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Cuihua He; Ke Wang; Zhongcheng Zhou

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EXPLICIT CONTROL LAW FOR A HEAT EQUATION BASED ON OUTPUT TRACKING

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Boundary control to track the output reference of the heat equation is considered. A control input is implemented at one boundary while requiring the other boundary to track the output reference. By introducing the error system and backstepping transformation, the control law is designed. The undetermined coefficient method and two identities are used to obtain the analytical solution of the kernel function by complex mathematical calculation. This establishes an explicit control law and ensures that the error system can effectively achieve the desired closed-loop stability. Simulation results validate the proposed theoretical results.

Keywords: backstepping, boundary control, output tracking

Classification: 93C20, 35K05, 93D15, 93D23

1. INTRODUCTION

The modeling and analysis of partial differential equations (PDEs) constitute essential methodologies for addressing complex engineering challenges [8]. In engineering practice, the selection of an appropriate output reference function is critical to achieving specific engineering objectives [2]. To strike a balance between computational efficiency and accuracy, this study adopts a quadratic polynomial as the approximate reference function, leveraging its appropriate approximation capabilities [18]. This choice not only ensures computational tractability but also establishes a foundational framework for subsequent investigations in this research.

Krstic et al. [12] developed a boundary control law by using PDE Backstepping, which offers a potent tool for designing the boundaries of PDE systems. The Backstepping method can effectively control the boundary of PDE systems [5, 10, 24] and coupled or cascaded ODE-PDE systems [16, 21, 23]. Moreover, its theoretical framework is essentially applicable to the local stabilization of nonlinear parabolic PDE systems [1, 14, 20].

In this pioneering work, Krstic and Smyshlyaev [12] elaborated extensively on the theoretical foundations and practical implications of stabilizing PDE feasible trajectories by boundary control, which remains a cornerstone of the field. PDE Backstepping is an effective method for rapid system stabilization. However, the primary challenge associated

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with this approach lies in solving the governing kernel equations required for deriving explicit control laws. Notably, these explicit control laws have demonstrated numerous advantages in practical applications, especially in enhancing real-time performance and improving control efficiency, as evidenced by studies in [11] and [13]. Consequently, this approach has had a favorable impact and imparted substantial benefits to engineering practice. The core goal of this paper is to study the quantitative characteristics of the backstepping transformation kernel equation in depth, so as to obtain the explicit control law. Yet, from a mathematical standpoint, the task of deriving these explicit control laws remains formidable.

In the field of control theory, significant advancements have been achieved in addressing control problems for strict-feedback systems under full state constraints in recent years. Notably, the adaptive optimal backstepping tracking control technique has been further developed to achieve stable system tracking without relying on the barrier Lyapunov function framework [15]. Additionally, an adaptive tracking control method driven by a predefined time-triggered mechanism has been proposed and investigated in [17], offering a novel perspective for the control of strict-feedback systems with full state constraints. Furthermore, building on the foundational contributions of Coron et al. [3, 4, 6, 7, 9, 22], these studies not only enrich the theoretical framework of system control but also provide a robust theoretical foundation and valuable insights for ongoing research. In this paper, we focus on a class of reaction-advection equations.

$$u_t(x,t) = u_{xx}(x,t) + \lambda u_x(x,t), x \in (0,1) \tag{1}$$

$$u_x(0,t) = 0 \tag{2}$$

$$u(1,t) = U(t) \tag{3}$$

with the expected reference output

$$u^r(0,t) = at^2 + bt + c, \tag{4}$$

where the temperature of the heat body is denoted by the signal $u(x,t) \in \mathbb{R}$, λ is a positive constant representing the convection coefficient, the control input is represented by the $U(t)$ and a, b, c are constants that are pre-determined. To ensure generality, the initial value $u(x,0)$ can be assumed to belong to $H^1(0,1)$, which is a Sobolev space with square integrable weak derivatives of order one. It is very important for the convection term $u_x(x,t)$ in system (1) when describing the heat diffusion.

The system described by equations (1) − (4) is an output regulation problem based on a certain class of physical scenarios. The challenge of output regulation is to develop a feedback control strategy to track the specified reference signal step by step and to suppress the undesired disturbance in the uncertain system step by step while guaranteeing the stability of the closed-loop system. Therefore, it is of great value and practical significance to study the PDE tracking control system by the backstepping method.

The control objective in this paper is to design a feedback control law such that

$$\lim_{t\to\infty} \mid u(0,t) - u^r(0,t) \mid = 0. \tag{5}$$

To solve this kind of tracking problem, first of all, we aim to utilize the temperature at $x = 0$ as determined by (4) to calculate the reference input at $x = 1$. In order to determine the reference input $u^r(1,t)$, it is necessary to construct the full-state trajectory $u^r(x,t)$ [12]. To achieve the progressive tracking of the reference signal or the progressive suppression of the undesired disturbance in the uncertain system, we introduce the error system $\varepsilon(x,t) = u(x,t) - u^r(x,t)$, which is expected to maintain the closed-loop stability of the error system. Backstepping is employed to obtain the control input $U(t)$, which achieves stability of the closed-loop error system.

The paper is structured as follows. Section 2 presents the trajectory generation of the full-state, introduces the error system, and obtains the kernel equation using standard variation. In Section 3, the kernel function in the standard transformation is solved, leading to the establishment of the explicit control law. The effectiveness of the control law is demonstrated through simulation in Section 4, and the paper is concluded in Section 5. Additionally, the appendix provides two crucial identities to prove the expression of the kernel function.

2. DESIGN OF CONTROL LAW

In this section, we first use the temperature at $x = 0$ to generate the full-state trajectory, then the error system is introduced, the kernel equation is obtained through the standard transformation and finally the control law is designed. The aforementioned procedure is the standard processing method, albeit with slightly complex calculations. For the sake of convenience, we have presented a concise process.

2.1. Trajectory generation

In order to determine the reference input $u^r(1,t)$, we first need to create the full-state trajectory $u^r(x,t)$, which must meet the conditions specified in (1), (2), and (4). Let us follow the idea of Taylor series expansion to search for the full-state trajectory equation. Let

$$u^r(x,t) = \sum_{n=0}^{\infty} a_n(t)\frac{x^n}{n!}, \tag{6}$$

where the $a_n(t)$ are time-varying coefficients that can be determined by equations (1), (2) and (4). Based on (6) and (4), the following equality holds

$$u^r(0,t) = a_0(t) = at^2 + bt + c. \tag{7}$$

Applying the boundary condition (2), it holds that

$$u^r_x(0,t) = a_1(t) = 0. \tag{8}$$

Next substituting (6) into (1), one knows that

$$\sum_{n=0}^{\infty} \dot{a}_n(t)\frac{x^n}{n!} = \frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} a_n(t)\frac{x^n}{n!} + \lambda\frac{\partial}{\partial x}\sum_{n=0}^{\infty} a_n(t)\frac{x^n}{n!}.$$

Hence, we get the recursive relation for $a_n(t)$

$$\dot{a}_n(t) = a_{n+2}(t) + \lambda a_{n+1}(t). \tag{9}$$

In terms of (7) – (9), it holds that

$$\begin{aligned}a_2(t) =& 2at+b, \quad a_3(t) = -\lambda(2at+b),\\ a_n(t) =& 2a(n-3)(-\lambda)^{n-4} + (2at+b)(-\lambda)^{n-2}, (n \geq 4).\end{aligned}$$

Therefore, the reference full-state trajectory is

$$\begin{aligned}u^r(x,t) =& at^2 + bt + c + (2at+b)\frac{x^2}{2!} - \lambda(2at+b)\frac{x^3}{3!}\\ &+ \sum_{n=4}^{\infty}\big(2a(n-3)(-\lambda)^{n-4} + (2at+b)(-\lambda)^{n-2}\big)\frac{x^n}{n!}.\end{aligned}$$

The reference input signal is

$$\begin{aligned}u^r(1,t) =& at^2 + bt + c + \frac{2at+b}{2!} - \frac{\lambda(2at+b)}{3!}\\ &+ \sum_{n=4}^{\infty}\frac{2a(n-3)(-\lambda)^{n-4} + (2at+b)(-\lambda)^{n-2}}{n!}. \end{aligned} \tag{10}$$

2.2. Error System

Introducing error system

$$\varepsilon(x,t) = u(x,t) - u^r(x,t),$$

then, the system (1) – (3) is converted into

$$\varepsilon_t(x,t) = \varepsilon_{xx}(x,t) + \lambda\varepsilon_x(x,t), x \in (0,1) \tag{11}$$

$$\varepsilon_x(0,t) = 0 \tag{12}$$

$$\varepsilon(1,t) = U(t) - u^r(1,t) \tag{13}$$

Let

$$\varepsilon(x,t) = \upsilon(x,t)e^{-\frac{\lambda}{2}x}, \tag{14}$$

substitute (14) into (11) – (13), then the error system is as follows

$$\upsilon_t(x,t) = \upsilon_{xx}(x,t) - \frac{\lambda^2}{4}\upsilon(x,t) \tag{15}$$

$$\upsilon_x(0,t) = \frac{\lambda}{2}\upsilon(0,t) \tag{16}$$

$$\upsilon(1,t) = \varepsilon(1,t)e^{\frac{\lambda}{2}}. \tag{17}$$

The corresponding control law is obtained by (13) and (17) as

$$U(t) = \upsilon(1,t)e^{-\frac{\lambda}{2}} + u^r(1,t), \tag{18}$$

where $\upsilon(1,t)$ is subsequently determined in Section 2.3.

If the system (15) – (17) can be stabilized by suitable control, it indicates that the error system can be stabilized to zero, which means that signal $u(x,t)$ can asymptotically track the signal $u^r(x,t)$. Next, we will design the control law to achieve the purpose of tracking the signal.

2.3. Kernel equations

The control law can be obtained by choosing a stable target system as follows

$$w_t(x,t) = w_{xx}(x,t), x \in (0,1) \tag{19}$$
$$w_x(0,t) = 0 \tag{20}$$
$$w(1,t) = 0. \tag{21}$$

Adopting the following standard backstepping transformation

$$w(x,t) = \upsilon(x,t) - \int_0^x k(x,y)\upsilon(y,t)\,\mathrm{d}y \tag{22}$$

to convert system (15) – (17) into target system (19) – (21) with undetermined kernel $k(x,y)$. Next, we will deduce the kernel equation of $k(x,y)$. By differentiating $w(x,t)$ in (22) twice with respect to x and once with respect to t, and then using integration by parts, we obtain

$$\begin{aligned} w_t(x,t) - w_{xx}(x,t) =& \big(2k'(x,x) - \frac{\lambda^2}{4}\big)\upsilon(x,t) + \big(\frac{\lambda}{2}k(x,0) - k_y(x,0)\big)\upsilon(0,t) \\ &+ \int_0^x (k_{xx}(x,y) - k_{yy}(x,y) + \frac{\lambda^2}{4}k(x,y))\upsilon(y,t)\,\mathrm{d}y, \end{aligned} \tag{23}$$

where $k'(x,x) = \frac{\mathrm{d}k(x,x)}{\mathrm{d}x}$.

According to the system (19) – (21), we can choose the kernel $k(x,y)$ satisfy

$$k_{xx}(x,y) - k_{yy}(x,y) = -\frac{\lambda^2}{4}k(x,y) \tag{24}$$
$$k_y(x,0) = \frac{\lambda}{2}k(x,0) \tag{25}$$
$$k'(x,x) = \frac{\lambda^2}{8}. \tag{26}$$

Further, according to $w_x(0,t) = 0$ and the transformation (22), we have $\upsilon_x(0,t) = k(0,0)\upsilon(0,t)$. In terms of the boundary condition (16), we know

$$k(0,0) = \frac{\lambda}{2}. \tag{27}$$

By using $w(1,t) = 0$, it holds that

$$\upsilon(1,t) = \int_0^1 k(1,y)\upsilon(y,t)\,\mathrm{d}y. \tag{28}$$

Then, the control law (18) is given as

$$U(t) = e^{-\frac{\lambda}{2}} \int_0^1 k(1,y)\upsilon(y,t)\,\mathrm{d}y + u^r(1,t). \tag{29}$$

3. ANALYTICAL SOLUTIONS

Solving the equations by the standard iteration method, the system (24) – (26) and (27) can be transformed into integral equations. Proving the existence of the kernel function $k(x, y)$ is straightforward, but finding its explicit mathematical solution is challenging. This challenge constitutes one of the key innovations of this work. As for the kernel equation, we can obtain the following lemma.

Lemma 3.1. For the kernel equation (24) – (26) and (27), there exists a unique classical solution, which can be expressed explicitly.

Proof. Let $\mu = x + y,\ \nu = x - y$ and define $k(x, y) := G(\mu, \nu)$, then, the equation (24) – (26) and (27) is converted into

$$G_{\mu\nu}(\mu,\nu) = -\frac{\lambda^2}{16}G(\mu,\nu) \tag{30}$$

$$G(\mu,\mu) = \frac{2}{\lambda}(G_\mu(\mu,\mu) - G_\nu(\mu,\mu)) \tag{31}$$

$$G_\mu(\mu,0) = \frac{\lambda^2}{16} \tag{32}$$

$$G(0,0) = \frac{\lambda}{2}. \tag{33}$$

By integrating (30) from 0 to ν with respect to ν, and then integrating from ν to μ with respect to μ, and utilizing (32), it holds that

$$G(\mu,\nu) = \frac{\lambda^2}{16}(\mu-\nu) + G(\nu,\nu) - \frac{\lambda^2}{16}\int_\nu^\mu\int_0^\nu G(s,\tau)\,\mathrm{d}\tau\mathrm{d}s. \tag{34}$$

The well-posedness of (34) can be proved by the same idea in [12]. Here, we use a different process to prove it for obtaining the explicit solution of (34) via the undetermined coefficient method.

We assume $G(\nu,\nu) = \sum_{k=0}^{\infty} c_k \frac{\nu^k}{k!}$ with c_k determined later, then,

$$G(\mu,\nu) = \frac{\lambda^2}{16}(\mu-\nu) + \sum_{k=0}^{\infty} c_k\frac{\nu^k}{k!} - \frac{\lambda^2}{16}\int_\nu^\mu\int_0^\nu G(s,\tau)\,\mathrm{d}\tau\mathrm{d}s. \tag{35}$$

We calculate the explicit solution of (35) by successive approximation method. First, we define the sequence functions as following

$$\begin{aligned} G_0(\mu,\nu) &= 0,\\ G_m(\mu,\nu) &= \frac{\lambda^2}{16}(\mu-\nu) + \sum_{k=0}^{\infty} c_k\frac{\nu^k}{k!} - \frac{\lambda^2}{16}\int_\nu^\mu\int_0^\nu G_{m-1}(s,\tau)\,\mathrm{d}\tau\mathrm{d}s, \end{aligned} \tag{36}$$

If the functions $\{G_{m+1}(\mu,\nu)\}$ converge uniformly, then we get

$$G(\mu,\nu) := \lim_{m\to\infty} G_m(\mu,\nu),$$

which is exactly the solution of the equation (35).

In order to derive the limit of the above sequence functions, we have the following recursive formula

$$\begin{aligned}\Delta G_0(\mu,\nu) =& G_1(\mu,\nu) - G_0(\mu,\nu),\\ \Delta G_m(\mu,\nu) =& G_{m+1}(\mu,\nu) - G_m(\mu,\nu) = -\frac{\lambda^2}{16}\int_\nu^\mu\int_0^\nu \Delta G_{m-1}(s,\tau)\,\mathrm{d}\tau\mathrm{d}s,\end{aligned}$$

then

$$G_m(\mu,\nu) = \sum_{j=0}^{m-1}\Delta G_j(\mu,\nu). \tag{37}$$

Next, we compute the sequence $\Delta G_m(\mu,\nu)$, and find out the general expression. Firstly

$$\Delta G_0(\mu,\nu) = G_1(\mu,\nu) - G_0(\mu,\nu) = \sum_{k=0}^{\infty} c_k\frac{\nu^k}{k!} + \frac{\lambda^2}{16}(\mu-\nu).$$

Secondly

$$\begin{aligned}\Delta G_1(\mu,\nu) =& -\frac{\lambda^2}{16}\int_\nu^\mu\int_0^\nu \Delta G_0(s,\tau)\,\mathrm{d}\tau\mathrm{d}s\\ =& -\frac{\lambda^2}{16}\sum_{k=0}^{\infty}\int_\nu^\mu\int_0^\nu c_k\frac{\tau^k}{k!}\mathrm{d}\tau\mathrm{d}s - (-\frac{\lambda^2}{16})^2\int_\nu^\mu\int_0^\nu (s-\tau)\,\mathrm{d}\tau\mathrm{d}s\\ =& -\frac{\lambda^2}{16}\sum_{k=0}^{\infty} c_k\frac{\nu^{k+1}(\mu-\nu)}{(k+1)!} - (-\frac{\lambda^2}{16})^2\frac{(\mu-\nu)\mu\nu}{2!},\end{aligned}$$

$$\begin{aligned}\Delta G_2(\mu,\nu) =& -\frac{\lambda^2}{16}\int_\nu^\mu\int_0^\nu \Delta G_1(s,\tau)\,\mathrm{d}\tau\mathrm{d}s\\ =&(-\frac{\lambda^2}{16})^2\sum_{k=0}^{\infty} c_k\int_\nu^\mu\int_0^\nu \frac{\tau^{k+1}(s-\tau)}{(k+1)!}\mathrm{d}\tau\mathrm{d}s - (-\frac{\lambda^2}{16})^3\int_\nu^\mu\int_0^\nu \frac{(s-\tau)s\tau}{2!}\mathrm{d}\tau\mathrm{d}s\\ =&(-\frac{\lambda^2}{16})^2\sum_{k=0}^{\infty} c_k\Big(\frac{\nu^{k+2}(\mu-\nu)^2}{2!(k+2)!} + \frac{\nu^{k+3}(\mu-\nu)}{1!(k+3)!}\Big) - (-\frac{\lambda^2}{16})^3\frac{(\mu-\nu)\mu^2\nu^2}{2!3!},\end{aligned}$$

and

$$\begin{aligned}\Delta G_3(\mu,\nu) =& -\frac{\lambda^2}{16}\int_\nu^\mu\int_0^\nu \Delta G_2(s,\tau)\,\mathrm{d}\tau\mathrm{d}s\\ =&(-\frac{\lambda^2}{16})^3\sum_{k=0}^{\infty} c_k\int_\nu^\mu\int_0^\nu \Big(\frac{\tau^{k+2}(s-\tau)^2}{2!(k+2)!} + \frac{\tau^{k+3}(s-\tau)}{1!(k+3)!}\Big)\mathrm{d}\tau\mathrm{d}s\\ &- (-\frac{\lambda^2}{16})^4\int_\nu^\mu\int_0^\nu \frac{(s-\tau)s^2\tau^2}{2!3!}\mathrm{d}\tau\mathrm{d}s\end{aligned}$$

$$=(-\frac{\lambda^2}{16})^3\sum_{k=0}^{\infty}c_k\Big(\frac{\nu^{k+3}(\mu-\nu)^3}{3!(k+3)!}+\frac{2\nu^{k+4}(\mu-\nu)^2}{2!(k+4)!}+\frac{2\nu^{k+5}(\mu-\nu)}{1!(k+5)!}\Big)$$
$$-(-\frac{\lambda^2}{16})^4\frac{(\mu-\nu)\mu^3\nu^3}{3!4!}.$$

By utilizing above equations for $\Delta G_0(\mu,\nu)$, $\Delta G_1(\mu,\nu)$, $\Delta G_2(\mu,\nu)$, $\Delta G_3(\mu,\nu)$, we can derive an expression of the general item $\Delta G_m(\mu,\nu)$, which is

$$\Delta G_m(\mu,\nu)=(-\frac{\lambda^2}{16})^m\sum_{k=0}^{\infty}c_k\sum_{i=0}^{m-1}\frac{b_{m,i}\nu^{k+m+i}(\mu-\nu)^{m-i}}{(k+m+i)!(m-i)!}-(-\frac{\lambda^2}{16})^{m+1}\frac{(\mu-\nu)\mu^m\nu^m}{m!(m+1)!},\tag{38}$$

where

$$b_{m,i}=\frac{(m-i)(m-1+i)!}{i!m!},(m=1,2,\cdots,\ i=0,1,\cdots,m-1).$$

The proof of (38) follows a similar mathematical induction as in [11]. For the sake of brevity, the core results are given in this paper, and the technical details are detailed in [11].

According to (37) and (38), we have

$$G_m(\mu,\nu)=\Delta G_0(\mu,\nu)+\sum_{j=1}^{m-1}\Delta G_j(\mu,\nu)$$
$$=\frac{\lambda^2}{16}(\mu-\nu)+\sum_{k=0}^{\infty}c_k\frac{\nu^k}{k!}+\sum_{j=1}^{m-1}(-\frac{\lambda^2}{16})^j\sum_{k=0}^{\infty}c_k\sum_{i=0}^{j-1}\frac{b_{j,i}\nu^{k+j+i}(\mu-\nu)^{j-i}}{(k+j+i)!(j-i)!}$$
$$-\sum_{j=1}^{m-1}(-\frac{\lambda^2}{16})^{j+1}\frac{(\mu-\nu)\mu^j\nu^j}{j!(j+1)!}$$
$$=\sum_{k=0}^{\infty}c_k\frac{\nu^k}{k!}-\sum_{j=0}^{m-1}(-\frac{\lambda^2}{16})^{j+1}\frac{(\mu-\nu)\mu^j\nu^j}{j!(j+1)!}+\sum_{k=0}^{\infty}c_k\sum_{j=1}^{m-1}(-\frac{\lambda^2}{16})^j\sum_{i=0}^{j-1}\frac{b_{j,i}\nu^{k+j+i}(\mu-\nu)^{j-i}}{(k+j+i)!(j-i)!},$$

where $b_{j,i}$ is given as before. Taking $m\to\infty$, we get the limited function $G(\mu,\nu)$ as

$$G(\mu,\nu)=\lim_{m\to\infty}G_m(\mu,\nu)=\Delta G_0(\mu,\nu)+\sum_{j=1}^{\infty}\Delta G_j(\mu,\nu)$$
$$=\sum_{k=0}^{\infty}c_k\frac{\nu^k}{k!}-\sum_{j=0}^{\infty}(-\frac{\lambda^2}{16})^{j+1}\frac{(\mu-\nu)\mu^j\nu^j}{j!(j+1)!}$$
$$+\sum_{k=0}^{\infty}c_k\sum_{j=1}^{\infty}(-\frac{\lambda^2}{16})^j\sum_{i=0}^{j-1}\frac{b_{j,i}\nu^{k+j+i}(\mu-\nu)^{j-i}}{(k+j+i)!(j-i)!}.\tag{39}$$

Then replacing the variables μ, ν with x, y, we know

$$k(x,y) = \sum_{k=0}^{\infty} c_k \frac{(x-y)^k}{k!} - 2y \sum_{j=0}^{\infty} (-\frac{\lambda^2}{16})^{j+1} \frac{(x^2-y^2)^j}{j!(j+1)!} + \sum_{k=0}^{\infty} c_k \sum_{j=1}^{\infty} (-\frac{\lambda^2}{16})^j \sum_{i=0}^{j-1} b_{j,i} \frac{(x-y)^{k+j+i}(2y)^{j-i}}{(k+j+i)!(j-i)!}. \tag{40}$$

Finally, we need to determine the undefined constant c_k. Taking the derivative of (40) with respect to y, and then taking $y = 0$, we have

$$k_y(x,0) = -\sum_{k=1}^{\infty} c_k \frac{x^{k-1}}{(k-1)!} - 2 \sum_{j=0}^{\infty} (-\frac{\lambda^2}{16})^{j+1} \frac{x^{2j}}{j!(j+1)!} + 2 \sum_{k=0}^{\infty} c_k \sum_{j=1}^{\infty} (-\frac{\lambda^2}{16})^j \frac{(2j-2)!}{(j-1)!j!} \frac{x^{k+2j-1}}{(k+2j-1)!}. \tag{41}$$

In terms of the equation (25), we know

$$\begin{aligned} \frac{\lambda}{2} \sum_{k=0}^{\infty} c_k \frac{x^k}{k!} &= -\sum_{k=1}^{\infty} c_k \frac{x^{k-1}}{(k-1)!} - 2 \sum_{j=0}^{\infty} (-\frac{\lambda^2}{16})^{j+1} \frac{x^{2j}}{j!(j+1)!} \\ &\quad + 2 \sum_{k=0}^{\infty} c_k \sum_{j=1}^{\infty} (-\frac{\lambda^2}{16})^j \frac{(2j-2)!}{(j-1)!j!} \frac{x^{k+2j-1}}{(k+2j-1)!} \\ &= -\sum_{k=1}^{\infty} c_k \frac{x^{k-1}}{(k-1)!} - 2 \sum_{j=0}^{\infty} (-\frac{\lambda^2}{16})^{j+1} \frac{x^{2j}}{j!(j+1)!} \\ &\quad + 2 \sum_{j=1}^{\infty} (-\frac{\lambda^2}{16})^j \frac{(2j-2)!}{(j-1)!j!} \sum_{k=0}^{\infty} c_k \frac{x^{k+2j-1}}{(k+2j-1)!}. \end{aligned} \tag{42}$$

Taking the derivative of $2n$ order with respect to x on both sides for (42), and then taking $x = 0$, we obtain

$$\frac{\lambda}{2} c_{2n} = -c_{2n+1} - 2(-\frac{\lambda^2}{16})^{n+1} \frac{(2n)!}{n!(n+1)!} + 2 \sum_{j=1}^{n} (-\frac{\lambda^2}{16})^j \frac{(2j-2)!}{(j-1)!j!} c_{2n+1-2j}. \tag{43}$$

Taking the derivative of $2n+1$ order with respect to x on both sides for (42), and then take $x = 0$, we have

$$\frac{\lambda}{2} c_{2n+1} = -c_{2n+2} + 2 \sum_{j=1}^{n+1} (-\frac{\lambda^2}{16})^j \frac{(2j-2)!}{(j-1)!j!} c_{2n+2-2j}. \tag{44}$$

Rearranging the terms in (43) and (44), we get

$$\begin{cases} c_{2n+1} = 2\sum_{j=1}^{n}(-\frac{\lambda^2}{16})^j \frac{(2j-2)!}{(j-1)!j!}c_{2n+1-2j} - 2(-\frac{\lambda^2}{16})^{n+1}\frac{(2n)!}{n!(n+1)!} - \frac{\lambda}{2}c_{2n}, \\ c_{2n+2} = 2\sum_{j=1}^{n+1}(-\frac{\lambda^2}{16})^j \frac{(2j-2)!}{(j-1)!j!}c_{2n+2-2j} - \frac{\lambda}{2}c_{2n+1}. \end{cases} \tag{45}$$

The main objective is to solve c_k, which becomes a general expression in (45). By using (27) and (40), we can determine c_0 by $k(0,0) = c_0 = \frac{\lambda}{2}$. Taking $x = 0$ in (42), we know $\frac{\lambda}{2}c_0 = -c_1 + \frac{\lambda^2}{8}$. Thus, $c_1 = \frac{\lambda^2}{8} - \frac{\lambda^2}{2^2} = -\frac{\lambda^2}{2^3}$. Now calculating multiple recurrence terms in terms of (45), we find out the general expression of c_k by summarizing.

Obviously,

$$\begin{aligned} c_2 =& 2\frac{-\lambda^2}{16}\frac{0!}{0!1!}c_0 - \frac{\lambda}{2}c_1 = 0, \\ c_3 =& 2\frac{-\lambda^2}{16}\frac{0!}{0!1!}c_1 - 2(\frac{-\lambda^2}{16})^2\frac{2!}{1!2!} - \frac{\lambda}{2}c_2 = \frac{2!}{1!2!}\frac{(-\lambda^2)^2}{2^7}, \\ c_4 =& 2\frac{-\lambda^2}{16}\frac{0!}{0!1!}c_2 + 2(\frac{-\lambda^2}{16})^2\frac{2!}{1!2!}c_0 - \frac{\lambda}{2}c_3 = 0, \end{aligned}$$

and

$$c_5 = 2\frac{-\lambda^2}{16}\frac{0!}{0!1!}c_3 + 2(\frac{-\lambda^2}{16})^2\frac{2!}{1!2!}c_1 - 2(\frac{-\lambda^2}{16})^3\frac{4!}{2!3!} - \frac{\lambda}{2}c_4 = \frac{4!}{2!3!}\frac{(-\lambda^2)^3}{2^{11}}.$$

Through the above calculation c_2, c_3, c_4, c_5, we can guess the general expression of c_n satisfying

$$\begin{cases} c_{2n} = 0, \\ c_{2n+1} = \frac{(2n)!}{n!(n+1)!}\frac{(-\lambda^2)^{n+1}}{2^{4n+3}}. \end{cases} \tag{46}$$

The inductive proof for c_n is very complicated, it also involves a lot of mathematical techniques and calculations, therefore, we present the proof in the Appendix part. Up to now, we proved the well-posedness in the classical sense of the kernel equation. □

Once the value of c_k has been determined, the value of $k(1, y)$ is then established as follows.

$$\begin{aligned} k(1,y) =& \sum_{k=0}^{\infty} c_k \frac{(1-y)^k}{k!} + \frac{\lambda y}{2}\frac{J_1(\frac{\sqrt{\lambda^2(1-y^2)}}{2})}{\sqrt{1-y^2}} \\ &+ \sum_{k=0}^{\infty} c_k \sum_{j=1}^{\infty}(-\frac{\lambda^2}{16})^j \sum_{i=0}^{j-1} b_{j,i}\frac{(1-y)^{k+j+i}(2y)^{j-i}}{(k+j+i)!(j-i)!}, \end{aligned}$$

where $J_1(x)$ belongs to a class of Bessel functions, the control $U(t)$ in equation (29) can be given explicitly. Next, we state the main theorem of this paper.

Theorem 3.2. For the control system (1), (2) and (3), taking the feedback control law (29), it can track out the reference output $u^r(0,t)$, that is to say, the state satisfies

$$\lim_{t\to\infty} \mid u(0,t) - u^r(0,t) \mid = 0. \tag{47}$$

Proof. Consider the following Lyapunov function

$$V(t) = \frac{1}{2}\int_0^1 w^2(x,t)\,\mathrm{d}x + \frac{1}{2}\int_0^1 w_x^2(x,t)\,\mathrm{d}x. \tag{48}$$

By employing the solution to the system of equations (19) − (21), computing the derivatives of the Lyapunov function, while utilizing equation $w_t(1,t) = 0$, and Poincaré inequality, we arrive at the conclusion that

$$\begin{aligned}\dot{V}(t) &= \int_0^1 w(x,t)w_t(x,t)dx + \int_0^1 w_x(x,t)w_{xt}(x,t)\,\mathrm{d}x = -\|w_x\|^2 - \|w_{xx}\|^2 \\ &\le -\frac{1}{2}\|w_x\|^2 - \frac{1}{2}\|w_x\|^2 \le -\frac{1}{8}\|w(t)\|^2 - \frac{1}{2}\|w_x\|^2 \le -\frac{1}{4}V(t).\end{aligned} \tag{49}$$

Thus

$$V(t) \le V(0)e^{-\frac{1}{4}t} = \frac{1}{2}\|w_0\|_{H^1}e^{-\frac{1}{4}t}.$$

From the transformation (22), it holds that

$$w(x,t) = \Pi v(x,t) := v(x,t) - \int_0^x k(x,y)v(y,t)\,\mathrm{d}y. \tag{50}$$

Then

$$\|w_0\|_{H^1} = \|\Pi v_0\|_{H^1} \le \|\Pi\|\|v_0\|_{H^1}, \tag{51}$$

where $\|\Pi\|$ presents the norm of operator Π from $H^1(0,1) \to H^1(0,1)$. Using (50) and (51), it yields

$$\begin{aligned}\|v(t)\|_{H^1} &= \|\Pi^{-1}w(t)\|_{H^1} \le \|\Pi^{-1}\|\|w(t)\|_{H^1} \le \|\Pi^{-1}\|\|w_0\|_{H^1}e^{-\frac{1}{4}t} \\ &\le \|\Pi^{-1}\|\|\Pi\|\|v_0\|_{H^1}e^{-\frac{1}{4}t} \le \zeta\|v_0\|_{H^1}e^{-\frac{1}{4}t},\end{aligned}$$

where $\zeta := \|\Pi^{-1}\|\|\Pi\|$.

Hence, we have showed the uniform convergence for $v(x,t)$, that is to say,

$$v(x,t) \to 0 \quad \text{uniformly as} \quad t \to \infty.$$

Based on H^1 stability of target system, trace theorem and Lemma 3.1, we obtain

$$\lim_{t\to\infty} \mid u(0,t) - u^r(0,t) \mid = 0.$$

□

Up to now, we have obtained the explicit control law (29), which can track the given object asymptotically by a boundary feedback control, which means that the boundary point implements progressive tracking out the reference signal $u^r(0,t)$.

4. SIMULATIONS

Next, we simulate a particular system for (1) − (4)

$$
\begin{aligned}
&u_t(x,t) = u_{xx}(x,t) + 5u_x(x,t), x \in (0,1),\\
&u_x(0,t) = 0, t > 0,\\
&u(1,t) = U(t), t > 0
\end{aligned}
$$

with the reference output $u^r(0,t) = -t^2 + 2t + 1$. Based on the above analysis, we can get the control law as

$$U(t) = e^{-\frac{5}{2}} \int_0^1 k(1,y)\upsilon(y,t)\,\mathrm{d}y + u^r(1,t). \tag{52}$$

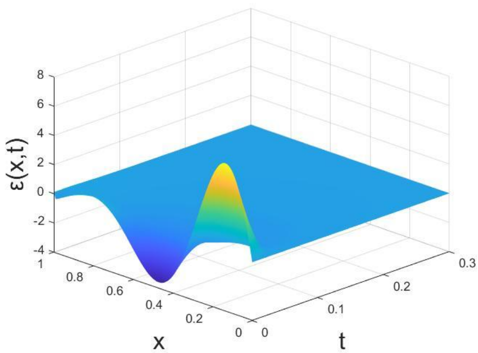

Fig. 1. The error signal $\varepsilon(x,t)$ of the closed loop system.

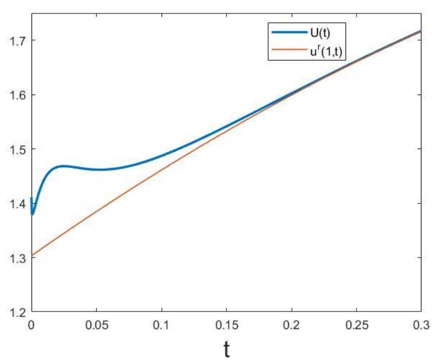

Fig. 2. The signal control $U(t)$ of the closed loop system and reference input $u^r(1,t)$.

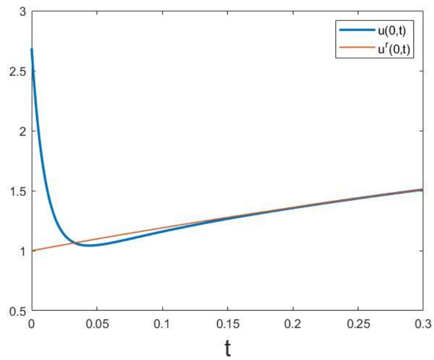

Fig. 3. The signal $u(0,t)$ of the closed loop system and the reference output $u^r(0,t)$.

The simulation results are shown in figures as follows. Figure 1 shows the error signal of the closed-loop system tends to zero under the input of control law (52), it indicates that signal $u(x,t)$ can asymptotically track the signal $u^r(x,t)$. Figure 2 shows the control law (52) tracks the reference input $u^r(1,t)$ asymptotically. Figure 3 clearly shows that the boundary temperature $u(0,t)$ tracks the reference output $u^r(0,t)$ asymptotically. The simulation results verify the validity of the theory results.

5. CONCLUSION

This paper considered the output tracking control problems for a heat equation with boundary control via constructing an explicit feedback control law. The control law is obtained by the normal backstepping method. The explicit control is given by using the undetermined coefficient method and two identity equality expressions. Finally, simulation results validate the proposed theoretical results. For practical purposes, this approach can be extended to include other types of systems, including linearly coupled or cascaded ODE-PDE systems, where the constraint of tracing the reference signal poses an equally interesting challenge. If a nonlinear term is added to the system, the problem will become more complex, and the design of the explicit control law will become a difficult task.

APPENDIX

In this part, we will prove two identity equalities hold for showing c_k satisfying (46), which is the key part for obtaining the explicit feedback control law.

Lemma 5.1. The following identity equality expression holds,

$$\sum_{q=1}^{n-1} \frac{1}{q+1} C_{2n-2q-2}^{n-q-1} C_{2q-2}^{q-1} = \frac{1}{12} C_{2n}^{n}, \tag{A.1}$$

where $c_n^k = \frac{n!}{k!(n-k)!}$.

Proof. Now we employ mathematical induction to prove that (A.1) holds for every positive integer n. In fact, verifying that (A.1) holds when $n = 2$ is straightforward. Assuming that (A.1) is valid for $m-1$, which means

$$I_{m-1} = \frac{1}{12} C_{2m}^{m}, \tag{A.2}$$

where I_{m-1} is defined by

$$I_{m-1} = \sum_{q=1}^{m-1} \frac{1}{q+1} C_{2m-2q-2}^{m-q-1} C_{2q-2}^{q-1}.$$

Next, we show that (A.1) holds for m. First, isolate the term in I_m that involves m, and then express it as a sum of two distinct terms, as detailed below

$$I_m = \sum_{q=1}^{m-1} \frac{1}{q+1} C_{2m-2q}^{m-q} C_{2q-2}^{q-1} + \frac{1}{m+1} C_{2m-2}^{m-1}. \tag{A.3}$$

We observe an interesting equality relation as follows

$$C_{2m-2q}^{m-q} = (-\frac{2}{m-q} + 4) C_{2m-2q-2}^{m-q-1}. \tag{A.4}$$

Next, we plan to substitute this equation (A.4) into the equation (A.3), further calculation can be obtained.

$$\begin{aligned} I_m = & -2 \sum_{q=1}^{m-1} \frac{1}{(q+1)(m-q)} C_{2m-2q-2}^{m-q-1} C_{2q-2}^{q-1} \\ & + \sum_{q=1}^{m-1} \frac{4}{q+1} C_{2m-2q-2}^{m-q-1} C_{2q-2}^{q-1} + \frac{1}{m+1} C_{2m-2}^{m-1}. \end{aligned} \tag{A.5}$$

At this juncture, we utilize equation (A.2) to deduce that the following equality holds

$$\sum_{q=1}^{m-1} \frac{4}{q+1} C_{2m-2q-2}^{m-q-1} C_{2q-2}^{q-1} = \frac{1}{3} C_{2m}^{m},$$

while noting that $\frac{1}{(q+1)(m-q)} = \frac{1}{m+1}(\frac{1}{q+1} + \frac{1}{m-q})$, then (A.5) can be written as

$$\begin{aligned} I_m = & \frac{1}{3} C_{2m}^{m} - \frac{2}{m+1} \sum_{q=1}^{m-1} \frac{1}{q+1} C_{2m-2q-2}^{m-q-1} C_{2q-2}^{q-1} \\ & - \frac{2}{m+1} \sum_{q=1}^{m-1} \frac{1}{m-q} C_{2m-2q-2}^{m-q-1} C_{2q-2}^{q-1} + \frac{1}{m+1} C_{2m-2}^{m-1}. \end{aligned} \tag{A.6}$$

Set $l = m - q$, then,

$$\sum_{q=1}^{m-1} \frac{1}{m-q} C_{2m-2q-2}^{m-q-1} C_{2q-2}^{q-1} = \sum_{l=1}^{m-1} \frac{1}{l} C_{2l-2}^{l-1} C_{2m-2l-2}^{m-l-1}. \tag{A.7}$$

According to the results

$$\sum_{l=1}^{m-1} \frac{1}{l} C_{2l-2}^{l-1} C_{2m-2l-2}^{m-l-1} = \frac{1}{2} C_{2m-2}^{m-1} \tag{A.8}$$

in [19]. Utilizing equations (A.7) and (A.8), we have

$$\frac{2}{m+1} \sum_{q=1}^{m-1} \frac{1}{m-q} C_{2m-2q-2}^{m-q-1} C_{2q-2}^{q-1} = \frac{1}{m+1} C_{2m-2}^{m-1}. \tag{A.9}$$

Substituting Equation (A.9) into (A.6) yields

$$I_m = \frac{1}{3} C_{2m}^{m} - \frac{2}{m+1} \sum_{q=1}^{m-1} \frac{1}{q+1} C_{2m-2q-2}^{m-q-1} C_{2q-2}^{q-1} = \frac{1}{3} C_{2m}^{m} - \frac{2}{m+1} I_{m-1}. \tag{A.10}$$

Further calculations using equation (A.2) yield

$$\frac{2}{m+1} I_{m-1} = \frac{1}{6(m+1)} C_{2m}^{m} \tag{A.11}$$

Finally, use (A.11) to obtain

$$\begin{aligned} I_m =& \frac{1}{3} C_{2m}^{m} - \frac{1}{6(m+1)} C_{2m}^{m} = \frac{1}{12} \frac{4m+2}{m+1} C_{2m}^{m} \\ =& \frac{1}{12} \frac{(2m+2)(2m+1)}{(m+1)(m+1)} \frac{(2m)!}{m!m!} = \frac{1}{12} C_{2m+2}^{m+1}, \end{aligned} \tag{A.12}$$

which finish the proof. □

Lemma 5.2. The following identity expression holds

$$\sum_{q=0}^{m} \frac{(2m-2q)!}{(m-q)!(m-q+1)!} \frac{(2q)!}{q!(q+1)!} = \frac{(2m+2)!}{(m+1)!(m+2)!} \tag{A.13}$$

Proof. The following two equations,

$$\begin{aligned} \frac{(2m-2q)!}{(m-q)!(m+1-q)!} &= \frac{(2m-2q)!(m+1-q)}{(m-q)!(m+1-q)!} - \frac{(2m-2q)!(m-q)}{(m-q)!(m+1-q)!} \\ &= C_{2m-2q}^{m-q} - C_{2m-2q}^{m-q-1} \end{aligned} \tag{A.14}$$

and

$$\frac{(2q)!}{q!(q+1)!} = \frac{(2q)!(q+1)}{q!(q+1)!} - \frac{(2q)!q}{q!(q+1)!} = C_{2q}^{q} - C_{2q}^{q-1} \tag{A.15}$$

clearly hold. Then using (A.14) and (A.15), we get

$$\begin{aligned}
&\sum_{q=0}^{m} \frac{(2m-2q)!}{(m-q)!(m+1-q)!} \cdot \frac{(2q)!}{q!(q+1)!} \\
=&\sum_{q=0}^{m} C_{2m-2q}^{m-q} C_{2q}^{q} - \sum_{q=0}^{m-1} C_{2m-2q}^{m-q-1} C_{2q}^{q} - \sum_{q=1}^{m} C_{2m-2q}^{m-q} C_{2q}^{q-1} + \sum_{q=1}^{m-1} C_{2m-2q}^{m-q-1} C_{2q}^{q-1}.
\end{aligned} \tag{A.16}$$

Next, we will proceed to calculate (A.16). Since the equation contains four terms, our solution strategy is to split it into four independent parts and solve them one by one, and finally add the results of these four parts to get the final answer. We find that the following four equations reveal a beautiful connection between combinatorial numbers

$$\begin{cases}
C_{2m-2q}^{m-q} = 4C_{2m-2q-2}^{m-q-1} - \dfrac{2}{m-q} C_{2m-2q-2}^{m-q-1} \\
C_{2m-2q}^{m-q-1} = 4C_{2m-2q-2}^{m-q-1} - \dfrac{6}{m-q+1} C_{2m-2q-2}^{m-q-1} \\
C_{2q}^{q} = 4C_{2q-2}^{q-1} - \dfrac{2}{q} C_{2q-2}^{q-1} \\
C_{2q}^{q-1} = 4C_{2q-2}^{q-1} - \dfrac{6}{q+1} C_{2q-2}^{q-1}
\end{cases}$$

Based on the four equations above, the first term of (A.16)

$$\begin{aligned}
&\sum_{q=0}^{m} C_{2m-2q}^{m-q} C_{2q}^{q} = 2C_{2m}^{m} + \sum_{q=1}^{m-1} C_{2m-2q}^{m-q} C_{2q}^{q} \\
=&2C_{2m}^{m} + \sum_{q=1}^{m-1} \left(4C_{2m-2q-2}^{m-q-1} - \frac{2}{m-q} C_{2m-2q-2}^{m-q-1}\right)\left(4C_{2q-2}^{q-1} - \frac{2}{q} C_{2q-2}^{q-1}\right) \\
=&2C_{2m}^{m} + 16\sum_{q=1}^{m-1} C_{2m-2q-2}^{m-q-1} C_{2q-2}^{q-1} - 8\sum_{q=1}^{m-1} \frac{1}{m-q} C_{2m-2q-2}^{m-q-1} C_{2q-2}^{q-1} \\
&-8\sum_{q=1}^{m-1} \frac{1}{q} C_{2m-2q-2}^{m-q-1} C_{2q-2}^{q-1} + 4\sum_{q=1}^{m-1} \frac{1}{(m-q)q} C_{2m-2q-2}^{m-q-1} C_{2q-2}^{q-1}.
\end{aligned} \tag{A.17}$$

the second term of (A.16)

$$\sum_{q=0}^{m-1} C_{2m-2q}^{m-q-1} C_{2q}^{q} = C_{2m}^{m-1} + \sum_{q=1}^{m-1} C_{2m-2q}^{m-q-1} C_{2q}^{q}$$

$$=C_{2m}^{m-1}+\sum_{q=1}^{m-1}\big(4C_{2m-2q-2}^{m-q-1}-\frac{6}{m-q+1}C_{2m-2q-2}^{m-q-1}\big)\big(4C_{2q-2}^{q-1}-\frac{2}{q}C_{2q-2}^{q-1}\big)$$

$$=C_{2m}^{m-1}+16\sum_{q=1}^{m-1}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}-24\sum_{q=1}^{m-1}\frac{1}{m-q+1}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}$$

$$-8\sum_{q=1}^{m-1}\frac{1}{q}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}+12\sum_{q=1}^{m-1}\frac{1}{(m-q+1)q}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1} \tag{A.18}$$

the third term of (A.16)

$$\sum_{q=1}^{m}C_{2m-2q}^{m-q}C_{2q}^{q-1}=C_{2m}^{m-1}+\sum_{q=1}^{m-1}C_{2m-2q}^{m-q}C_{2q}^{q-1}$$

$$=C_{2m}^{m-1}+\sum_{q=1}^{m-1}\big(4C_{2m-2q-2}^{m-q-1}-\frac{2}{m-q}C_{2m-2q-2}^{m-q-1}\big)\big(4C_{2q-2}^{q-1}-\frac{6}{q+1}C_{2q-2}^{q-1}\big)$$

$$=C_{2m}^{m-1}+16\sum_{q=1}^{m-1}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}-8\sum_{q=1}^{m-1}\frac{1}{m-q}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}$$

$$-24\sum_{q=1}^{m-1}\frac{1}{q+1}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}+12\sum_{q=1}^{m-1}\frac{1}{(m-q)(q+1)}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}, \tag{A.19}$$

the fourth term of (A.16)

$$\sum_{q=1}^{m-1}C_{2m-2q}^{m-q-1}C_{2q}^{q-1}$$

$$=\sum_{q=1}^{m-1}\big(4C_{2m-2q-2}^{m-q-1}-\frac{6}{m-q+1}C_{2m-2q-2}^{m-q-1}\big)\big(4C_{2q-2}^{q-1}-\frac{6}{q+1}C_{2q-2}^{q-1}\big)$$

$$=16\sum_{q=1}^{m-1}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}-24\sum_{q=1}^{m-1}\frac{1}{m-q+1}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}$$

$$-24\sum_{q=1}^{m-1}\frac{1}{q+1}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}+36\sum_{q=1}^{m-1}\frac{1}{(m-q+1)(q+1)}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}. \tag{A.20}$$

Substituting Eqs. (A.17) – (A.20) into the (A.16) and simplifying it, we have

$$\sum_{q=0}^{m}\frac{(2m-2q)!}{(m-q)!(m+1-q)!}\cdot\frac{(2q)!}{q!(q+1)!}$$

$$=2C_{2m}^{m}-2C_{2m}^{m-1}+4\sum_{q=1}^{m-1}\frac{1}{(m-q)q}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}$$

$$
\begin{aligned}
&-12\sum_{q=1}^{m-1}\frac{1}{(m-q+1)q}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}\\
&-12\sum_{q=1}^{m-1}\frac{1}{(m-q)(q+1)}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}\\
&+36\sum_{q=1}^{m-1}\frac{1}{(m-q+1)(q+1)}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}\\
=&2C_{2m}^{m}-2C_{2m}^{m-1}+\frac{4}{m}\sum_{q=1}^{m-1}(\frac{1}{m-q}+\frac{1}{q})C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}\\
&-\frac{12}{m+1}\sum_{q=1}^{m-1}(\frac{1}{m-q+1}+\frac{1}{q})C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}\\
&-\frac{12}{m+1}\sum_{q=1}^{m-1}(\frac{1}{m-q}+\frac{1}{q+1})C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}\\
&+\frac{36}{m+2}\sum_{q=1}^{m-1}(\frac{1}{m-q+1}+\frac{1}{q+1})C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1},
\end{aligned}
\tag{A.21}
$$

To simplify the equation (A.21), we note that the following two expressions hold under setting $l=m-q$,

$$
\sum_{q=1}^{m-1}\frac{1}{m-q}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}=\sum_{l=1}^{m-1}\frac{1}{l}C_{2l-2}^{l-1}C_{2m-2l-2}^{m-l-1},
\tag{A.22}
$$

and

$$
\sum_{q=1}^{m-1}\frac{1}{m-q+1}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}=\sum_{l=1}^{m-1}\frac{1}{l+1}C_{2l-2}^{l-1}C_{2m-2l-2}^{m-l-1}.
\tag{A.23}
$$

Therefore, substituting (A.22) and (A.23) into (A.21), and simplifying it, we obtain

$$
\begin{aligned}
&\sum_{q=0}^{m}\frac{(2m-2q)!}{(m-q)!(m+1-q)!}\cdot\frac{(2q)!}{q!(q+1)!}\\
=&2C_{2m}^{m}-2C_{2m}^{m-1}+(\frac{8}{m}-\frac{24}{m+1})\sum_{q=1}^{m-1}\frac{1}{q}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}\\
&+(\frac{72}{m+2}-\frac{24}{m+1})\sum_{q=1}^{m-1}\frac{1}{q+1}C_{2m-2q-2}^{m-q-1}C_{2q-2}^{q-1}.
\end{aligned}
\tag{A.24}
$$

Substituting the (A.1) and (A.8) into (A.24), we get

$$
\sum_{q=0}^{m}\frac{(2m-2q)!}{(m-q)!(m+1-q)!}\cdot\frac{(2q)!}{q!(q+1)!}
$$

$$=2C_{2m}^{m} - 2C_{2m}^{m-1} + (\frac{8}{m} - \frac{24}{m+1}) \cdot \frac{1}{2}C_{2m-2}^{m-1} + (\frac{72}{m+2} - \frac{24}{m+1}) \cdot \frac{1}{12}C_{2m}^{m}. \tag{A.25}$$

After a simple calculation, we can confirm that the following expression is true

$$2C_{2m}^{m} - 2C_{2m}^{m-1} + (\frac{8}{m} - \frac{24}{m+1}) \cdot \frac{1}{2}C_{2m-2}^{m-1} = 0 \tag{A.26}$$

then, (A.25) can be simplified as

$$\begin{aligned}&\sum_{q=0}^{m} \frac{(2m-2q)!}{(m-q)!(m+1-q)!} \cdot \frac{(2q)!}{q!(q+1)!}\\ =&\frac{1}{12}(\frac{72}{m+2} - \frac{24}{m+1})C_{2m}^{m} = \frac{(2m+2)!}{(m+1)!(m+2)!},\end{aligned} \tag{A.27}$$

which shows (A.13) holds. □

Next, we prove the formula (46) holds by mathematical induction in terms of Lemma 5.1 and Lemma 5.2.

P r o o f.

First, let's prove the even terms results. In fact, verifying that (46) holds when $n = 2$ is straightforward. Assuming that (46) is valid for $2m$, we show that even term of (46) holds for $2m + 2$.

Using the recursive formula (45),

$$\begin{aligned}c_{2m+2} =&2\sum_{q=1}^{m+1}(-\frac{\lambda^2}{16})^q \frac{(2q-2)!}{(q-1)!q!} c_{2m+2-2q} - \frac{\lambda}{2}c_{2m+1}\\ =&2(-\frac{\lambda^2}{16})^{m+1}\frac{(2m)!}{m!(m+1)!}c_0 - \frac{\lambda}{2} \cdot \frac{(2m)!}{m!(m+1)!} \cdot \frac{(-\lambda^2)^{m+1}}{2^{4m+3}}\\ =&2\frac{(-\lambda^2)^{m+1}}{2^{4n+4}} \cdot \frac{\lambda}{2} \cdot \frac{(2m)!}{m!(m+1)!} - \frac{\lambda}{2} \cdot \frac{(-\lambda^2)^{m+1}}{2^{4m+3}} \cdot \frac{(2m)!}{m!(m+1)!} = 0.\end{aligned}$$

Therefore, the even term solution of equation (46) is valid.

Second, let's prove the odd terms results. It is easy to verify that odd terms of (46) holds for $n = 3$. Suppose that (46) holds for $2m + 1$.

$$c_{2m+1} = \frac{(2m)!}{m!(m+1)!}\frac{(-\lambda^2)^{m+1}}{2^{4m+3}}, \quad m = 1, 2, \cdots \tag{A.28}$$

We show that the odd terms of (46) holds for $2m + 3$. Using the recursive formula (45), we obtain

$$c_{2m+3} =2\sum_{q=1}^{m+1}(-\frac{\lambda^2}{16})^q \frac{(2q-2)!}{(q-1)!q!} c_{2m+3-2q} - 2(\frac{-\lambda^2}{16})^{m+2}\frac{(2m+2)!}{(m+1)!(m+2)!} - \frac{\lambda}{2}c_{2m+2}. \tag{A.29}$$

From the assumption (A.28), we know that the following equation holds

$$c_{2m+3-2q} = \frac{(2m+2-2q)!}{(m+1-q)!(m+2-q)!}\frac{(-\lambda^2)^{m+2-q}}{2^{4(m+1-q)+3}}. \tag{A.30}$$

Substituting the equation (A.30) into (A.29), the calculation yields

$$c_{2m+3} = \frac{(-\lambda^2)^{m+2}}{2^{4m+7}}\Big(2\sum_{q=0}^{m}\frac{(2m-2q)!}{(m-q)!(m-q+1)!}\frac{(2q)!}{q!(q+1)!} - \frac{(2m+2)!}{(m+1)!(m+2)!}\Big) \tag{A.31}$$

By Lemma 5.2, we obtain

$$c_{2m+3} = \frac{(2m+2)!}{(m+1)!(m+2)!}\frac{(-\lambda^2)^{m+2}}{2^{4m+7}},$$

which show the results of the odd term in (46) also hold.

Therefore, (46) hold for any n. □

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Cuihua He, School of Big Data and Artificial Intelligence, Chengdu Technological University, Chengdu, 611730, Sichuan. P. R. China.
e-mail: hchua2@163.com

Ke Wang, School of Big Data and Artificial Intelligence, Chengdu Technological University, Chengdu, 611730, Sichuan. P. R. China.
e-mail: wke1@cdtu.edu.cn

Zhongcheng Zhou, Corresponding author. School of Mathematics and Statistics, Southwest University, Chongqing, 400700. P. R. China.
e-mail: zhouzc@amss.ac.cn