Rubén Becerril-Borja

Sequential games with turn selection process and fuzzy utility functions

Kybernetika, Vol. 61 (2025), No. 5, 647–665

Persistent URL: `http://dml.cz/dmlcz/153208`

KYBERNETIKA — VOLUME *61* (2025), NUMBER 5, PAGES 647–665

SEQUENTIAL GAMES WITH TURN SELECTION PROCESS AND FUZZY UTILITY FUNCTIONS

Rubén Becerril-Borja

A particular group of models of sequential games is studied where the order of the turns is not known beforehand by the players, and where the utility functions for each player are fuzzy numbers. For these models, a series of results are proven to show the existence of equilibria under two criteria, and a brief application is described where it usually is not possible to give utilities a precise value, hence, where fuzzy numbers are adequate.

Keywords: sequential game, risk sensitive, turn selection process, fuzzy numbers, fuzzy utility functions

Classification: 91A06, 91A10, 91A18, 91A50

1. INTRODUCTION

Game theory focuses on studying situations where the choices of the individuals involved affect everyone's utility. Many interactions that we have every day can be thought in this way, so it is of great interest to be able to think about how to go about them.

In the classical theory ([7, 9, 13] among many others), we can group games according to how the decisions are taken: simultaneously or sequentially. In simultaneous games, every player makes their decision at the same time, or if not exactly so, it would seem like it since they don't know about other player's choices. In sequential games, players make their choices in order and the players that choose later may know about some of the choices made previously to inform their own. Moreover, whichever group a game is on, players must decide how to maximize their utility, subject to the fact that each player can only change their own choices.

In the case of sequential games, many changes have been made to accommodate situations where some randomness may happen, leading to adding a player called Nature, so the initial structure could be different. Later on, since only changing the structure of the game at the beginning didn't fit other interesting situations, led to the study of what's known as stochastic games, where random events may happen between periods of time where decisions are made, though for this, the approach was to study simultaneous games that are played in sequence after which the utility of the whole game is calculated as some sort of discounted sum (more so in the case of infinite stochastic games).

DOI: 10.14736/kyb-2025-5-0647

Previously, a model that maintains the sequential structure without introducing simultaneous games was studied where players do not know previous to the game in which periods of time they will make a decision ([1, 2]). Only until it's their turn to choose do they learn about it, making it so they are able to make a decision according to what they have learned up to that point, and not being able to game the model by expecting to be able to make a specific choice in some point in the future, rather, by only knowing they have a certain probability they might make a decision in the future. In such a case, it was possible to show that an equilibrium exists, though in many cases it's not easy to compute due to the added complexity.

In the work presented here, a generalization of that model is presented, by means of taking fuzzy utility functions. The idea behind this approach is that many times, utility functions in games are not backed up by something real, being rather just numbers that are put in place so they reflect very general order relations between the utilities obtained by different choices. Or in other instances, such information may only be known with some certainty, for example, when companies study their opposition, they might have an idea of what the costs and profits of the opposition are, but not with total certainty. Moreover, in such situations, it is not in our best interest to consider a random process, since in many situations the utilities are not random, but they are not well known to every player. In such cases, working with fuzzy utility functions gives more freedom to model the situations. As can be seen in [10], this is an idea that has been considered for a while in many models of game theory.

Such fuzzy utility functions will be founded on the fuzzy theory proposed by Zadeh [14]. The utility functions that will be used are of the trapezoidal kind, which gives a first approach into how to generalize them into this framework (see [4, 8]). To have an order defined for the fuzzy utility functions and be able to define an equilibrium, we will use what is known as the average ranking of the trapezoidal numbers (following ideas from [5, 6]). Other approaches include using other averaging methods that also obtain non-fuzzy numbers that can be compared directly and may be of interest in future work, to see their advantages and disadvantages.

The structure of the paper is as follows: Section 2 defines the preliminaries for the model, as well as establishing the base model to be studied. Section 3 introduces the basics of fuzzy theory that will be used to define the utility functions for our model. In section 4 we give the results that will guarantee the existence of at least an equilibrium in our model for the average ranking criterion. Section 5 gives the corresponding results to ensure at least an equilibrium for games with utility functions based on a trapezoidal number for each player under the fuzzy expected utility criterion. Section 6 shows a basic application of our model where it is easy to make the computations and show the equilibrium. Finally, in Section 7 we give conclusions and some future work that can occur from what has been discussed.

2. PRELIMINARIES

Throughout the article we use standard notation, as in [13]. A **game** consists of the following elements:

- A set $I = \{1, 2, \ldots, N\}$ of **players**.

- A finite set of **pure strategies** S_i for each $i \in I$.
- A real-valued **utility function** $u_i \colon \Sigma \to \mathbb{R}$ for each $i \in I$, with $\Sigma = S^T \times \cdots \times S^1$ where each S^t is the finite set of all the strategies available for any of the one of the players in I at time t.

In this article we study sequential games (also called dynamic games), therefore a **horizon** of play $T \in \mathbb{N}$ is also required, which will tell us the number of decision points in the game. A **decision point** will be the equivalent of what is known as an information set, except the game also has to choose which player will act at that point. This is because the change made in this paper to models of games is introducing uncertainty for players regarding the order in which they will make a choice. This modification is in part due to how some models are made in which it is determined that players have to decide in a given order, but in reality that might not be the case, and the order is completely different or can even be considered to be random for all intents and purposes.

From the basic elements described above we can obtain for each player $i \in I$ their **set of mixed strategies** M_i, which is made of the probability distributions that have the set S_i as their support. A **profile of mixed strategies** $x = (x_1, x_2, \ldots, x_N)$ is defined as a vector made of strategies $x_i \in M_i$ for each player i, that is, a profile of mixed strategies x describes the strategies followed by all players in the game. The set of profiles of mixed strategies is denoted here by M.

If there is a profile $x = (x_1, x_2, \ldots, x_N)$, and $\tilde{x}_i \in M_i$, it is possible to combine them to make the profile

$$(\tilde{x}_i, x_{-i}) = (x_1, x_2, \ldots, x_{i-1}, \tilde{x}_i, x_{i+1}, \ldots, x_N),$$

that is, to replace in x the strategy x_i corresponding to player i with the strategy $\tilde{x}_i$.

Let us notice that each player can be chosen to make a move at every decision point in the game, therefore forcing every player to select a strategy for each possible decision point. Players learn who has made a move in previous decision points, and it may or may not be that they also learn the actions other players have made. In either case we assume players have **perfect recall** and as such their decision is conditioned by the actions (or possible actions) of all players. P_i is defined as the set of **plans of conditioned strategies** for player i, which can be defined as the strategies $(s_i \mid r^1, \ldots, r^{k-1}) \in P_i$ where player i observes the actions $(r^1, \ldots, r^{k-1})$ taken before the current turn $k \in T$. In an analogous manner it is possible to consider **mixed plans of conditioned strategies** for player i, $x_i(s_i \mid r^1, \ldots, r^{k-1})$ at each decision point $k \in T$, which consist of probability distributions that have P_i as their support; the set of mixed conditioned strategies for player i is denoted by Q_i. Finally, it is possible to define **profiles of plans of (mixed) conditioned strategies** as the vectors that consider (mixed) plans of conditioned strategies for each player $i \in I$. The set of these is denoted by P (resp. Q). When referring to the sets of profiles of plans of (mixed) conditioned strategies of players other than i, we denote them by P_{-i} (resp. Q_{-i}).

It would be desirable to define a concept of solution for the games, that is, profiles that satisfy some properties. Such a property is that once the strategies for all players are chosen, no player would like to deviate from their choice. A profile that satisfies this is called an equilibrium. These equilibria shall be defined for each of the models in the following section.

For the games that will be studied in the rest of the article, it is considered that players can be chosen to make many or few decisions in the game, anywhere from 0 to T. Therefore the information of how many decisions as well as when in the game players will be making them is hidden from every player until they reach each of the decision points in the game. Many variations on this idea can be made, as shown in [1]. For purposes of sumamrizing the ideas that can be applied to all of such variations, we work with the following model.

A **sequential game without a predetermined order of turns** is a game with a horizon of play T and a set of probability densities $\mathcal{P} = \{p_1, \ldots, p_N\}$, where $p_i(m)$ is the probability according to player i that player m is chosen at each decision point.

In other words, the games that will be studied all have the characteristic that before the game, it is not known what player will decide when. As the game goes on and it is required for a decision to be made, the game decides, via a randomized process (or what at least seems to be for the players), who gets to move and then the player chosen acts. Notice that it is allowed for each player to have their own model for the **turn selection process**, which are known to every player, which also means that, if it is not explicitly required that no player has an advantage, if such advantage exists, it is not known to the other players, so each considers their density to be the most adequate model for the turn selection process. As such, it can be assumed that however players come up with their probability distribution model, they consider it to be the most adequate.

In order to work with these games, and be able to give a reasonable idea of how players should choose their strategies, the expected utility is the way to go. The **expected utility** of each player i when the profile of plans of mixed conditioned strategies $x \in Q$ is played and u_i is player i's utility is given by

$$E_i(x, u_i) = \sum_{n^1 \in I} \sum_{s^1 \in S_{n^1}} \cdots \sum_{n^T \in I} \sum_{s^T \in S_{n^T}} u_i(s^T, \ldots, s^1) \\ \times x_n^T(s^T \mid s^{T-1}, \ldots, s^1) p_i(n^T) \cdots x_{n^1}(s^1) p_i(n^1).$$

As it stands, fixed probabilities for selecting players are considered but it is possible to adapt the model to accept variable probabilities at each stage. One thing that should be noticed is that these probabilities are known by every player, so it is known how each player views the turn selection process. An **equilibrium**, therefore, is a profile $x^* \in Q$ such that for every player $i \in I$ and every plan of mixed conditioned strategies $x_i \in Q_i$ it holds that

$$E_i(x^*, u_i) \geq E_i((x_i, x^*_{-i}), u_i).$$

3. SOME PREREQUISITES ON FUZZY THEORY

Let Λ be a non-empty set. A **fuzzy set** Γ on Λ is defined in terms of its **membership function** Γ' which assigns to each element of Λ a real value from the interval $[0, 1]$. The

α-cut of Γ, denoted Γ_α is defined as

$$\Gamma_\alpha = \{x \in \Lambda \mid \Gamma'(x) \geq \alpha\},$$

for $0 \leq \alpha \leq 1$. The set Γ_0 is the closure of $\{x \in \Lambda \mid \Gamma'(x) > 0\}$, denoted by $\mathrm{cl}\{x \in \Lambda \mid \Gamma'(x) > 0\}$.

A **fuzzy number** Γ is a fuzzy set defined on the set of real numbers $\mathbb{R}$ (that is, we take $\Lambda = \mathbb{R}$) which satisfies

1. Γ' is normal, that is, there exists $x \in \mathbb{R}$ with $\Gamma'(x) = 1$.
2. Γ' is convex, that is, Γ_α is convex for all $\alpha \in [0, 1]$.
3. Γ' is upper-semicontinuous.
4. Γ_0 is compact.

The set of fuzzy numbers will be denoted by $\mathcal{F}(\mathbb{R})$.

A fuzzy number Γ is called a **trapezoidal fuzzy number** if its membership function has the form

$$\Gamma'(x) = \begin{cases} 0 & \text{if } x \leq \ell, \\ \dfrac{x-\ell}{m-\ell} & \text{if } \ell < x \leq m, \\ 1 & \text{if } m < x \leq n, \\ \dfrac{p-x}{p-n} & \text{if } n < x \leq p, \\ 0 & \text{if } p < x, \end{cases}$$

where $\ell \leq m \leq n \leq p$ are real numbers. We denote such trapezoidal fuzzy number as (ℓ, m, n, p).

Remark 3.1. For a trapezoidal fuzzy number $\Gamma = (\ell, m, n, p)$ the corresponding α-cuts are the intervals $\Gamma_\alpha = [(m-\ell)\alpha + \ell, p - (p-n)\alpha]$ for $\alpha \in [0, 1]$.

Let Γ and Υ be fuzzy numbers. If $\star$ denotes the addition or the scalar multiplication, then the fuzzy number $\Gamma \star \Upsilon$ has the membership function

$$(\Gamma \star \Upsilon)'(u) = \sup_{u = x \star y} \min\{\Gamma'(x), \Upsilon'(y)\}.$$

Therefore, the next result holds true for trapezoidal fuzzy numbers.

Lemma 3.2. If $\Gamma = (a, b, c, d)$ and $\Upsilon = (e, f, g, h)$ are two trapezoidal fuzzy numbers, and $\lambda > 0$, then

1. $\lambda\Gamma = (\lambda a, \lambda b, \lambda c, \lambda d)$.
2. $\Gamma +^* \Upsilon = (a+e, b+f, c+g, d+h)$. This also holds for a sum of finitely many trapezoidal fuzzy numbers.

Let the set of closed bounded intervals be denoted by $\mathbb{D}$. For $\Psi = [a, b], \Phi = [c, d] \in \mathbb{D}$ we define the metric

$$d = (\Psi, \Phi) = \max\{|a - c|, |b - d|\},$$

so $(\mathbb{D}, d)$ is a complete metric space. In a similar fashion, if $\Gamma \in \mathcal{F}(\mathbb{R})$, then Γ_α is a compact set since its membership function is upper semicontinuous and has compact support. Let $\Gamma, \Upsilon \in \mathcal{F}(\mathbb{R})$, and define

$$d^*(\Gamma, \Upsilon) = \sup_{\alpha \in [0,1]} d(\Gamma_\alpha, \Upsilon_\alpha),$$

which is a metric in $\mathcal{F}(\mathbb{R})$.

A sequence $(\Gamma_h)_{h=0}^{\infty}$ of fuzzy numbers is **convergent** to the fuzzy number Γ, which we denote

$$\lim_{h \to \infty}{}^* \Gamma_t = \Gamma,$$

if $d^*(\Gamma_h, \Gamma) \to 0$ when $h \to \infty$.

Let $\Gamma, \Upsilon \in \mathcal{F}(\mathbb{R})$, with α-cuts $\Gamma_\alpha = [w_\alpha, x_\alpha]$ and $\Upsilon_\alpha = [y_\alpha, z_\alpha]$ for $\alpha \in [0, 1]$. Then we say that $\Gamma \leq^* \Upsilon$ if and only if $w_\alpha \leq y_\alpha$ and $x_\alpha \leq z_\alpha$ for every $\alpha \in [0, 1]$. This way, $\leq^*$ is a partial order on $\mathcal{F}(\mathbb{R})$.

We define another comparison that will be used. Given a trapezoidal number $\Gamma = (a, b, c, d)$, we define its **average ranking** as

$$R(\Gamma) = \frac{a + b + c + d}{4},$$

which defines a real number that represents Γ; and we say for two trapezoidal numbers Γ and Υ that $\Gamma \leq \Upsilon$ if

$$R(\Gamma) \leq R(\Upsilon).$$

Considering what is done in [11] and [12] we define a fuzzy random variable and its expectation. We consider $\mathcal{C}(\mathbb{R})$ to be the class of non-empty compact subsets of $\mathbb{R}$.

Given a measurable space $(\Omega, \mathcal{A})$ and the measurable space of real numbers $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we define a **fuzzy random variable** (associated with $(\Omega, \mathcal{A})$) as a function $X^* \colon \Omega \to \mathcal{F}(\mathbb{R})$ such that for each $\alpha \in [0, 1]$, the α-cut function $X^*_\alpha(\omega) = (X^*(\omega))_\alpha$ satisfies

$$\mathrm{Gr}(X^*_\alpha) = \{(\omega, x) \in \Omega \times \mathbb{R} \mid x \in (X^*(\omega))_\alpha\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}),$$

where $\mathcal{A} \otimes \mathcal{B}(\mathbb{R})$ is the product σ-algebra associated with $\Omega \times \mathbb{R}$.

In order to define the expectation of a random variable we consider only integrably bounded random variables. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Then X^* is an **integrably bounded random variable** (with respect of $(\Omega, \mathcal{A}, P)$) if there exists a function $h \colon \Omega \to \mathbb{R}$, $h \in L^1(\Omega, \mathcal{A}, P)$ such that $|x| \leq h(\omega)$ for all $(\omega, x) \in \Omega \times \mathbb{R}$ with $x \in (X^*(\omega))_0 = X^*_0(\omega)$.

With this, let $(\Omega, \mathcal{A}, P)$ be a probability space and X^* be an integrably bounded random variable. The **fuzzy expected value** of X^* (in the sense of Aumann) is the unique fuzzy set of $\mathbb{R}$, denoted by $E^*(X^*)$ that satisfies for each $\alpha \in [0, 1]$:

$$(E^*(X^*))_\alpha = \left\{ \int_\Omega f(\omega)\, \mathrm{d}P(\omega) \,\middle|\, f \colon \Omega \to \mathbb{R}, f \in L^1(\Omega, \mathcal{A}, P), f(\omega) \in (X^*(\omega))_\alpha \text{ a.s.} \right\}.$$

4. GAMES WITH FUZZY UTILITY FUNCTIONS USING AVERAGE RANKING

In our case we will be considering the model presented before but with trapezoidal fuzzy utility functions. That is, $u_i\colon \Sigma \to \mathcal{F}(\mathbb{R})$. Therefore, we need to define an expected utility operator in this framework. Given $x \in Q$, and $u_i = (a_i, b_i, c_i, d_i)$, the **fuzzy expected utility** of x is defined as:

$$E_i^*(x, u_i) = \left(E_i(x, a_i), E_i(x, b_i), E_i(x, c_i), E_i(x, d_i)\right),$$

and an **equilibrium** is defined in an analogous way, with $x^* \in Q$ be such that for every player $i \in I$ and every plan of mixed conditioned strategies $x_i \in Q_i$ we have

$$E_i^*(x^*, u_i) \geq E_i^*((x^*_{-i}, x_i), u_i).$$

With this, we can show the following results.

Proposition 4.1. The average ranking of the fuzzy expected utility function is continuous in each player's plan of strategies.

P r o o f . Notice that each component in the fuzzy expected utility function can be split in the following way:

$$\begin{aligned}
E_i(x, A) = &\left(\sum_{s^1 \in S_i} \cdots \sum_{s^T \in S_i} A(s^T, \ldots, s^1) x_i(s^T \mid s^{T-1}, \ldots, s^1) p_i(i) \cdots x_i(s^1) p_i(i) \right) \\
&+ \left(\sum_{n^1 \in I\setminus\{i\}} \sum_{s^1 \in S_{n^1}} \sum_{s^2 \in S_i} \cdots \sum_{s^T \in S_i} A(s^T, \ldots, s^1) x_i(s^T \mid s^{T-1}, \ldots, s^1) \right. \\
&\quad \times p_i(i) \cdots x_i(s^2 \mid s^1) p_i(i) x_{n^1}(s^1) p_i(n^1) + \cdots \\
&\quad + \sum_{s^1 \in S_i} \cdots \sum_{s^{T-1} \in S_i} \sum_{n^T \in I\setminus\{i\}} \sum_{s^T \in S_{n^T}} A(s^T, \ldots, s^1) x_{n^T}(s^T \mid s^{T-1}, \ldots, s^1) \\
&\quad \left. \times\ p_i(n^T) x_i(s^{T-1} \mid s^{T-2}, \ldots, s^1) p_i(i) \cdots x_i(s^1) p_i(i) \right) + \cdots \\
&+ \left(\sum_{n^1 \in I\setminus\{i\}} \sum_{s^1 \in S_{n^1}} \cdots \sum_{n^T \in I\setminus\{i\}} \sum_{s^T \in S_{n^T}} A(s^T, \ldots, s^1) \right. \\
&\quad \left. \times\ x_{n^T}(s^T \mid s^{T-1}, \ldots, s^1) p_i(n^T) \cdots x_{n^1}(s^1) p_i(n^1) \right),
\end{aligned}$$

where the terms in the first bracket are those in which player i was selected T times, the terms in the second bracket are those in which player i was selected $T-1$ times, and so on, ending with the terms in the last bracket in which player i was selected 0 times. It can be easily seen that each of these terms is continuous in the plan of strategies for player i, so the whole sum is continuous in the plan of strategies of player i. This works for A being either a_i, b_i, c_i or d_i. As such, the whole fuzzy expected utility function is continuous. Finally, the average ranking as a sum of each component of the fuzzy number is also continuous on each component, which are continuous. □

The next proposition follows in the same way as in [2].

Proposition 4.2. The set of profiles of conditioned strategies Q is a non-empty, compact and convex subset of some $\mathbb{R}^q$.

We define for each player i the best response correspondence BR_i to the partial profile $x_{-i} \in Q_{-i}$ as

$$BR_i(x_{-i}) = \{x_i \in Q_i \mid E_i^*(x_{-i}, x_i) \geq E_i^*(x_{-i}, y_i) \text{ for all } y_i \in Q_i\},$$

and the best response correspondence $BR\colon Q \to Q$ as

$$BR(x) = (BR_1(x_{-1}), BR_2(x_{-2}), \ldots, BR_N(x_{-N})).$$

Proposition 4.3. The best response correspondence BR is a non-empty correspondence with a closed graph.

Proof. Since the average ranking of the expected utility function is a continuous function defined on a compact set, it must achieve its maximum at some point $x_i \in Q_i$ for each $x_{-i} \in Q_{-i}$ for each player i. Therefore, BR_i is non-empty for each player i, and this means $BR(x)$ is a non-empty correspondence for every $x \in Q$.

Now let $(x_h)_{h=1}^{\infty}$ be a sequence of profiles of conditioned strategies and $(x'_h)_{h=1}^{\infty}$ the best responses, that is $x'_h \in BR(x_h)$ for every h. Let $x^* = \lim_{h\to\infty} x_h$ and $x'^* = \lim_{h\to\infty} x'_h$. For player i, we have that $x'_{h,i} \in BR(x_{h,-i})$, which means that

$$E_i^*((x_{h,-i}, x'_{h,i}), u_i) \geq E_i^*((x_{h,-i}, y_i), u_i),$$

for every $y_i \in Q_i$. Taking limits on both sides, by the continuity of E_i^* we have that

$$\lim_{h\to\infty} E_i^*((x_{h,-i}, x'_{h,i}), u_i) \geq \lim_{h\to\infty} E_i^*((x_{h,-i}, y_i), u_i),$$

and interchanging the order of limits and sums

$$E_i^*((x_{-i}^*, x_i'^*), u_i) \geq E_i^*((x_{-i}^*, y_i), u_i),$$

for every $y_i \in Q_i$, which means that $x_i'^* \in BR_i(x_{-i}^*)$ for every player i, implying $x'^* \in BR(x^*)$. □

Now we prove that the best response correspondence is convex.

Proposition 4.4. The best response correspondence BR is convex.

Proof. Let $x_i, y_i \in BR_i(x_{-i})$. Then the expected utility of the convex combination $\alpha x_i + (1-\alpha)y_i$ for $\alpha \in [0,1]$ when the other players use the profile x_{-i} can be written as

$$E_i((\alpha x_i + (1-\alpha)y_i, x_{-i}), A) = \Bigg(\sum_{s^1\in S_i} \cdots \sum_{s^T \in S_i} A(s^T, \dots, s^1)$$
$$\times (\alpha x_i(s^T \mid s^{T-1}, \dots, s^1) + (1-\alpha)y_i(s^T \mid s^{T-1}, \dots, s^1))p_i(i) \cdots$$
$$\times (\alpha x_i(s^1) + (1-\alpha)y_i(s^1))p_i(i) \Bigg)$$
$$+ \Bigg(\sum_{n^1 \in I\setminus\{i\}} \sum_{s^1 \in S_{n^1}} \sum_{s^2\in S_i} \cdots \sum_{s^T\in S_i} A(s^T, \dots, s^1) \cdots$$
$$\times (\alpha x_i(s^T \mid s^{T-1}, \dots, s^1) + (1-\alpha)y_i(s^T \mid s^{T-1}, \dots, s^1))p_i(i) \cdots$$
$$\times (\alpha x_i(s^2 \mid s^1) + (1-\alpha)y_i(s^2 \mid s^1))p_i(i) x_{n^1}(s^1)p_i(n^1) + \cdots$$
$$+ \sum_{s^1\in S_i} \cdots \sum_{s^{T-1}\in S_i} \sum_{n^T \in I\setminus I} \sum_{s^T \in S_{n^T}} A(s^T, \dots, s^1) x_{n^T}(s^T \mid s^{T-1}, \dots, s^1) p_i(n^T) \cdots$$
$$\times (\alpha x_i(s^{T-1} \mid s^{T-2}, \dots, s^1) + (1-\alpha)y_i(s^{T-1} \mid s^{T-2}, \dots, s^1))p_i(i) \Bigg) + \cdots$$
$$+ \Bigg(\sum_{n^1 \in I\setminus\{i\}} \sum_{s^1 \in S_{n^1}} \sum_{s^1 \in S_{n^1}} \cdots \sum_{n^T \in I\setminus\{i\}} \sum_{s^T \in S_{n^T}} A(s^T, \dots, s^1)$$
$$\times x_{n^T}(s^T \mid s^{T-1}, \dots, s^1) p_i(n^T) \cdots x_{n^1}(s^1) p_i(n^1) \Bigg)$$
$$= \Bigg(\sum_{s^1\in S_i} \cdots \sum_{s^T \in S_i} A(s^T, \dots, s^1)$$
$$\times (\alpha x_i(s^T \mid s^{T-1}, \dots, s^1) + (1-\alpha)y_i(s^T \mid s^{T-1}, \dots, s^1))p_i(i) \cdots$$
$$\times (\alpha x_i(s^1) + (1-\alpha)y_i(s^1))p_i(i) \Bigg)$$
$$+ \Bigg(\sum_{n^1 \in I\setminus\{i\}} \sum_{s^1 \in S_{n^1}} \sum_{s^2\in S_i} \cdots \sum_{s^T\in S_i} A(s^T, \dots, s^1) \cdots$$
$$\times (\alpha x_i(s^T \mid s^{T-1}, \dots, s^1) + (1-\alpha)y_i(s^T \mid s^{T-1}, \dots, s^1))p_i(i) \cdots$$
$$\times (\alpha x_i(s^2 \mid s^1) + (1-\alpha)y_i(s^2 \mid s^1))p_i(i)(\alpha x_{n^1}(s^1) + (1-\alpha)x_{n^1}(s^1))p_i(n^1) + \cdots$$

$$+ \sum_{s^1 \in S_i} \cdots \sum_{s^{T-1} \in S_i} \sum_{n^T \in I \setminus I} \sum_{s^T \in S_{n^T}} A(s^T, \dots, s^1)$$
$$\times (\alpha x_{n^T}(s^T \mid s^{T-1}, \dots, s^1) + (1-\alpha) x_{n^T}(s^T \mid s^{T-1}, \dots, s^1)) p_i(n^T) \cdots$$
$$\left. \times (\alpha x_i(s^{T-1} \mid s^{T-2}, \dots, s^1) + (1-\alpha) y_i(s^{T-1} \mid s^{T-2}, \dots, s^1)) p_i(i) \right) + \cdots$$
$$+ \left(\sum_{n^1 \in I \setminus \{i\}} \sum_{s^1 \in S_{n^1}} \sum_{s^1 \in S_{n^1}} \cdots \sum_{n^T \in I \setminus \{i\}} \sum_{s^T \in S_{n^T}} A(s^T, \dots, s^1) \right.$$
$$\times (\alpha x_{n^T}(s^T \mid s^{T-1}, \dots, s^1) + (1-\alpha) x_{n^T}(s^T \mid s^{T-1}, \dots, s^1)) p_i(n^T) \cdots$$
$$\left. \times (\alpha x_{n^1}(s^1) + (1-\alpha) x_{n^1}(s^1)) p_i(n^1)) \right)$$
$$= \alpha E_i((x_i, x_{-i}), A) + (1-\alpha) E_i((y_i, x_{-i}), A),$$

where A can be either of a_i, b_i, c_i or d_i, and since both x_i and y_i are best responses for i given the other players choose x_{-i}, then $E_i^*((x_i, x_{-i}), u_i) \geq E_i^*((y_i, x_{-i}), u_i)$ and $E_i^*((y_i, x_{-i}), u_i) \geq E_i^*((x_i, x_{-i}), u_i)$. Therefore

$$\begin{aligned} E_i^*((\alpha x_i + (1-\alpha) y_i), u_i) &= \alpha E_i^*((x_i, x_{-i}), u_i) + (1-\alpha) E_i^*((y_i, x_{-i}), u_i) \\ &= E_i^*((x_i, x_{-i}), u_i), \end{aligned}$$

so $\alpha x_i + (1-\alpha) y_i \in BR_i(x_{-i})$. This follows for all players, so BR is a convex correspondence. □

The previous results show that BR satisfies Kakutani's fixed point theorem, so at least one fixed point exists for BR. And precisely, those fixed points are the equilibria of the game, since we want to maximize the expected utility for each player i given what the other players are choosing, but since a player's strategy is already a best response, they cannot improve any further. Therefore we have the following result.

Theorem 4.5. Every sequential game with finite strategy sets, finite horizon and turn selection process with fuzzy trapezoidal utility functions has at least one equilibrium under the average ranking criterion.

5. GAMES WITH FUZZY UTILITIES FUNCTIONS USING α-CUTS

Now we will consider the fuzzy expected value as our expectation operator. In this case we apply this operator to our utility function, and the associated probability measure is derived from both the selection process as well as from the profile of mixed strategies each player uses. In this regard, to indicate it, we will apply the fuzzy expected value to $u_i(x)$.

Similarly as before, we will say $x^* \in Q$ is a **fuzzy equilibrium** if for every player $i \in I$ and every plan of mixed conditioned strategies $x_i \in Q_i$ we have

$$E^*(u_i(x^*)) \geq^* E^*(u_i(x^*_{-i}, x_i)),$$

whenever $u_i(x^*)$ and $u_i(x^*_{-i}, x_i)$ are comparable with this order.

We define as well for each player i the best response correspondence BR^*_i to the partial profile $x_{-i} \in Q_{-i}$ as

$$BR^*_i(x_{-i}) = \{x_i \in Q_i \mid E^*(u_i(x_{-i}, x_i)) \geq^* E^*(u_i(x_{-i}, y_i)) \text{ for all } y_i \in Q_i\},$$

and the best response correspondence $BR^*\colon Q \to Q$ as

$$BR^*(x) = (BR^*_1(x_{-1}), BR^*_2(x_{-2}), \ldots, BR^*_N(x_{-N})).$$

Proposition 5.1. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let X be a nonnegative discrete random variable associated to $(\Omega, \mathcal{A}, P)$ such that $E(X)$ exists. Then $\tilde{X} = X(a, b, c, d)$ is a fuzzy random variable associated to $(\Omega, \mathcal{A}, P)$ and

$$E^*(\tilde{X}) = E(X)(a, b, c, d).$$

Proof. Let X be a nonnegative discrete random variable with finite or denumerable range denoted by $\{x_1, x_2, \ldots\}$. Then $\{X = x_j\} = \{\omega \in \Omega \mid X(\omega) = x_j\}$, for $j = 1, 2, \ldots$. Let $\Gamma = (a, b, c, d)$ with α-cuts $\Gamma_\alpha = [s(\alpha), t(\alpha)]$ where $s(\alpha) = \alpha a + (1 - \alpha)b$, $t(\alpha) = \alpha c + (1 - \alpha)d$. Fix $\alpha \in [0, 1]$. Define for $\omega \in \Omega$ the multifunction

$$\tilde{X}_\alpha(\omega) = (\tilde{X}(\omega))_\alpha = (X(\omega)\Gamma)_\alpha = X(\omega)[s(\alpha), t(\alpha)].$$

We can see that

$$\begin{aligned} \mathrm{Gr}(\tilde{X}_\alpha) &= \{(\omega, x) \in \Omega \times \mathbb{R} \mid x \in \tilde{X}_\alpha(\omega)\} \\ &= \{(\omega, x) \in \Omega \times \mathbb{R} \mid x \in X(\omega)[s(\alpha), t(\alpha)]\} \\ &= \bigcup_{j=1}^{\infty} (\{X = x_j\} \times x_j[s(\alpha), t(\alpha)]). \end{aligned}$$

Therefore, $\mathrm{Gr}(\tilde{X}_\alpha) \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R})$. Since this holds for all $\alpha \in [0, 1]$, then $\tilde{X}$ is a fuzzy random variable. Now, given $\omega \in \Omega$, we have that

$$(\tilde{X}(\omega))_0 = \tilde{X}_0(\omega) = X(\omega)[a, d].$$

Define $h\colon \Omega \to \mathbb{R}$ for $\omega \in \Omega$ as

$$h(\omega) = X(\omega)d.$$

Then it's easy to see that

$$|x| \leq h(\omega),$$

for all $(\omega, x) \in \Omega \times \mathbb{R}$ with $x \in \tilde{X}_0(\omega) = X(\omega)[a, d]$. Also, we have that $E(h) = dE(X)$ which is finite. Therefore, $\tilde{X}$ is an integrably bounded fuzzy random variable with respect of $(\Omega, \mathcal{A}, P)$.

Now, we have that there is a unique fuzzy expected value $E^*(\tilde{X})$ such that for each $\alpha \in [0, 1]$

$$(E^*(\tilde{X}))_\alpha = E(X)[s(\alpha), t(\alpha)],$$

which for each α is exactly

$$(E(X)(a,b,c,d))_\alpha,$$

the α-cut of the trapezoidal number $E(X)(a,b,c,d)$, so

$$E^*(\tilde{X}) = E(X)(a,b,c,d).$$

□

Proposition 5.2. The best response correspondence BR^* is non-empty with closed graph.

Proof. Similarly to Proposition 4.3, since the expected utility function is a continuous function defined on a compact set, it achieves its maximum at some point in Q_i for every $x_{-i} \in Q_{-i}$, which implies BR_i^* is non-empty for let $(x_h)_{h=1}^\infty$ be a sequence of profiles of conditioned strategies and $(x'_h)_{h=1}^\infty$ the best responses, so $x'_h \in BR^*(x_h)$ for every h. Let $x^* = \lim_{h\to\infty} x_h$ and $x'^* = \lim_{h\to\infty} x'_h$. For player i, we have that $x'_{h,i} \in BR^*(x_{h,-i})$, which means that

$$E^*(u_i(x_{h,-i}, x'_{h,i})) \geq E^*(u_i(x_{h,-i}, y_i)),$$

for every $y_i \in Q_i$. Taking limits on both sides, given that we know that $E_i(x)$ is a continuous operator (see [3]) then E^* is also continuous, given the expression obtained in 5.1, we have that

$$\lim_{h\to\infty} E^*(u_i(x_{h,-i}, x'_{h,i})) \geq \lim_{h\to\infty} E^*(u_i(x_{h,-i}, y_i)),$$
$$\lim_{h\to\infty} E_i((x_{h,-i}, x'_{h,i}))(A,B,C,D) \geq \lim_{h\to\infty} E_i((x_{h,-i}, y_i))(A,B,C,D),$$

where utilities for these games can be given in terms of the trapezoidal number (A,B,C,D). We can interchange limits and sums, so

$$E_i((x^*_{h,-i}, x'^*_{h,i}))(A,B,C,D) \geq E_i((x^*_{h,-i}, y_i))(A,B,C,D),$$

for every $y_i \in Q_i$, that is, $x'^*_i \in BR_i^*(x^*_{-i})$ for every player i, which implies that $x'^* \in BR^*(x^*)$. □

Proposition 5.3. The best response correspondence BR^* is convex.

Proof. Let $x_i, y_i \in BR_i^*(x_{-i})$. Then the fuzzy expected utility of the convex combination $\beta x_i + (1-\beta)y_i$ for $\beta \in [0,1]$ when the other players use the profile $x_{-i} \in Q_{-i}$ is

$$E^*(u_i(\beta x_i + (1-\beta)y_i, x_{-i})) = E_i(\beta x_i + (1-\beta)y_i, x_{-i})(a,b,c,d),$$

and by convexity of the expected utility operator (see [3]) we have the last expression is equal to

$$(\beta E_i(x_i, x_{-i}) + (1-\beta)E_i(y_i, x_{-i}))(a,b,c,d) = \beta E^*(u_i(x_i, x_{-i})) + (1-\beta)E^*(u_i(y_i, x_{-i})).$$

Now, since x_i is a best response to x_{-i}, then in particular

$$E^*(u_i(x_i, x_{-i})) \geq E^*(u_i(y_i, x_{-i})),$$

and

$$E^*(u_i(y_i, x_{-i})) \geq E^*(u_i(x_i, x_{-i})),$$

so we have that

$$\begin{aligned} E^*(u_i(\beta x_i + (1-\beta)y_i, x_{-i})) &= \beta E^*(u_i(x_i, x_{-i})) + (1-\beta)E^*(u_i(y_i, x_{-i})) \\ &= E^*(u_i(x_i, x_{-i})), \end{aligned}$$

which means $(\beta x_i + (1-\beta)y_i, x_{-i}) \in BR_i^*(x_{-i})$. As this is true for all players i, then BR^* is a convex correspondence. □

Finally, using Kakutani's theorem once again, as the fixed points of BR^* are precisely those that correspond to a fuzzy equilibrium, we can state the main result of this section.

Theorem 5.4. Every sequential game with finite strategy sets, finite horizon and turn selection process with fuzzy utility functions of the form $U(a, b, c, d)$ for some trapezoidal number (a, b, c, d) has at least one fuzzy equilibrium under the fuzzy expectation criterion.

6. AN APPLICATION

We present the following example, in which we consider a situation where the probabilities of the turn selection process and the available strategy sets for the players change, conditioned on the choices made in previous turns.

6.1. Using the average ranking criterion

Two teams must choose between two new recruits. As usual, they've tried to measure the benefit that the new teammates would have, but there is a certain degree of uncertainty, due to non-measurable characteristics. As such, the first team considers the first athlete to give a utility of $(0, 1, 4, 5)$, whereas the second athlete gives a utility of $(1, 2, 4, 6)$. However, the second team considers the first athlete to give a utility of $(1, 3, 4, 5)$ and the second athlete to give a utility of $(0, 3, 5, 6)$. The utility obtained by each team after making their picks is the sum of the utilities of the athletes they have chosen. Moreover, in each of the two periods, the teams have a probability of being chosen. The probability of team 1 being selected in the first period is $1/3$ and of team 2 being selected is $2/3$. If the first athlete is chosen in the first selection, then the team that picked him has its probability reduced by a half for being selected in the second period (the probability they lose gets added to the other team), whereas if the second athlete is chosen in the first selection, the probability of the team is reduced by a third. In the second period, the team chosen gets the athlete that's left over.

To find the equilibrium, we observe that both players are modelling the turn selection process with the same probability distribution, which is

$$p(1) = 1/3, \qquad p(2) = 2/3,$$

and

$$\begin{aligned} p(1 \mid A_1) &= 1/6, & p(2 \mid A_1) &= 5/6, \\ p(1 \mid B_1) &= 1/9, & p(2 \mid B_1) &= 8/9, \\ p(1 \mid A_2) &= 2/3, & p(2 \mid A_2) &= 1/3, \\ p(1 \mid B_2) &= 7/9, & p(2 \mid B_2) &= 2/9, \end{aligned}$$

where A_j indicates that in the first turn, player j chose the first athlete, and B_j indicates that in the first turn, team j chose the second athlete.

Now, since we have that the utility function is the fuzzy sum of the utilities obtained from the athletes, then we can compute the expected utility, and substitute the corresponding utility function in each term. Observe also that, since the second pick is automatic, $x_j(K \mid \cdot) = 1$ for $K \in \{A_j, B_j\}$, whereas any other mixed strategy is equal to zero.

For team 1, the expected utility for each coordinate of its utility is:

$$\begin{aligned} E_1(x, a_1) &= \frac{9}{27} - \frac{5}{18} x(A_1) + \frac{4}{9} x(A_2), \\ E_1(x, b_1) &= \frac{11}{9} - \frac{7}{27} x(A_1) + \frac{38}{27} x(A_2), \\ E_1(x, c_1) &= \frac{32}{9} + \frac{2}{27} x(A_1) + \frac{104}{27} x(A_2), \\ E_1(x, d_1) &= \frac{43}{9} - \frac{5}{27} x(A_1) + \frac{142}{27} x(A_2), \end{aligned}$$

whereas for team 2

$$\begin{aligned} E_2(x, a_2) &= \frac{4}{9} + \frac{14}{27} x(A_2) - \frac{8}{27} x(A_1), \\ E_2(x, b_2) &= \frac{30}{9} + \frac{2}{9} x(A_2) - \frac{1}{18} x(A_1), \\ E_2(x, c_2) &= \frac{46}{9} - \frac{4}{27} x(A_2) + \frac{11}{54} x(A_1), \\ E_2(x, d_2) &= \frac{56}{9} - \frac{2}{27} x(A_2) + \frac{5}{27} x(A_1). \end{aligned}$$

Finally, we compute the average ranking for both players' expected utility:

$$\begin{aligned} R(E_1^*(x, u_1)) &= \frac{89}{36} - \frac{35}{216} x(A_1) + \frac{74}{27} x(A_2), \\ R(E_2^*(x, u_2)) &= \frac{34}{9} + \frac{7}{54} x(A_2) + \frac{1}{108} x(A_1), \end{aligned}$$

and observe that team 1 can maximize their average ranking if $x(A_1) = 0$, while team 2 can maximize their average ranking if $x(A_2) = 1$, meaning that team 1, if selected to make the first choice, will pick the second athlete, whereas team 2, if selected to make the first choice, will pick the first athlete.

6.2. Comparing the average ranking criterion to the fuzzy expected utility criterion

Now, we take the same structure of two teams vying to recruit two athletes, but this time, the utilities obtained by each team are as follows: The first team considers the first athlete to give them a utility of $(0, 2, 4, 10)$, whereas the second athlete would give them an utility of $(0, 3, 6, 15)$. The second team considers the first athlete gives them a utility of $(1, 3, 4, 6)$ whereas the second athlete gives them a utility of $(0.5, 1.5, 2, 3)$. As before, team 1 has probability of being selected 1/3, and therefore, team 2 has probability 2/3 of being selected. And if the first athlete is chosen, then that team has its probability reduced by half, whereas the second athlete reduces the probabilities by a third.

We have the same probabilities as before

$$\begin{aligned} &p(1) = 1/3, && p(2) = 2/3, \\ &p(1 \mid A_1) = 1/6, && p(2 \mid A_1) = 5/6, \\ &p(1 \mid B_1) = 1/9, && p(2 \mid B_1) = 8/9, \\ &p(1 \mid A_2) = 2/3, && p(2 \mid A_2) = 1/3, \\ &p(1 \mid B_2) = 7/9, && p(2 \mid B_2) = 2/9. \end{aligned}$$

Under the average ranking criterion, we compute the expected utility for each coordinate for team 1

$$\begin{aligned} E_1(x, a_1) &= 0, \\ E_1(x, b_1) &= \frac{19}{9} - \frac{13}{54}x(A_1) + \frac{64}{27}x(A_2), \\ E_1(x, c_1) &= \frac{38}{9} - \frac{13}{27}x(A_1) + \frac{128}{27}x(A_2), \\ E_1(x, d_1) &= \frac{95}{9} - \frac{65}{54}x(A_1) + \frac{320}{27}x(A_2), \end{aligned}$$

and for team 2

$$\begin{aligned} E_2(x, a_2) &= \frac{7}{9} + \frac{8}{27}x(A_2) - \frac{17}{108}x(A_1), \\ E_2(x, b_2) &= \frac{21}{9} + \frac{8}{9}x(A_2) - \frac{17}{36}x(A_1), \\ E_2(x, c_2) &= \frac{28}{9} + \frac{32}{27}x(A_2) - \frac{17}{27}x(A_1), \\ E_2(x, d_2) &= \frac{42}{9} + \frac{16}{9}x(A_2) - \frac{85}{9}x(A_1). \end{aligned}$$

From here, we can compute the average ranking

$$\begin{aligned} R(E_1^*(x, u_1)) &= \frac{38}{9} - \frac{13}{27}x(A_1) + \frac{128}{27}x(A_2), \\ R(E_2^*(x, u_2)) &= \frac{49}{18} + \frac{28}{27}x(A_2) - \frac{289}{108}x(A_1). \end{aligned}$$

which to maximize implies that team 1 should play $x(A_1) = 0$, whereas team 2 should play $x(A_2) = 1$, meaning that team 1 if chosen in the first pick will choose the second athlete, whereas if team 2 is chosen in the first pick will choose the first athlete.

Now we use the fuzzy expected utility criterion. For this, we note that for team 1

$$E^*(u_1(x)) = E(X)(a_1, b_1, c_1, d_1),$$

with X the random variable that takes values on $\Omega = \{0, 2, 3, 5\}$ and $(a_1, b_1, c_1, d_1) = (0, 1, 2, 5)$. Therefore, we have that

$$E(X) = 0p(X = 0) + 2p(X = 2) + 3p(X = 3) + 5p(X = 5),$$

where we have the probabilities

$$\begin{aligned}
p(X = 0) &= p(2 \mid A_2)p(2)x(A_2) + p(2 \mid B_2)p(2)x(B_2),\\
p(X = 2) &= p(1 \mid B_2)p(2)x(B_2) + p(2 \mid A_1)p(1)x(A_1),\\
p(X = 3) &= p(1 \mid A_2)p(2)x(A_2) + p(2 \mid B_1)p(1)x(B_1),\\
p(X = 5) &= p(1 \mid A_1)p(1)x(A_1) + p(1 \mid B_1)p(1)x(B_1),
\end{aligned}$$

so the expected utility of X is

$$\begin{aligned}
E(X) &= 2[p(1 \mid B_2)p(2)x(B_2) + p(2 \mid A_1)p(1)x(A_1)]\\
&\quad + 3[p(1 \mid A_2)p(2)x(A_2) + p(2 \mid B_1)p(1)x(B_1)]\\
&\quad\quad + 5[p(1 \mid A_1)p(1)x(A_1) + p(1 \mid B_1)p(1)x(B_1)]\\
&= 2\left[\frac{7}{9}\cdot\frac{2}{3}(1 - x(A_2)) + \frac{5}{6}\cdot\frac{1}{3}x(A_1)\right]\\
&\quad + 3\left[\frac{2}{3}\cdot\frac{2}{3}x(A_2) + \frac{8}{9}\cdot\frac{1}{3}(1 - x(A_1))\right]\\
&\quad\quad + 5\left[\frac{1}{6}\cdot\frac{1}{3}x(A_1) + \frac{1}{9}\cdot\frac{1}{3}(1 - x(A_1))\right]\\
&= \frac{19}{9} - \frac{13}{54}x(A_1) + \frac{8}{27}x(A_2),
\end{aligned}$$

and the fuzzy expected utility is

$$E^*(u_1(x)) = \left[\frac{19}{9} - \frac{13}{54}x(A_1) + \frac{8}{27}x(A_2)\right](0, 1, 2, 5).$$

For team 2 we have that

$$E^*(u_2(x)) = E(Y)(a_2, b_2, c_2, d_2),$$

with Y the random variable that takes values on $\Omega = \{0, 1, 2, 3\}$ and $(a_2, b_2, c_2, d_2) = (0.5, 1.5, 2, 3)$. Therefore we have that

$$E(Y) = 0p(Y = 0) + 1p(Y = 1) + 2p(Y = 2) + 5p(Y = 5),$$

where we have the probabilities

$$\begin{aligned}
p(Y=0) &= p(1 \mid A_1)p(1)x(A_1) + p(1 \mid B_1)p(1)x(B_1),\\
p(Y=1) &= p(2 \mid A_1)p(1)x(A_1) + p(1 \mid B_2)p(2)x(B_2),\\
p(Y=2) &= p(2 \mid B_1)p(1)x(B_1) + p(1 \mid A_2)p(2)x(A_2),\\
p(Y=3) &= p(2 \mid A_2)p(2)x(A_2) + p(2 \mid B_2)p(2)x(B_2),
\end{aligned}$$

so the expected utility of Y is

$$\begin{aligned}
E(Y) = 1\,&[p(2 \mid A_1)p(1)x(A_1) + p(1 \mid B_2)p(2)x(B_2)]\\
&+ 2\,[p(2 \mid B_1)p(1)x(B_1) + p(1 \mid A_2)p(2)x(A_2)]\\
&\quad + 3\,[p(2 \mid A_2)p(2)x(A_2) + p(2 \mid B_2)p(2)x(B_2)]\\
&= 1\left[\frac{5}{6}\cdot\frac{1}{3}x(A_1) + \frac{7}{9}\cdot\frac{2}{3}(1 - x(A_2))\right]\\
&\quad + 2\left[\frac{8}{9}\cdot\frac{1}{3}(1 - x(A_1)) + \frac{2}{3}\cdot\frac{2}{3}x(A_2)\right]\\
&\quad\quad + 3\left[\frac{1}{3}\cdot\frac{2}{3}x(A_2) + \frac{2}{9}\cdot\frac{2}{3}(1 - x(A_2))\right]\\
&= \frac{14}{9} + \frac{16}{27}x(A_2) - \frac{17}{54}x(A_1),
\end{aligned}$$

and the fuzzy expected utility is

$$E^*(u_2(x)) = \left[\frac{14}{9} + \frac{16}{27}x(A_2) - \frac{17}{54}x(A_1)\right](0.5, 1.5, 2, 3).$$

We can see that in this case as well, to maximize the corresponding fuzzy expected utilities, $x(A_1) = 0$ whereas $x(A_2) = 1$, which means that team 1 should choose the second athlete if selected in the first turn, and team 2 should choose the first athlete if selected in the first turn.

Note that the obtained expressions to determine this for each criterion have different coefficients, even up to multiples, so this means that both approaches, though they gave the same strategy for each player, are not necessarily interchangeable.

7. CONCLUSIONS

In this paper a generalization of a model with turn selection process has been described in which utility functions are fuzzy, allowing for models where players may not know exactly when they are choosing before the game, and also where players may not know exactly their or their opponents utility functions, but have an idea of what they may be, so the uncertainty doesn't have to be modelled by a random variable. Moreover, the model introduced is a sequential model, which is a natural way in which situations in real life happen. For this model, we were able to prove the existence of an equilibrium, though not a clear way of finding such equilibria.

As we can see in the first example, it is possible to model situations where the utilities are not well known, though in that case, are very similar, since both teams have similar appreciations of the athletes, but they may lack the knowledge of certain aspects, which leads to the fuzzy utilities. In the case of the example, the probabilities with which each team is chosen in the first turn did not change the fact that the second athlete is perceived by both teams as better, in spite of them losing a good chunk of chance to be chosen in the second period. This also leads to the possibility of using these models to create situations where we can modify the behaviors of the players, by changing the probabilities with which they can be chosen in the different turns.

From the second example we can see that if the games are of a specific structure, it is possible to study them as well under the fuzzy expected utility criterion and though we got the same strategy of choosing in the first turn athlete 2 for team 1 and athlete 1 for team 2, the expressions obtained are completely different and therefore it is not possible to interchange the approaches. Whether there is a relation between both approaches is an interesting question that could be studied in the future.

Further work includes considering changes on other aspect, including types for players, as well as other types of fuzzy utility functions to consider even cases where players may be prone or adverse to risk as in [3]. Other lines of work could include studying the model as an optimization problem to establish a way to find approximate solutions.

ACKNOWLEDGEMENT

This work was partially supported by CONAHCyT-Mexico under Grant No. CF-2023-I-1362. The author would also like to thank the associate editor and referee for the comments that helped improve the presentation of the paper.

(Received December 16, 2024)

REFERENCES

[1] R. Becerril-Borja and R. Montes-de-Oca: Sequential games with finite horizon and turn selection process: Finite strategy sets case. In: Proc. 5th the International Conference on Operations Research and Enterprise Systems (ICORES 2016) (2016), pp. 44–50. DOI:10.5220/0005696400440050

[2] R. Becerril-Borja and R. Montes-de-Oca: A family of models for finite sequential games without a predetermined order of turns. In: Operations Research and Enterprise Systems: 5th International Conference ICORES 2016, Rome 2016, Revised Selected Papers, (B. Vitoriano and G. H. Parlier, eds.), pp. 35–51. Springer, Cham 2017. DOI:10.1007/978-3-319-53982-9_3

[3] R. Becerril-Borja and R. Montes-de-Oca: Incomplete information and risk sensitive analysis of sequential games without a predetermined order of turns. Kybernetika *57* (2021), 312–331. DOI:10.14736/kyb-2021-2-0312

[4] O. K. Borj, N. Ramezani, and T. M. R. Akbarzade: Using decision trees and a-cuts for solving matrix games with fuzzy payoffs. In: 2014 Iranian Conference on Intelligent Systems 2014, pp. 1–6. DOI:10.1109/iraniancis.2014.6802591

[5] K. Carrero-Vera, H. Cruz-Suárez, and R. Montes-de-Oca: Markov decision processes on finite spaces with fuzzy total rewards. Kybernetika *58* (2022), 180–199. DOI:10.14736/kyb-2022-2-0180

[6] H. Cruz-Suárez, R. Montes-de-Oca, and R. I. Ortega-Gutiérrez: An extended version of average Markov decision processes on discrete spaces under fuzzy environment. Kybernetika *59* (2023), 160–178. DOI:10.14736/kyb-2023-1-0160

[7] D. Fudenberg and J. Tirole: Game Theory. The MIT Press, Boston 1991.

[8] D.-F. Li: A fast approach to compute fuzzy values of matrix games with payoffs of triangular fuzzy numbers. European J. Oper. Res. *223* (2012), 421–429. DOI:10.1016/j.ejor.2012.06.020

[9] M. Maschler, E. Solan, and S. Zamir: Game Theory. Cambridge University Press, Cambridge 2020. DOI10.26226/morressier.60546d5e30a2e980041f1f70

[10] I. Nishizaki and M. Sakawa: Fuzzy and Multiobjective Games for Conflict Resolution. Physica-Verlag, Heidelberg, 2001.

[11] M. L. Puri and D. A. Ralescu: Fuzzy random variable. J. Math. Anal. Appl. *114* (1986), 409–422. DOI:10.1016/0022-247X(86)90093-4

[12] S. Rezvani and M. Molani: Representation of trapezoidal fuzzy numbers with shape function. Ann. Fuzzy Math. Inform. *8* (2014), 89–112.

[13] S. Tadelis: Game Theory: An Introduction. Princeton University Press, Princeton 2013.

[14] L. Zadeh: Fuzzy sets. Inform. Control *8* (1965), 338–353. DOI:10.1016/S0019-9958(65)90241-X

Rubén Becerril-Borja, Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa, Av. Ferrocarril San Rafael Atlixco 186, Col. Leyes de Reforma 1a Sección, 09310, Mexico City. Mexico.

e-mail: ruben@xanum.uam.mx