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A NOTE ON THE UNIFORMITY OF STRONG SUBREGULARITY AROUND THE REFERENCE POINT

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This paper investigates strong metric subregularity around the reference point as introduced by H. Gfrerer and J. V. Outrata in [9]. In the setting of Banach spaces, we analyse its stability under Lipschitz continuous perturbations and establish its uniformity over compact sets. Our results ensure that the property is preserved under small Lipschitz perturbations, which is crucial for maintaining robustness in variational analysis. Furthermore, we apply the developed theory to parametric inclusion problems. The analysis demonstrates that the uniformity of strong metric subregularity provides a theoretical foundation for addressing stability issues in parametrized optimization and control applications.

Keywords: strong metric subregularity, Lipschitz continuity, uniformity, sum stability

Classification: 49J53, 49J52, 90C33

1. INTRODUCTION

Stability and robustness of solutions in optimization and control are central issues in variational analysis, where understanding local solution behavior and its sensitivity to perturbations is essential. Recently, inspired by the Newton method, H. Gfrerer and J. V. Outrata [8] introduced the semismooth* method, bringing renewed attention to the concept of strong metric subregularity, e. g., [3]. The property was extended by the same authors to strong metric subregularity around the reference, which enables the analysis of solution stability and the convergence of the method, see [2].

The aim of this study is to demonstrate that strong metric subregularity is preserved and even becomes uniform using Lipschitz continuous perturbations. Building upon these foundations, we prove the existence of uniform constants and domains that ensure the subregularity property holds uniformly across compact sets. Such uniformity plays a crucial role in the robustness of numerical methods and path-following techniques for solving parametric inclusion problems, e. g., [1, 5, 6].

In the sections that follow, we first establish the basic notation and necessary definitions within the framework of Banach spaces. We then present the main theoretical results, demonstrating that local strong metric subregularity can be extended to uniform subregularity on compact sets. This extension not only extends the theoretical

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foundations of variational analysis but also opens up new avenues for the construction and analysis of efficient algorithms in optimization, control, and economic modelling.

2. PRELIMINARIES

Throughout the whole paper, we assume that X and Y are Banach spaces and P is a metric space. The closed ball and open ball of radius δ centred at a point $x \in X$ are defined respectively as

$$\mathbb{B}_X[x,\delta] := \{u \in X : \|x-u\| \le \delta\} \quad \text{and} \quad \mathbb{B}_X(x,\delta) = \{u \in X : \|x-u\| < \delta\}.$$

The distance from a point $x \in X$ to a set A is denoted by $\operatorname{dist}(x, A)$ and is defined as the shortest distance between x and any point in A, expressed as $\operatorname{dist}(x, A) = \inf_{u \in A} \|u - x\|$.

The graph of a set-valued mapping F, represented as $\operatorname{gph} F$, comprises all pairs (x, y) such that $y \in F(x)$. Additionally, the domain of F, denoted by $\operatorname{dom} F$, includes all points x for which the set $F(x)$ is nonempty, indicating the extent of the definition of F. The inverse of a set-valued mapping F, denoted by F^{-1}, is defined such that $y \in F(x)$ implies $x \in F^{-1}(y)$. This is expressed as $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$.

In modern variational analysis, examining the regularity of set-valued mappings is essential for interpreting various mathematical models, especially in fields such as optimization, control theory, and economics. The regularity of these mappings refers to the characteristics that determine the local behaviour of the mapping around a point in its domain. Here, we focus solely on properties that are relevant to our research.

Definition 2.1. Let $F : X \rightrightarrows Y$ be a set-valued mapping and let $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ be a given point. We say that F is:

(i) *metrically subregular* at $(\bar{x}, \bar{y})$ if there exists $\kappa \ge 0$ along with some neighborhood U of $\bar{x}$ such that

$$\operatorname{dist}(x, F^{-1}(\bar{y})) \le \kappa \operatorname{dist}(\bar{y}, F(x)) \quad \text{for each} \quad x \in U;$$

(ii) *strongly metrically subregular* at $(\bar{x}, \bar{y})$ if there exists $\kappa \ge 0$ and a neighborhood U of $\bar{x}$ such that

$$\|x - \bar{x}\| \le \kappa \operatorname{dist}(\bar{y}, F(x)) \quad \text{for each} \quad x \in U;$$

(iii) (strongly) *metrically subregular around* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if there is a neighborhood W of $(\bar{x}, \bar{y})$ and $\kappa \ge 0$ such that at each $(x, y) \in \operatorname{gph} F \cap W$ there is a neighborhood U of x such that

$$\|u - x\| \le \kappa \operatorname{dist}(y, F(u)) \quad \text{for each} \quad u \in U.$$

Strong metric subregularity around the reference point was first introduced in [9]; also see [7, 10] for the additional properties.

Remark 2.2. In contrast to the subregularity notions above—which fix one argument at $\bar{y}$ or allow perturbations only "around" the graph—one can also consider the full metric regularity and its strong variant:

(a) *Metric regularity.* F is said to be metrically regular at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if there exist constants $\kappa > 0$ and $\delta > 0$ such that for each $(x, y) \in I\!B_X[\bar{x}, \delta] \times I\!B_Y[\bar{y}, \delta]$ we have

$$\operatorname{dist}\big(x,\, F^{-1}(y)\big) \leq \kappa \operatorname{dist}\big(y,\, F(x)\big).$$

(b) *Strong metric regularity.* F is strongly metrically regular at $(\bar{x}, \bar{y})$ if, in addition, the inverse F^{-1} admits a single-valued localization $s\colon V \to U$ around $\bar{y}$ (with $s(\bar{y}) = \bar{x}$) which is Lipschitz continuous: there exist $\kappa > 0$ and neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ such that

$$F^{-1}(y) \cap U = \{\, s(y) \,\} \quad \text{and} \quad \|s(y_1) - s(y_2)\| \leq \kappa \|y_1 - y_2\| \quad \text{for each} \quad y_1, y_2 \in V.$$

Moreover, (strong) metric regularity at $(\bar{x}, \bar{y})$ implies (strong) metric subregularity around $(\bar{x}, \bar{y})$ with the same constant κ and uniform neighborhoods.

The following example presents set-valued mappings that exhibit strong metric subregularity property at every point on their graph.

Example 2.3. (i) Define a set-valued mapping $F : \mathbb{R} \rightrightarrows \mathbb{R}$ given by

$$\operatorname{gph} F := \big\{ \big(x, x\big), \big(x, -x\big) : x \in [-1, 1] \big\}.$$

The graph of F is in Figure 1(a).

(ii) Define a set-valued mapping $F : \mathbb{R} \rightrightarrows \mathbb{R}$ given by

$$F(x) := \begin{cases} \sqrt[2]{x} + 1 & \text{for} \quad x > 0, \\ [-1, 1] & \text{for} \quad x = 0, \\ \sqrt[2]{x} - 1 & \text{for} \quad x < 0. \end{cases}$$

The graph of F is in Figure 1(b).

(iii) Define a set-valued mapping $F : \mathbb{R} \rightrightarrows \mathbb{R}$ given by

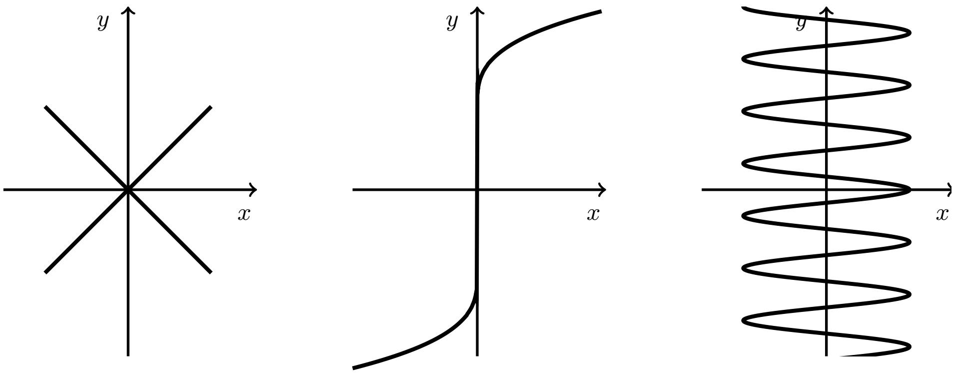
$$\operatorname{gph} F := \big\{ \big(\cos(x), x \big) : x \in \mathbb{R} \big\}.$$

The graph of F is in Figure 1(c).

Definition 2.4. Let $F : X \rightrightarrows Y$ be a set-valued mapping and let $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. We say that

(i) F is *calm* at $(\bar{x}, \bar{y})$ if there exist $\mu \geq 0$ and a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ such that

$$F(x) \cap V \subset F(\bar{x}) + \mu \|x - \bar{x}\| I\!B_Y \quad \text{for all } x \in U;$$



(a) (b) (c)

Fig. 1. Graphs of set-valued mappings having strong metric subregularity around every point of their graph from Example 2.3.

(ii) F is *isolated calm* at $(\bar{x}, \bar{y})$ if there exist $\mu \geq 0$ and a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ such that

$$F(x) \cap V \subset \{\bar{y}\} + \mu\|x - \bar{x}\| I\!B_Y \quad \text{for all} \quad x \in U.$$

Remark 2.5. Note that a single-valued mapping $f : X \longrightarrow Y$ is (isolated) calm at $\bar{x}$ if there exist $\mu \geq 0$ and a neighborhood U of $\bar{x}$ such that

$$\|f(x) - f(\bar{x})\| \leq \mu\|x - \bar{x}\| \quad \text{for each} \quad x \in U.$$

It is known that strong metric subregularity is stable under a calm single-valued perturbation [3].

Theorem 2.6. Let α, κ, μ be positive constants such that $\kappa\mu < 1$. Consider a mapping $F : X \rightrightarrows Y$ which is strongly subregular at $(\bar{x}, \bar{y})$ with the constant κ and a neighborhood $I\!B_X[\bar{x}, \alpha]$, and a function $g : X \longrightarrow Y$ which is calm at $\bar{x}$ with constant μ and a neighborhood $I\!B_X[\bar{x}, \beta]$. Then $g + F$ is strongly metrically subregular at $(\bar{x}, \bar{y} + g(\bar{x}))$ with the constant $\kappa/(1 - \kappa\mu)$ and the neighborhood $I\!B_X[\bar{x}, \alpha]$.

It should be noted that metric subregularity does not necessarily hold when a calm single-valued perturbation is applied. The example below illustrates this with a counterexample.

Example 2.7. Define two functions f and g such that

$$f(x) = \begin{cases} 0, & x \leq 0, \\ x, & x > 0, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x^2, & x \leq 0, \\ -x^2, & x > 0, \end{cases}$$

where f is metrically subregular at 0 and g is calm at 0.

Since $|g(x)| \le |x|$ near 0, g is (isolated) calm. Now, consider the function $h := f + g$:

$$h(x) = \begin{cases} x^2, & x \le 0, \\ x - x^2, & x > 0. \end{cases}$$

Metric subregularity of h at 0 would require the existence of $\kappa > 0$ such that

$$|x| \le \kappa\, |h(x)|$$

in a neighborhood of 0. For $x < 0$, we have $h(x) = x^2$, so

$$|x| \le \kappa |x^2| \quad \to \quad \frac{1}{|x|} \le \kappa.$$

This inequality fails as the left-hand side goes to infinity when $x \downarrow 0$.

Using ideas from [3], the following proposition establishes the stability of strong metric subregularity at the reference point under set-valued perturbations.

Proposition 2.8. Consider a set-valued mapping $F : X \rightrightarrows Y$ and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Assume that there are $\kappa > 0$ and $\alpha > 0$ such that F is strongly metrically subregular at $(\bar{x}, \bar{y})$ with the constant κ and the neighborhood $I\!B_X[\bar{x}, \alpha]$. Let $\mu > 0$ be such that $\kappa\mu < 1$ and let $\kappa' > \kappa/(1 - \kappa\mu)$.

Then for each $\beta \in (0, \alpha]$ and for each set-valued mapping $G : X \rightrightarrows Y$ satisfying

$$G(\bar{x}) = \{\bar{z}\} \quad \text{and} \quad G(x) \subset \{\bar{z}\} + \mu\|x - \bar{x}\| I\!B_Y \quad \text{for each} \quad x \in I\!B_X[\bar{x}, \beta], \tag{1}$$

we have that the mapping $G + F$ is strongly metrically subregular at $(\bar{x}, \bar{y} + \bar{z})$ with the constant κ' and the neighborhood $I\!B_X[\bar{x}, \beta]$.

Proof. Fix any $\mu > 0$, $\beta > 0$, and any mapping G as in the conclusion. Fix any $x \in I\!B_X[\bar{x}, \beta]$.

If $G(x) + F(x) = \emptyset$, we are done. If not, fix any $z \in G(x)$, then

$$\|z - \bar{z}\| \le \mu\|x - \bar{x}\|.$$

Then

$$\begin{aligned} \|x - \bar{x}\| &\le \kappa \operatorname{dist}(\bar{y}, F(x)) \le \kappa \operatorname{dist}(\bar{y} + \bar{z}, z + F(x)) + \kappa\|z - \bar{z}\| \\ &\le \kappa \operatorname{dist}(\bar{y} + \bar{z}, z + F(x)) + \kappa\mu\|x - \bar{x}\|. \end{aligned}$$

Taking into account that $\kappa\mu < 1$ and $\frac{\kappa}{1-\kappa\mu} < \kappa'$ and that z is fixed arbitrary, we obtain

$$\|x - \bar{x}\| \le \tfrac{\kappa}{1-\kappa\mu} \operatorname{dist}(\bar{y} + \bar{z}, G(x) + F(x)) \le \kappa' \operatorname{dist}(\bar{y} + \bar{z}, G(x) + F(x)).$$

□

Note that the property in (1) is stronger than isolated calmness at $(\bar{x}, \bar{z})$. Moreover, since $G(\bar{x})$ is a singleton and G is upper semicontinuous at $\bar{x}$, it follows that F and G are sum-stable in the sense of [11].

3. UNIFORMITY OF STRONG SUBREGULARITY

We are investigating uniformity of strong metric subregularity around the reference point on compact subsets of Banach spaces of mappings which are defined as a sum of a single-valued (possibly nonsmooth) mapping and a set-valued mapping. We are following ideas of the proofs from [4, Section 2].

First, we present a statement concerning perturbed strong metric subregularity on a set.

Theorem 3.1. Consider a set-valued mappings $F : X \rightrightarrows Y$ and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Assume that there are positive constants a, b, and κ such that for each $(x, y) \in \big(\mathbb{B}_X[\bar{x}, a] \times \mathbb{B}_Y[\bar{y}, b]\big) \cap \operatorname{gph} F$ there is $r > 0$ such that for each $u \in \mathbb{B}_X[x, r]$ we have

$$\|u - x\| \le \kappa \operatorname{dist}(y, F(u) \cap \mathbb{B}_Y[\bar{y}, b]). \tag{2}$$

Let $\mu > 0$ be such that $\kappa\mu < 1$ and let $\kappa' > \kappa/(1 - \kappa\mu)$. Then for every positive α and β such that $2\alpha \le a$ and $2\beta + \mu\alpha \le b$ and for every mapping $g : X \longrightarrow Y$ satisfying

$$\|g(\bar{x})\| \le \beta \quad \text{and} \quad \|g(x) - g(u)\| \le \mu\|x - u\| \quad \text{for each} \quad x, u \in \mathbb{B}_X[\bar{x}, \alpha],$$

we have that for each $y \in \mathbb{B}_Y[\bar{y}, \beta]$ and each $x \in (g + F)^{-1}(y) \cap \mathbb{B}_X[\bar{x}, \alpha]$ there is $r \in (0, \alpha]$ such that each $u \in \mathbb{B}_X[x, r]$ and each $v \in (g + F)(u) \cap \mathbb{B}_Y[\bar{y}, \beta]$ we have

$$\|u - x\| \le \kappa'\|y - v\|.$$

Proof. Fix any $\alpha > 0$ and $\beta > 0$ and any mapping g as in the conclusion. Then fix any $y \in \mathbb{B}_Y[\bar{y}, \beta]$. Fix any $x \in (g + F)^{-1}(y) \cap \mathbb{B}_X[\bar{x}, \alpha]$ and find a corresponding $r \in (0, \alpha]$ such that (2), for each $u \in \mathbb{B}_X[x, r]$, holds. Fix any $u \in \mathbb{B}_X[x, r]$. Then $u \in \mathbb{B}_X[\bar{x}, a]$ since

$$\|u - \bar{x}\| \le \|u - x\| + \|x - \bar{x}\| \le r + \alpha \le 2\alpha \le a.$$

Note that $\mathbb{B}_Y[\bar{y}, \beta] \subset \mathbb{B}_Y[\bar{y} + g(x), b]$, since for each $v \in \mathbb{B}_Y[\bar{y}, \beta]$ we have

$$\|\bar{y} + g(x) - v\| \le \|\bar{y} - v\| + \|g(x) - g(\bar{x})\| + \|g(\bar{x})\| \le \beta + \mu r + \beta \le 2\beta + \mu\alpha \le b;$$

so is $y - g(x) \in \mathbb{B}_Y[\bar{y}, b]$. If $(g(u) + F(u)) \cap \mathbb{B}_Y[\bar{y}, \beta] = \emptyset$, we are done. If not, fix any $v \in (g(u) + F(u)) \cap \mathbb{B}_Y[\bar{y}, \beta]$. Then

$$\begin{aligned} \|u - x\| &\le \kappa \operatorname{dist}(y - g(x), F(u) \cap \mathbb{B}_Y[\bar{y}, b]) \le \kappa \operatorname{dist}(y - g(u), F(u) \\ &\qquad \cap \mathbb{B}_Y[\bar{y}, b]) + \kappa\|g(u) - g(x)\| \\ &\le \kappa \operatorname{dist}(y, (g(u) + F(u)) \cap \mathbb{B}_Y[\bar{y} + g(u), b]) + \kappa\mu\|u - x\| \\ &\le \kappa \operatorname{dist}(y, (g(u) + F(u)) \cap \mathbb{B}_Y[\bar{y}, \beta]) + \kappa\mu\|u - x\| \\ &\le \kappa\|y - v\| + \kappa\mu\|u - x\|. \end{aligned}$$

Taking into account that $\kappa\mu < 1$ and $\frac{\kappa}{1-\kappa\mu} < \kappa'$, we obtain

$$\|u - x\| \le \tfrac{\kappa}{1-\kappa\mu}\|y - v\| \le \kappa'\|y - v\|.$$

□

Theorem 3.1 not only refines [4, Theorem 2.3] by extending pointwise strong metric subregularity (around the reference point) to uniform strong metric subregularity around the reference point, but also complements the parametric stability results for generalized equations in [4, Section 2]. As a consequence, we obtain the following immediate corollaries on strong subregularity in a neighborhood of the reference.

We will now demonstrate that strong metric subregularity around each point of a compact set implies uniform strong metric subregularity. In other words, it is possible to find the same constant and neighborhood for all points within this set.

Theorem 3.2. Let $\Omega \subset P \times X$ be a compact set. Consider a set-valued mapping $F : X \rightrightarrows Y$ and a continuous single-valued mapping $f : P \times X \longrightarrow Y$ such that for each $(t, \bar{x}) \in \Omega$ we have:

(i) the mapping $X \ni x \longmapsto G_t(x) := f(t,x) + F(x)$ is strongly metrically subregular around $(\bar{x}, 0)$;

(ii) for each $\mu > 0$ there is $\alpha > 0$ such that for each $x, u \in \mathbb{B}_X[\bar{x}, \alpha]$ and each $s \in \mathbb{B}_P[t, \alpha]$ we have

$$\|f(s,u) - f(t,u) - (f(s,x) - f(t,x))\| \leq \mu \|x - u\|.$$

Then:

(iii) there are positive constants κ and a, b such that for each $(t, \bar{x}) \in \Omega$ the mapping G_t is strongly metrically subregular around $(\bar{x}, 0)$ with the constant κ and neighborhoods $\mathbb{B}_X[\bar{x}, a]$ and $\mathbb{B}_Y[0, b]$;

(iv) there are $\kappa > 0$ and $a > 0$ such that for each $(t, x) \in \Omega$ the mapping G_t is strongly metrically subregular at $(x, 0)$ with the constant κ' and the neighborhood $\mathbb{B}_X[x, a]$.

P r o o f. We are showing only (iii). The proof of (iv) follows similarly from Proposition 2.8, see also [4, Theorem 2.6]. Fix any $(t, \bar{x}) \in \Omega$. Find positive a, b, and κ, such that for each $(x, y) \in \big(\mathbb{B}_X[\bar{x}, a] \times \mathbb{B}_Y[0, b]\big) \cap \operatorname{gph} G_t$ there is $r > 0$ such that for each $u \in \mathbb{B}_X[x, r]$ we have

$$\|u - x\| \leq \kappa \operatorname{dist}(y, G_t(u)).$$

Let $\mu := 1/(2\kappa)$ and $\kappa' := 3\kappa$. Then $\kappa\mu < 1$ and $\kappa' > 2\kappa = \kappa/(1 - \kappa\mu)$. Find $\alpha \in (0, b/(2\mu))$ such that each $x, u \in \mathbb{B}_X[\bar{x}, 2\alpha]$ and each $s \in \mathbb{B}_P[t, \alpha]$ we have

$$\|f(s,u) - f(t,u) - (f(s,x) - f(t,x))\| \leq \mu \|x - u\|.$$

Let $\beta := b/4$. Then $2\beta + \mu\alpha < b/2 + b/2 = b$. Since f is continuous, there is $r' \in (0, \alpha/2]$ such that

$$\|f(s, \bar{x}) - f(t, \bar{x})\| \leq \beta \quad \text{for each} \quad s \in \mathbb{B}_P[t, r'].$$

Fix any $(s, x) \in \big(\mathbb{B}_P[t, r'] \times \mathbb{B}_X[\bar{x}, r']\big) \cap \Omega$. Define a mapping $g : X \longrightarrow Y$ such that

$$g(u) := f(s,u) - f(t,u) \quad \text{for} \quad u \in X.$$

Then $G_s = g + G_t$ and for each $x, u \in I\!B_X[\bar{x}, 2\alpha]$ we have

$$\|g(x) - g(u)\| \leq \mu\|x - u\| \quad \text{and} \quad \|g(\bar{x})\| \leq \beta.$$

Theorem 3.1, with $G := G_t$ and $\bar{y} := 0$, implies that for each $y \in I\!B_Y[0, \beta]$ and each $x \in (g + G_t)^{-1}(y) \cap I\!B_X[\bar{x}, \alpha]$ there is $r \in (0, \alpha]$ such that for each $u \in I\!B_X[x, r]$ and each $v \in (g + G_t)(u) \cap I\!B_X[0, \beta]$ we have

$$\|u - x\| \leq \kappa' \|y - v\|.$$

We are showing that for each $(x, y) \in \big(I\!B_X[\bar{x}, \alpha/3] \times I\!B_Y[0, \beta/3]\big) \cap \operatorname{gph} G_s$ there is $r > 0$ such that for each $u \in I\!B_X[x, r]$ we have

$$\|u - x\| \leq \kappa' \operatorname{dist}(y, G_s(u)).$$

Fix any such (x, y) and find a corresponding $r \in (0, 2\kappa'\beta/3]$ as in the claim and fix any $u \in I\!B_X[x, r]$. Thus $x \in (g + G_t)^{-1}(y) \cap I\!B_X[\bar{x}, \alpha]$. Fix any $v \in G_s(u)$. If $\|v\| \leq \beta$, using the claim, we get $\|u - x\| \leq \kappa'\|y - v\|$.

If $\|v\| > \beta$, then $\|y - v\| \geq \|v\| - \|y\| > \beta - \beta/3 = 2/3\beta$ and so

$$\|u - x\| \leq r \leq 2\kappa'\beta/3 < \kappa'\|y - v\|.$$

To sum up, we show that for each $(t, \bar{x}) \in \Omega$ there are constants $\kappa' > 0$, $\alpha > 0$, $\beta > 0$, and $r' \in (0, \alpha/2)$ such that for each $(s, u) \in \big(I\!B_P[t, r'] \times I\!B_X[\bar{x}, r']\big) \cap \Omega$ and each $(x, y) \in \big(I\!B_X[u, \alpha] \times I\!B_Y[0, \beta]\big) \cap \operatorname{gph} G_s$ there is $r > 0$ such that for each $v \in I\!B_X[x, r]$ we have

$$\|v - x\| \leq \kappa' \operatorname{dist}(y, G_s(v)).$$

Note that $I\!B_X[u, r'] \subset I\!B_X[\bar{x}, \alpha]$, then G_s is strongly metrically subregular around $(x, 0)$ with the constant κ' and neighborhoods $I\!B_X[x, \alpha]$ and $I\!B_Y[0, \beta]$.

So κ', α, β, and r' depends only on the choice $(\bar{t}, \bar{x}) \in \Omega$. Then from open covering $\cup_{z=(t,x)\in\Omega}\big(I\!B_P(t, r'_z) \times I\!B_X(x, r'_z)\big)$ of compact set Ω find a finite subcovering $\mathcal{O}_i := \big(I\!B_P(t_i, r'_i) \times I\!B_X(x_i, r'_i)\big)$ for $i = 1, 2, 3, \ldots, N$. Let $a := \min\{\alpha_i : i = 1, 2, 3, \ldots, N\}$, $b := \min\{\beta_i : i = 1, 2, 3, \ldots, N\}$, and $\kappa := \max\{\kappa'_i : i = 1, 2, 3, \ldots, N\}$. For any $(t, x) \in \Omega$ there is an index $i \in \{1, 2, 3, \ldots, N\}$ such that $(t, x) \in \mathcal{O}_i$. Hence the mapping G_t is strongly metrically subregular around $(x, 0)$ with the constant κ and neighborhoods $I\!B_X[x, a]$ and $I\!B_Y[0, b]$. □

Note that if f is continuously differentiable, then condition (ii) is satisfied and that while (i) asserts pointwise subregularity at each $(t, \bar{x}) \in \Omega$, statement (iii) furnishes uniform constants on whole Ω.

Example 3.3. Consider a single-valued mapping $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by

$$g(x, y) := (x, y, x^2 + y^2) \quad \text{for} \quad x, y \in \mathbb{R}.$$

Then

$$g^{-1}(x, y, v) = (x, y) \quad \text{for} \quad x, y, v \in \mathbb{R} \quad \text{with} \quad v = x^2 + y^2.$$

Then for each $\bar{x}, \bar{y}, \bar{v} \in \mathbb{R}$ with $\bar{v} = \bar{x}^2 + \bar{y}^2$ and each $x, y \in \mathbb{R}$ we have

$$\left\|(x,y) - g^{-1}(\bar{x},\bar{y},\bar{v})\right\| = \|(x,y) - (\bar{x},\bar{y})\| \leq \left\|(x,y,x^2+y^2) - (\bar{x},\bar{y},\bar{v})\right\| \\ = \|g(x,y) - (\bar{x},\bar{y},\bar{v})\|.$$

So g is strongly metrically regular at each point $(\bar{x}, \bar{y})$ of its graph. So g is strongly metrically regular around each point $(\bar{x}, \bar{y})$ of its graph.

Choose any compact subset $\Omega \subset \{(g(x), x) \in \mathbb{R}^3 \times \mathbb{R}^2 : x \in \mathbb{R}^2\}$ and for $t \in \mathbb{R}^3$ define mapping $G_t : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $G_t \colon X \to Y, \quad G_t(x) = g(x) - t \quad \text{for} \quad x \in \mathbb{R}^2$. Then assumptions of Theorem 3.2 are satisfied for this setting with $F \equiv 0$.

Theorem 3.4. Let $\Omega \subset P \times X$ be a compact set. Consider a set-valued mapping $F : X \rightrightarrows Y$ and a continuous single-valued mapping $f : P \times X \longrightarrow Y$ such that for each $(t, \bar{x}) \in \Omega$ we have:

(i) the mapping $X \ni x \longmapsto G_t(x) := f(t,x) + F(x)$ is strongly metrically subregular **at** $(\bar{x}, 0)$;

(ii) for each $\mu > 0$ there is $\alpha > 0$ such that for each $x, u \in \mathbb{B}_X[\bar{x}, \alpha]$ and each $s \in \mathbb{B}_P[t, \alpha]$ we have

$$\|f(s,u) - f(t,u) - (f(s,x) - f(t,x))\| \leq \mu\|x - u\|.$$

Then there are $\kappa > 0$ and $c > 0$ such that for each $(t, x) \in \Omega$ the mapping G_t is strongly metrically subregular **at** $(x, 0)$ with the constant κ and the neighborhood $\mathbb{B}_X[x, c]$.

Proof. The proof is similar to the proof of Theorem 3.2, but instead of using Theorem 3.1, we apply Proposition 2.8. □

The following statement guarantees that uniform strong metric subregularity is preserved along continuous solution trajectories for the widely studied parametric generalized equation, for $T > 0$, given by

$$p(t) \in f(t, x(t)) + F(x(t)) \quad \text{for each} \quad t \in [0, T],$$

where $p : [0,T] \longrightarrow Y, x : [0,T] \longrightarrow X$, $f : [0,T] \times X \longrightarrow Y$, and $F : X \rightrightarrows Y$.

This means that as one follows a continuous path within the domain, the property of strong metric subregularity remains consistent and uniform, providing stability in the behaviour of the system

Theorem 3.5. Let $T > 0$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ and a continuous single-valued mapping $f : [0,T] \times X \longrightarrow Y$, and two continuous mappings $x : [0,T] \longrightarrow X$ and $p : [0,T] \longrightarrow Y$ such that

(i) for each $t \in [0,T]$ the mapping $X \ni x \longmapsto G_t(x) := f(t,x) + F(x)$ is strongly metrically subregular around $(x(t), p(t))$;

(ii) for each $t \in [0,T]$ and each $\mu > 0$ there is $\delta > 0$ such that for each $x, u \in I\!B_X[x(t), \delta]$ and each $s \in (t-\delta, t+\delta)$ we have

$$\|f(s,u) - f(t,u) - (f(s,x) - f(t,x))\| \le \mu \|x - u\|.$$

Then:

(iii) there are positive constants a, b, and κ such that for each $t \in [0,T]$ the mapping G_t is strongly metrically subregular around $(x(t), p(t))$ with the constant κ and neighborhoods $I\!B_X[x(t), a]$ and $I\!B_Y[p(t), b]$;

(iv) there are $c > 0$ and $\kappa' > 0$ such that for each $t \in [0,T]$ the mapping G_t is strongly metrically subregular at $(x(t), p(t))$ with the constant κ' and neighborhood $I\!B_X[x(t), c]$.

Proof. For (iii), apply Theorem 3.2, and for (iv), apply Theorem 3.4, both with $P := [0,T] \times Y$, the compact set $\Omega := \bigcup_{t \in [0,T]} (t, p(t), x(t))$, and the function $f(t,x) := f(p,x) - y$ for $t = (p,y) \in P$ and $x \in X$. □

Example 3.6. Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be defined by

$$F(x) \ := \ \bigcup_{i=1}^{k} G_i(x),$$

where, for each $i \in \{1, 2, \ldots, k\}$, $G_i\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximally monotone, i.e. for every $x, u \in \mathbb{R}^n$ and every $y \in G_i(x)$, $v \in G_i(u)$, one has

$$\big\langle y - v,\ x - u \big\rangle \ \ge \ 0,$$

and moreover, if a monotone mapping (satisfying the previous property) $\widetilde{G}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfies $\operatorname{gph} G_i \subset \operatorname{gph} \widetilde{G}$, then necessarily $G_i = \widetilde{G}$. Further, consider a single-valued mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, which is strongly monotone, i.e., there is $c > 0$ such that

$$\langle f(x) - f(u), x - u \rangle \ge c\|x - u\|^2 \quad \text{for each} \quad x, u \in \mathbb{R}^n.$$

By [12, 12.54 Proposition], for each $i \in \{1, 2, \ldots, k\}$ the mapping $\mathbb{R}^n \ni x \longmapsto (f + G_i)^{-1}(x)$ is single-valued and Lipschitz continuous on $\mathbb{R}^n$ with the constant $\frac{1}{c}$. Then the mapping $\mathbb{R}^n \ni x \longrightarrow (f + F)^{-1}(x) = \bigcup_{i=1}^{k} (f + G_i)^{-1}(x)$; therefore the mapping is union of k Lipschitz continuous mappings with the constant $\frac{1}{c}$. Hence the mapping $f + F$ is strongly metrically subregular around each point of its graph. Fix any $T > 0$ and any continuous mapping $p : [0,T] \longrightarrow \mathbb{R}^n$. Fix any $i \in \{1, 2, \ldots, k\}$ and let

$$x(t) := (f + G_i)^{-1}(p(t)) \quad \text{for each} \quad t \in [0,T].$$

Since the mapping $(f + G_i)^{-1}$ is single-valued and Lipschitz continuous on $\mathbb{R}^n$, then the mapping $[0,T] \ni t \longrightarrow x(t)$ is continuous on $[0,T]$ and we have

$$p(t) \in f(x(t)) + F(x(t)) \quad \text{for each} \quad t \in [0,T].$$

Assumptions of Theorem 3.5, with $f(t,x) := f(x)$ for $x \in \mathbb{R}^n$ and $t \in [0,T]$, are satisfied.

4. CONCLUSION

In this paper, we have provided a comprehensive study of strong metric subregularity and its uniformity in Banach spaces. Our investigation has led to the following key contributions. We extended the local concept of strong metric subregularity to a uniform version over compact sets, thereby enabling the use of a common constant and neighborhood for all points in a given compact set.

These findings not only reinforce the theoretical underpinnings of variational analysis but also offer new avenues for the development of robust computational schemes in the context of generalized equations. Future work may explore further extensions of these uniformity results to broader classes of perturbations and more general frameworks, thereby enhancing both the theory and practical applications in optimization and control.

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