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Bojana Stojčetović

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ON THE NUMBER OF ITERATIONS IN THE ALTERNATING METHOD FOR INTEGER MATRICES IN MAX-ALGEBRA

Bojana Stojčetović

The Alternating Method in max-algebra is an efficient approach for solving two-sided max-linear systems of the form $A \otimes x = B \otimes y$, where A, B are matrices and x, y are vectors of compatible sizes. This iterative procedure typically begins with a randomly chosen initial vector. In the case when matrices A and B are integer matrices and one is finite while the other has at least one finite element in each row and in each column, and provided that the initial vector is also an integer vector, an upper bound on the number of iterations can be determined. This paper proposes starting the Alternating Method with a vector selected based on the matrix elements of $\tilde{A} = (-A^\top) \otimes A$, where A is a finite matrix of the given system, instead of using a randomly selected vector. This choice of initial vector aims to minimize the number of iterations in the Alternating Method. We have proved that, with the proposed choice of initial vector, the number of iterations is bounded above by the expression containing the maximum element of matrix $\tilde{A}$. From this statement, we derive additional conclusions regarding this bound. Finally, we compare the number of iterations in the Alternating Method when it starts from a randomly chosen vector versus when it starts from the vector we propose in this study.

Keywords: max-algebra, Alternating Method, two-sided systems, integer matrices

Classification: 15A80, 15A24

1. INTRODUCTION

Max-algebra is an analogue of linear algebra developed over the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, equipped with two binary operations that we call max-algebraic addition ($\oplus$) and max-algebraic multiplication ($\otimes$), respectively. The structure $(\overline{\mathbb{R}}, \oplus, \otimes)$ is a (commutative idempotent) semiring.

Throughout this paper, when we say "max-algebra" we mean "max-plus algebra". Also, some authors refer to max-algebra as "extremal algebra" [30], "path algebra" [11] or "schedule algebra" [21]. "Tropical algebra" has recently been used as a common synonym for "max-algebra", though distinctions may still apply depending on context.

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The development of max-algebra began with the papers of R. A. Cuninghame-Green [17], B. A. Carre [11], N. N. Vorobyov [30], M. Gondran and M. Minoux, [22], B. Giffler [21], and others. To date, a substantial body of literature has addressed this topic. For detailed treatments of max-algebra, see [1, 5, 9, 14, 16].

The strong interest in max-algebra stems from the fact that it allows nonlinear problems to be viewed and solved using linear-like operations. Its applications are numerous, like, for example, in discrete event systems, scheduling problems, synchronization problems, traffic light control, and transportation, among others (see [4, 12, 19, 20, 23]).

From the very beginning of the development of max-algebra, those questions that are the main ones in the conventional linear algebra were considered: solving systems of equations and inequalities, linear independence, the problem of eigenvalues and eigenvectors, dimension and rank of matrices, among other properties.

When it comes to max-algebraic systems of equations and inequalities, these systems arise in practical applications. In this regard, let us consider the following two motivational examples, the first of which is the Multi-Machine Interactive Production Process (MMIPP) [18] and which is the basis for other models.

Example 1.1. Let n machines take part in the production of m products P_i, $i = \overline{1, m}$, by producing components. We start from the assumption that each of the n machines can work on all products simultaneously and that all actions on a machine start as soon as it starts working. Let a_{ij} be the time required for the jth machine to produce a component for product P_i ($i = \overline{1, m}$, $j = \overline{1, n}$). Denote by x_j starting time of machine j. Then, product P_i will be ready at time

$$\max(x_1 + a_{i1}, \ldots, x_n + a_{in}).$$

If $b_1, \ldots, b_m$ are target completion times, then we come to the system of equations

$$\max(x_1 + a_{i1}, \ldots, x_n + a_{in}) = b_i, \qquad i = \overline{1, m}.$$

Using the notation $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$ for $a, b \in \mathbb{R}$ extended to matrices and vectors in the same way as in linear algebra, this system can be written compactly as

$$A \otimes x = b, \tag{1}$$

which is a *one-sided max-linear system.* The matrix A is called the *production matrix.*

The problem of solving such systems in max-algebra was considered in the earliest published papers in this field, see [15, 17, 30, 31]. This question has also been discussed later, for example, in [9].

Example 1.2. As part of a wider MMIPP, suppose that t other machines produce components for products Q_i, $i = \overline{1, m}$, and the duration and starting times are b_{ij} and y_j, respectively. The synchronization problem is to find starting times of all $n + t$ machines, so that each pair (P_i, Q_i) is completed at the same time. The corresponding system of equations now has the form

$$\max(x_1 + a_{i1}, \ldots, x_n + a_{in}) = \max(y_1 + b_{i1}, \ldots, y_t + b_{it}), \qquad i = \overline{1, m}.$$

In the max-algebraic notation, we have

$$A \otimes x = B \otimes y. \tag{2}$$

We call such systems *two-sided max-linear systems*. These systems (considered, among other papers, in [13]) can be reformulated as

$$A \otimes x = B \otimes x, \tag{3}$$

a two-sided systems that were studied in the papers of Butkovič, such as [6] and [7], as well as in the later papers by the same author. The solution set of (3) is finitely generated [8].

In [10], the Stepping Stone Method for solving systems of the form (3) was presented. On the other hand, one of the principal methods for solving systems of the form (2) is the Alternating Method [13]. This method typically begins with a randomly chosen initial vector and generates sequences that converge to the (finite) solution of the given system (if such a solution exists). Given that system (2) can be transformed into system (3), these two methods can be compared [3].

This paper focuses on the Alternating Method.

It is shown in [13] that if A and B are integer matrices, with one being finite and the other has at least one finite element in each row and in each column, and if the initial vector is also an integer, then we can estimate the upper bound on the number of iterations in the Alternating Method (for simplicity, we use the term 'bound' to refer to the upper bound). Based on that estimation, this paper proposes that the Alternating Method starts not from a randomly chosen vector, but from the vector chosen based on the elements of the matrix $\tilde{A}$. This matrix is computed as the max-algebraic product of $-A^\top$ and A, where A is the finite matrix of the given two-sided system. This proposal aims to reduce the number of required iterations in the Alternating Method as much as possible. To calculate this bound, we use the properties of the operations $\oplus$ and $\otimes$ that apply to numbers, vectors, and matrices, without needing tools specific to max-algebra, such as maximum cycle mean, max-algebraic eigenvalues/eigenvectors, or max-algebraic permanent (see Sect. 1.6 in [5] for more details).

This paper is organized as follows. After this introductory section, in Section 2, we provide the basics of max-algebra and describe the properties that are applicable here, which will be used throughout the present study. In Section 3, we briefly describe (one-sided and two-sided) max-linear systems. We also provide a description of the Alternating Method, along with the corresponding algorithm. The main results are presented in Section 4, with the key theorem being Theorem 4.1. We propose a method for defining the initial vector in the Alternating Method, and we calculate the bound on the number of iterations for that initial vector. Additionally, we outline the conditions under which the number of iterations matches the bound and describe when it may be less. Examples supporting the claims are provided at appropriate places in the text. In the final part of the paper, in Section 5, we compare whether the Alternating Method has a smaller bound on the number of iterations when starting with a randomly chosen initial vector or with the vector we propose. The numerical results of this comparison are presented and discussed in this section.

2. PRELIMINARIES

Over the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, we can define two binary operations, denoted as $\oplus$ and $\otimes$, as follows:

$$a \oplus b = \max\{a, b\} \qquad \text{and} \qquad a \otimes b = a + b.$$

The structure $(\overline{\mathbb{R}}, \oplus, \otimes)$ is a semiring known as *max-plus algebra* or simply *max-algebra*. Operations $\oplus$ and $\otimes$ are called *max-algebraic addition* and *max-algebraic multiplication*, respectively. This terminology is justified because many properties from conventional linear algebra carry over max-algebra, if we replace "+" with "$\oplus$" and "$\times$" with "$\otimes$". In terms of operations priority, there is no difference compared to linear algebra, i. e., max-algebraic multiplication has higher priority than max-algebraic addition. *Zero* and *unity*, i. e. *additive* and *multiplicative identity* for the operations $\oplus$ and $\otimes$ are elements $-\infty$ and 0, respectively. Both max-algebraic operations are commutative and associative, and $\otimes$ is distributive over $\oplus$, which is easy to prove. In addition, the operation $\oplus$ in $\overline{\mathbb{R}}$ is not invertible.

In the following, we will denote $-\infty$ by ε (it will also be the notation for every vector or matrix whose elements are all equal to $-\infty$). Besides, we will denote by M and N the sets $\{1, \dots, m\}$ and $\{1, \dots, n\}$, respectively, where m, n are natural numbers.

The basic max-algebraic operations are extended to matrices as follows: if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\overline{\mathbb{R}}$, then we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$, for all $i \in M$, $j \in N$ and $C = A \otimes B$ if $c_{ij} = \bigotimes_k (a_{ik} \otimes b_{kj}) = \max_k(a_{ik} + b_{kj})$, for all $i \in M$, $j \in N$.
The operation $\oplus$, when it comes to matrices and vectors, is commutative and associative, while $\otimes$ is only associative. Also, for matrices A, B and C of compatible sizes, *distributive laws* hold:

$$(A \oplus B) \otimes C = A \otimes C \oplus B \otimes C \qquad \text{and} \qquad A \otimes (B \oplus C) = A \otimes B \oplus A \otimes C. \tag{4}$$

The *transpose* of matrix A is denoted $A^\top$. In addition to the transposed matrix, the matrix $A^* = -A^\top$ is also of great importance in max-algebra. This is called the *conjugate matrix* [14] of A. In fact, this matrix partially replaces the role of the inverse matrix (which does not exist in the general case in max-algebra, making many linear algebra procedures inapplicable here). As we shall see, A^* plays a very important role in solving max-linear systems. It is easy to show that the max-algebraic product of the matrix A and its conjugate matrix is a square matrix with a zero diagonal (matrix $A \otimes A^*$, as well as $A^* \otimes A$, has other characteristic properties (described in [29]) that are not essential for the purposes of this paper. Therefore, here we only state the property that we will use: the existence of a zero diagonal in matrices $A \otimes A^*$ and $A^* \otimes A$).

Definition 2.1. The matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{m \times n}$ is called a *column (row)* $\mathbb{R}-$*astic matrix* [15], if A does not contain a column (row) whose elements are all equal to ε, that is, if $\sum_{i \in M}^{\oplus} a_{ij} \in \mathbb{R}$ for all $j \in N$ (if $\sum_{j \in N}^{\oplus} a_{ij} \in \mathbb{R}$ for all $i \in M$).

In simpler terms, a matrix is column (row) $\mathbb{R}-$astic if there is at least one finite element in each column (row) of that matrix.

Definition 2.2. The matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{m\times n}$ is a *double* $\mathbb{R}-$*astic* matrix if it has at least one finite element in each column and row, that is, if it is both column and row $\mathbb{R}$-astic.

The matrix A is *finite*, if none of its elements is ε. The same holds for vectors and scalars.

For matrices $A, B \in \overline{\mathbb{R}}^{m\times n}$ ordering is defined pointwise:

$$A \le B \quad \Leftrightarrow \quad a_{ij} \le b_{ij}, \qquad i \in M, j \in N.$$

Many problems in conventional algebra are solved using inverse operations, such as subtraction instead of addition. Since max-algebraic addition is not invertible, many standard algebraic procedures are not applicable. To overcome this problem, we define a *dual pair of operations* $(\oplus', \otimes')$:

$$a \oplus' b = \min\{a, b\} \quad \text{and} \quad a \otimes' b = a + b,$$

for all $a, b \in \overline{\overline{\mathbb{R}}} = \overline{\mathbb{R}} \cup \{+\infty\}$. Operations $(\oplus', \otimes')$ are extended to matrices and vectors in the same way as $(\oplus, \otimes)$. Here, we adopt the following convention:

$$(-\infty) \otimes (+\infty) = (+\infty) \otimes (-\infty) = -\infty$$

and

$$(-\infty) \otimes' (+\infty) = (+\infty) \otimes' (-\infty) = +\infty.$$

3. MAX-ALGEBRAIC SYSTEMS OF EQUATIONS

A system of the form

$$A \otimes x = b, \tag{5}$$

where $A \in \overline{\mathbb{R}}^{m\times n}$ and $b \in \overline{\mathbb{R}}^{m}$, is called a *one-sided max-linear system.* This system is closely related to the corresponding max-algebraic system of inequalities:

$$A \otimes x \le b. \tag{6}$$

There are two basic approaches for solving one-sided systems: combinatorial and algebraic.

The *combinatorial method* ensures that for each linear system, there exists a finite set and a subset collection such that solvability is equivalent to a set covering condition, while the unique solvability is equivalent to minimum set covering (see [9, 31]).

In the *algebraic approach*, the following relation plays a crucial role [14]: for $A \in \overline{\mathbb{R}}^{m\times n}$, $b \in \overline{\mathbb{R}}^{m}$ and $x \in \overline{\mathbb{R}}^{n}$ it holds

$$A \otimes x \le b \qquad \Leftrightarrow \qquad x \le A^* \otimes' b. \tag{7}$$

It turns out that $A^* \otimes' b$ is always a solution to the system of inequalities (6) (it is its greatest solution), while a system of equations (5) has a solution if and only if $A^* \otimes' b$ is its solution. Due to the importance of this solution, we call it the *principal solution*

[15] and denote it by $\overline{x}$. This result can be extended to matrix equations and matrix inequalities of more general form, that is, for $A \otimes X = B$ and $A \otimes X \leq B$, where $A \in \overline{\mathbb{R}}^{m\times n}$, $B \in \overline{\mathbb{R}}^{m\times k}$ and $\overline{X} = A^* \otimes' B$.

A system of the form

$$A \otimes x \oplus c = B \otimes x \oplus d, \tag{8}$$

where $A, B \in \overline{\mathbb{R}}^{m\times n}$ and $c, d \in \overline{\mathbb{R}}^{m}$, is called a *two-sided max-linear system.* The solution set of this system is finitely generated [8].

For $c = d = \varepsilon$, we have the system

$$A \otimes x = B \otimes x, \tag{9}$$

which is called *homogeneous system.* Otherwise, the system is *non-homogeneous.*

For $A \in \overline{\mathbb{R}}^{m\times n}$ and $B \in \overline{\mathbb{R}}^{m\times k}$, a system of the form

$$A \otimes x = B \otimes y \tag{10}$$

represents a special homogeneous system called *system with separated variables.*

It is also important to note the following: non-homogeneous systems can be transformed into homogeneous systems, and homogeneous systems can be transformed into systems with separated variables.

3.1. Alternating Method

The Alternating Method, as presented in [13], is an algorithm for solving two-sided systems with separated variables. Without loss of generality, we assume that the matrices A and B are double $\mathbb{R}-$astic. Considering that the principal solution $\overline{x}$ plays a significant role at one-sided systems, the idea is to use it to solve two-sided systems as well.

Therefore, in the Alternating Method, we start with a randomly chosen initial vector $x(0)$, calculate $A\otimes x(0)$, and then find $y(0)$ as the principal solution for $B\otimes y = A\otimes x(0)$, i. e., $y(0) = B^* \otimes' (A \otimes x(0))$. Next, for the obtained $y(0)$, we calculate $x(1)$ as the principal solution for $A \otimes x(1) = B \otimes y(0)$, then $y(1)$ as the principal solution for $B \otimes y(1) = A \otimes x(1)$, and so on. In this way, we get sequences $\{x(t)\}_{t=0}^{\infty}$ and $\{y(t)\}_{t=0}^{\infty}$ that converge, for any initial vector, to a final solution whenever a final solution exists.

The following presents the algorithm of the Alternating Method.

Algorithm 1 Alternating Method

Input: $A \in \mathbb{R}^{m\times n}$, $B \in \overline{\mathbb{R}}^{m\times k}$, $x \in \mathbb{R}^n$.
Output: A solution (x, y) to $A \otimes x = B \otimes y$ or an indication that there is no solution.

1: $r = 0$, $x(0) = x$.
2: $y = B^* \otimes' (A \otimes x)$, $y(r) = y$.
3: If $r \geq 1$, $y(r) < y(r-1)$, STOP ('no solution').
4: $x = A^* \otimes' (B \otimes y)$, $x(r+1) = x$.
5: If $x(r+1) < x(r)$, STOP ('no solution').
6: $r = r + 1$.
7: If $A \otimes x = B \otimes y$, STOP.
8: Go to 2.

In the next theorem, without loss of generality, it is assumed that A is finite and B is double $\mathbb{R}-$astic matrix.

Theorem 3.1. (Butkovič [13]) If $A \in \mathbb{Z}^{m\times n}$, $B \in \overline{\mathbb{Z}}^{m\times k}$ and the Alternating Method starts with $x(0) \in \mathbb{Z}^n$, then it will terminate after at most

$$\Omega = (n-1)(1 + x(0)^* \otimes A^* \otimes A \otimes x(0)) \tag{11}$$

iterations.

The proof, as well as additional features of the Alternating Method, can be found in [13].

4. CHOICE OF INITIAL VECTOR

We now present the key theorem of this paper, which establishes a method for choosing the initial vector and determining the upper bound on the number of iterations.
As in the previous sections, we assume that A is finite and B is a double $\mathbb{R}-$astic matrix.

Theorem 4.1. Let $A \in \mathbb{Z}^{m\times n}$ and $B \in \overline{\mathbb{Z}}^{m\times k}$. If we define the initial column vector $x(0)$ as

$$x(0) = (x_i(0)) = \left(\max_j \{\tilde{a}_{ij}\}\right), \quad i,j \in N, \tag{12}$$

then the number of iterations in the Alternating Method will be less than or equal to

$$\Omega = (n-1)\left(1 + \max_{i,j} \{\tilde{a}_{ij}\}\right), \quad i,j \in N, \tag{13}$$

where $\tilde{A} = (\tilde{a}_{ij}) = A^* \otimes A$.

Proof. Based on Theorem 3.1, the number of iterations in the Alternating Method for $A \in \mathbb{Z}^{m\times n}$, $B \in \overline{\mathbb{Z}}^{m\times k}$ and an arbitrary initial vector $x(0) \in \mathbb{Z}^n$ is less than or equal to (11). We first observe that the matrix $\tilde{A} = A^* \otimes A \in \mathbb{R}^{n\times n}$, as defined in (11), has a zero diagonal.

Let us denote $T = x(0)^* \otimes A^* \otimes A \otimes x(0) = x(0)^* \otimes \tilde{A} \otimes x(0)$, where we define the vector $x(0)$ as in (12). We now prove that $T \leq \max_{i,j} \{\tilde{a}_{ij}\}$. Considering the dimensions of the matrix $\tilde{A}$ and the vector $x(0)$, it is clear that T will be an integer.

From (12), it follows that $x_i(0)$, for each $i \in N$, is the maximal element in the ith row of $\tilde{A}$. Thus, $x_i(0) \geq \tilde{a}_{ij}$, for every $j \in N$, that is:

$$x_i(0) \geq \tilde{a}_{ij}, \qquad \Leftrightarrow \qquad -x_i(0) + \tilde{a}_{ij} \leq 0, \qquad \text{for every } i,j \in N. \tag{14}$$

Since $\tilde{A}$ is a square matrix with a zero diagonal, all coordinates of the vector $x(0)$ are non-negative:

$$x_i(0) \geq 0, \qquad \text{for every } i \in N. \tag{15}$$

Let $x^*(0) = -x(0)^\top$. Then, for each $i \in N$, the entries of $x^*(0)$ satisfy $x_i^*(0) = -x_i(0)$. So, from (15) we have that $x_i^*(0) \leq 0$, for every $i \in N$.

In the following, we will use the notations:

$$x_i^*(0) + \tilde{a}_{ij} = p_{ij}, \qquad i, j \in N, \tag{16}$$

and

$$\max_i \{p_{ij}\} = q_j, \quad j \in N. \tag{17}$$

From (14), we obtain the following expression for p_{ij}:

$$p_{ij} = x_i^*(0) + \tilde{a}_{ij} = -x_i(0) + \tilde{a}_{ij} \leq 0, \quad i, j \in N. \tag{18}$$

Since all the values are non-positive, each q_j is also non-positive for every $j \in N$.

Using the properties of $\otimes$ and the introduced notations, we have:

$$\begin{aligned} T &= x^*(0) \otimes A^* \otimes A \otimes x(0) = x^*(0) \otimes \tilde{A} \otimes x(0) = \left(x^*(0) \otimes \tilde{A}\right) \otimes x(0) \\ &= (\max\{p_{11}, p_{21} \dots, p_{n1}\}, \dots, \max\{p_{1n}, p_{2n}, \dots, p_{nn}\}) \otimes \begin{pmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{pmatrix} \\ &= \left(\max_i\{p_{i1}\}, \dots, \max_i\{p_{in}\}\right) \otimes \begin{pmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{pmatrix} \\ &= (q_1, \dots, q_n) \otimes \begin{pmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{pmatrix}. \end{aligned}$$

Given that all coordinates of the vector q are non-positive and for $x_i(0)$ (15) holds, it follows that the iteration bound T further satisfies:

$$\begin{aligned} T &= (q_1, \dots, q_n) \otimes \begin{pmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{pmatrix} && (19) \\ &= \max\{q_1 + x_1(0), \dots, q_n + x_n(0)\} && (20) \\ &\leq \max\{x_1(0), \dots, x_n(0)\} = \max_i\{x_i(0)\} && (21) \\ &= \max_i\left(\max_j\{\tilde{a}_{ij}\}\right) = \max_{i,j}\{\tilde{a}_{ij}\}. && (22) \end{aligned}$$

This establishes that $T \leq \max_{i,j}\{\tilde{a}_{ij}\}$, completing the proof. □

Given the initial vector defined in Theorem 4.1, we conclude that, in the worst case, T equals the greatest element of the matrix $\tilde{A}$, although this value may be reduced depending on the other elements of this matrix. However, it remains inconclusive whether Ω would be smaller for arbitrary initial vectors (a comparison and discussion of results is given in Section 5).

The proposed initial vector enables the iteration bound to be calculated using only standard max-algebraic operations, without requiring tools such as maximum cycle mean or max-algebraic eigenvalue analysis.

Corollary 4.2. In the trivial case where $A = \rho \otimes O$, with O being a matrix whose entries are all 0 and $\rho \in \mathbb{Z}$ (observe that $\tilde{A}$ also consists of entries equal to 0, i.e. $\max_{i,j} \{\tilde{a}_{ij}\} = 0$), the number of iterations in the Alternating Method, with the choice of the initial vector specified in Theorem 4.1, will be bounded by

$$(n-1)(1+0) = n-1.$$

In the nontrivial case, the smallest possible upper bound for the number of iterations is $2(n-1)$, because the smallest value that $\max_{i,j} \{\tilde{a}_{ij}\}$ can take is 1, since A is an integer matrix (clearly, $\max_{ij} \{\tilde{a}_{ij}\}$ cannot be negative because $\tilde{A}$ has a zero diagonal). Thus, the number of iterations will be less than or equal to

$$(n-1)(1+1) = 2(n-1).$$

Example 4.3. Let the matrix A be given by:

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -9 & 4 & 6 \end{pmatrix}.$$

Then:

$$A^* = \begin{pmatrix} -1 & -2 & 9 \\ -3 & 1 & -4 \\ -5 & -4 & -6 \end{pmatrix} \quad \text{and} \quad \tilde{A} = \begin{pmatrix} 0 & 13 & 15 \\ 3 & 0 & 5 \\ -2 & -2 & 0 \end{pmatrix}.$$

If we choose the vector $x(0)$ as in (12), we get

$$x(0) = \begin{pmatrix} 15 & 5 & 0 \end{pmatrix}^\top.$$

We compute:

$$T = x^*(0) \otimes \tilde{A} \otimes x(0) = \begin{pmatrix} -15 & -5 & 0 \end{pmatrix} \otimes \begin{pmatrix} 18 & 18 & 13 \end{pmatrix}^\top = 13.$$

We obtained that $T = 13 \leq 15 = \max_{i,j} \{\tilde{a}_{ij}\}$. Hence, the total number of iterations is bounded above by

$$(3-1)(1+13) = 28.$$

The following are some values of T obtained using randomly selected initial vectors:

$$\begin{aligned} x(0) &= \begin{pmatrix} -2 & -4 & 3 \end{pmatrix}^\top && \Rightarrow && T = 15; \\ x(0) &= \begin{pmatrix} 3 & 9 & -2 \end{pmatrix}^\top && \Rightarrow && T = 20; \\ x(0) &= \begin{pmatrix} 9 & 20 & 4 \end{pmatrix}^\top && \Rightarrow && T = 31; \\ x(0) &= \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}^\top && \Rightarrow && T = 15. \end{aligned}$$

The smallest value of T is achieved when the initial vector is chosen as in (12), though this may not generalize to all cases.

Lemma 4.4. Let $A \in \mathbb{Z}^{m\times n}$ and $B \in \overline{\mathbb{Z}}^{m\times k}$ be given matrices. Let each column of $\tilde{A} = A^* \otimes A$ contains the maximal element of some row of that matrix. If this condition holds and the initial vector $x(0)$ is defined as in (12), then the bound on the number of iterations in the Alternating Method will be equal to

$$(n-1)\left(1+\max_{i,j}\{\tilde{a}_{ij}\}\right), \quad i,j \in N.$$

P r o o f. Following the notation of Theorem 4.1, inequality (21) becomes an equality if q is the zero vector, i.e., if $q_j = 0$, for all $j \in N$.

The coordinates of the vector $x^*(0)$ are the negative maxima of the rows of $\tilde{A}$. When calculating the sth element of the vector q (for some $s \in N$), these coordinates are added (i.e., max-algebraically multiplied) with the elements of the sth column of $\tilde{A}$. By the condition of the lemma, each column of this matrix contains maximal element of some row. Let $\tilde{a}_{ts}$ denote the element of the sth column that is the maximum of the tth row: $\tilde{a}_{ts} = \max\{\tilde{a}_{t1}, \tilde{a}_{t2}, \ldots, \tilde{a}_{tn}\}$, for some $t \in N$. However, considering (12), this implies that

$$\tilde{a}_{ts} = x_t(0) = -x_t^*(0).$$

From here and from (18), for fixed index $j = s \in N$, we have:

$$\begin{aligned} q_s &= \max_i\{p_{is}\} = \max\{p_{1s}, \ldots, p_{ts}, \ldots, p_{ns}\} = \max\{p_{1s}, \ldots, x_t^*(0) + \tilde{a}_{ts}, \ldots, p_{ns}\} \\ &= \max\{p_{1s}, \ldots, -\tilde{a}_{ts} + \tilde{a}_{ts}, \ldots, p_{ns}\} = \max\{p_{1s}, \ldots, 0, \ldots, p_{ns}\} \\ &= 0. \end{aligned}$$

Hence, all entries of the vector q are zero in this case. This completes the proof. □

Remark 4.5. If the row-wise maximum is attained by a single element, then, since $\tilde{A}$ is a square matrix, in each column there will be exactly one maximal element of some row. On the other hand, the row-wise maximum may also be attained by multiple elements simultaneously. In such cases, when calculating the coordinates of q, within the corresponding maximum, more than one element will be equal to zero. Nonetheless, this does not affect the correctness of the statement or the proof.

Example 4.6. Let the matrix A be given:

$$A = \begin{pmatrix} -1 & -3 & 0 \\ -2 & 2 & 1 \\ 1 & 0 & -1 \end{pmatrix}. \qquad \text{Then:} \qquad \tilde{A} = \begin{pmatrix} 0 & \boxed{4} & 3 \\ 2 & 0 & \boxed{3} \\ \boxed{2} & 1 & 0 \end{pmatrix}.$$

The row-wise maximal elements of the matrix $\tilde{A}$ appear in distinct columns.

If we define the initial vector as in (12), we get

$$x(0) = \begin{pmatrix} 4 & 3 & 2 \end{pmatrix}^\top.$$

Now, we have:

$$\begin{aligned} T = x^*(0) \otimes \tilde{A} \otimes x(0) &= \begin{pmatrix} -4 & -3 & -2 \end{pmatrix} \otimes \begin{pmatrix} 0 & 4 & 3 \\ 2 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = 4 = \max_{i,j} \{\tilde{a}_{ij}\}. \end{aligned}$$

As established in Lemma 4.4, all entries of the vector q are zero in this case.

Lemma 4.4 provides a necessary condition for $T < \max_{i,j} \{\tilde{a}_{ij}\}$.

Corollary 4.7. Let the initial vector be chosen as in (12). The value of T can be smaller than $\max_{i,j} \{\tilde{a}_{ij}\}$ only if $\tilde{A}$ has at least one column that does not contain the maximal element of any row.

Example 4.8. The row-wise maximal elements of matrix $\tilde{A}$ from Example 4.3 lie in the same column, i. e., $\tilde{A}$ has two columns that does not contain a maximal element of any row. Since this condition holds, the value of T is less than $\max_{i,j} \{\tilde{a}_{ij}\}$ in this case.

However, this condition is not sufficient: a reduction in the value of T is not guaranteed even if $\tilde{A}$ has a column lacking a row-maximal element.

5. NUMERICAL RESULTS

We now compare the number of iterations in two cases: using a random initial vector and using the proposed initial vector from (12). For this purpose, we generated 10,000 random 3×3 matrices whose entries were taken from three different intervals to ensure statistically reliable results. For each matrix $A(i)$, $i \in [1,\ 10,000]$, we generated 10 random vectors, calculated the bound on the number of iterations for each of those vectors, and then compared the obtained results with the bound we got for the vector chosen as proposed in this paper.

Here, we present results for the first 10 matrices and 10 corresponding random vectors, whose entries are from $[-10, 10]$. In the table below, $A(i)$ denotes randomly generated matrices, where $i = \overline{1, 10}$. With $\Omega(i,j)$ we denote the value of Ω (calculated by (11)) for the matrix $A(i)$ and random initial vector $x_j^{(i)}(0)$, $j = \overline{1, 10}$:

$$\begin{aligned} \Omega(i,j) &= (n-1)\left(1 + \left(x_j^{(i)}(0)\right)^* \otimes (A(i))^* \otimes A(i) \otimes \left(x_j^{(i)}(0)\right)\right) \\ &= 2\left(1 + \left(x_j^{(i)}(0)\right)^* \otimes \tilde{A}(i) \otimes \left(x_j^{(i)}(0)\right)\right), \end{aligned}$$

for all $i = \overline{1, 10}$ and $j = \overline{1, 10}$. The initial vector from (12) will be denoted by $x(0)$, and the corresponding bound for that initial vector will be denoted by $\Omega(i)$:

$$\Omega(i) = 2\left(1 + x(0)^* \otimes (A(i))^* \otimes A(i) \otimes x(0)\right), \qquad \text{for every } i \in N. \tag{23}$$

To calculate the bound $\Omega(i,j)$ we apply formula (11) (we have no other option in the case of random initial vector); from the other side, if we choose initial vector like in (12), we can calculate $\Omega(i)$ not only by (11), but also by (13). However, as shown in Section 4 (Corollary 4.7), substituting the proposed vector into (11) can produce a smaller bound than the one obtained via (13). Therefore, here we have used (11) to compute $\Omega(i)$.

In the following, "best (better) result" refers to the smallest (smaller) bound on the number of iterations.
R_1 denotes the number of random vectors for which $x(0)$ yields a better result. $R2$ equals 1 if $x(0)$ outperforms more than half of the 10 random vectors; otherwise, it equals 0. $R3$ equals 1 if $x(0)$ produces the best result among all random vectors; otherwise, 0. $R4$ equals 1 if $x(0)$ gives the second-best result (i. e., better than 9 out of 10 random vectors); otherwise, 0.

A(i)	x(0)	$x_j^i(0)$	Ω(i,j)	Ω(i)	R1	R2	R3	R4
$\begin{pmatrix}-10&1&-3\\3&-10&-6\\1&3&9\end{pmatrix}$	$\begin{pmatrix}11\\13\\9\end{pmatrix}$	$\begin{pmatrix}7\\6\\6\end{pmatrix}\begin{pmatrix}-10\\-5\\-6\end{pmatrix}\begin{pmatrix}-5\\-10\\2\end{pmatrix}\begin{pmatrix}-10\\5\\-5\end{pmatrix}\begin{pmatrix}3\\8\\-9\end{pmatrix}\begin{pmatrix}2\\8\\-2\end{pmatrix}\begin{pmatrix}0\\-10\\2\end{pmatrix}\begin{pmatrix}1\\-3\\-7\end{pmatrix}\begin{pmatrix}10\\0\\3\end{pmatrix}\begin{pmatrix}-1\\8\\-6\end{pmatrix}$	28, 32, 36, 52, 42, 34, 46, 34, 46, 40	26	10	1	1	0
$\begin{pmatrix}-10&-4&5\\-1&9&2\\6&10&6\end{pmatrix}$	$\begin{pmatrix}15\\9\\7\end{pmatrix}$	$\begin{pmatrix}7\\-9\\9\end{pmatrix}\begin{pmatrix}1\\-8\\0\end{pmatrix}\begin{pmatrix}-5\\4\\0\end{pmatrix}\begin{pmatrix}-6\\-1\\3\end{pmatrix}\begin{pmatrix}10\\-10\\0\end{pmatrix}\begin{pmatrix}-5\\-7\\-4\end{pmatrix}\begin{pmatrix}-6\\-8\\-9\end{pmatrix}\begin{pmatrix}0\\-7\\6\end{pmatrix}\begin{pmatrix}5\\10\\-3\end{pmatrix}\begin{pmatrix}4\\6\\-3\end{pmatrix}$	54, 34, 40, 48, 38, 32, 24, 44, 40, 32	18	10	1	1	0
$\begin{pmatrix}-2&-2&5\\2&-3&-5\\5&7&-9\end{pmatrix}$	$\begin{pmatrix}7\\7\\16\end{pmatrix}$	$\begin{pmatrix}-3\\10\\7\end{pmatrix}\begin{pmatrix}-8\\2\\3\end{pmatrix}\begin{pmatrix}-8\\-4\\7\end{pmatrix}\begin{pmatrix}-6\\0\\-1\end{pmatrix}\begin{pmatrix}0\\1\\9\end{pmatrix}\begin{pmatrix}4\\-5\\-10\end{pmatrix}\begin{pmatrix}-10\\-10\\6\end{pmatrix}\begin{pmatrix}-9\\-9\\-6\end{pmatrix}\begin{pmatrix}10\\-3\\4\end{pmatrix}\begin{pmatrix}-6\\9\\4\end{pmatrix}$	38, 36, 44, 34, 32, 56, 46, 26, 40, 42	32	8	1	0	0
$\begin{pmatrix}-3&2&-4\\-4&-8&0\\-6&-3&-3\end{pmatrix}$	$\begin{pmatrix}5\\8\\6\end{pmatrix}$	$\begin{pmatrix}7\\10\\-4\end{pmatrix}\begin{pmatrix}9\\10\\10\end{pmatrix}\begin{pmatrix}-2\\-4\\7\end{pmatrix}\begin{pmatrix}0\\-5\\-10\end{pmatrix}\begin{pmatrix}3\\-10\\10\end{pmatrix}\begin{pmatrix}-5\\3\\6\end{pmatrix}\begin{pmatrix}5\\-7\\-10\end{pmatrix}\begin{pmatrix}-6\\-6\\3\end{pmatrix}\begin{pmatrix}6\\-7\\2\end{pmatrix}\begin{pmatrix}10\\-2\\3\end{pmatrix}$	40, 16, 38, 22, 56, 30, 32, 34, 34, 32	16	9	1	0	1
$\begin{pmatrix}-6&-7&-9\\5&4&-2\\8&-3&-4\end{pmatrix}$	$\begin{pmatrix}0\\11\\12\end{pmatrix}$	$\begin{pmatrix}-8\\-1\\-8\end{pmatrix}\begin{pmatrix}-10\\1\\-4\end{pmatrix}\begin{pmatrix}8\\-1\\-2\end{pmatrix}\begin{pmatrix}8\\4\\-5\end{pmatrix}\begin{pmatrix}-10\\-4\\10\end{pmatrix}\begin{pmatrix}-4\\2\\4\end{pmatrix}\begin{pmatrix}3\\-4\\-1\end{pmatrix}\begin{pmatrix}-2\\5\\-2\end{pmatrix}\begin{pmatrix}-5\\-3\\-7\end{pmatrix}\begin{pmatrix}1\\0\\-2\end{pmatrix}$	26, 22, 44, 50, 34, 10, 36, 26, 28, 30	20	9	1	0	1
$\begin{pmatrix}-2&6&-4\\-8&4&-5\\-6&1&-6\end{pmatrix}$	$\begin{pmatrix}12\\0\\10\end{pmatrix}$	$\begin{pmatrix}-1\\9\\-8\end{pmatrix}\begin{pmatrix}8\\10\\6\end{pmatrix}\begin{pmatrix}0\\8\\4\end{pmatrix}\begin{pmatrix}1\\2\\-10\end{pmatrix}\begin{pmatrix}-3\\-8\\6\end{pmatrix}\begin{pmatrix}1\\5\\-1\end{pmatrix}\begin{pmatrix}9\\-8\\9\end{pmatrix}\begin{pmatrix}0\\7\\-3\end{pmatrix}\begin{pmatrix}-10\\5\\2\end{pmatrix}\begin{pmatrix}-4\\5\\-8\end{pmatrix}$	54, 28, 40, 44, 24, 32, 20, 40, 54, 46	10	10	1	1	0
$\begin{pmatrix}8&1&-8\\9&4&4\\-5&-7&-8\end{pmatrix}$	$\begin{pmatrix}0\\7\\16\end{pmatrix}$	$\begin{pmatrix}9\\1\\-9\end{pmatrix}\begin{pmatrix}-1\\7\\-4\end{pmatrix}\begin{pmatrix}-8\\-5\\-6\end{pmatrix}\begin{pmatrix}6\\7\\8\end{pmatrix}\begin{pmatrix}-6\\6\\0\end{pmatrix}\begin{pmatrix}10\\-2\\3\end{pmatrix}\begin{pmatrix}-8\\3\\5\end{pmatrix}\begin{pmatrix}-6\\-3\\-7\end{pmatrix}\begin{pmatrix}-7\\6\\8\end{pmatrix}\begin{pmatrix}0\\-10\\-3\end{pmatrix}$	68, 40, 28, 28, 30, 46, 20, 34, 24, 38	26	8	1	0	0
$\begin{pmatrix}10&-10&9\\4&-2&-4\\-10&-2&2\end{pmatrix}$	$\begin{pmatrix}12\\20\\8\end{pmatrix}$	$\begin{pmatrix}-3\\3\\7\end{pmatrix}\begin{pmatrix}4\\2\\7\end{pmatrix}\begin{pmatrix}3\\-1\\2\end{pmatrix}\begin{pmatrix}2\\5\\-8\end{pmatrix}\begin{pmatrix}-9\\0\\2\end{pmatrix}\begin{pmatrix}2\\8\\-7\end{pmatrix}\begin{pmatrix}-1\\2\\8\end{pmatrix}\begin{pmatrix}9\\-9\\-6\end{pmatrix}\begin{pmatrix}-1\\-1\\-8\end{pmatrix}\begin{pmatrix}8\\-5\\1\end{pmatrix}$	46, 48, 48, 36, 46, 34, 50, 76, 40, 66	32	10	1	1	0
$\begin{pmatrix}-5&-3&7\\2&7&10\\-3&-2&2\end{pmatrix}$	$\begin{pmatrix}12\\10\\0\end{pmatrix}$	$\begin{pmatrix}-8\\-3\\5\end{pmatrix}\begin{pmatrix}2\\-9\\10\end{pmatrix}\begin{pmatrix}-3\\4\\-1\end{pmatrix}\begin{pmatrix}2\\-4\\-9\end{pmatrix}\begin{pmatrix}-5\\2\\-10\end{pmatrix}\begin{pmatrix}0\\9\\8\end{pmatrix}\begin{pmatrix}10\\4\\7\end{pmatrix}\begin{pmatrix}3\\-8\\-3\end{pmatrix}\begin{pmatrix}-3\\4\\-5\end{pmatrix}\begin{pmatrix}2\\9\\7\end{pmatrix}$	50, 58, 28, 12, 24, 40, 26, 30, 24, 34	14	9	1	0	1
$\begin{pmatrix}9&9&-10\\1&2&2\\8&3&-10\end{pmatrix}$	$\begin{pmatrix}1\\5\\19\end{pmatrix}$	$\begin{pmatrix}-1\\4\\-1\end{pmatrix}\begin{pmatrix}-7\\4\\-5\end{pmatrix}\begin{pmatrix}-1\\-6\\-7\end{pmatrix}\begin{pmatrix}-9\\2\\3\end{pmatrix}\begin{pmatrix}-1\\-6\\-10\end{pmatrix}\begin{pmatrix}7\\10\\4\end{pmatrix}\begin{pmatrix}6\\-10\\-6\end{pmatrix}\begin{pmatrix}10\\8\\-9\end{pmatrix}\begin{pmatrix}-1\\3\\-10\end{pmatrix}\begin{pmatrix}-1\\5\\-1\end{pmatrix}$	48, 56, 50, 36, 56, 50, 62, 76, 64, 50	38	9	1	0	1

Tab. 1. Representation of results for the first 10 randomly generated matrices.

In the table below, $R2'$, $R3'$ and $R4'$ denote the total number of ones observed for $R2$, $R3$ and $R4$, respectively. Total values obtained from 10,000 matrices are:

	$[-5,5]$	$[-10,10]$	$[-100,100]$	Percentage
$R2'$	9860	9912	9939	99%
$R3'$	3067	3832	4497	38%
$R4'$	2780	2714	2620	27%

Tab. 2. Total obtained values for 10, 000 matrices.

From the results presented, we conclude that the proposed initial vector yields better performance in 99 % of the cases. According to $R3$, the percentage of cases where the proposed vector gives the best result is 38 %, and the percentage that it gives the second best result is 27 %. Hence, the percentage that the solution obtained for the proposed vector choice will be the best or second best solution is 65 %.

Therefore, the obtained results support our choice of the initial vector, increasing the likelihood that the bound on the number of iterations in the Alternating Method will be minimized.

CONCLUSIONS

In this study, we proposed a vector with which the Alternating Method should start, and we determined an upper bound on the number of iterations for that initial vector. We then compared the upper bound for random initial vectors with the upper bound for the initial vector we proposed. The obtained results justify our choice of the initial vector. As a continuation of this work, future research could compare the exact number of iterations (and not just the upper bound) after which the Alternating Method for random vectors and for the vector proposed here ends.

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REFERENCES

[1] M. Akian, R. Bapat, and S. Gaubert: Max-plus algebra. In: Handbook of linear algebra, Chapter 25 (L. Hogben, R. Brualdi, A. Greenbaum, and R. Mathias, eds.), Chapman and Hall, 2007. DOI:10.1201/9781420010572

[2] A. Aminu, S. Olowo, I. Sulaiman, N. Abu Bakar, and M. Mamat: On application of max-plus algebra to synchronized discrete event system. Math. Statist. *9* (2021), 2, 81–92. DOI:10.13189/ms.2021.090201

[3] A. Aminu and P. Butkovič: Comparison of methods for solving two-sided systems in max-algebra. University of Birmingham, preprint 2008/8. www.academia.edu/73728997

[4] F. Baccelli, G. Cohen, G. Olsder, and J. Quadrat: Synchronization and linearity: An algebra for discrete event systems. J. Oper. Res. Soc. *45* (1994), 118–119.DOI:10.2307/2583959

[5] P. Butkovič: Max-linear Systems: Theory and Algorithms. Springer Verlag, London 2010.DOI:10.1007/978-1-84996-299-5

[6] P. Butkovič: On certain properties of the systems of linear extremal equations. Ekon. Mat. Obzor *14* (1978), 72–78. https://web.mat.bham.ac.uk/P.Butkovic/Pubs.html

[7] P. Butkovič: Solution of systems of linear extremal equations. Ekon. Mat. Obzor *17* (1981), 402–416. https://web.mat.bham.ac.uk/P.Butkovic/Pubs.html

[8] P. Butkovič and G. Hegedus: An elimination method for finding all solutions of the system of linear equations over an extremal algebra. Ekon. Mat. Obzor *20* (1984), 203–215. https://web.mat.bham.ac.uk/P.Butkovic/Pubs.html

[9] P. Butkovič: Max-algebra: the linear algebra of combinatorics? Linear Algebra Appl. *367* (2003), 313–335.DOI:10.1016/S0024-3795(02)00655-9

[10] P. Butkovič: A strongly polynomial algorithm for solving two-sided linear systems in max-algebra. Discrete Appl. Math. *154* (2006), 437–446.DOI:10.1016/j.dam.2005.09.008

[11] B A. Carre: An algebra for network routing problems. J. Institute of Mathematics and Its Applications *7* (1971), 273–294.DOI:10.1093/imamat/7.3.273

[12] G. Cohen, S. Gaubert, E. Mancinelli, J. P. Quadrat, and E. Rofman: On traffc light control of regular towns. Math. Notae, Boletin del Instituto de Matematica "Beppo Levi", Ano XLIII, (2005), 51–62. https://inria.hal.science/inria-00072311

[13] R. A. Cuninghame-Green and P. Butkovič: The equation $A \otimes x = B \otimes y$ over $(\max, +)$. Theoret. Comput. Sci. *293* (2003), 3–12.DOI:10.1016/S0304-3975(02)00228-1

[14] R. A. Cuninghame-Green: Projections in minimax algebra. Math. Program. *10* (1976), 1, 111–123.DOI:10.1007/BF01580656

[15] R. A. Cuninghame-Green: Minimax Algebra. Lecture Notes Econom. Math. Systems *166*, Springer, Berlin 1979. DOI:10.1007/978-3-642-48708-8

[16] R. A. Cuninghame-Green: Minimax algebra and applications. Advances Imaging Electron Physics *90* (1994), 1–121.DOI:10.1016/S1076-5670(08)70083-1

[17] R. A. Cuninghame-Green: Process synchronisation in a steelworks - a problem of feasibility. In: Proc. 2nd Int. Conf. on Operational Research (Banbury and Maitland, eds.), English University Press (1960), pp. 323–328.

[18] R. A. Cuninghame-Green: Describing industrial processes with interference and approximating their steady-state behaviour. Oper. Res. Quarterly *13* (1962), 95–100.DOI:10.2307/3007584

[19] S. Gaubert: Methods and Applications of $(\max, +)$ Linear Algebra. Research Report RR3088, INRIA 1997.DOI:10.1007/BFb0023465

[20] S. Gaubert: Nonlinear Perron–Frobenius theory and discrete event systems. Actes du colloque Modelisation de Systmes Ractifs (MSR'05) JESA *39* (2005), 175–190.DOI:10.3166/jesa.39.175-190

[21] B. Giffler: Scheduling general production systems using schedule algebra. Naval Res. Logist. Quarterly *10* (1963), 237–255.DOI:10.1002/nav.3800100119

[22] M. Gondran and M. Minoux: Valeurs propres et vecteur propres dans les dioides et leur interpretation en theorie des graphes. Bulletin de la Direction des Etudes et Recherches Serie C Mathematiques et Informatiques *2* (1977), 25–41.

[23] B. Heidergott, G. J. Olsder, and J. van der Woude: Max-plus at work, Modelling and Analysis of Synchronized Systems: A course on Max-Plus Algebra and Its Applications. Princeton University Press, New Jersey 2006.DOI:10.1515/9781400865239

[24] D. Jones: On two-sided Max-Linear equations. Discr. Appl. Math. *254* (2019), 146–160.DOI:10.1016/j.dam.2018.06.011

[25] N. Krivulin and S. Sergeev: Minimizing maximum lateness in two-stage projects by tropical optimization. Kybernetika *58* (2022), 5, 816-841.DOI:10.14736/kyb-2022-5-0816

[26] P. Li: On sparsity of approximate solutions to max-plus linear systems. Kybernetika *60* (2024), 3, 412–425.DOI:10.14736/kyb-2024-3-0412

[27] S. Sergeev: Alternating Method for homogeneous systems of equations over max-algebra. School of Mathematics pre-print, University of Birmingham, *18* (2009). https://api.semanticscholar.org/CorpusID:1966959

[28] B. Stojčetović: Max-algebraic systems of equations – from basic to higher level. In: The Fourth International Conference on Education in Mathematics, Physics and Related Science, Skopje 2023.

[29] B. Stojčetović, J. Ignjatović, and M. Ćirić: Pseudo skew-symmetric matrices over max-plus algebras. Accepted for publication in Filomat.

[30] N. N. Vorobyov: Extremal algebra of positive matrices. Elektronische Informationsverarbeitung und Kybernetik *3* (1967), 39–71 (in Russian).

[31] K. Zimmermann: Extremalni algebra. Vyzkumna publikace Ekonomicko-matematicke laboratore pri Ekonomickem ustavu CSAV 46, Praha 1976. (in Czech).

Bojana Stojčetović, University of Priština in Kosovska Mitrovica, Faculty of Sciences and Mathematics, Lole Ribara 29, 38220 Kosovska Mitrovica. Serbia.
e-mail: bojana.stojcetovic@pr.ac.rs