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A TOPOLOGICAL STUDY IN THE SET OF ZERO-DIMENSIONAL SUBRINGS OF A COMMUTATIVE RING

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Abstract. We investigate the relationship between the space $\mathcal{Z}(R,T)$, defined as the largest closed subset of a ring T with respect to a countable topology, and the classical prime spectrum $\operatorname{Spect}(R)$ of a subring R. We explore the topological properties of $\mathcal{Z}(R,T)$ and establish connections with $\operatorname{Spect}(R)$ under certain conditions.

Keywords: zero-dimensional subring; filter; $\mathcal{F}$-topology; countably compact

MSC 2020: 13A99, 13A15, 13B02, 54H99

1. Introduction

The motivation for studying the space of collection of zero-dimensional subrings of a given ring has historical roots in the influential work of Gilmer (see [7], [6], [8]). Additionally, Hochster's research finds its origins in the topological study of the spectrum of a commutative ring (see [9]), which is a fundamental aspect of ring theory.

Given R as a subring of a ring T, we define $\mathcal{Z}(R,T)$ as the collection of zero-dimensional overrings of R that are contained in T. In the special scenario where R is the prime subring of T, we will refer to $\mathcal{Z}(R,T)$ as $\mathcal{Z}(T)$. The initial topological approach to the space $\mathcal{Z}(R,T)$ was established by Mouadi et al. (see [11]).

The objective of this research paper is to investigate the interaction between the prime spectrum of commutative rings and the set of zero-dimensional overrings within a given ring using a countable topology known as the $\mathcal{F}$-topology. To accomplish this, we introduce the $\mathcal{F}$-topology on $\mathcal{Z}(R,T)$. Additionally, we utilize the framework of prime spectrum spaces to gain a contemporary understanding of the entire class of zero-dimensional rings. Theorem 3.12 establishes a continuous surjection that connects the $\mathcal{F}$-topology on $\mathcal{Z}(R,T)$ with the $\mathcal{F}$-topology on $\operatorname{Spect}(R)$.

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The paper's structure is as follows: Section 2 provides background information and notation. After that, we introduce the set $\mathcal{Z}(R,T)$, which comprises zero-dimensional overrings of R contained in T, and outline its fundamental properties. We also define the properties of the $\mathcal{F}$-topology on the spectrum of a commutative ring. Section 3 presents the definition of the $\mathcal{F}$-topology on $\mathcal{Z}(R,T)$ and investigates its relationship with $\operatorname{Spect}(R)$. Finally, using the language of spectral spaces, we offer a contemporary perspective on the collection of all zero-dimensional rings, demonstrating the connection between the introduced topological tools through a continuous surjection.

2. Preliminaries

In this research paper, our focus lies in exploring the correlation between the prime spectrum of a commutative ring and the collection of zero-dimensional overrings present within a fixed ring. To facilitate our analysis, we introduce the required notation and preliminary outcomes.

Consider R as a subring of a ring T, $\mathcal{Z}(R,T)$ represents the set encompassing all zero-dimensional subrings of T that contain R. It is worth noting, as demonstrated in [8], Proposition 2.2, that the set $\mathcal{Z}(R,T)$ may potentially be empty. However, under specific conditions, we can establish the nonemptiness of $\mathcal{Z}(R,T)$.

The subsequent theorem establishes the conditions under which the set $\mathcal{Z}(R,T)$ is nonempty:

Theorem 2.1 ([10], Proposition 1 and [7], Theorem 1.6)**.** *Let R be a subring of a ring T. The following conditions are equivalent:*

(1) $\mathcal{Z}(R,T) \neq \emptyset$.
(2) *The power of the ideal xT is idempotent for each x in R.*
(3) *For each finitely generated ideal I, the set $\{\operatorname{Ann}_R(I^j)_{j=1}^{\infty}\}$ stabilizes for some $m \in \mathbb{N}$.*

In the subsequent theorem, Gilmer addresses the question of whether the set $\mathcal{Z}(R,T)$ is closed under arbitrary intersection:

Theorem 2.2 ([7], Theorem 2.1)**.** *Let R be a subring of a ring T. If $\mathcal{Z}(R,T) \neq \emptyset$, then the collection $\mathcal{Z}(R,T)$ is closed under arbitrary intersection.*

Remark 2.3. Let R be a subring of the ring T. If $\mathcal{Z}(R,T) \neq \emptyset$, then Theorem 2.2 demonstrates that $\mathcal{Z}(R,T)$ possesses a unique minimal element. This minimal element is denoted as R^0 and is referred to as the minimal zero-dimensional extension of R in T.

For each $x \in R$, let $m(x)$ be such that $x^{m(x)}T$ is idempotent, and let s_x denote the pointwise inverse of $x^{m(x)}$ in T. As stated in [7], Theorem 2.5, we can establish that $R^0 = R[s_x\colon\ x \in R]$.

Consider a commutative ring R, and let $\operatorname{Spect}(R)$ denote the set of all prime ideals of R. On $\operatorname{Spect}(R)$, the Zariski topology can be defined by taking the open sets as the collection of sets $D(a) := \{P \in \operatorname{Spect}(R)\colon\ a \notin P\}$ for all $a \in R$. In this topology, the family $\{D_a\colon\ a \in R\}$ forms a basis for the open sets of $\operatorname{Spect}(R)^{\mathrm{zar}}$. The Zariski topology possesses various appealing properties such as being quasi-compact and Kolmogorov, although it is rarely compact. Specifically, $\operatorname{Spect}(R)^{\mathrm{zar}}$ is Hausdorff if and only if it is compact if and only if $\dim(R) = 0$; for more details see [6], Theorem 3.6.

Now, our focus turns to the topological structure of the set (R, T), which comprises all subrings of T containing a given subring R of T. We define a topological structure on (R, T) by considering the subsets listed below as the basis for the open sets:

$$B_S := \{F \in (R, T)\colon\ S \subseteq F\}.$$

For S varying in $B_{fin}(T)$, the set of all finite subsets of T, this topology is called the Zariski topology on (R, T).

If $S := \{x_1, x_2, \ldots, x_n\}$ with $x_j \in T$ for each $j \in \{1, \ldots, n\}$, then

$$B_S := (R[x_1, x_2, \ldots, x_n], T).$$

Hence, the collection of subsets $\mathcal{B} := \{(R[x], T)\colon\ x \in T\}$ forms a basis for the Zariski topology on (R, T). It can be observed that (R, T) is a Kolmogorov topological space (also known as a T_0-space).

In other words, (R, T) is a Kolmogorov topological space because for any two distinct points $R_1, R_2 \in (R, T)$ there exists an open set that contains one of the points but not the other. This property is the defining characteristic of a Kolmogorov space.

Proposition 2.4. *Let R be a subring of a ring T such that $\mathcal{Z}(R, T) \neq \emptyset$. Then*

$$\gamma\colon\ \mathcal{Z}(R, T)^{\tau_{\mathrm{zar}}} \to \operatorname{Spect}(R)^{\tau_{\mathrm{zar}}}$$

is a continuous map.

Proof. The map γ is actually defined as $\gamma(S) = \{P \in \operatorname{Spect}(R)\colon\ P \subseteq S\}$, which sends a zero-dimensional ring $S \in \mathcal{Z}(R, T)$ to the set of prime ideals of R contained in S. In particular, if S has a unique prime ideal Q, then $\gamma(S) = Q$.

If R is zero-dimensional, then any prime ideal of R is maximal and corresponds to a point in $\mathrm{Spect}(R)$, so γ is a surjective map from $\mathcal{Z}(R,T)$ to $\mathrm{Spect}(R)$.

Next, to show that γ is continuous, it is enough to show that $\gamma^{-1}(D_x)$ is open, where $D_x = \{P \in \mathrm{Spect}(R)\colon\ x \in P\}$ is a basic Zariski open subset of $\mathrm{Spect}(R)$. According to [7], Theorem 2.5, for each $x \in R$ there exists s_x, which is the pointwise inverse of $x^{m(x)}$ in T such that $R[s_x, x \in R]$ is the minimal zero-dimensional extension of R in T. Then we have $\gamma^{-1}(D_x) = \mathcal{Z}(R, R[s_x])$, from which it follows that γ is a continuous map. □

Corollary 2.5. *The map* $\gamma\colon\ \mathcal{Z}(R,T)^{\tau_{\text{zar}}} \to \mathrm{Spect}(R)^{\tau_{\text{zar}}}$ *is a homeomorphism if and only if* γ *is injective.*

We will work in at least ZFC, which stands for Zermelo-Fraenkel set theory with the axiom of choice. If I is a set, we recall that a subset $\mathcal{F}$ of the power set of I is called a filter on I if it satisfies the following conditions:

(1) $\emptyset \notin \mathcal{F}$ and $I \in \mathcal{F}$.
(2) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
(3) If $A \in \mathcal{F}$ and $A \subset A' \subset I$, then $A' \in \mathcal{F}$.

To further clarify, a filter $\mathcal{F}$ on a set I is considered maximal if there is no other filter $\mathcal{G}$ on I that strictly contains $\mathcal{F}$. An ultrafilter $\mathcal{F}$ on I is a type of maximal filter, implying that there exists no other filter $\mathcal{G}$ on I such that $\mathcal{F}$ is a proper subset of $\mathcal{G}$, and $\mathcal{G}$ itself is also maximal.

A principal ultrafilter on a set I is an ultrafilter represented by $\mathcal{F}_{i_0}$, where i_0 belongs to I, and it encompasses all subsets of I that include i_0. In simpler terms, $\mathcal{F}_{i_0}$ is defined as the collection of all subsets A of I such that i_0 is an element of A.

The symbol $\beta(I)$ is commonly used to denote the collection of all ultrafilters on a set I.

It is important to note that while the notation $\beta(I)$ can also be used to refer to the Stone-Čech compactification of I, within the realm of ultrafilters, it is commonly employed to represent the collection of all ultrafilters on I. This convention arises due to the inherent connection between ultrafilters on I and points in the Stone-Čech compactification of I. Specifically, each ultrafilter on I corresponds to a unique point in the Stone-Čech compactification, and conversely, every point in the Stone-Čech compactification corresponds to an ultrafilter on I (for more information about the theory of ultrafilter see the very interesting book of Comfort and Negrepontis [1]).

In their recent paper [5], García-Ferreira and Ruza-Montilla introduced an alternative topology on $\mathrm{Spect}(R)$. For this topology, consider a sequence $(P_n)_{n\in\mathbb{N}}$ of prime ideals in $\operatorname{Spect}(R)$, and let $\mathcal{F}$ be an ultrafilter on $\mathbb{N}$. Then the set

$$\mathcal{F}\text{-}\lim_{n\in\mathbb{N}} P_n := \{a \in R\colon \{n \in \mathbb{N}\colon a \in P_n\} \in \mathcal{F}\}$$

plays a significant role.

It can be easily demonstrated that $\mathcal{F}\text{-}\lim_{n\in\mathbb{N}} P_n$ is a prime ideal. Suppose $ab \in \mathcal{F}\text{-}\lim_{n\in\mathbb{N}} P_n$. Then $A = \{n \in \mathbb{N}\colon ab \in P_n\} \in \mathcal{F}$. We have $A = \{n \in \mathbb{N}\colon a \in P_n\} \cup \{n \in \mathbb{N}\colon b \in P_n\}$. Since $\mathcal{F}$ is an ultrafilter, it follows that either $a \in \mathcal{F}\text{-}\lim_{n\in\mathbb{N}} P_n$ or $b \in \mathcal{F}\text{-}\lim_{n\in\mathbb{N}} P_n$. Hence, $\mathcal{F}\text{-}\lim_{n\in\mathbb{N}} P_n$ is a prime ideal. This notion of $\mathcal{F}$-limit of collections of prime ideals has been crucial in constructing the $\mathcal{F}$-topology on $\operatorname{Spect}(R)$. In the case where $\{P_n\}_{n\in\mathbb{N}}$ is a sequence from $C \subset \operatorname{Spect}(R)$ and $\mathcal{F}$ is a principal ultrafilter on $\mathbb{N}$, it is observed that $\mathcal{F}\text{-}\lim_{n\in\mathbb{N}} P_n = P_k$ for some $P_k \in C$ [5], Section 2. However, if $\mathcal{F}$ is a nonprincipal ultrafilter, it is not evident that the prime ideal $\mathcal{F}\text{-}\lim_{n\in\mathbb{N}} P_n$ should belong to C. This motivates the following definition.

Definition 2.6. Consider a collection $\{P_i\}_{i\in I}$ of prime ideals in $\operatorname{Spect}(R)$, and let $\mathcal{F}$ be an ultrafilter on I. A set C is said to be $\mathcal{F}$-closed in $\operatorname{Spect}(R)$ if for each collection $\{P_i\}_{i\in I}$ in C we have that $\mathcal{F}\text{-}\lim_{i\in I} P_i \in C$.

According to [5], Theorem 4.2, the $\mathcal{F}$-closed subsets of $\operatorname{Spect}(R)$ form a topology on the set $\operatorname{Spect}(R)$, known as the $\mathcal{F}$-topology on $\operatorname{Spect}(R)$. We denote the set $\operatorname{Spect}(R)$ equipped with the $\mathcal{F}$-topology as $\operatorname{Spect}(R)^{\tau_{\mathcal{F}}}$.

One of the key outcomes presented in the recent article by García-Ferreira and Ruza-Montilla in [5] is the following result.

Theorem 2.7 ([5], Theorem 4.4). *Consider a commutative ring R. If $\mathcal{F}$ is a nonprincipal ultrafilter on $\mathbb{N}$, then the $\mathcal{F}$-topology $\tau_{\mathcal{F}}$ on* $\operatorname{Spect}(R)$ *is countably compact.*

3. The $\mathcal{F}$-topology

Let us start by revisiting an important and pertinent definition.

Definition 3.1. Consider a ring T, an infinite set I, an ultrafilter $\mathcal{F}$ on I, and a collection of subrings $\{R_i\}_{i\in I}$ of T. We can define the $\mathcal{F}$-limit of this collection of subrings as follows:

$$\mathcal{F}\text{-}\lim_{i\in I} R_i := \{a \in T\colon \{i \in I\colon a \in R_i\} \in \mathcal{F}\}.$$

We observe that $\mathcal{F}\text{-}\lim_{i\in I} R_i$ is also a subring of T, and the following properties hold:

$$\mathcal{F}\text{-}\lim_{i\in I} R_i = \bigcup_{X\in\mathcal{F}} \bigcap_{i\in X} R_i.$$

Now, we state, without providing the proofs here, some straightforward and well-known properties. For detailed proofs see [5].

Proposition 3.2. *Let T be a ring, I an infinite set, $\mathcal{F}$ an ultrafilter on I, and $\{R_i\}_{i\in I}$ a collection of subrings of T. In this context, the following properties hold:*

(1) $\mathcal{F}_{\{k\}}\text{-}\lim_{i\in I} R_i = R_k$ *for each $k\in I$, and each principal ultrafilter $\mathcal{F}_{\{k\}}$ on I.*

(2) *If $J\in\mathcal{F}$, then*

$$\mathcal{F}\text{-}\lim_{i\in I} R_i = \mathcal{F}\mid_J\text{-}\lim_{i\in J} R_i,$$

where $\mathcal{F}\mid_J = \{A\subseteq J\colon\ A\in\mathcal{F}\}$.

(3) *Let Γ be an infinite set, and let $\sigma\colon\ \Delta\to\Gamma$ be a surjective function. For each $j\in\Gamma$, let $T_j = R_i$ if $\sigma(i) = j$. Then we have the equality*

$$\mathcal{F}\text{-}\lim_{i\in\Delta} R_i = \mathcal{C}\text{-}\lim_{j\in\Gamma} T_j,$$

where $\sigma(\mathcal{F}) = \{\sigma[F]\colon\ F\in\mathcal{F}\} = \mathcal{C}$.

Proposition 3.3. *Let R be a subring of a ring T. Suppose $X\subseteq\mathcal{Z}(R,T)$, I is an infinite set, $\mathcal{F}$ is an ultrafilter on I, and $\{R_i\colon\ i\in I\}\subseteq X$. Consider the map*

$$\pi\colon\ \beta(I)\to X,$$
$$\mathcal{F}\to\mathcal{F}\text{-}\lim_{i\in I} R_i.$$

Then the following holds:

(1) $\{R_i\colon\ i\in I\}\subseteq \mathrm{Im}(\pi)$.

(2) *The map $\pi\colon\ \beta(I)\to X$ is a surjection if and only if for every $\mathcal{F}\in\beta(I)$, the set X is stable under $\mathcal{F}$-limits.*

Proof. (1) For each R_k, if we consider the principal ultrafilter $\mathcal{F}_k$ and by Proposition 3.2, we have $\mathcal{F}_k\text{-}\lim_{i\in I} R_i = R_k$.

(2) To generalize the result, let X be a set that is stable under $\mathcal{F}$-limit for every collection $\{R_i\}_{i\in I}$ in X, where $\mathcal{F}$ is an ultrafilter on I. According to Proposition 3.3, there exists a surjective map $\pi\colon\ X\to Y$, where Y is the set of $\mathcal{F}$-limits of collections in X. This surjection arises from the fact that X is stable under $\mathcal{F}$-limit. □

E x a m p l e 3.4. With the notation of the previous Proposition 3.3 and by Theorem 3.5, if we consider $X = \mathcal{Z}(R,T)$, the set of zero-dimensional subrings of T containing R, then $\pi\colon\ \mathcal{Z}(R,T) \to Y$ is a continuous surjection. The continuity of π follows from the fact that the $\mathcal{F}$-limit operation is a continuous operation on $\mathcal{Z}(R,T)$, as established in Theorem 3.5.

The motivation behind investigating the topology on $\mathcal{Z}(R,T)$ arises from the following theorem.

Theorem 3.5. *Let R be a subring of a ring T, and assume that $\mathcal{Z}(R,T) \neq \emptyset$. If $R_i \in \mathcal{Z}(R,T)$ for every $i \in I$, and $\mathcal{F}$ is an ultrafilter on I, then the ring $\mathcal{F}\text{-}\lim\limits_{i\in I} R_i$ is also a zero-dimensional ring.*

P r o o f. According to [11], Proposition 4.2, it can be shown that $\mathcal{F}\text{-}\lim\limits_{i\in I} R_i$ is a direct limit of zero-dimensional rings. Consequently, the conclusion follows immediately from [10], Introduction. □

Definition 3.6. Let R be a subring of a ring T, and let $\mathcal{F}$ be an ultrafilter on an infinite set I. We define an $\mathcal{F}$-closed set $C \subseteq (R,T)$ as a set that satisfies the following property:

For every collection $\{R_i\}_{i\in I}$ in C, we have that $\mathcal{F}\text{-}\lim\limits_{i\in I} R_i \in C$.

We shall introduce a novel topology on the collection of subrings of a given commutative ring T.

Theorem 3.7. *Consider a subring R of a ring T, and let $\mathcal{F}$ be an ultrafilter on $\mathbb{N}$. We define the $\mathcal{F}$-topology on (R,T) as the collection of all $\mathcal{F}$-closed subsets, which forms the family of closed sets for this topology. We denote this topology as $\tau_{\mathcal{F}}$.*

P r o o f. It can be observed that the empty set and the entire set (R,T) are trivially $\mathcal{F}$-closed subsets. Now, let C_1 and C_2 be two $\mathcal{F}$-closed subsets of (R,T), and consider their intersection $C = C_1 \cap C_2$.

For any sequence $\{R_n\}_{n\in\mathbb{N}}$ in C, it also lies in both C_1 and C_2 since $C \subseteq C_1$ and $C \subseteq C_2$. Since C_1 and C_2 are $\mathcal{F}$-closed, we have $\mathcal{F}\text{-}\lim\limits_{n\in\mathbb{N}} R_n \in C_1$ and $\mathcal{F}\text{-}\lim\limits_{n\in\mathbb{N}} R_n \in C_2$. Therefore, $\mathcal{F}\text{-}\lim\limits_{n\in\mathbb{N}} R_n \in C_1 \cap C_2 = C$. Hence, C is also an $\mathcal{F}$-closed subset of (R,T).

Indeed, we have shown that the collection of $\mathcal{F}$-closed subsets of (R,T) possesses the properties required for a family of closed sets in a topology. Therefore, it defines a topology on (R,T), which we denote by $\tau_{\mathcal{F}}$. This topology, known as the $\mathcal{F}$-topology, is characterized by having the $\mathcal{F}$-closed sets as its closed sets. □

Lemma 3.8. *Consider a subring R of a ring T and a nonprincipal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. If we have an infinite set $\{R_n\}_{n\in\mathbb{N}}$ contained in (R,T), then the $\mathcal{F}\text{-}\lim\limits_{n\in\mathbb{N}} R_n$ serves as an accumulation point for the sequence $\{R_n\}_{n\in\mathbb{N}}$ within the $\mathcal{F}$-topology $\tau_{\mathcal{F}}$.*

P r o o f. The proof is similar to the proof of [4], Lemma 4.3. □

Consider R as a subring of a ring T. Denote by $\mathcal{A}(R,T)$ and $\mathcal{DU}(R,T)$, respectively, the sets of Artinian subrings of T that contain R and directed unions of Artinian subrings of T that contain R. In general, the intersection of two Artinian rings need not be Artinian (see [7]). Our focus will be on $\mathcal{Z}(R,T)$, which is a significant space of interest. It is the largest closed subset of (R,T) under the $\mathcal{F}$-topology $\tau_{\mathcal{F}}$ that contains both $\mathcal{A}(R,T)$ and $\mathcal{DU}(R,T)$, as stated in Theorem 3.5.

Naturally, we begin by comparing the $\mathcal{F}$-topology $\tau_{\mathcal{F}}$ on $\mathcal{Z}(R,T)$ with the usual topology.

Theorem 3.9. *Let R be a subring of a ring T such that $\mathcal{Z}(R,T)\neq\emptyset$. Then:*

(1) *The $\mathcal{F}$-topology $\tau_{\mathcal{F}}$ is finer than the Zariski topology on $\mathcal{Z}(R,T)$.*
(2) *The $\mathcal{F}$-topology $\tau_{\mathcal{F}}$ is a Hausdorff topology on $\mathcal{Z}(R,T)$.*
(3) *The $\mathcal{F}$-topology $\tau_{\mathcal{F}}$ is countably compact.*

P r o o f. (1) Since $\mathcal{B} := \{\mathcal{Z}(R[x],T)\colon x\in T\}$ is a base of $\mathcal{Z}(R,T)$ endowed with the Zariski topology, it is enough to prove that $C := \mathcal{Z}(R,T)\setminus\mathcal{Z}(R[x],T)$ is $\mathcal{F}$-closed for every $x\in T$. Assume, by contradiction, that there exists an ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that $\mathcal{F}\text{-}\lim\limits_{n\in\mathbb{N}} R_n \notin C$ for each sequence $\{R_n\}_{n\in\mathbb{N}}$ in C. Let $x\in\mathcal{F}\text{-}\lim\limits_{n\in\mathbb{N}} R_n$. Then $\{n\in\mathbb{N}\colon x\in R_n\}\in\mathcal{F}$, by the definition of C, and from the fact that $\emptyset\notin\mathcal{F}$, we have a contradiction.

(2) According to statement (1), the basic open sets of the Zariski topology on $\mathcal{Z}(R,T)$ are both open and closed in the $\mathcal{F}$-topology. This implies that the $\mathcal{F}$-topology is finer than a certain topology, which can be defined as the coarsest topology for which the sets $\mathcal{Z}(R[x],T)$ are both open and closed for every $x\in T$. Moreover, this topology is Hausdorff. To see this, consider two distinct elements V_1 and V_2 of $\mathcal{Z}(R,T)$. Without loss of generality, assume that there exists $y\in V_1\setminus V_2$. By the definition of the topology mentioned above, the sets $\mathcal{Z}(R,T)\setminus\mathcal{Z}(R[y],T)$ and $\mathcal{Z}(R[y],T)$ are disjoint open neighborhoods of V_1 and V_2, respectively. This confirms that the $\mathcal{F}$-topology is finer than the mentioned topology.

(3) Indeed, by statement (2) of Theorem 3.9, we know that the $\mathcal{F}$-topology on $\mathcal{Z}(R,T)$ is a Hausdorff topology. Now, consider any infinite subset A of $\mathcal{Z}(R,T)$. We want to show that A has an accumulation point in the $\mathcal{F}$-topology. By Lemma 3.8, we know that every infinite subset of $\mathcal{Z}(R,T)$ has a limit point in the Zariski topology. Since the $\mathcal{F}$-topology is finer than the Zariski topology, every limit point of A in the

Zariski topology is also a limit point of A in the $\mathcal{F}$-topology. Therefore, A has an accumulation point in the $\mathcal{F}$-topology, which shows that the $\mathcal{F}$-topology on $\mathcal{Z}(R,T)$ is countably compact. □

Remark 3.10. The space $\mathcal{Z}(R,T)^{\tau_\mathcal{F}}$ is countably compact since every countable open cover has a finite subcover, which follows from the fact that $\mathcal{Z}(R,T)^{\tau_\mathcal{F}}$ is a subspace of the Stone-Cech compactification of the discrete space $\mathcal{Z}(R,T)$. This compactness property ensures that any countable sequence in $\mathcal{Z}(R,T)$ has an accumulation point in $\mathcal{Z}(R,T)^{\tau_\mathcal{F}}$.

However, in general, $\mathcal{Z}(R,T)^{\tau_\mathcal{F}}$ is not compact because not every open cover has a finite subcover. The reason for this is that the $\mathcal{F}$-topology can be quite fine, allowing for infinite families of sets that intersect only at a single point. Consequently, an open cover that includes all of these sets cannot be reduced to a finite subcover. This lack of compactness highlights the subtle difference between countable compactness and compactness in the context of the $\mathcal{F}$-topology on $\mathcal{Z}(R,T)$.

In other words, Theorem 3.11 states that the countability, spectral property, and existence of Frechet limits in $\mathcal{Z}(R,T)$ are equivalent. This result provides a characterization of when $\mathcal{Z}(R,T)$ possesses these properties, allowing us to study the interplay between countability, spectralness, and convergence behavior in the $\mathcal{F}$-topology.

Theorem 3.11. *Let R be a subring of a ring T such that $\mathcal{Z}(R,T) \neq \emptyset$, and let $\mathcal{F}_r$ be the Frechet ultrafilter on $\mathbb{N}$. Then the following conditions are equivalent:*

(1) *The set $\mathcal{Z}(R,T)$ is countable.*
(2) *The set $\mathcal{Z}(R,T)$ is a spectral space.*
(3) *The limit $\mathcal{F}_r\text{-}\lim\limits_{n\in\mathbb{N}} R_n$ exists for any sequence $\{R_n\colon n \in \mathbb{N}\} \subseteq \mathcal{Z}(R,T)$.*

Proof. (1) ⇒ (2). According to [2], Theorem 3.10.3, every countably compact, countable Hausdorff space is compact. Therefore since $\mathcal{Z}(R,T)^{\tau_\mathcal{F}}$ is countably compact and Hausdorff, it is compact. Inspired by the idea given in [5], by [11], Lemma 4.4, the $\mathcal{F}$-topology and ultrafilter topology are the same. Therefore by [3], Corollary 3.3, $\mathcal{Z}(R,T)$ is a spectral space.

(2) ⇒ (3). Let $\mathcal{Z}(R,T) \simeq \operatorname{Spect}(S)$ for a ring S and let $\varphi\colon \mathcal{Z}(R,T) \to \operatorname{Spect}(S)$. If $\{P_n\colon n \in \mathbb{N}\} \subseteq \operatorname{Spect}(S)$, then according to [2], Theorem 3.10.3 (v), $\mathcal{F}\text{-}\lim\limits_{n\in\mathbb{N}} P_n$ exists for each nonprincipal ultrafilter on $\mathbb{N}$. On the other hand, $\varphi^{-1}(\mathcal{F}\text{-}\lim\limits_{n\in\mathbb{N}} P_n) = \mathcal{F}\text{-}\lim\limits_{n\in\mathbb{N}} \varphi^{-1}(P_n)$. Thus, it suffices to choose $R_n := \varphi^{-1}(P_n)$, since $\mathcal{F}\text{-}\lim\limits_{n\in\mathbb{N}} R_n$ exists for every nonprincipal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. Then $\mathcal{F}_r\text{-}\lim\limits_{n\in\mathbb{N}} R_n$ exists by applying [4], Definition 1.1.

(3) $\Rightarrow$ (1) Let $\mathcal{F}_r$ be a nonprincipal ultrafilter on $\mathbb{N}$ and assume that $\mathcal{F}_r\text{-}\lim_{n\in\mathbb{N}} R_n$ exists. Then by [11], Lemma 4.4, the $\mathcal{F}$-topology and the ultrafilter topology are the same, where $\mathcal{F}$ is any nonprincipal ultrafilter on $\mathbb{N}$. Therefore $\mathcal{F}\text{-}\lim_{n\in\mathbb{N}} R_n$ also exists for every nonprincipal ultrafilter $\mathcal{F}$ on $\mathbb{N}$.

Moreover, since $\mathcal{Z}(R,T)^{\tau_\mathcal{F}}$ is $\mathcal{F}$-compact for each nonprincipal ultrafilter $\mathcal{F}$, according to [12], Theorem 2.9, $\mathcal{Z}(R,T)^{\tau_\mathcal{F}}$ is a compact space. Then by [2], Theorem 3.10.3, $\mathcal{Z}(R,T)^{\tau_\mathcal{F}}$ is a countably compact, countable Hausdorff space and hence is compact. Therefore $\mathcal{F}\text{-}\lim_{n\in\mathbb{N}} R_n$ exists for every nonprincipal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. $\square$

Our focus now shifts to the study of the map γ when the spaces $\mathcal{Z}(R,T)$ and $\operatorname{Spect}(R)$ are equipped with the $\mathcal{F}$-topology.

Theorem 3.12. *Let T be a ring and R a subring of T such that $\mathcal{Z}(R,T) \neq \emptyset$. Then the surjective map $\gamma\colon \mathcal{Z}(R,T)^{\tau_\mathcal{F}} \to \operatorname{Spect}(R)^{\tau_\mathcal{F}}$ is continuous and closed.*

P r o o f. According to Theorem 3.9, $\mathcal{Z}(R,T)$ is a Hausdorff space. By straightforward topological arguments, it is enough to show that γ is continuous. Let C be an $\mathcal{F}$-closed subset of $\operatorname{Spect}(R)^{\tau_\mathcal{F}}$, and let $\{S_n\colon n \in \mathbb{N}\} \subseteq \gamma^{-1}(C)$ be a sequence, where $\mathcal{F}$ is an ultrafilter on $\mathbb{N}$. Then it suffices to show that $\mathcal{F}$-limit of S_n belongs to $\gamma^{-1}(C)$. According to Theorem 3.5, we have $\mathcal{F}\text{-}\lim_{n\in\mathbb{N}} S_n \in \mathcal{Z}(R,T)$, then $R \subseteq \mathcal{F}\text{-}\lim_{n\in\mathbb{N}} S_n$. On the other hand, for each $(P_n) \in C$ we can also consider the ideal

$$\mathcal{F}\text{-}\lim_{n\in\mathbb{N}} P_n = \{a \in R\colon \{n \in \mathbb{N}\colon a \in P_n\} \in \mathcal{F}\},$$

which is a prime ideal of R. By [8], there exists a prime ideal Q of $\mathcal{F}\text{-}\lim_{n\in\mathbb{N}} S_n$ such that $Q \cap R = \mathcal{F}\text{-}\lim_{n\in\mathbb{N}} P_n$. Since by [5], C is an $\mathcal{F}$-closed subset, we have $\gamma(\mathcal{F}\text{-}\lim_{n\in\mathbb{N}} S_n) = \mathcal{F}\text{-}\lim_{n\in\mathbb{N}} P_n \in C$, and so $\mathcal{F}\text{-}\lim_{n\in\mathbb{N}} S_n \in \gamma^{-1}(C)$. Therefore, we deduce that $\gamma^{-1}(C)$ is a closed subset of $\mathcal{Z}(R,T)^{\tau_\mathcal{F}}$, hence the conclusion. $\square$

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