and $\varpi_6(t) = (d_2 - d)\varpi_4(t)$. Thus, for all matrices S_1, S_2 in $\mathbb{R}^{16n\times 2n}$, the following equality holds using the matrices g_1 and g_2 described in (27),

$$2\psi^T \left(S_1 g_1(d) + S_2 g_2(d)\right) \psi = 0,$$

therefore, the derivative of $V_1(\xi_t)$ is given by

$$\dot{V}_1\left(\hat{\xi}_t\right) = \psi^T \operatorname{He}\left(L_1^T(d) P L_0 + S_1 g_1(d) + S_2 g_2(d)\right) \psi, \tag{43}$$

the derivatives of $V_2(\xi_t)$, yields,

$$\begin{aligned} \dot{V}_2\left(\hat{\xi}_t\right) &= \hat{\xi}^T(t) Z_1 \hat{\xi}(t) - \hat{\xi}^T(t-d_1) Z_1 \hat{\xi}(t-d_1) \\ &\quad + \hat{\xi}^T(t-d_1) Z_2 \hat{\xi}(t-d_1) - \hat{\xi}^T(t-d_2) Z_2 \hat{\xi}(t-d_2) \\ &= \psi^T \hat{Z}, \end{aligned} \tag{44}$$

the derivative of $V_3(\hat{\xi}_t, \dot{\hat{\xi}}_t)$ gives,

$$\dot{V}_3(\hat{\xi}_t, \dot{\hat{\xi}}_t) = d_1^2 \dot{\hat{\xi}}(t)^T Q_1 \dot{\hat{\xi}}(t) - d_1 \int_{t-d_1}^{t} \dot{\hat{\xi}}(s)^T Q_1 \dot{\hat{\xi}}(s)\, \mathrm{d}s \tag{45}$$

$$+ d_{21}^2\dot{\hat{\xi}}(t)^T Q_2 \dot{\hat{\xi}}(t) - d_{21}\int_{t-d_2}^{t-d} \dot{\hat{\xi}}(s)^T Q_2 \dot{\hat{\xi}}(s)\,\mathrm{d}s$$
$$- d_{21}\int_{t-d}^{t-d_1} \dot{\hat{\xi}}(s)^T Q_2 \dot{\hat{\xi}}(s)\,\mathrm{d}s,$$

with $N = 2$, apply Lemma 2.2.

$$-\int_{t-d_1}^{t} \dot{\hat{\xi}}(s)^T Q_1 \dot{\hat{\xi}}(s)\,\mathrm{d}s \le -\frac{1}{d_1}\Upsilon_2^T\Gamma_2^T\hat{Q}_1\Gamma_2\Upsilon_2, \tag{46}$$

where

$$\Upsilon_2 = \begin{bmatrix} \hat{\xi}^T(t) & \hat{\xi}^T(t-d_1) & \frac{1}{d_1}\int_{-d_1}^{0}\hat{\xi}^T(s)\,\mathrm{d}s & \frac{1}{d_1}\int_{-d_1}^{0}F_1(s)\hat{\xi}^T(s)\,\mathrm{d}s \end{bmatrix}^T,$$

then

$$\begin{aligned}\Upsilon_2^T\Gamma_2^T &= \psi^T\begin{bmatrix} \hat{e}_1^T - \hat{e}_2^T & \hat{e}_1^T + \hat{e}_2^T - 2\hat{e}_5^T & \hat{e}_1^T - \hat{e}_2^T - 6\hat{e}_6^T \end{bmatrix}\\ &= \psi^T\mathcal{K}_0^T.\end{aligned} \tag{47}$$

Substituting (47) into (46) gives,

$$-\int_{t-d_1}^{t} \dot{\hat{\xi}}(s)^T Q_1 \dot{\hat{\xi}}(s)\,\mathrm{d}s \le -\frac{1}{d_1}\psi^T\mathcal{K}_0^T\hat{Q}_1\mathcal{K}_0\psi, \tag{48}$$

using Lemma 2.3 with $N = 2$,

$$-\int_{t-d}^{t-d_1} \dot{\hat{\xi}}^{\mathrm{T}}(s) Q_2 \dot{\hat{\xi}}(s)\,\mathrm{d}s \le 2\Upsilon_2^T\Gamma_2^T S_1\psi + (d-d_1)\psi^T S_1^T\tilde{Q}_2 S_1\psi. \tag{49}$$

Following the same procedure in (47), one has

$$\begin{aligned}\Upsilon_2^T\Gamma_2^T &= \psi^T\begin{bmatrix} \hat{e}_1^T - \hat{e}_3^T & \hat{e}_1^T + \hat{e}_3^T - 2\hat{e}_7^T & \hat{e}_1^T - \hat{e}_3^T + 6\hat{e}_7^T - 12\hat{e}_8^T \end{bmatrix}\\ &= \psi^T\mathcal{K}_1^T.\end{aligned} \tag{50}$$

Substituting (50) into (49) gives,

$$\begin{aligned}-\int_{t-d}^{t-d_1} \dot{\hat{x}}^{\mathrm{T}}(s) Q_2 \dot{\hat{\xi}}(s)\,\mathrm{d}s &\le 2\psi^T\mathcal{K}_1^T\bar{S}_1^T\psi + (d-d_1)\psi^T\bar{S}_1\tilde{Q}_2\bar{S}_1^T\psi\\ &\le \psi^T\,\mathrm{He}\left(\bar{S}_1\mathcal{K}_1\right)\psi + (d-d_1)\psi^T\bar{S}_1\tilde{Q}_2\bar{S}_1^T\psi\\ &\le -\psi^T\Big(\mathrm{He}\left(S_1\mathcal{K}_1\right) - (d-d_1)S_1\tilde{Q}_2S_1^T\Big)\psi\\ &\le -\psi^T\Lambda_1(d)\psi.\end{aligned} \tag{51}$$

Following the same procedure in (51), one has

$$-\int_{t-d_2}^{t-d} \dot{\hat{\xi}}^{\mathrm{T}}(s) Q_2 \dot{\hat{\xi}}(s)\,\mathrm{d}s \le -\psi^T\Big(\mathrm{He}\left(S_2\mathcal{K}_2\right) - (d-d_1)S_2\tilde{Q}_2S_2^T\Big)\psi \tag{52}$$

$$\leq -\psi^T \Lambda_2(d)\psi.$$

Substituting (48), (51) and (52) into (45) gives,

$$\dot{V}_3(\hat{\xi}_t, \dot{\hat{\xi}}_t) \leq \psi^T \Big(\hat{e}_{15}^T (d_1^2 Q_1 + d_{21}^2 Q_2)\hat{e}_{15} - \mathcal{K}_0^T \hat{Q}_1 \mathcal{K}_0 - d_{21}(\Lambda_1(d) + \Lambda_2(d)) \Big)\psi. \quad (53)$$

From equations (43), (44) and (53), we have,

$$\begin{aligned} \dot{V}(\hat{\xi}_t, \dot{\hat{\xi}}_t) &\leq \dot{V}_1(\hat{\xi}_t) + \dot{V}_2(\hat{\xi}_t) + \dot{V}_3(\hat{\xi}_t, \dot{\hat{\xi}}_t) \\ &= \psi^T \Omega(d)\psi, \end{aligned} \quad (54)$$

where

$$\begin{aligned} \Omega(d) = \text{He}\left(L_1^T(d) P L_0 + S_1 g_1(d) + S_2 g_2(d)\right) + \hat{Z} + \hat{e}_1^T (d_1^2 Q_1 + d_{21}^2 Q_2)\hat{e}_1 \\ - \mathcal{K}_0^T \hat{Q}_1 \mathcal{K}_0 - d_{21}(\Lambda_1(d) + \Lambda_2(d)). \end{aligned}$$

If g is the matrix defined in (21), then $g\psi(t) = 0$ can be easily verified. It follows that, for any invertible matrix V,

$$2\psi^T(t)(\bar{E}^T V g)\psi(t) = 0, \quad (55)$$

$\bar{E}$ is defined in (21). Summing (55) with (54) leads to,

$$\dot{V}(\hat{\xi}_t, \dot{\hat{\xi}}_t) \leq \psi^T(t)\Big(\Omega(d) + 2\bar{E}^T V g\Big)\psi(t). \quad (56)$$

First, we analyse the asymptotic stability of the system (4) in the absence of perturbation, i. e. for $w(t) = 0$. In this case, equation (56) is rewritten as follows:

$$\dot{V}(\hat{\xi}_t, \dot{\hat{\xi}}_t) \leq \psi_0^T(t)\Big(\Omega(d) + 2\bar{E}^T V g_0\Big)\psi_0(t). \quad (57)$$

where

$$\begin{aligned} \psi_0 &= \text{col}\left\{\varpi_1(t), \varpi_2(t), \varpi_3(t), \varpi_4(t), \varpi_5(t), \varpi_6(t), \dot{\hat{\xi}}(t)\right\}, \\ g_0 &= \hat{A}\hat{e}_1 + \hat{A}_d \hat{e}_3 - \hat{e}_{15}. \end{aligned} \quad (58)$$

Then for $w(t) = 0$, based on the Lyapunov–Krasovskii theorem [12], asymptotic stability is guaranteed if $\Omega(d) + 2\bar{E}^T V g_0 < 0$. According to the Schur complement, this condition is obtained if the following LMIs are satisfied,

$$\begin{bmatrix} \Xi_0(d_1) & d_{21}N_2 \\ d_{21}N_2^T & -\hat{Q}_2 \end{bmatrix} < 0, \quad \begin{bmatrix} \Xi_0(d_2) & d_{21}N_1 \\ d_{21}N_1^T & -\hat{Q}_2 \end{bmatrix} < 0, \quad (59)$$

where

$$\Xi_0(\theta) = \Phi_0(\theta) + 2\bar{E}^T V g_0, \quad (60)$$

and

$$\Phi_0(\theta) = \mathrm{He}\left(L_1^T(\theta)PL_0 + S_1 g_1(\theta) + S_2 g_2(\theta)\right) + \hat{Z} + \hat{e}_{15}^T\left(d_1^2 Q_1 + d_{21}^2 Q_2\right)\hat{e}_{15} \\ - \mathcal{K}_0^T \hat{Q}_1 \mathcal{K}_0 + d_{21}\,\mathrm{He}\left(N_1\mathcal{K}_1 + N_2\mathcal{K}_2\right), \tag{61}$$

Next, assuming that $\varrho(t) = 0, t \in [-d_2, 0]$, we consider the performance index (5) of system (4). From (56) one obtains

$$\begin{aligned}\mathcal{J}(\hat{y}(t), w(t)) &= \int_0^\infty \left[\hat{y}^T(t)\hat{y}(t) - \delta^2 w^T(t)w(t) + \dot{V}(\hat{\xi}_t, \dot{\hat{\xi}}_t)\right] \mathrm{d}t - V(\hat{\xi}_t, \dot{\hat{\xi}}_t)|_{t\to\infty} \\ &\leqslant \int_0^\infty \left[\hat{y}^T(t)\hat{y}(t) - \delta^2 w^T(t)w(t) + \dot{V}(\hat{\xi}_t, \dot{\hat{\xi}}_t)\right] \mathrm{d}t \\ &= \int_0^\infty \psi^T(t)\Pi(d)\psi(t)\,\mathrm{d}t,\end{aligned} \tag{62}$$

where

$$\Pi(d) = \Omega(d) + 2\bar{E}^T V g - \delta^2 \hat{e}_{16}^T \hat{e}_{16} + \hat{e}_1^T \hat{e}_1.$$

By using Schur complement, we get that $\Pi(d) < 0$ if LMIs (24) and (25) are satisfied. Then, system (1) is asymptotically stable with H_∞ performance level δ if conditions (24) and (25) are satisfied. The proof is complete. □

3.2. H_∞ dynamic output feedback controller design

In this subsection, Theorem 3.1 is extended to the design of an H_∞ DOFC for system (1). The corresponding results are summarized in the following theorem.

Theorem 3.2. Given positive scalars $d_1, d_2, \delta > 0$, for any disturbance $w(t) \in L_2[0, \infty)$, the system (4) is asymptotically stable while ensuring an H_∞ performance index δ. If there exist $P \in \mathbf{S}_+^{5n}$, $Q_1, Q_2, Z_1, Z_2 \in \mathbf{S}_+^n, S_1, S_2 \in \mathbf{R}^{16n\times 2n}$, and any matrix $N_1, N_2 \in \mathbf{R}^{16n\times 3n}$, $F \in \mathbf{R}^{n\times n}, E \in \mathbf{R}^{n\times n}$ are symmetric matrices, $H \in \mathbf{R}^{n\times n}, U \in \mathbf{R}^{n\times n}$ are any nonsingular matrices, and matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \overline{A}_d$, such that

$$\begin{bmatrix} \Psi(d_1)+ & d_{21}N_2 \\ d_{21}N_2^T & -\hat{Q}_2 \end{bmatrix} < 0, \tag{63}$$

$$\begin{bmatrix} \Psi(d_2) & d_{21}N_1 \\ d_{21}N_1^T & -\hat{Q}_2 \end{bmatrix} < 0, \tag{64}$$

are valid, for all $\theta \in \mathbf{R}$,

$$\Psi(\theta) = \Phi(\theta) + 2\bar{E}^T \hat{g}, \tag{65}$$

$\Phi(\theta)$, $2\bar{E}^T\hat{g}$ are defined respectively in (27) and (73).
The controller gain matrices are designed as follows,

$$A_c = H^{-1}\left(\mathcal{A} - EAF - EBD_cCF\right)U^{-T} - B_cCFU^{-T} - H^{-1}EBC_c, \tag{66}$$

$$B_c = H^{-1}\left(\mathcal{B} - EBD_c\right), \tag{67}$$
$$C_c = \left(\mathcal{C} - D_cCF\right)U^{-T}, \tag{68}$$
$$D_c = \mathcal{D}. \tag{69}$$

Proof. First, we introduce the following coordinate transformation where the stability of the new system is the same as that of system (4),

$$\varphi(t) = \left(RV^T\right)^{-1}\hat{\xi}(t).$$

Using equation (4), we obtain

$$RV^T\varphi(t) = \hat{\xi}(t), \tag{70}$$
$$RV^T\dot{\varphi}(t) = \dot{\hat{\xi}}(t),$$
$$RV^T\dot{\varphi}(t) = \hat{A}RV^T\varphi(t) + \hat{A}_dRV^T\varphi(t-d(t)) + \hat{B}\omega(t). \tag{71}$$

Then, according to the previous equation, g defined in equation (21) becomes

$$\hat{g} = \hat{A}RV^T\hat{e}_1 + \hat{A}_dRV^T\hat{e}_3 - RV^T\hat{e}_{15} + \hat{B}\hat{e}_{16}. \tag{72}$$

The non-linear term $(2\bar{E}^TVg)$ in Theorem 3.1, which calculates the controller gain, is then as follows,

$$2\bar{E}^TV\hat{g} = 2\bar{E}^TV\left(\hat{A}RV^T\hat{e}_1 + \hat{A}_dRV^T\hat{e}_3 - RV^T\hat{e}_{15} + \hat{B}\hat{e}_{16}\right), \tag{73}$$

we define

$$R^{-1} = \begin{bmatrix} E & * \\ H^T & \hat{E} \end{bmatrix},\ R = \begin{bmatrix} F & * \\ U^T & \hat{F} \end{bmatrix},\ V = \begin{bmatrix} I & 0 \\ E & H \end{bmatrix}. \tag{74}$$

where $F \in \mathbf{R}^{n\times n}, \hat{F} \in \mathbf{R}^{n\times n}, E \in \mathbf{R}^{n\times n}, \hat{E} \in \mathbf{R}^{n\times n}$ are symmetric matrices and $H \in \mathbf{R}^{n\times n}, U \in \mathbf{R}^{n\times n}$ are any nonsingular matrices. It follows that: $FE + UH^T = I$ and $U^TE + \hat{F}H^T = 0$. We replace the matrices R and V in equation (73),

$$V\hat{A}RV^T = V\begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix} RV^T = \begin{bmatrix} AF + B\mathcal{C} & A + B\mathcal{D}C \\ \mathcal{A} & EA + \mathcal{B}C \end{bmatrix},$$
$$V\hat{A}_dRV^T = V\begin{bmatrix} A_d & 0 \\ 0 & 0 \end{bmatrix} RV^T = \begin{bmatrix} A_dF & A_d \\ \overline{A}_d & EA_d \end{bmatrix},$$
$$VRV^T = \begin{bmatrix} F & * \\ I & E \end{bmatrix},$$
$$V\hat{B} = V\begin{bmatrix} BD_c + B \\ B_c \end{bmatrix} = \begin{bmatrix} B\mathcal{D} + B \\ \mathcal{B} + EB \end{bmatrix},$$

where

$$\overline{A}_d = EA_dF,$$

$$\begin{aligned}
\mathcal{A} =& EAF + EBD_cCF + HB_cCF + EBC_cU^T + HA_cU^T,\\
\mathcal{B} =& EBD_c + HB_c,\\
\mathcal{C} =& D_cCF + C_cU^T,\\
\mathcal{D} =& D_c.
\end{aligned}$$

This results in the inequalities (63) and (64). Furthermore, if (63) and (64) are satisfied, the controller gains are determined by equations (66), (67), (68) and (69). □

4. ILLUSTRATIVE EXAMPLES

In this section, three numerical cases are examined and the results are compared with previous studies to illustrate the effectiveness of the proposed method.

Example 4.1. Consider the system studied in [32], where the system and input matrices are as follows

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

For different values of d_1, the maximum upper bounds on the delay d_2 obtained from Theorem 3.1 are calculated and listed in Table 1, together with the results provided in other works. Table 1 shows that the results obtained in this paper are less conservative than the others in the literature.

d_1	0.1	0.4	0.7	1
[16]	2.26	2.29	2.34	2.40
[32]	2.27	2.30	2.36	2.43
Theorem 3.1	2.37	2.98	3.75	3.80

Tab. 1: Admissible upper bound of d_2 for different d_1.

Example 4.2. Consider system (1) with

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.3 & 1 \end{bmatrix}.$$

The system described above, in the absence of control, is unstable because the eigenvalues of the system matrix have positive real parts ($eig(A) = [0; 1]$). We then apply the controller proposed in this work to stabilize the closed-loop system using the following simulation parameters: Time -varying delay $d(t) = 0.5 + 0.345|\sin(2.6087t)|$, $d_1 = 0.5$, $d_2 = 1.591$, the external disturbance $w(t) = 0$, the initial functions $\xi(t) = \varrho(t) = [1\ \ 2]^T$ and based on Theorem 3.2 the gain matrices are given by:

$$A_c = \begin{bmatrix} -0.0277 & -0.3046 \\ -0.2599 & -2.8590 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0.0841 \\ 0.7980 \end{bmatrix}, \quad C_c = \begin{bmatrix} -0.0012 & 0.0083 \end{bmatrix}, \quad D_c = \begin{bmatrix} -1.0023 \end{bmatrix}.$$

Fig. 1 illustrates the system state trajectories with the proposed controller, showing that these trajectories converge to zero. This demonstrates the effective performance of the controller designed in this work. Furthermore, the simulation results validate the theoretical findings and confirm the effectiveness of the proposed method. To demonstrate that Theorem 3.2 is less conservative than existing techniques, we calculate the admissible upper limit of d_2 for different values of d_1 and compare the results with those in [34], [26] and [7], as presented in Table 2.

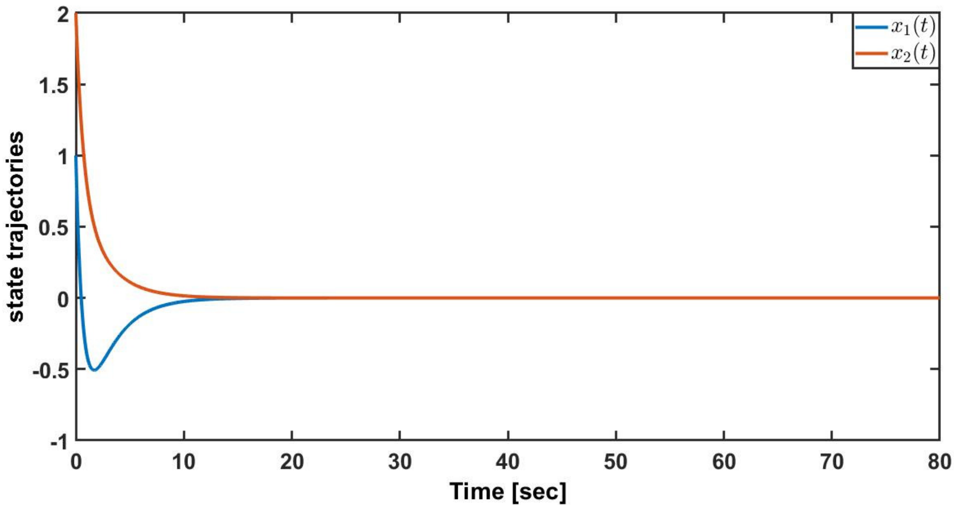

Fig. 1: Closed-loop system state $\xi(t)$ with controller.

Method	d_1	d_2	d_1	d_2
[34]	0.2	0.428	0.5	0.549
[26]	0.2	0.846	0.5	0.863
[7]	0.2	0.963	0.5	0.967
Theorem 3.2	0.2	1.798	0.5	1.887

Tab. 2: Admissible upper bound of d_2 for different d_1.

Example 4.3. We also consider the following dynamics of an additional system (this example is used in [26] and [7] without considering external disturbances).

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -0.9 & -0.1 \\ 0 & 0 & -0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.3 & 1 & 0.2 \end{bmatrix}.$$

We consider a disturbance $w(t) \in L_2[0, \infty)$ given by: $w(t) = \exp(-0.4t)\sin(2t)$.

Based on Theorem 3.2, we increase the values of the delay d_1 to determine the corresponding upper limit of the delay d_2. The results are presented in Table 3, clearly demonstrating the superiority of the proposed approach compared to [26] and [7].
Fig. 2 shows the state trajectories with the stabilizing controller gain:

$$A_c = \begin{bmatrix} 0.0365 & -0.3228 & -0.2939 \\ 0.0429 & -2.2444 & -1.4241 \\ 0.1748 & -2.1185 & -1.7386 \end{bmatrix}, \quad B_c = \begin{bmatrix} -0.0329 \\ -0.2146 \\ -0.1400 \end{bmatrix},$$
$$C_c = \begin{bmatrix} -0.0029 & 0.0142 & 0.0027 \end{bmatrix}, \quad D_c = \begin{bmatrix} -0.9991 \end{bmatrix}.$$

The controller gain is derived using Theorem 3.2 with the following simulation parameters: $\delta = 0.3$, $d_1 = 0.5$, $d_2 = 1.61$, $d(t) = 0.5 + 0.467|\sin(1.071t)|$ and initial functions $\varrho(t) = [6\ \ 4\ \ 2]^T$.

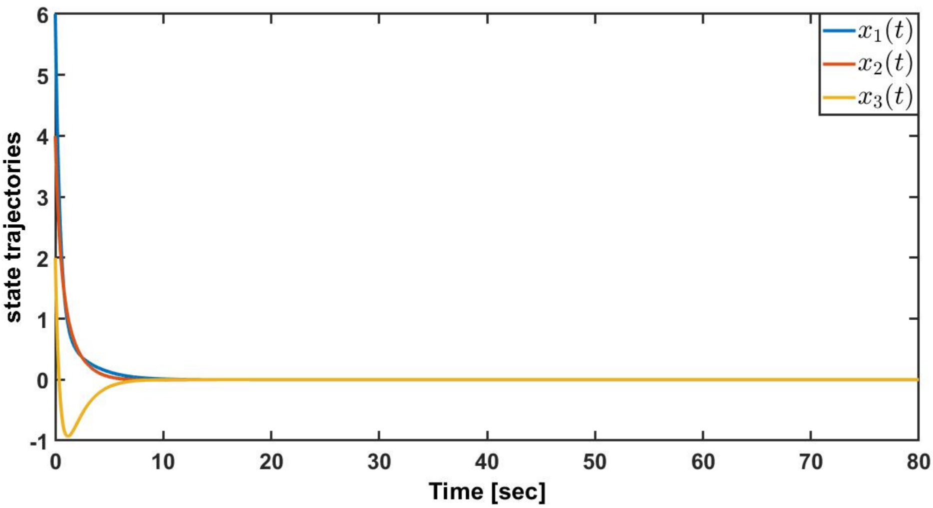

Fig. 2: State responses of the closed-loop system.

Method	d_1	d_2	d_1	d_2
[26]	0	0.851	0.5	0.902
[7]	0	0.930	0.5	0.967
Theorem 3.2	0	1.579	0.5	1.679

Tab. 3: Admissible upper bound of d_2 for different d_1.

5. CONCLUSION

This work proposes a new approach to stability and control design adapted to linear systems with time-varying delay and external disturbance. By using the generalized free-matrix-based integral inequality to handle the integral term of the LK function, a new and improved LK function is obtained. Then, by solving a specific nonlinearity problem, a new LMI condition is derived to determine the gains of the DOFC. Finally, numerical examples are provided to demonstrate the effectiveness of the results.

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