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CONSTRUCTIONS OF UNINORMS WITH ORDINAL SUM UNDERLYING T-NORMS (T-CONORMS) ON BOUNDED LATTICES

Hua-Wen Liu

Uninorms are a special type of aggregation operators proposed by Yager and Rybalov in 1996, and since then, there have been numerous research achievements on uninorms on the unit real interval. In 2015, the concept of uninorms was extended to a more general algebraic structure - bounded lattices. This article aims to study the construction of uninorms on bounded lattices. We first provide the construction methods of uninorms on bounded lattices by using ordinal sum t-norms or ordinal sum t-conorms. Then, we clarify that the new methods are the extensions of some construction methods in literature. Finally, some illustrative examples for the new constructions of uninorms on bounded lattices are provided. This study is the first attempt to construct using the ordinal sum underlying operators and it will open up new ideas for in-depth analysis of the structure of uninorms on bounded lattices.

Keywords: aggregation operators, uninorms, t-norms, ordinal sums, bounded lattices

Classification: 03E72, 03B52, 03G10, 94D05

1. INTRODUCTION

The concept of t-norms was initially proposed by Karl Menger [18] within the framework of statistical metric spaces, with the aim of generalizing triangular inequalities. In the 1960s, Schweizer and Sklar [21, 22] provided the axioms of t-norms, as they are commonly used today. Later, it was proven that t-norms can be as the interpretation of conjunctions in multi-valued logic, especially in fuzzy logic. T-conorms are the dual of t-norms. Uninorms on the real unit interval were proposed by Yager and Rybalov [27] in 1996, and later analyzed in detail by Fodor et al. [13], who provided the structure of uninorms and showed that uninorms are a generalization and unification of t-norms and t-conorms. Later, this kind of operators has been proven to have a wide range of applications, such as fuzzy set theory, fuzzy logic, fuzzy decision making and expert systems [9, 28, 29]. Therefore, it is important to conduct intensive study on uninorms.

A series of research results have been achieved about uninorms on the real unit interval [10, 19]. In 2015, Karaçal and Mesiar [16] extended the concept of uninorms to a more general algebraic structure - bounded lattices, which further expanded the application fields of uninorms and also pushed the theoretical research about uninorms to

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a new level. Due to the much more complex nature of bounded lattices compared to the real unit interval, it means that uninorms on bounded lattices have richer internal structures. Therefore, further in-depth exploration about uninorms on bounded lattices is a meaningful and highly challenging task. So far, a series of techniques or methods have been achieved to construct uninorms on bounded lattices, such as the methods based on t-(sub)norms/t-(sub)conorms or closure/interior operators [1 − 7, 11, 14 − 16, 20, 24 − 26, 30, 31]. It is worth mentioning that most of the existing methods for constructing uninorms are only based on the existence of underlying t-norms and t-conorms on bounded lattices, and do not take into account their internal structure. As a result, the structures of the constructed uninorms on specific subregions are simple. For example, in 2021, Zhang et al. [31] introduced a subclass $\mathcal{U}_{\min} \cup \mathcal{U}_{\max}$ of uninorms on bounded lattices, which covers the vast majority of uninorms in literature. However, the members in this subclass have simple structures in specific subregions, such as the member in $\mathcal{U}_{\max}$ which only takes the value x on the subdomain $I_e \times [0, e[$. For another example, Sun and Liu [24] in 2022 introduced a subclass of uninorms on bounded lattices: $\mathcal{U}_{\wedge} \cup \mathcal{U}_{\vee}$, where $\mathcal{U}_{\min} \subseteq \mathcal{U}_{\wedge}$ and $\mathcal{U}_{\max} \subseteq \mathcal{U}_{\vee}$. Several methods constructing uninorms in $\mathcal{U}_{\wedge} \setminus \mathcal{U}_{\min}$ or in $\mathcal{U}_{\vee} \setminus \mathcal{U}_{\max}$ have also emerged in recent literature (such as [5, 6, 30], etc.). However, these methods still do not take into account the internal structure of the t-norms and t-conorms used, resulting in simple strucutre of the uninorms obtained in specific subregions.

Actually, we know as above mentioned that the uninorms on bounded lattices should have richer structures, so it is necessary to consider the internal structure of t-norms and t-conorms when we construct uninorms on bounded lattices. Based on this motivation, this article intends to make a first attempt in this regard, and plans to use t-norms and t-conorms with ordinal sum structures to construct uninorms on bounded lattices. In other words, we will construct the uninorms with ordinal sum underlying t-norms or t-conorms. Our work will demonstrate that by considering the internal structure of t-norms and t-conorms used, we can capture the uninorms with more complex internal structures on bounded lattices. This is not only beneficial for analyzing the structure of uninorms on bounded lattices, but also provides a larger operator selection space for the design and analysis of many intelligent systems, thereby improving the flexibility and accuracy of the systems.

The remainder of this article is organized as follows. Section 2 lists some symbols and recalls some concepts and results about uninorms on bounded lattices. Sections 3 and 4 are the main parts of this article, where we propose the methods for constructing uninorms on bounded lattices based on ordinal sum t-norms and ordinal sum t-conorms respectively. We illustrate our new construction methods can cover some of existing methods in literature, and provide two examples to demonstrate the effectiveness of the new methods. Section 5 shows some concluding remarks and future work.

2. PRELIMINARIES

In this section, we first recall some necessary concepts and results regarding uninorms on bounded lattices, and list necessary symbols that will be used in the rest of this article. For more detailed knowledge, we refer the reader to the literature [8, 16, 17, 23, 27].

Definition 2.1. (Davey and Priestley [8]) Let $(P, \leq)$ and $(Q, \leq)$ be (disjoint) partially ordered sets (posets for short). The linear sum $P \oplus Q$ is defined by taking the following partial order on $P \cup Q$: $x \leq y$ if and only if $x, y \in P$ and $x \leq y$ in P, or $x, y \in Q$ and $x \leq y$ in Q, or $x \in P$ and $y \in Q$.

Definition 2.2. (Davey and Priestley [8]) Let $(P, \leq)$ be a poset and $S \subseteq P$. An element $x \in P$ is an upper bound of S if $s \leq x$ for all $s \in S$. A lower bound is defined dually. The set of all upper bounds of S is denoted by S^u and the set of all lower bounds by S^l, i. e., $S^u = \{x \in P | (\forall s \in S) s \leq x\}$ and $S^l = \{x \in P | (\forall s \in S) s \geq x\}$.

Definition 2.3. (Davey and Priestley [8])

(i) A poset $(L, \leq)$ is called a lattice if any two elements x, y in L have the greatest lower bound denoted by $x \wedge y$ and the least upper bound denoted by $x \vee y$.

(ii) A lattice $(L, \wedge, \vee)$ is called bounded if it has the top element 1 and bottom element 0, i. e., there exist two elements 1 and 0 in L such that $0 \leq x \leq 1$ for all $x \in L$.

Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice. For $a, b \in L$, if a and b are incomparable, we use the notation $a \parallel b$. We denote the set of all incomparable elements with a as I_a, i. e., $I_a = \{x \in L | x \parallel a\}$. For $a, b \in L$ and $a \leq b$, we define the following intervals: $[a, b] = \{x \in L | a \leq x \leq b\}$, $]a, b] = \{x \in L | a < x \leq b\}$, $[a, b[= \{x \in L | a \leq x < b\}$, $]a, b[= \{x \in L | a < x < b\}$.

Definition 2.4. (Karaçal and Mesiar [16]) Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice. We call function $U : L^2 \to L$ a uninorm if it is commutative, associative, and increasing with respect to both variables, and has a neutral element $e \in L$, i. e. $U(e, x) = x$ for all $x \in L$. Especially, the uninorm U reduces to a t-norm if $e = 1$ and a t-conorm if $e = 0$.

Proposition 2.5. (Karaçal and Mesiar [16]) Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and U be a uninorm on L with the neutral element $e \in L \setminus \{0, 1\}$. The following statements are true:

(i) If T is the restriction of U on $[0, e]^2$, then T is a t-norm on $[0, e]$;

(ii) If S is the restriction of U on $[e, 1]^2$, then S is a t-conorm on $[e, 1]$.

We call the T and S in Proposition 2.5 the underlying t-norm and underlying t-conorm of uninorm U, respectively.

Proposition 2.6. (Karaçal and Mesiar [16]) Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and U be a uninorm on L with the neutral element $e \in L \setminus \{0, 1\}$. Then we have the following statements:

(i) $x \wedge y \leq U(x, y) \leq x \vee y$ for $(x, y) \in [0, e] \times [e, 1] \cup [e, 1] \times [0, e]$;

(ii) $U(x, y) \leq x$ for $(x, y) \in L \times [0, e]$;

(iii) $U(x, y) \leq y$ for $(x, y) \in [0, e] \times L$;

(*iv*) $x \leq U(x,y)$ for $(x,y) \in L \times [e,1]$;

(*ii*) $y \leq U(x,y)$ for $(x,y) \in [e,1] \times L$.

In 2021-2022, Zhang et al. [31] and Sun et al. [24] proposed the following classes of uninorms on bounded lattices, respectively.

Definition 2.7. (Zhang et al. [31]) Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and U be a uninorm on L with a neutral element $e \in L \setminus \{0,1\}$. We say that U belongs to $\mathcal{U}_{\min}$ if $U(x,y) = y$ for all $(x,y) \in]e,1] \times L \setminus [e,1]$, and U belongs to $\mathcal{U}_{\max}$ if $U(x,y) = y$ for all $(x,y) \in [0,e[\times L \setminus [0,e]$.

Definition 2.8. (Sun and Liu [24]) Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and U be a uninorm on L with a neutral element $e \in L \setminus \{0,1\}$. We say that U belongs to $\mathcal{U}_{\wedge}$ if $U(x,y) = x \wedge y$ for all $(x,y) \in [0,e[\times [e,1] \cup [e,1] \times [0,e[$, and U belongs to $\mathcal{U}_{\vee}$ if $U(x,y) = x \vee y$ for all $(x,y) \in [0,e] \times]e,1] \cup]e,1] \times [0,e]$.

It is clear from above two definitions that $\mathcal{U}_{\min} \subseteq \mathcal{U}_{\wedge}$ and $\mathcal{U}_{\max} \subseteq \mathcal{U}_{\vee}$.

Recently, for the sake of reducing the complexity in the proof of associativity of commutative binary functions, Ji [15] proposed an effectively simplified program.

Definition 2.9. (Ji [15]) Let S be a nonempty set, A, B, C be subsets of S and H a binary operation on S. If for all permutations $[X, Y, Z]$ of $\{A, B, C\}$, the equality $H(H(x,y),z) = H(x, H(y,z))$ for all $x \in X, y \in Y, z \in Z$ always holds, then we call H alternating associative on (A, B, C).

Proposition 2.10. (Ji [15]) Let S be a nonempty set, A, B, C be subsets of S and H a commutative binary operation on S.

(*i*) If $H(H(x,y),z) = H(x, H(y,z)) = H(H(x,z),y)$ for all $x \in A, y \in B, z \in C$, then H is alternating associative on (A, B, C).

(*ii*) If $H(H(x,y),z) = H(x, H(y,z))$ for all $x \in A, y \in A, z \in B$, then H is alternating associative on (A, A, B).

(*iii*) If $H(H(x,y),z) = H(x, H(y,z))$ for all $x \in A, y \in B, z \in B$, then H is alternating associative on (A, B, B).

In 2006, Saminger defined the ordinal sum of a family of t-norms on bounded lattices as follows.

Definition 2.11. (Saminger [23]) Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and I a linearly ordered index set. Assume that $(]a_i, b_i[)_{i \in I}$ is a family of pairwise disjoint subintervals of L and T_i is a t-norm on $[a_i, b_i]$ for any $i \in I$, then the ordinal sum $T = \{< a_i, b_i, T_i >\}_{i \in I}$ is defined by

$$T(x,y) = \begin{cases} T_i(x,y) & \text{if } x, y \in [a_i, b_i], i \in I, \\ x \wedge y & \text{otherwise.} \end{cases} \tag{1}$$

It is noted that the ordinal sum T given by equation (1) need not be a t-norm, in general.

Proposition 2.12. (Dvořák and Holčapek [12], Saminger [23]) Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice. Assume that $L = \bigcup_{i=1}^{n} [a_{i-1}, a_i]$, where $0 = a_0 < a_1 < \cdots < a_n = 1$, and T_i is a t-norm on $[a_{i-1}, a_i]$ for any $i \in \{1, \ldots, n\}$, then the ordinal sum T of $\{T_i\}_{i\in\{1,\ldots,n\}}$ defined by equation (1) is a t-norm on L.

By the duality between t-norms and t-conorms, we can easily obtain the corresponding definition and results regarding the ordinal sum of a family of t-conorms on bounded lattices.

3. CONSTRUCTIONS OF UNINORMS WITH ORDINAL SUM UNDERLYING T-NORMS

This section will provide a method for constructing uninorms on bounded lattices by means of putting the ordinal sum t-norms as their underlying t-norms.

Theorem 3.1. Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and $e \in L \setminus \{0, 1\}$. If the following conditions hold,

(*i*) the ordinal sum $T = \{< a_{i-1}, a_i, T_i >\}_{i\in\{1,2,\ldots,n\}}$ of $\{T_i\}_{i\in\{1,2,\ldots,n\}}$ is a t-norm on $[0, e]$, where $[0, e] = \bigoplus_{i=1}^{n} [a_{i-1}, a_i]$ and T_i is a t-norm on $[a_{i-1}, a_i]$ for each $i \in \{1, 2, \ldots, n\}$;

(*ii*) function S is a t-conorm on $[e, 1]$ and the sublattice $[e, 1]$ has a unique atom e' such that $S(e', z) = z$ for all $z \in]e, 1]$;

(*iii*) $I_e \subseteq [0, e[^u \cap]e, 1]^l$ and $I_e = \bigoplus_{i=0}^{n} A_{n-i}$ with $A_i \cap A_j = \emptyset$ $(i \neq j)$ for all $i, j \in \{0, 1, \ldots, n\}$, where A_i is closed w.r.t the operation $\vee$ for each $i \in \{1, 2, \ldots, n-1\}$ and A_n is closed w.r.t the operation $\wedge$,

then the function $U_1 : L^2 \to L$ defined by the following is a uninorm with neutral element e.

$$U_1(x,y) = \begin{cases} T(x,y) & \text{if } (x,y) \in [0,e]^2, \\ S(x,y) & \text{if } (x,y) \in [e,1]^2, \\ x \wedge y & \text{if } (x,y) \in (\bigcup_{i=1}^{n} A_i \times [0,a_i[) \cup (\bigcup_{i=1}^{n} [0,a_i[\times A_i) \cup A_n^2, \\ x & \text{if } (x,y) \in (\bigcup_{i=0}^{n-1} A_i \times [a_i,e[) \cup I_e \times \{e\}, \\ y & \text{if } (x,y) \in (\bigcup_{i=0}^{n-1} [a_i,e[\times A_i) \cup \{e\} \times I_e. \\ x \vee y & \text{otherwise.} \end{cases} \tag{2}$$

Proof. Firstly, the function U_1 is obviously well-defined so it suffices to prove that U_1 is a uninorm. It is clear that U_1 is commutative and its neutral element is e. We only need to prove the associativity and monotonicity below.

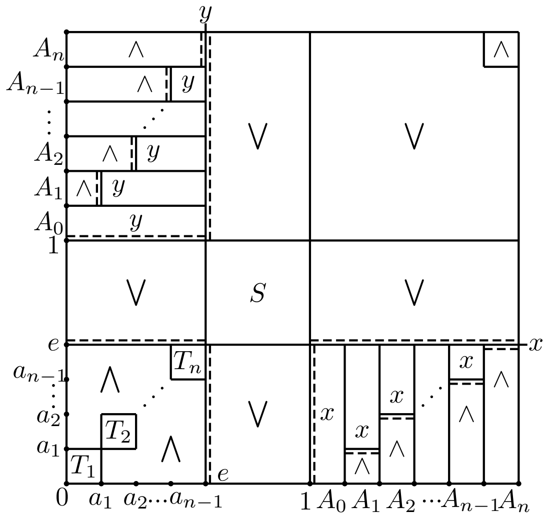

Fig. 1. The uninorm U_1 in Theorem 3.1.

1° Monotonicity. Assume that $x \le y$, then it is clear that the case $(x, y) \in I_e \times [0, e[\cup]e, 1] \times I_e \cup]e, 1] \times [0, e[$ is impossible. From condition (iii) we know that the case $(x, y) \in \bigcup_{i=0}^{n-1}(A_i \times \bigcup_{k=i+1}^{n} A_k)$ with $x \le y$ is also impossible. Thus, only the following cases need to be verified.

Case 1 : $z \in [0, e[$.
The expression of U_1 on $L \times [0, e[$ is as follows:

$$U_1(x,z) = \begin{cases} T(x,z) & \text{if } (x,z) \in [0,e] \times [0,e[, \\ x \vee z & \text{if } (x,z) \in]e,1] \times [0,e[, \\ x & \text{if } (x,z) \in \bigcup_{i=0}^{n-1}(A_i \times [a_i, e[), \\ x \wedge z & \text{if } (x,z) \in \bigcup_{i=1}^{n}(A_i \times [0, a_i[). \end{cases}$$

It is clear that $U_1(x, z) \le U_1(y, z)$ holds when $(x, y) \in [0, e]^2 \cup]e, 1]^2 \cup (\bigcup_{i=0}^{n} A_i^2)$ with $x \le y$. We now check other subcases with $x \le y$.

Case 1.1 : $(x, y) \in [0, e[\times]e, 1]$.
$U_1(x, z) = T(x, z) \le x \le y \vee z = U_1(y, z)$.

Case 1.2 : $(x, y) \in [0, e[\times I_e$.
It follows from $U_1(x, z) = T(x, z) \le x \wedge z$ and $U_1(y, z) = y$ or $y \wedge z$ that $U_1(x, z) \le U_1(y, z)$ holds.

Case 1.3 : $(x, y) \in I_e \times]e, 1]$.
It follows from $U_1(x, z) = x$ or $x \wedge z$ and $U_1(y, z) = y$ or $y \vee z$ that $U_1(x, z) \le U_1(y, z)$ holds.

Case 1.4 : $(x, y) \in \bigcup_{i=1}^{n} \bigcup_{j=0}^{i-1} (A_i \times A_j)$.
If $j = 0$, i.e., $(x, y) \in A_i \times A_0,\ i \in \{1, 2, \dots, n\}$, then it follows from $U_1(x, z) = x$ or $x \wedge z$ and $U_1(y, z) = y$ that $U_1(x, z) \leq U_1(y, z)$ holds.
If $i = n$, i.e., $(x, y) \in A_n \times A_j,\ j \in \{0, 1, \dots, n-1\}$, then it follows from $U_1(x, z) = x \wedge z$ and $U_1(y, z) = y$ or $y \wedge z$ that $U_1(x, z) \leq U_1(y, z)$ holds.
If $i \neq n, j \neq 0$, then we need to consider the following subcases due to $a_j < a_i$.
If $z \in [0, a_j[$, then $U_1(x, z) = x \wedge z \leq y \wedge z = U_1(y, z)$.
If $z \in [a_j, a_i[$, then $U_1(x, z) = x \wedge z \leq x \leq y = U_1(y, z)$.
If $z \in [a_i, e[$, then $U_1(x, z) = x \leq y = U_1(y, z)$.

Case 2 : $z \in]e, 1]$.
The expression of U_1 on $L \times]e, 1]$ is as follows:

$$U_1(x, z) = \begin{cases} x \vee z & \text{if } (x, z) \in ([0, e[\cup I_e) \times]e, 1], \\ S(x, z) & \text{if } (x, z) \in [e, 1] \times]e, 1]. \end{cases}$$

Clearly, we have $U_1(x, z) \leq U_1(y, z)$ when $(x, y) \in [0, e[^2 \cup [e, 1]^2 \cup I_e^2$ and $x \leq y$. We need to check other subcases with $x \leq y$.

Case 2.1 : $(x, y) \in ([0, e[\cup I_e) \times]e, 1]$.
$U_1(x, z) = x \vee z \leq y \vee z \leq S(y, z) = U_1(y, z)$.

Case 2.2 : $(x, y) \in [0, e[\times I_e$.
$U_1(x, z) = x \vee z \leq y \vee z = U_1(y, z)$.

Case 3 : $z = e$.
It is trivial in this case because $U_1(x, e) = x$ for all $x \in L$.

Case 4 : $z \in A_0$.
The expression of U_1 on $L \times A_0$ is as follows:

$$U_1(x, z) = \begin{cases} z & \text{if } (x, z) \in [0, e] \times A_0, \\ x \vee z & \text{if } (x, z) \in (]e, 1] \cup I_e) \times A_0. \end{cases}$$

It is obvious that $U_1(x, z) \leq U_1(y, z)$ holds when $(x, y) \in [0, e]^2 \cup]e, 1]^2 \cup I_e^2$ and $x \leq y$. We check other subcases with $x \leq y$.

Case 4.1 : $(x, y) \in [0, e] \times (]e, 1] \cup I_e)$.
$U_1(x, z) = z \leq y \vee z = U_1(y, z)$.

Case 4.2 : $(x, y) \in I_e \times]e, 1]$.
$U_1(x, z) = x \vee z \leq y \vee z = U_1(y, z)$.

Case 5 : $z \in A_i,\ i = 1, 2, \dots, n-1$.
The expression of U_1 on $L \times A_i$ is as follows:

$$U_1(x, z) = \begin{cases} x \wedge z & \text{if } (x, z) \in [0, a_i[\times A_i, \\ z, & \text{if } (x, z) \in [a_i, e] \times A_i, \\ x \vee z & \text{if } (x, z) \in (]e, 1] \cup I_e) \times A_i. \end{cases}$$

It is obvious that $U_1(x, z) \leq U_1(y, z)$ holds for the case of $(x, y) \in [0, a_i[^2 \cup [a_i, e]^2 \cup]e, 1]^2 \cup I_e^2$ with $x \leq y$. We now check the other subcases with $x \leq y$.

Case 5.1 : $(x, y) \in [0, a_i[\times[a_i, e]$.
$U_1(x, z) = x \wedge z \leq x \leq y = U_1(y, z)$.

Case 5.2 : $(x, y) \in [0, e] \times (]e, 1] \cup I_e)$.
Since $U_1(x, z) = x \wedge z$ or z, and $U_1(y, z) = y \vee z$, we have $U_1(x, z) \leq U_1(y, z)$.

Case 5.3 : $(x, y) \in I_e\times]e, 1]$.
$U_1(x, z) = x \vee z \leq y \vee z = U_1(y, z)$.

Case 6 : $z \in A_n$.

The expression of U_1 on $L \times A_n$ is as follows:

$$U_1(x,z) = \begin{cases} x \wedge z & \text{if } (x,z) \in ([0,e[\cup A_n) \times A_n, \\ z, & \text{if } (x,z) \in \{e\} \times A_n, \\ x \vee z & \text{if } (x,z) \in (]e,1] \cup (I_e \setminus A_n)) \times A_n. \end{cases}$$

It is clear that $U_1(x, z) \leq U_1(y, z)$ holds when $(x, y) \in [0, e[^2 \cup]e, 1]^2 \cup (I_e \setminus A_n)^2 \cup A_n^2$ and $x \leq y$. We now check the other subcases with $x \leq y$.

Case 6.1 : $(x, y) \in [0, e]\times]e, 1]$.
If $x \in [0, e[$, then $U_1(x, z) = x \wedge z \leq y \vee z = U_1(y, z)$.
If $x = e$, then $U_1(x, z) = z \leq y \vee z = U_1(y, z)$.

Case 6.2 : $(x, y) \in [0, e[\times I_e$.
Since $U_1(x, z) = x \wedge z$ and $U_1(y, z) = y \vee z$ or $y \wedge z$, we have $U_1(x, z) \leq U_1(y, z)$.

Case 6.3 : $(x, y) \in I_e\times]e, 1]$.
Since $U_1(x, z) = x \vee z$ or $x \wedge z$, and $U_1(y, z) = y \vee z$ we have $U_1(x, z) \leq U_1(y, z)$.

Case 6.4 : $(x, y) \in A_n \times (I_e \setminus A_n)$.
$U_1(x, z) = x \wedge z \leq y \vee z = U_1(y, z)$.

2° Associativity. For any $x, y, z \in L$, we need to prove $U_1(U_1(x, y), z) = U_1(x, U_1(y, z))$. Firstly, if one of x, y and z equals e, then the equation clearly holds. In addition, since the case of $x, y, z \notin I_e$ is direct, we let at least one of x, y and z belongs to I_e in our following discussion.

Case 1 : One of x, y and z belongs to I_e. Without loss of generality, we let $x \in I_e$.

Case 1.1 : $(x, y, z) \in A_0 \times [0, e[\times[0, e[$.
$U_1(U_1(x, y), z) = U_1(x, z) = x = U_1(x, T(y, z)) = U_1(x, U_1(y, z))$.

Case 1.2 : $(x, y, z) \in A_i \times [0, e[\times[0, e[,\ i = 1, 2, \ldots, n - 1$.
If $y \in [0, a_i[$, then it follows from $A_i \subseteq [0, a_i]^u$ that $U_1(U_1(x, y), z) = U_1(x \wedge y, z) = U_1(y, z) = T(y, z) = x \wedge T(y, z) = U_1(x, U_1(y, z))$ and $U_1(U_1(x, z), y) = T(z, y)$ because if $z \in [0, a_i[$ then $U_1(U_1(x, z), y) = U_1(x \wedge z, y) = T(z, y)$ and if $z \in [a_i, e[$ then $U_1(U_1(x, z), y) = U_1(x, y) = x \wedge y = y = T(z, y)$.
If $y \in [a_i, e[, z \in [0, a_i[$, then it follows from $A_i \subseteq [0, a_i]^u$ that $U_1(U_1(x, y), z) = U_1(x, z) = z = T(y, z) = x \wedge T(y, z) = U_1(x, U_1(y, z))$.
If $y \in [a_i, e[, z \in [a_i, e[$, then $U_1(U_1(x, y), z) = U_1(x, z) = x = U_1(x, T(y, z)) = U_1(x, U_1(y, z))$.

Case 1.3 : $(x, y, z) \in A_n \times [0, e[\times[0, e[$.
Since $A_n \subseteq [0, a_i]^u$, we have $U_1(U_1(x, y), z) = U_1(x \wedge y, z) = U_1(y, z) = T(y, z) = x \wedge T(y, z) = U_1(x, T(y, z)) = U_1(x, U_1(y, z))$.

Case 1.4 : $(x, y, z) \in A_0 \times [0, e[\times]e, 1]$.
Since $U_1(U_1(x, y), z) = U_1(x, z) = z = U_1(x, z) = U_1(x, y \vee z) = U_1(x, U_1(y, z))$ and $U_1(U_1(x, z), y) = U_1(z, y) = z$, we obtain $U_1(U_1(x, y), z) = U_1(x, U_1(y, z)) = U_1(U_1(x, z), y)$.

Case 1.5 : $(x, y, z) \in A_i \times [0, e[\times]e, 1],\ i = 1, 2, \dots, n-1$.
If $y \in [0, a_i[$, then it follows from $A_i \subseteq [0, a_i[^u$ and $A_i \subseteq]e, 1]^l$ that $U_1(U_1(x, y), z) = U_1(x \wedge y, z) = U_1(y, z) = y \vee z = z = x \vee z = U_1(x, z) = U_1(x, y \vee z) = U_1(x, U_1(y, z))$, and $U_1(U_1(x, z), y) = U_1(z, y) = z$.
If $y \in [a_i, e[$, then $U_1(U_1(x, y), z) = U_1(x, z) = z = U_1(x, y \vee z) = U_1(x, U_1(y, z))$, and $U_1(U_1(x, z), y) = U_1(z, y) = z$.

Case 1.6 : $(x, y, z) \in A_n \times [0, e[\times]e, 1]$.
It follows from $A_n \subseteq [0, e[^u$ and $A_n \subseteq]e, 1]^l$ that $U_1(U_1(x, y), z) = U_1(x \wedge y, z) = U_1(y, z) = y \vee z = z = x \vee z = U_1(x, z) = U_1(x, U_1(y, z))$, and $U_1(U_1(x, z), y) = U_1(z, y) = z$.

Case 1.7 : $(x, y, z) \in I_e \times]e, 1]\times]e, 1]$.
It follows from $I_e \subseteq]e, 1]^l$ that $U_1(U_1(x, y), z) = U_1(x \vee y, z) = U_1(y, z) = S(y, z) = x \vee S(y, z) = U_1(x, S(y, z)) = U_1(x, U_1(y, z))$.

Case 2 : Two of x, y and z belong to I_e. Without loss of generality, we let $x, y \in I_e$.

Case 2.1 : $(x, y, z) \in A_0 \times A_0 \times [0, e[$.
A natural consequence of condition $I_e = A_n \oplus A_{n-1} \oplus \cdots \oplus A_0$ is that $x \vee y \in A_0 \cup]e, 1]$ for all $x, y \in A_0$. Thus, we have $U_1(U_1(x, y), z) = U_1(x \vee y, z) = x \vee y = U_1(x, y) = U_1(x, U_1(y, z))$.

Case 2.2 : $(x, y, z) \in A_0 \times A_0 \times]e, 1]$.
If $x \vee y \in A_0$, then $U_1(U_1(x, y), z) = U_1(x \vee y, z) = x \vee y \vee z = z$ because $A_0 \subseteq]e, 1]^l$.
If $x \vee y \in]e, 1]$, then $U_1(U_1(x, y), z) = U_1(x \vee y, z) = S(x \vee y, z)$. It follows from $A_0 \subseteq]e, 1]^l$ that $x < t, y < t$, i. e., $x \vee y \leq t$ for all $t \in]e, 1]$. According to condition (ii) we know that $x \vee y = \bigwedge_{t \in]e, 1]} t = e'$ and $S(x \vee y, z) = z$, i. e., we obtain $U_1(U_1(x, y), z) = z$.
Also due to $U_1(x, U_1(y, z)) = U_1(x, y \vee z) = U_1(x, z) = z$. Thus, we have $U_1(U_1(x, y), z) = U_1(x, U_1(y, z))$.

Case 2.3 : $(x, y, z) \in A_i \times A_i \times [0, e[,\ i = 1, 2, \dots, n-1$.
If $z \in [0, a_i[$, then it follows from $A_i \subseteq [0, a_i[^u$ and $x \vee y \in A_i$ that $U_1(U_1(x, y), z) = U_1(x \vee y, z) = (x \vee y) \wedge z = x \wedge z = U_1(x, z) = U_1(x, y \wedge z) = U_1(x, U_1(y, z))$.
If $z \in [a_i, e[$, then $U_1(U_1(x, y), z) = U_1(x \vee y, z) = x \vee y = U_1(x, y) = U_1(x, U_1(y, z))$.

Case 2.4 : $(x, y, z) \in A_i \times A_i \times]e, 1]\ i = 1, 2, \dots, n-1$.
It follows from $x \vee y \in A_i$ and $A_i \subseteq]e, 1]^l$ that $U_1(U_1(x, y), z) = U_1(x \vee y, z) = (x \vee y) \vee z = x \vee z = U_1(x, z) = U_1(x, y \vee z) = U_1(x, U_1(y, z))$.

Case 2.5 : $(x, y, z) \in A_n \times A_n \times [0, e[$.

It follows from $x \wedge y \in A_n$ and $A_n \subseteq [0,e[^u$ that $U_1(U_1(x,y),z) = U_1(x \wedge y, z) = (x \wedge y) \wedge z = z = U_1(x,z) = U_1(x, y \wedge z) = U_1(x, U_1(y,z))$.

Case 2.6 : $(x,y,z) \in A_n \times A_n \times]e,1]$.
Due to $x \wedge y \in A_n$ and $A_n \subseteq]e,1]^l$, we obtain $U_1(U_1(x,y),z) = U_1(x \wedge y, z) = (x \wedge y) \vee z = z = U_1(x,z) = U_1(x, U_1(y,z))$.

Case 2.7 : $(x,y,z) \in A_0 \times A_i \times [0,e[,\ i = 1,2,\ldots,n-1$.
From the conditions $A_i \subseteq [0,a_i[^u$ and $I_e = \bigoplus_{i=0}^{n} A_{n-i}$, we get $U_1(U_1(x,y),z) = U_1(x \vee y, z) = U_1(x,z) = x = x \vee y = U_1(x,y) = U_1(U_1(x,z),y)$.
If $z \in [0,a_i[$, then it follows from $A_i \subseteq [0,a_i[^u$ that $U_1(x,U_1(y,z)) = U_1(x, y \wedge z) = U_1(x,z) = x$. If $z \in [a_i,e[$, then $U_1(x,U_1(y,z)) = U_1(x,y) = x \vee y = x$.
From above we know $U_1(U_1(x,y),z) = U_1(x,U_1(y,z)) = U_1(U_1(x,z),y)$.

Case 2.8 : $(x,y,z) \in A_0 \times A_n \times [0,e[$.
From the conditions $A_0 \subseteq [0,e[^u$ and $I_e = \bigoplus_{i=0}^{n} A_{n-i}$, we get $U_1(U_1(x,y),z) = U_1(x \vee y, z) = U_1(x,z) = x = U_1(x, y \wedge z) = U_1(x, U_1(y,z))$ and $U_1(U_1(x,z),y) = U_1(x,y) = x \vee y = x$, i. e., $U_1(U_1(x,y),z) = U_1(x,U_1(y,z)) = U_1(U_1(x,z),y)$.

Case 2.9 : $(x,y,z) \in A_0 \times A_i \times]e,1], i = 1,2,\ldots,n$.
From the conditions $A_i \subseteq]e,1]^l$ and $I_e = \bigoplus_{i=0}^{n} A_{n-i}$, we get $U_1(U_1(x,y),z) = U_1(x \vee y, z) = U_1(x,z) = z = U_1(x, y \vee z) = U_1(x, U_1(y,z))$ and $U_1(U_1(x,z),y) = U_1(x \vee z, y) = U_1(z,y) = z$, i. e., $U_1(U_1(x,y),z) = U_1(x,U_1(y,z)) = U_1(U_1(x,z),y)$.

Case 2.10 : $(x,y,z) \in \bigcup_{i=1}^{n-2}(A_i \times (\bigcup_{k=i+1}^{n-1} A_k) \times [0,e[)$.
According to the conditions $A_i \subseteq [0,a_i[^u$ $(i = 1,2,\ldots,n-1)$ and $I_e = \bigoplus_{i=0}^{n} A_{n-i}$, we make the following discussion.
If $z \in [0,a_i[$, then $U_1(U_1(x,y),z) = U_1(x \vee y, z) = U_1(x,z) = x \wedge z = z = y \wedge z = U_1(z,y) = U_1(U_1(x,z),y)$ and $U_1(x,U_1(y,z)) = U_1(x, y \wedge z) = U_1(x,z) = z$, i. e., $U_1(U_1(x,y),z) = U_1(x,U_1(y,z)) = U_1(U_1(x,z),y)$.
If $z \in [a_i,e[$, then $U_1(U_1(x,y),z) = U_1(x \vee y, z) = U_1(x,z) = x = x \vee y = U_1(x,y) = U_1(U_1(x,z),y)$.

If $z \in [a_i,a_k[$, then $U_1(x,U_1(y,z)) = U_1(x, y \wedge z) = U_1(x,z) = x$.

If $z \in [a_k,e[$, then $U_1(x,U_1(y,z)) = U_1(x,y) = x \vee y = x$.

Thus, we have $U_1(U_1(x,y),z) = U_1(x,U_1(y,z)) = U_1(U_1(x,z),y)$.

Case 2.11 : $(x,y,z) \in A_i \times A_n \times [0,e[,\ i = 1,2,\ldots,n-1$.
According to the conditions $A_i \subseteq [0,a_i[^u$ $(i = 1,2,\ldots,n)$ and $I_e = \bigoplus_{i=0}^{n} A_{n-i}$, we discuss as follows.
If $z \in [0,a_i[$, then $U_1(U_1(x,y),z) = U_1(x,z) = z = y \wedge z = U_1(z,y) = U_1(U_1(x,z),y)$ and $U_1(x,U_1(y,z)) = U_1(x, y \wedge z) = U_1(x,z) = z$, i. e., $U_1(U_1(x,y),z) = U_1(x,U_1(y,z)) = U_1(U_1(x,z),y)$.
If $z \in [a_i,e[$, then $U_1(U_1(x,y),z) = U_1(x,z) = x = U_1(x,y) = U_1(U_1(x,z),y)$ and $U_1(x,U_1(y,z)) = U_1(x, y \wedge z) = U_1(x,z) = x$, i. e., we have $U_1(U_1(x,y),z) = U_1(x,U_1(y,z)) = U_1(U_1(x,z),y)$.

Case 2.12 : $(x,y,z) \in \bigcup_{i=1}^{n-1}(A_i \times (\bigcup_{k=i+1}^{n} A_k) \times]e,1])$.

According to the conditions $I_e \subseteq]e,1]^l$ and $I_e = \bigoplus_{i=0}^{n} A_{n-i}$, we have $U_1(U_1(x,y),z) = U_1(x \vee y, z) = U_1(x,z) = x \vee z = z = U_1(x, y \vee z) = U_1(x, U_1(y,z))$ and $U_1(U_1(x,z),y) = U_1(x \vee z, y) = U_1(z,y) = z$, i. e., $U_1(U_1(x,y),z) = U_1(x,U_1(y,z)) = U_1(U_1(x,z),y)$.

Case 3 : All of x, y and z belong to I_e.

Case 3.1 : $(x,y,z) \in A_i \times A_i \times A_i, i = 0,1,\ldots,n$.
If $i = 0$, i. e., $(x,y,z) \in A_0 \times A_0 \times A_0$, then since $x \vee y, y \vee z \in A_0 \cup]e,1]$, we have $U_1(U_1(x,y),z) = U_1(x \vee y, z) = x \vee y \vee z = U_1(x, U_1(y,z))$.
If $i \in \{1,2,\ldots,n-1\}$, then it follows from $x \vee y, y \vee z \in A_i$ that $U_1(U_1(x,y),z) = U_1(x \vee y, z) = x \vee y \vee z = U_1(x,U_1(y,z))$.
If $i = n$, then it follows from $x \wedge y, y \wedge z \in A_n$ that $U_1(U_1(x,y),z) = U_1(x \wedge y, z) = x \wedge y \wedge z = U_1(x, U_1(y,z))$.

Case 3.2 : $(x,y,z) \in A_i \times A_i \times \bigcup_{k=0}^{i-1} A_k, i = 1,2,\ldots,n$.
If $i \in \{1,2,\ldots,n-1\}$, then from $x \vee y \in A_i$ and $I_e = \bigoplus_{i=0}^{n} A_{n-i}$, we have $U_1(U_1(x,y),z) = U_1(x \vee y, z) = x \vee y \vee z = z = U_1(x,z) = U_1(x,U_1(y,z))$.
If $i = n$, then it follows from $x \wedge y \in A_n$ and $I_e = \bigoplus_{i=0}^{n} A_{n-i}$, that $U_1(U_1(x,y),z) = U_1(x \wedge y, z) = (x \wedge y) \vee z = z = U_1(x,z) = U_1(x, y \vee z) = U_1(x,U_1(y,z))$.

Case 3.3 : $(x,y,z) \in A_i \times A_i \times \bigcup_{k=i+1}^{n} A_k, i = 0,1,\ldots,n-1$.
If $i = 0$, then $(x,y,z) \in A_0 \times A_0 \times \bigcup_{k=1}^{n} A_k$. Since $x \vee y \in A_0 \cup]e,1]$, we have $U_1(U_1(x,y),z) = U_1(x \vee y, z) = x \vee y \vee z = x \vee y = U_1(x,y) = U_1(x, y \vee z) = U_1(x, U_1(y,z))$.
If $i \in \{1,2,\ldots,n-1\}$, then from $x \vee y \in A_i$ and $I_e = \bigoplus_{i=0}^{n} A_{n-i}$, we have $U_1(U_1(x,y),z) = U_1(x \vee y, z) = x \vee y \vee z = x \vee y = U_1(x,y) = U_1(x, y \vee z) = U_1(x, U_1(y,z))$.

Case 3.4 : $(x,y,z) \in \bigcup_{i=0}^{n-2} (A_i \times \bigcup_{j=i+1}^{n-1} (A_j \times \bigcup_{k=j+1}^{n} A_k))$.
It follows from $I_e = \bigoplus_{i=0}^{n} A_{n-i}$ that $U_1(U_1(x,y),z) = U_1(x \vee y, z) = U_1(x,z) = x \vee z = x = x \vee y = U_1(x,y) = U_1(x, y \vee z) = U_1(x,U_1(y,z))$ and $U_1(U_1(x,z),y) = U_1(x \vee z, y) = U_1(x,y) = x$. Therefore, $U_1(U_1(x,y),z) = U_1(x,U_1(y,z)) = U_1(U_1(x,z),y)$ holds in this case.

□

Clearly, uninorm U_1 belongs to $\mathcal{U}_\vee \setminus \mathcal{U}_{\max}$. If we add such a restrictive condition in Theorem 3.1: A_0 is closed w.r.t $\vee$, then from above proof we can get the following immediate corollary.

Corollary 3.2. Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and $e \in L \setminus \{0,1\}$. If the following conditions hold,

(*i*) the function S is a t-conorm on $[e,1]$ and the ordinal sum $T = \{< a_{i-1}, a_i, T_i >\}_{i \in \{1,2,\ldots,n\}}$ of $\{T_i\}_{i \in \{1,2,\ldots,n\}}$ is a t-norm on $[0,e]$, where $[0,e] = \bigoplus_{i=1}^{n} [a_{i-1}, a_i]$ and T_i is a t-norm on $[a_{i-1}, a_i]$ for each $i \in \{1,2,\ldots,n\}$;

(*ii*) $I_e \subseteq [0,e[^u \cap]e,1]^l$ and $I_e = \bigoplus_{i=0}^{n} A_{n-i}$ with $A_i \cap A_j = \emptyset$ $(i \neq j)$ for all $i, j \in \{0,1,\ldots,n\}$, where A_i is closed w.r.t the operation $\vee$ for each $i \in \{0,1,\ldots,n-1\}$ and A_n is closed w.r.t the operation $\wedge$, then the function $U_1 : L^2 \to L$ defined by formula (2) is a uninorm with neutral element e.

In order to facilitate comparison with existing conclusions in the literature, special emphasis is placed here on the case of $n = 1$ in Theorem 3.1.

Corollary 3.3. Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0, 1\}$, T and S be t-norm and t-conorm on $[0, e]$ and $[e, 1]$ respectively. Assume that, $I_e = A_1 \oplus A_0$ and $A_0 \cap A_1 = \emptyset$. If the following conditions hold,

(*i*) $A_1 \subseteq [0, e[^u, A_0 \subseteq]e, 1]^l$;

(*ii*) $x \wedge y \in A_1$ for all $x, y \in A_1$;

(*iii*) sublattice $[e, 1]$ has a unique atom e' such that $S(e', z) = z$ for all $z \in]e, 1]$,

then the function $U_{1.1} : L^2 \to L$ defined by formula (3) is a uninorm with neutral element e.

$$U_{1.1}(x, y) = \begin{cases} T(x, y) & \text{if } (x, y) \in [0, e]^2, \\ S(x, y) & \text{if } (x, y) \in [e, 1]^2, \\ x \wedge y & \text{if } (x, y) \in A_1 \times [0, e[\cup [0, e[\times A_1 \cup A_1^2, \\ x & \text{if } (x, y) \in A_0 \times [0, e[\cup I_e \times \{e\}, \\ y & \text{if } (x, y) \in [0, e[\times A_0 \cup \{e\} \times I_e. \\ x \vee y & \text{otherwise.} \end{cases} \tag{3}$$

The following corollary indicates that if the conditions in Corollary 3.3 are slightly changed, the function $U_{1.1}$ defined by (3) can be still guaranteed to be a uninorm on L.

Corollary 3.4. Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0, 1\}$, T and S be t-norm and t-conorm on $[0, e]$ and $[e, 1]$ respectively. Assume that, $I_e = A_1 \oplus A_0$ and $A_0 \cap A_1 = \emptyset$. If the following conditions hold,

(*i*) $A_1 \subseteq [0, e[^u, A_0 \subseteq]e, 1]^l$;

(*ii*) sublattice $[e, 1]$ has a unique atom e' such that $S(e', z) = z$ for all $z \in]e, 1]$; sublattice $[0, e]$ has a unique coatom e'' such that $T(e'', z) = z$ for all $z \in [0, e[$,

then the function $U_{1.1} : L^2 \to L$ defined by formula (3) is a uninorm with neutral element e.

Proof. It is easy to know from formula (3) that $U_{1.1}$ is commutative and its neutral element is e. The proof of monotonicity is completely similar to the proof of Theorem 3.1. The following only proves the associativity of $U_{1.1}$.

For any $x, y, z \in L$, we need to prove $U_{1.1}(U_{1.1}(x, y), z) = U_{1.1}(x, U_{1.1}(y, z))$. Firstly, if one of x, y and z equals e, then the equation clearly holds. In addition, since the case of $x, y, z \notin I_e$ is direct, we let at least one of x, y and z belongs to I_e in our following discussion.

From the assumptions $I_e = A_1 \oplus A_0$, $A_1 \subseteq [0, e[^u$ and $A_0 \subseteq]e, 1]^l$, we know $I_e \subseteq [0, e[^u$ and $I_e \subseteq]e, 1]^l$.

Case 1 : One of x, y and z belongs to I_e.

The proof is same as Theorem 3.1, so it is omitted.

Case 2 : Two of x, y and z belong to I_e. Without loss of generality, we let $x, y \in I_e$.

For the subcases of $(x, y, z) \in A_0 \times A_i \times ([0, e[\cup]e, 1]),\ i = 0, 1$, the proof is same as Theorem 3.1. The following is to prove other subcases.

Case 2.1 : $(x, y, z) \in A_1 \times A_1 \times [0, e[$.

The condition $I_e = A_1 \oplus A_0$ can lead to such a fact $x \wedge y \in A_1 \cup [0, e[$ for all $x, y \in A_1$. So we now need to make the following discussion.

If $x \wedge y \in A_1$, then it follows from $A_1 \subseteq [0, e[^u$ that $U_{1.1}(U_{1.1}(x, y), z) = U_{1.1}(x \wedge y, z) = (x \wedge y) \wedge z = z$.

If $x \wedge y \in [0, e[$, then $U_{1.1}(U_{1.1}(x, y), z) = U_{1.1}(x \wedge y, z) = T(x \wedge y, z)$. It follows from $A_1 \subseteq [0, e[^u$ that $x > t, y > t$, i. e., $x \wedge y \geq t$ for all $t \in [0, e[$. According to condition (ii) we know that $x \wedge y = \bigvee_{t \in [0,e[} t = e''$ and $T(x \wedge y, z) = z$, i. e., we obtain $U_{1.1}(U_{1.1}(x, y), z) = z$.

Also due to $U_{1.1}(x, U_{1.1}(y, z)) = U_{1.1}(x, y \wedge z) = U_{1.1}(x, z) = z$, we get

$$U_{1.1}(U_{1.1}(x, y), z) = U_{1.1}(x, U_{1.1}(y, z)).$$

Case 2.2 : $(x, y, z) \in A_1 \times A_1 \times]e, 1]$.

If $x \wedge y \in A_1$, then it follows from $A_1 \subseteq]e, 1]^l$ that $U_{1.1}(U_{1.1}(x, y), z) = U_{1.1}(x \wedge y, z) = (x \wedge y) \vee z = z$.

If $x \wedge y \in [0, e[$, then $U_{1.1}(U_{1.1}(x, y), z) = U_{1.1}(x \wedge y, z) = (x \wedge y) \vee z = z$.

Also due to $U_{1.1}(x, U_{1.1}(y, z)) = U_{1.1}(x, y \vee z) = U_{1.1}(x, z) = z$, we get

$$U_{1.1}(U_{1.1}(x, y), z) = U_{1.1}(x, U_{1.1}(y, z)).$$

Case 3 : All of x, y and z belong to I_e.

For the subcases of $(x, y, z) \in A_0 \times A_0 \times A_i,\ i = 0, 1$, the proof is same as Theorem 3.1. We now prove other subcases.

Case 3.1 : $(x, y, z) \in A_1 \times A_1 \times A_1$.

It follows from $x \wedge y \in A_1 \cup [0, e[$ for all $x, y \in A_1$ and $U_{1.1} = \wedge$ on $A_1 \times ([0, e[\cup A_1)$ that $U_{1.1}(U_{1.1}(x, y), z) = U_{1.1}(x \wedge y, z) = x \wedge y \wedge z = U_{1.1}(x, U_{1.1}(y, z))$.

Case 3.2 : $(x, y, z) \in A_1 \times A_1 \times A_0$.

If $x \wedge y \in A_1$, then it follows from $I_e = A_1 \oplus A_0$ that $U_{1.1}(U_{1.1}(x, y), z) = U_{1.1}(x \wedge y, z) = (x \wedge y) \vee z = z$.

If $x \wedge y \in [0, e[$, then $U_{1.1}(U_{1.1}(x, y), z) = U_{1.1}(x \wedge y, z) = z$.

Also due to $U_{1.1}(x, U_{1.1}(y, z)) = U_{1.1}(x, y \vee z) = U_{1.1}(x, z) = z$, we know $U_{1.1}(U_{1.1}(x, y), z) = U_{1.1}(x, U_{1.1}(y, z))$ holds in this case.

□

If $A_1 = \emptyset$ is taken in Corollary 3.3 or 3.4, then the following corollary can be implied.

Corollary 3.5. Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0,1\}$, T and S be t-norm and t-conorm on $[0,e]$ and $[e,1]$ respectively. If the following conditions hold,

(*i*) $I_e \subseteq]e,1]^l$;

(*ii*) sublattice $[e,1]$ has a unique atom e' such that $S(e',z) = z$ for all $z \in]e,1]$,

then the function $U_{1.2} : L^2 \to L$ defined by formula (4) is a uninorm with neutral element e.

$$U_{1.2}(x,y) = \begin{cases} T(x,y) & \text{if } (x,y) \in [0,e]^2, \\ S(x,y) & \text{if } (x,y) \in [e,1]^2, \\ x & \text{if } (x,y) \in I_e \times [0,e], \\ y & \text{if } (x,y) \in [0,e] \times I_e. \\ x \vee y & \text{otherwise.} \end{cases} \tag{4}$$

Obviously, $U_{1.2}$ belongs to $\mathcal{U}_{\max}$.

It should be noted that if $A_1 = \emptyset$ is taken in Corollary 3.4, it can be inferred from the proof that condition "sublattice $[0,e]$ has a unique coatom e'' such that $T(e'',z) = z$ for all $z \in [0,e[$" in Corollary 3.4 is redundant.

Remark 3.6. (i) If $S = \vee$ is taken in Corollary 3.5, then the conditions (i) and (ii) in Corollary 3.5 are obviously redundant. At this point, $U_{1.2}$ is the U_t provided in [4] (see Theorem 1 of [4]), and is also the uninorm when $\Uparrow(x) = x$ is taken in [20] (see Theorem 4.1 of [20]).

(ii) If $x \vee y \in I_e$ for all $x, y \in I_e$ is restricted in Corollary 3.5, then the condition (ii) becomes redundant. At this point, $U_{1.2}$ is the uninorm obtained in [11] (see Corollary 3.2 of [11]).

If we take $A_0 = \emptyset$ in Corollary 3.3, then the following corollary can be ensured.

Corollary 3.7. Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0,1\}$, T and S be t-norm and t-conorm on $[0,e]$ and $[e,1]$ respectively. If the following conditions hold,

(*i*) $I_e \subseteq [0,e[^u, I_e \subseteq]e,1]^l$;

(*ii*) $x \wedge y \in I_e$ for all $x, y \in I_e$,

then the function $U_{1.3} : L^2 \to L$ defined by formula (5) is a uninorm with neutral element e.

$$U_{1.3}(x,y) = \begin{cases} T(x,y) & \text{if } (x,y) \in [0,e]^2, \\ S(x,y) & \text{if } (x,y) \in [e,1]^2, \\ x \wedge y & \text{if } (x,y) \in I_e \times [0,e[\cup [0,e[\times I_e \cup I_e^2, \\ x & \text{if } (x,y) \in I_e \times \{e\}, \\ y & \text{if } (x,y) \in \{e\} \times I_e. \\ x \vee y & \text{otherwise.} \end{cases} \tag{5}$$

It is noted that the method proposed in this corollary is same as the method in [5] (see Theorem 10 of [5]) and it is also the method when $int(x) = x$ is taken in [30] (see Corollary 4.7 of [30]).

If we take $A_0 = \emptyset$ in Corollary 3.4, then the following corollary can be inferred and it is also the method provided in [25] (see Theorem 6 of [25]).

Corollary 3.8. Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0, 1\}$, T and S be a t-norm and a t-conorm on $[0, e]$ and $[e, 1]$ respectively. If the following conditions hold,

(*i*) $I_e \subseteq [0, e[^u, I_e \subseteq]e, 1]^l$;

(*ii*) sublattice $[0, e]$ has a unique coatom e'' such that $T(e'', z) = z$ for all $z \in [0, e[$,

then the function $U_{1.3} : L^2 \to L$ defined by formula (5) is a uninorm with neutral element e .

Example 3.9. Consider the lattice L_1 drawn in Figure 2. A t-norm T on $[0, e]$ is taken as $T = \{< a_{i-1}, a_i, T_i >\}_{i \in \{1,2,3\}}$, where t-norms T_1, T_2 and T_3 are as follows:

$$T_1(x, y) = \begin{cases} x \wedge y & \text{if } a_1 \in \{x, y\}, \\ x \wedge y \wedge g_1 & \text{otherwise}, \end{cases} \tag{6}$$

$$T_2(x, y) = \begin{cases} x \wedge y & \text{if } a_2 \in \{x, y\}, \\ a_1 & \text{otherwise}, \end{cases} \tag{7}$$

$$T_3(x, y) = \begin{cases} x \wedge y & \text{if } e \in \{x, y\}, \\ a_2 & \text{otherwise}. \end{cases} \tag{8}$$

A t-conorm S on $[e, 1]$ is defined by

$$S(x, y) = \begin{cases} x \vee y & \text{if } e, e' \in \{x, y\}, \\ 1 & \text{otherwise}. \end{cases} \tag{9}$$

From Figure 2 we know $I_e = A_3 \oplus A_2 \oplus A_1 \oplus A_0$, where $A_0 = \{c_1, c_2, c_3\}, A_1 = \{d_1, d_2\}$, $A_2 = \{e_1, e_2\}, A_3 = \{f_1, f_2\}$. It is easy to verify that the conditions in Theorem 3.1 are met. By using the formula (2), the function U defined by Table 1 is a uninorm on L_1 having neutral element e.

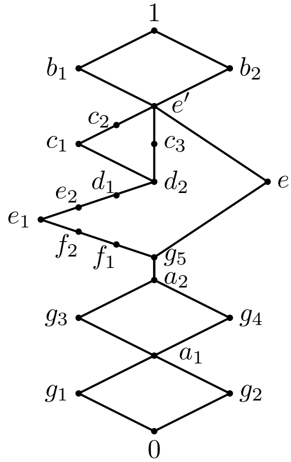

Fig. 2. The lattice L_1 in Example 3.9.

U	0	g_1	g_2	a_1	g_3	g_4	a_2	g_5	e	e'	b_1	b_2	1	c_1	c_2	c_3	d_1	d_2	e_1	e_2	f_1	f_2
0	0	0	0	0	0	0	0	0	0	e'	b_1	b_2	1	c_1	c_2	c_3	0	0	0	0	0	0
g_1	0	g_1	0	g_1	g_1	g_1	g_1	g_1	g_1	e'	b_1	b_2	1	c_1	c_2	c_3	g_1	g_1	g_1	g_1	g_1	g_1
g_2	0	0	0	g_2	g_2	g_2	g_2	g_2	g_2	e'	b_1	b_2	1	c_1	c_2	c_3	g_2	g_2	g_2	g_2	g_2	g_2
a_1	0	g_1	g_2	a_1	a_1	a_1	a_1	a_1	a_1	e'	b_1	b_2	1	c_1	c_2	c_3	d_1	d_2	a_1	a_1	a_1	a_1
g_3	0	g_1	g_2	a_1	a_1	a_1	g_3	g_3	g_3	e'	b_1	b_2	1	c_1	c_2	c_3	d_1	d_2	g_3	g_3	g_3	g_3
g_4	0	g_1	g_2	a_1	a_1	a_1	g_4	g_4	g_4	e'	b_1	b_2	1	c_1	c_2	c_3	d_1	d_2	g_4	g_4	g_4	g_4
a_2	0	g_1	g_2	a_1	g_3	g_4	a_2	a_2	a_2	e'	b_1	b_2	1	c_1	c_2	c_3	d_1	d_2	e_1	e_2	a_2	a_2
g_5	0	g_1	g_2	a_1	g_3	g_4	a_2	a_2	g_5	e'	b_1	b_2	1	c_1	c_2	c_3	d_1	d_2	e_1	e_2	g_5	g_5
e	0	g_1	g_2	a_1	g_3	g_4	a_2	g_5	e	e'	b_1	b_2	1	c_1	c_2	c_3	d_1	d_2	e_1	e_2	f_1	f_2
e'	e'	e'	e'	e'	e'	e'	e'	e'	e'	e'	b_1	b_2	1	e'	e'	e'	e'	e'	e'	e'	e'	e'
b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	1	1	1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1
b_2	b_2	b_2	b_2	b_2	b_2	b_2	b_2	b_2	b_2	b_2	1	1	1	b_2	b_2	b_2	b_2	b_2	b_2	b_2	b_2	b_2
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
c_1	c_1	c_1	c_1	c_1	c_1	c_1	c_1	c_1	c_1	e'	b_1	b_2	1	c_1	c_2	e'	c_1	c_1	c_1	c_1	c_1	c_1
c_2	c_2	c_2	c_2	c_2	c_2	c_2	c_2	c_2	c_2	e'	b_1	b_2	1	c_2	c_2	e'	c_2	c_2	c_2	c_2	c_2	c_2
c_3	c_3	c_3	c_3	c_3	c_3	c_3	c_3	c_3	c_3	e'	b_1	b_2	1	e'	e'	c_3	c_3	c_3	c_3	c_3	c_3	c_3
d_1	0	g_1	g_2	d_1	d_1	d_1	d_1	d_1	d_1	e'	b_1	b_2	1	c_1	c_2	c_3	d_1	d_2	d_1	d_1	d_1	d_1
d_2	0	g_1	g_2	d_2	d_2	d_2	d_2	d_2	d_2	e'	b_1	b_2	1	c_1	c_2	c_3	d_2	d_2	d_2	d_2	d_2	d_2
e_1	0	g_1	g_2	a_1	g_3	g_4	e_1	e_1	e_1	e'	b_1	b_2	1	c_1	c_2	c_3	d_1	d_2	e_1	e_2	e_1	e_1
e_2	0	g_1	g_2	a_1	g_3	g_4	e_2	e_2	e_2	e'	b_1	b_2	1	c_1	c_2	c_3	d_1	d_2	e_2	e_2	e_2	e_2
f_1	0	g_1	g_2	a_1	g_3	g_4	a_2	g_5	f_1	e'	b_1	b_2	1	c_1	c_2	c_3	d_1	d_2	e_1	e_2	f_1	f_2
f_2	0	g_1	g_2	a_1	g_3	g_4	a_2	g_5	f_2	e'	b_1	b_2	1	c_1	c_2	c_3	d_1	d_2	e_1	e_2	f_2	f_2

Tab. 1. The uninorm U in Example 3.9

The following example shows that the conditions in Theorem 3.1 are indispensable.

Example 3.10.

(i) We now show that the function U given by formula (2) may not necessarily be a uninorm if the t-norm T on $[0, e]$ is not in the form of ordinal sums.

Consider the lattice L_2 drawn in Figure 3. Assume that a t-norm T on $[0, e]$ is defined by the following:

$$T(x, y) = \begin{cases} x \wedge y & \text{if } e \in \{x, y\}, \\ 0 & \text{otherwise}, \end{cases} \tag{10}$$

and S is a t-conorm on $[e, 1]$ such that $S(e', z) = z$ for all $z \in]e, 1]$. It follows from Figure 3 that the other conditions in Theorem 3.1 are satisfied, where $A_0 = \{c\}, A_1 = \{d\}, A_2 = \{f\}$. But the function U defined by the formula (2) is not a uninorm on L_2, because $U(U(d, g_3), g_1) = U(d, g_1) = d \wedge g_1 = g_1$, while $U(d, U(g_3, g_1)) = U(d, 0) = 0$.

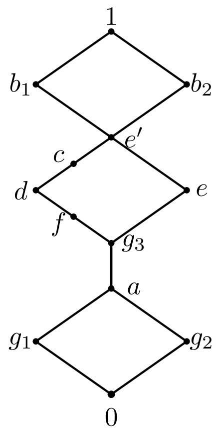

Fig. 3. The lattice L_2 in Example 3.10 (*i*)

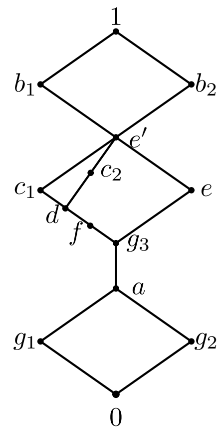

Fig. 4. The lattice L_3 in Example 3.10 (*ii*)

(*ii*) The condition "$S(e', z) = z$ for all $z \in]e, 1]$" in Theorem 3.1 cannot be removed.

Consider the lattice L_3 drawn in Figure 4. Assume that $T = \{< 0, a, T_1 >, < a, e, T_2 >\}$ is a t-norm on $[0, e]$, where T_1 and T_2 are t-norms on $[0, a]$ and $[a, e]$ respectively; and a t-conorm S on $[e, 1]$ is defined by

$$S(x, y) = \begin{cases} x \vee y & \text{if } e \in \{x, y\}, \\ 1 & \text{otherwise}. \end{cases} \tag{11}$$

From Figure 4 we know that other conditions in Theorem 3.1 are satisfied, where $A_0 = \{c_1, c_2\}, A_1 = \{d\}, A_2 = \{f\}$. The function U defined by the formula (2) is not a uninorm on L_3, because $U(U(c_1, c_2), b_1) = U(e', b_1) = 1$, while $U(c_1, U(c_2, b_1)) = U(c_1, c_2 \vee b_1) = U(c_1, b_1) = b_1$.

(*iii*) The condition "$I_e = \bigoplus_{i=0}^{n} A_{n-i}$ with $A_i \cap A_j = \emptyset$ $(i \neq j)$" in Theorem 3.1 cannot be omitted.

If we take $A_1 = \{d_1, e_1\}$ and $A_2 = \{d_2, e_2\}$ in Example 3.9, then from Figure 2 we know that $I_e = \cup_{i=0}^{3} A_{3-i}$ but $I_e \neq \bigoplus_{i=0}^{3} A_{3-i}$ because $e_1 < e_2$. The function U is defined by formula (2), then by the monotonicity of U we have $e_1 = U(e_1, g_3) \leq U(e_2, g_3) = e_2 \wedge g_3 = g_3$ which is contradiction.

(iv) The condition "$I_e \subseteq [0, e[^u$" in Theorem 3.1 cannot be deleted.

Consider the lattice L_4 drawn in Figure 5. Assume that a t-norm T on $[0, e]$ is taken as an ordinal sum $T = \{< 0, a_1, T_1 >, < a_1, a_2, T_2 >, < a_2, e, T_3 >\}$, where T_1 is defined by the following:

$$T_1(x, y) = \begin{cases} x \wedge y & \text{if } a_1 \in \{x, y\}, \\ 0 & \text{otherwise}, \end{cases} \tag{12}$$

T_2 and T_3 are t-norms on $[a_1, a_2]$ and $[a_2, e]$ respectively; while S is a t-conorm on $[e, 1]$ such that $S(e', z) = z$ for all $z \in]e, 1]$. It follows from Figure 5 that $I_e \nsubseteq [0, e[^u$ and other conditions in Theorem 3.1 are satisfied, where $A_0 = \{c\}, A_1 = \{d\}, A_2 = \{f\}$ and $A_3 = \{h\}$. We get that the function U defined by the formula (2) is not a uninorm on L_4, because $U(U(f, g_3), g_1) = U(f \wedge g_3, g_1) = T_1(g_1, g_1) = 0$, while $U(f, U(g_3, g_1)) = U(f, g_1) = g_1$.

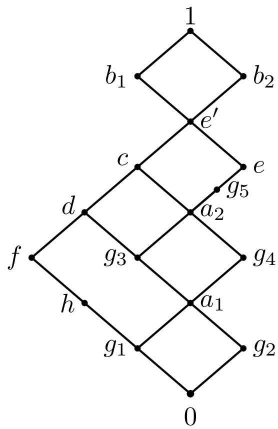

Fig. 5. The lattice L_4 in Example 3.10 (iv)

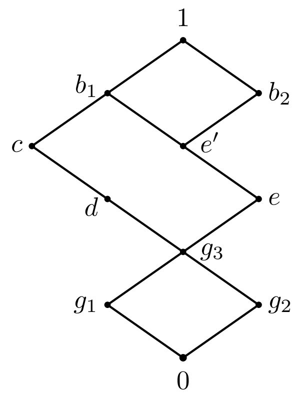

Fig. 6. The lattice L_5 in Example 3.10 (v)

(v) The condition "$I_e \subseteq]e, 1]^l$" in Theorem 3.1 cannot be deleted.

Consider the lattice L_5 drawn in Figure 6. Assume that T is a t-norm on $[0, e]$ and S is a t-conorm on $[e, 1]$ such that $S(e', z) = z$ for all $z \in]e, 1]$. It follows from Figure 6 that $I_e \nsubseteq]e, 1]^l$ and other conditions in Theorem 3.1 are satisfied, where $A_0 = \{c\}$ and $A_1 = \{d\}$. The function U defined by the formula (2) is not a uninorm on L_5, because $U(U(d, g_3), e') = U(d \wedge g_3, e') = U(g_3, e') = e'$, while $U(d, U(g_3, e')) = U(d, e') = d \vee e' = b_1$.

(vi) The condition "A_i is closed w.r.t the operation $\vee$ for each $i \in \{1, 2, \ldots, n-1\}$" in Theorem 3.1 cannot be removed.

Consider the lattice L_6 drawn in Figure 7. Assume that a t-norm T on $[0, e]$ is taken as an ordinal sum $T = \{< 0, a, T_1 >, < a, e, T_2 >\}$, where T_1 and T_2 are t-norms on $[0, a]$ and $[a, e]$ respectively; while S is a t-conorm on $[e, 1]$ such that $S(e', z) = z$ for all $z \in]e, 1]$. From Figure 7 we know that $f_1 \vee f_2 \notin A_1$ for $f_1, f_2 \in A_1$ and other conditions in Theorem 3.1 are satisfied, where $A_0 = \{c\}, A_1 = \{f_1, f_2\}$ and $A_2 = \{d\}$. The function U defined by the formula (2) is not a uninorm on L_6, because $U(U(f_1, f_2), g_1) = U(f_1 \vee f_2, g_1) = U(c, g_1) = c$, while $U(f_1, U(f_2, g_1)) = U(f_1, g_1) = g_1$.

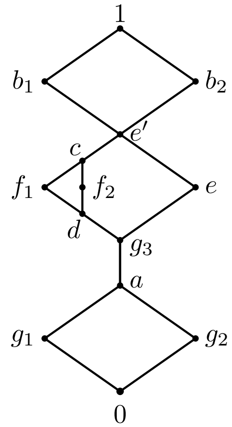

Fig. 7. The lattice L_6 in Example 3.10 (*vi*)

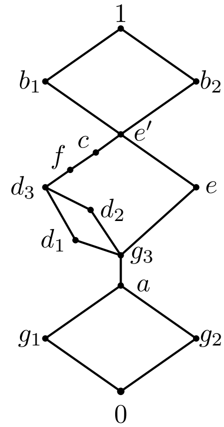

Fig. 8. The lattice L_7 in Example 3.10 (*vii*)

(*vii*) The condition "A_n is closed w.r.t the operation $\wedge$" in Theorem 3.1 cannot be omitted.

Consider the lattice L_7 drawn in Figure 8. Assume that a t-norm T on $[0, e]$ is taken as an ordinal sum $T = \{< 0, a, T_1 >, < a, e, T_2 >\}$, where T_1 and T_2 are t-norms on $[0, a]$ and $[a, e]$ respectively, and

$$T_2(x, y) = \begin{cases} x \wedge y & \text{if } e \in \{x, y\}, \\ a & \text{otherwise}; \end{cases} \tag{13}$$

while S is a t-conorm on $[e, 1]$ such that $S(e', z) = z$ for all $z \in]e, 1]$. From Figure 8 we know that $d_1 \wedge d_2 \notin A_2$ for $d_1, d_2 \in A_2$ and other conditions in Theorem 3.1 are satisfied, where $A_0 = \{c\}, A_1 = \{f\}$ and $A_2 = \{d_1, d_2, d_3\}$. The function U defined by the formula (2) is not a uninorm on L_7, because $U(U(d_1, d_2), g_3) = U(d_1 \wedge d_2, g_3) = T_2(g_3, g_3) = a$, while $U(d_1, U(d_2, g_3)) = U(d_1, g_3) = g_3$.

4. CONSTRUCTIONS OF UNINORMS WITH ORDINAL SUM UNDERLYING T-CONORMS

Completely similar to the discussion in the previous section, we will in this section provide a method for constructing uninorms with ordinal sum underlying t-conorms.

Theorem 4.1. Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and $e \in L \backslash \{0, 1\}$. If the following conditions hold,

(*i*) function T is a t-norm on $[0, e]$ and the sublattice $[0, e]$ has a unique coatom e'' such that $T(e'', z) = z$ for all $z \in [0, e[$;

(*ii*) the ordinal sum $S = \{< b_{i-1}, b_i, S_i >\}_{i\in\{1,2,\ldots,n\}}$ of $\{S_i\}_{i\in\{1,2,\ldots,n\}}$ is a t-conorm on $[e,1]$, where $[e,1] = \bigoplus_{i=1}^{n}[b_{i-1}, b_i]$ and S_i is a t-conorm on $[b_{i-1}, b_i]$ for each $i \in \{1,2,\ldots,n\}$;

(*iii*) $I_e \subseteq [0,e[^u \cap]e,1]^l$ and $I_e = \bigoplus_{i=0}^{n} B_{n-i}$ with $B_i \cap B_j = \emptyset$ $(i \neq j)$ for all $i,j \in \{0,1,\ldots,n\}$, where B_i is closed w.r.t the operation $\wedge$ for each $i \in \{1,2,\ldots,n-1\}$ and B_0 is closed w.r.t the operation $\vee$,

then the function $U_2 : L^2 \to L$ defined by the following is a uninorm with neutral element e.

$$U_2(x,y) = \begin{cases} T(x,y) & \text{if } (x,y) \in [0,e]^2, \\ S(x,y) & \text{if } (x,y) \in [e,1]^2, \\ x \vee y & \text{if } (x,y) \in (\bigcup_{i=0}^{n-1} B_i \times]b_i,1]) \cup (\bigcup_{i=0}^{n-1}]b_i,1] \times B_i) \cup B_0^2, \\ x & \text{if } (x,y) \in (\bigcup_{i=1}^{n} B_i \times]e,b_i]) \cup I_e \times \{e\}, \\ y & \text{if } (x,y) \in (\bigcup_{i=1}^{n}]e,b_i] \times B_i) \cup \{e\} \times I_e, \\ x \wedge y & \text{otherwise.} \end{cases} \tag{14}$$

The proof is similar to that of Theorem 3.1, so we omit it.

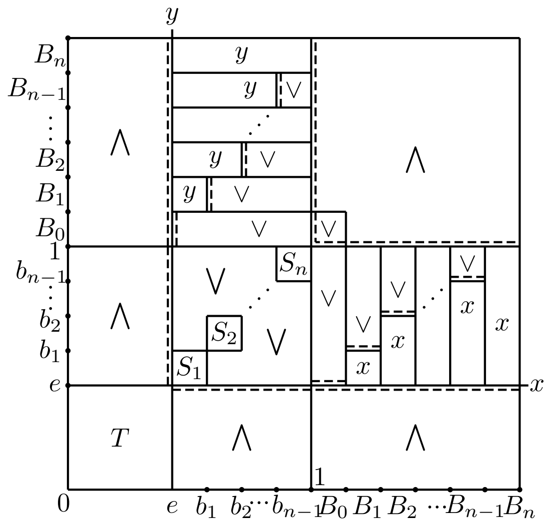

Fig. 9. The uninorm U_2 in Theorem 4.1.

Clearly, uninorm U_2 belongs to $\mathcal{U}_\wedge \setminus \mathcal{U}_{\min}$. If we add such a restrictive condition in Theorem 4.1: B_n is closed w.r.t $\wedge$, then we have the following immediate corollary.

Corollary 4.2. Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and $e \in L \setminus \{0,1\}$. If the following conditions hold,

(i) the function T is a t-norm on $[0,e]$ and the ordinal sum $S = \{< b_{i-1}, b_i, S_i >\}_{i\in\{1,2,\ldots,n\}}$ of $\{S_i\}_{i\in\{1,2,\ldots,n\}}$ is a t-conorm on $[e,1]$, where $[e,1] = \bigoplus_{i=1}^{n}[b_{i-1}, b_i]$ and S_i is a t-conorm on $[b_{i-1}, b_i]$ for each $i \in \{1,2,\ldots,n\}$;

(ii) $I_e \subseteq [0,e[^u \cap]e,1]^l$ and $I_e = \bigoplus_{i=0}^{n} B_{n-i}$ with $B_i \cap B_j = \emptyset$ $(i \neq j)$ for all $i,j \in \{0,1,\ldots,n\}$, where B_i is closed w.r.t the operation $\wedge$ for each $i \in \{1,2,\ldots,n\}$ and B_0 is closed w.r.t the operation $\vee$,

then the function $U_2 : L^2 \to L$ defined by formula (14) is a uninorm with neutral element e.

We here consider the case of $n = 1$ in Theorem 4.1 for the sake of facilitating comparison with the results in literature.

Corollary 4.3. Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0,1\}$, T and S be t-norm and t-conorm on $[0,e]$ and $[e,1]$ respectively. Assume that, $I_e = B_1 \oplus B_0$ and $B_0 \cap B_1 = \emptyset$. If the following conditions hold,

(i) $B_1 \subseteq [0,e[^u, B_0 \subseteq]e,1]^l$;

(ii) $x \vee y \in B_0$ for all $x, y \in B_0$;

(iii) sublattice $[0,e]$ has a unique coatom e'' such that $T(e'', z) = z$ for all $z \in [0,e[$,

then the function $U_{2.1} : L^2 \to L$ defined by the following formula is a uninorm with neutral element e.

$$U_{2.1}(x,y) = \begin{cases} T(x,y) & \text{if } (x,y) \in [0,e]^2, \\ S(x,y) & \text{if } (x,y) \in [e,1]^2, \\ x \vee y & \text{if } (x,y) \in B_0 \times]e,1] \cup]e,1] \times B_0 \cup B_0^2, \\ x & \text{if } (x,y) \in B_1 \times]e,1] \cup I_e \times \{e\}, \\ y & \text{if } (x,y) \in]e,1] \times B_1 \cup \{e\} \times I_e. \\ x \wedge y & \text{otherwise.} \end{cases} \tag{15}$$

The function $U_{2.1}$ can be still guaranteed to be a uninorm on L if we make appropriate changes for the conditions in Corollary 4.3.

Corollary 4.4. Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0,1\}$, T and S be t-norm and t-conorm on $[0,e]$ and $[e,1]$ respectively. Assume that, $I_e = B_1 \oplus B_0$ and $B_0 \cap B_1 = \emptyset$. If the following conditions hold,

(i) $B_1 \subseteq [0,e[^u, B_0 \subseteq]e,1]^l$;

(ii) sublattice $[e,1]$ has a unique atom e' such that $S(e', z) = z$ for all $z \in]e,1]$; sublattice $[0,e]$ has a unique coatom e'' such that $T(e'', z) = z$ for all $z \in [0,e[$,

then the function $U_{2.1} : L^2 \to L$ defined by formula (15) is a uninorm with neutral element e.

If we take $B_0 = \emptyset$ in Corollary 4.3 or 4.4, the following corollary can be obtained.

Corollary 4.5. Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0, 1\}$, T and S be t-norm and t-conorm on $[0, e]$ and $[e, 1]$ respectively. If the following conditions hold,

(*i*) $I_e \subseteq [0.e[^u$;

(*ii*) sublattice $[0, e]$ has a unique coatom e'' such that $T(e'', z) = z$ for all $z \in [0, e[$,

then the function $U_{2.2} : L^2 \to L$ defined by formula (16) is a uninorm with neutral element e.

$$U_{2.2}(x, y) = \begin{cases} T(x, y) & \text{if } (x, y) \in [0, e]^2, \\ S(x, y) & \text{if } (x, y) \in [e, 1]^2, \\ x & \text{if } (x, y) \in I_e \times [e, 1], \\ y & \text{if } (x, y) \in [e, 1] \times I_e. \\ x \wedge y & \text{otherwise.} \end{cases} \tag{16}$$

Obviously, $U_{2.2}$ belongs to $\mathcal{U}_{\min}$.

Remark 4.6. (i) If $T = \wedge$ is taken in Corollary 4.5, then the conditions (i) and (ii) in Corollary 4.5 are obviously redundant. At this point, $U_{2.2}$ is the U_s provided in [4] (see Theorem 1 of [4]), and is also the uninorm when $\Downarrow (x) = x$ is taken in [20] (see Theorem 5.6 of [20]).

(ii) If $x \wedge y \in I_e$ for all $x, y \in I_e$ is restricted in Corollary 4.5, then the condition (ii) becomes redundant. $U_{2.2}$ at this point is the uninorm obtained in [11] (see Corollary 3.4 of [11]).

If we take $B_1 = \emptyset$ in Corollary 4.3, then the condition (iii) in Corollary 4.3 becomes redundant. Thus, the following corollary can be implied and it is also the method in [5] (see Theorem 7 of [5]).

Corollary 4.7. Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0, 1\}$, T and S be t-norm and t-conorm on $[0, e]$ and $[e, 1]$ respectively. If the following conditions hold,

(*i*) $I_e \subseteq [0, e[^u, I_e \subseteq]e, 1]^l$;

(*ii*) $x \vee y \in I_e$ for all $x, y \in I_e$,

then the function $U_{2.3} : L^2 \to L$ defined by formula (17) is a uninorm with neutral element e.

$$U_{2.3}(x, y) = \begin{cases} T(x, y) & \text{if } (x, y) \in [0, e]^2, \\ S(x, y) & \text{if } (x, y) \in [e, 1]^2, \\ x \vee y & \text{if } (x, y) \in I_e \times]e, 1] \cup]e, 1] \times I_e \cup I_e^2, \\ x & \text{if } (x, y) \in I_e \times \{e\}, \\ y & \text{if } (x, y) \in \{e\} \times I_e. \\ x \wedge y & \text{otherwise.} \end{cases} \tag{17}$$

If we take $B_1 = \emptyset$ in Corollary 4.4, then the following corollary can be ensured and it is also the method provided in [25] (see Corollary 1 of [25]).

Corollary 4.8. Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0, 1\}$, T and S be t-norm and t-conorm on $[0, e]$ and $[e, 1]$ respectively. If the following conditions hold,

(*i*) $I_e \subseteq [0, e[^u, I_e \subseteq]e, 1]^l$;

(*ii*) sublattice $[e, 1]$ has a unique atom e' such that $S(e', z) = z$ for all $z \in]e, 1]$,

then the function $U_{2.3} : L^2 \to L$ defined by formula (17) is a uninorm with neutral element e .

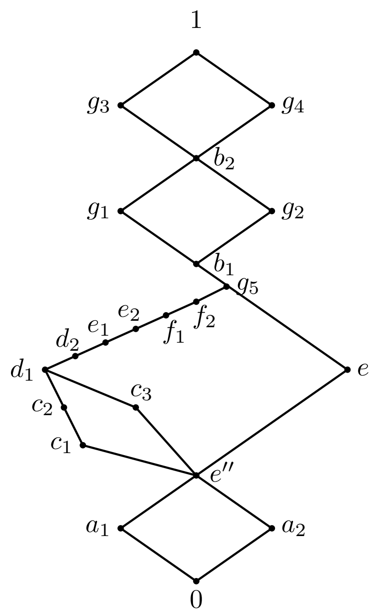

Fig. 10. The lattice L_8 in Example 4.9.

Example 4.9. Consider the lattice L_8 drawn in Figure 10. A t-conorm S on $[e, 1]$ is taken as $S = \{< b_{i-1}, b_i, S_i >\}_{i \in \{1,2,3\}}$, where t-conorms S_1, S_2 and S_3 are as follows:

$$S_1(x, y) = \begin{cases} x \vee y & \text{if } e \in \{x, y\}, \\ b_1 & \text{otherwise,} \end{cases} \tag{18}$$

$$S_2(x, y) = \begin{cases} x \vee y & \text{if } b_1 \in \{x, y\}, \\ b_2 & \text{otherwise,} \end{cases} \tag{19}$$

$$S_3(x, y) = \begin{cases} x \vee y & \text{if } b_2 \in \{x, y\}, \\ x \vee y \vee g_3 & \text{otherwise.} \end{cases} \tag{20}$$

A t-norm T on $[0, e]$ is defined by

$$T(x, y) = \begin{cases} x \wedge y & \text{if } e, e'' \in \{x, y\}, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

From Figure 10 we know $I_e = B_3 \oplus B_2 \oplus B_1 \oplus B_0$, where $B_3 = \{c_1, c_2, c_3\}, B_2 = \{d_1, d_2\}, B_1 = \{e_1, e_2\}, B_0 = \{f_1, f_2\}$. It is easy to verify that the conditions in Theorem 4.1 are met. By using the formula (14), the function U defined by Table 2 is a uninorm on L_8 having neutral element e.

U	0	a_1	a_2	e''	e	g_5	b_1	g_1	g_2	b_2	g_3	g_4	1	c_1	c_2	c_3	d_1	d_2	e_1	e_2	f_1	f_2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a_1	0	0	0	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_2	0	0	0	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2
e''	0	a_1	a_2	e''	e''	e''	e''	e''	e''	e''	e''	e''	e''	e''	e''	e''	e''	e''	e''	e''	e''	e''
e	0	a_1	a_2	e''	e	g_5	b_1	g_1	g_2	b_2	g_3	g_4	1	c_1	c_2	c_3	d_1	d_2	e_1	e_2	f_1	f_2
g_5	0	a_1	a_2	e''	g_5	b_1	b_1	g_1	g_2	b_2	g_3	g_4	1	c_1	c_2	c_3	d_1	d_2	e_1	e_2	g_5	g_5
b_1	0	a_1	a_2	e''	b_1	b_1	b_1	g_1	g_2	b_2	g_3	g_4	1	c_1	c_2	c_3	d_1	d_2	e_1	e_2	b_1	b_1
g_1	0	a_1	a_2	e''	g_1	g_1	g_1	b_2	b_2	b_2	g_3	g_4	1	c_1	c_2	c_3	d_1	d_2	g_1	g_1	g_1	g_1
g_2	0	a_1	a_2	e''	g_2	g_2	g_2	b_2	b_2	b_2	g_3	g_4	1	c_1	c_2	c_3	d_1	d_2	g_2	g_2	g_2	g_2
b_2	0	a_1	a_2	e''	b_2	b_2	b_2	b_2	b_2	b_2	g_3	g_4	1	c_1	c_2	c_3	d_1	d_2	b_2	b_2	b_2	b_2
g_3	0	a_1	a_2	e''	g_3	g_3	g_3	g_3	g_3	g_3	g_3	1	1	c_1	c_2	c_3	g_3	g_3	g_3	g_3	g_3	g_3
g_4	0	a_1	a_2	e''	g_4	g_4	g_4	g_4	g_4	g_4	1	1	1	c_1	c_2	c_3	g_4	g_4	g_4	g_4	g_4	g_4
1	0	a_1	a_2	e''	1	1	1	1	1	1	1	1	1	c_1	c_2	c_3	1	1	1	1	1	1
c_1	0	a_1	a_2	e''	c_1	c_1	c_1	c_1	c_1	c_1	c_1	c_1	c_1	c_1	c_1	e''	c_1	c_1	c_1	c_1	c_1	c_1
c_2	0	a_1	a_2	e''	c_2	c_2	c_2	c_2	c_2	c_2	c_2	c_2	c_2	c_1	c_2	e''	c_2	c_2	c_2	c_2	c_2	c_2
c_3	0	a_1	a_2	e''	c_3	c_3	c_3	c_3	c_3	c_3	c_3	c_3	c_3	e''	e''	c_3	c_3	c_3	c_3	c_3	c_3	c_3
d_1	0	a_1	a_2	e''	d_1	d_1	d_1	d_1	d_1	d_1	g_3	g_4	1	c_1	c_2	c_3	d_1	d_1	d_1	d_1	d_1	d_1
d_2	0	a_1	a_2	e''	d_2	d_2	d_2	d_2	d_2	d_2	g_3	g_4	1	c_1	c_2	c_3	d_1	d_2	d_2	d_2	d_2	d_2
e_1	0	a_1	a_2	e''	e_1	e_1	e_1	g_1	g_2	b_2	g_3	g_4	1	c_1	c_2	c_3	d_1	d_2	e_1	e_1	e_1	e_1
e_2	0	a_1	a_2	e''	e_2	e_2	e_2	g_1	g_2	b_2	g_3	g_4	1	c_1	c_2	c_3	d_1	d_2	e_1	e_2	e_2	e_2
f_1	0	a_1	a_2	e''	f_1	g_5	b_1	g_1	g_2	b_2	g_3	g_4	1	c_1	c_2	c_3	d_1	d_2	e_1	e_2	f_1	f_2
f_2	0	a_1	a_2	e''	f_2	g_5	b_1	g_1	g_2	b_2	g_3	g_4	1	c_1	c_2	c_3	d_1	d_2	e_1	e_2	f_2	f_2

Tab. 2. The uninorm U in Example 4.9

The following example shows that the conditions in Theorem 4.1 are indispensable.

Example 4.10.

(i) The condition "$T(e'', z) = z$ for all $z \in [0, e[$" in Theorem 4.1 cannot be removed.

Consider the lattice L_9 drawn in Figure 11. Assume that the ordinal sum $S = \{< e, b, S_1 >, < b, 1, S_2 >\}$ is a t-conorm on $[e, 1]$, where S_1 and S_2 are t-conorms on $[e, b]$ and $[b, 1]$ respectively; and a t-norm T on $[0, e]$ is defined by formula (10), which means that the condition "$T(e'', z) = z$ for all $z \in [0, e[$" does not hold. From Figure 11 we know that the other conditions in Theorem 4.1 are satisfied, where $B_0 = \{f\}, B_1 = \{d\}$ and $B_2 = \{c_1, c_2\}$. The following fact means that the function U defined by formula (14) is not a uninorm: $U(U(c_1, c_2), a_1) = U(e'', a_1) = 0$, while $U(c_1, U(c_2, a_1)) = U(c_1, c_2 \wedge a_1) = U(c_1, a_1) = a_1$.

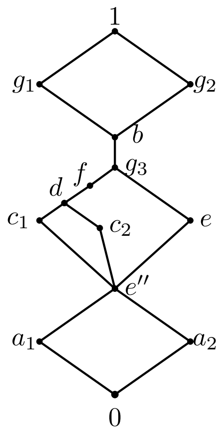

Fig. 11. The lattice L_9 in Example 4.10 (i)

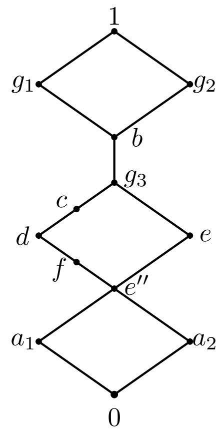

Fig. 12. The lattice L_{10} in Example 4.10 (ii)

(ii) We now show that the function U given by formula (14) may not necessarily be a uninorm if the t-conorm S on $[e,1]$ is not in the form of ordinal sums.

Consider the lattice L_{10} drawn in Figure 12. Assume that T is a t-norm on $[0,e]$ satisfying $T(e'',z)=z$ for all $z\in[0,e[$ and S is a t-conorm on $[e,1]$ defined by formula (11). It follows from Figure 12 that the other conditions in Theorem 4.1 are satisfied, where $B_0=\{c\}, B_1=\{d\}, B_2=\{f\}$. But the function U defined by the formula (14) is not a uninorm on L_{10}, because $U(U(d,g_3),g_1)=U(d,g_1)=d\vee g_1=g_1$, while $U(d,U(g_3,g_1))=U(d,1)=1$.

(iii) The condition "$I_e=\bigoplus_{i=0}^{n} B_{n-i}$ with $B_i\cap B_j=\emptyset$ $(i\neq j)$" in Theorem 4.1 cannot be omitted.

If we take $B_1=\{d_1,e_1\}$ and $B_2=\{d_2,e_2\}$ in Example 4.9, then from Figure 10 we know that $I_e=\cup_{i=0}^{3}B_{3-i}$ but $I_e\neq\bigoplus_{i=0}^{3}B_{3-i}$ because $e_1<e_2$. For the function U defined by formula (14), we have $U(e_1,g_1)=e_1\vee g_1=g_1>e_2=U(e_2,g_1)$ which means that the monotonicity of U does not hold.

(iv) The condition "$I_e\subseteq[0,e[^{u}$" in Theorem 4.1 cannot be deleted.

Consider the lattice L_{11} drawn in Figure 13. Assume that S is a t-conorm on $[e,1]$ and T is a t-norm on $[0,e]$ such that $T(e'',z)=z$ for all $z\in[0,e[$. It follows from Figure 13 that $I_e\nsubseteq[0,e[^{u}$ and other conditions in Theorem 4.1 are satisfied, where $B_0=\{c\}$ and $B_1=\{d\}$. The function U defined by the formula (14) is not a uninorm on L_{11}, because $U(U(c,g_3),e'')=U(c\vee g_3,e'')=U(g_3,e'')=g_3\wedge e''=e''$, while $U(c,U(g_3,e''))=U(c,e'')=c\wedge e''=a_1$.

(v) The condition "$I_e\subseteq]e,1]^{l}$" in Theorem 4.1 cannot be deleted.

Consider the lattice L_{12} drawn in Figure 14. Assume that a t-conorm S on $[e,1]$ is taken as an ordinal sum $S=\{<e,b_1,S_1>,<b_1,b_2,S_2>,<b_2,1,S_3>\}$, where S_3 is defined by

$$S_3(x,y)=\begin{cases}x\vee y & \text{if } b_2\in\{x,y\},\\ 1 & \text{otherwise,}\end{cases}\tag{22}$$

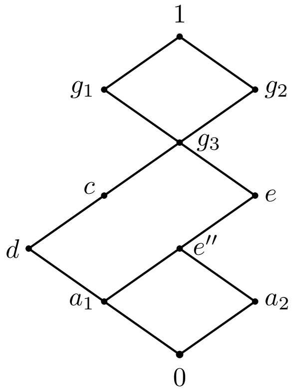

Fig. 13. The lattice L_{11} in Example 4.10 (iv)

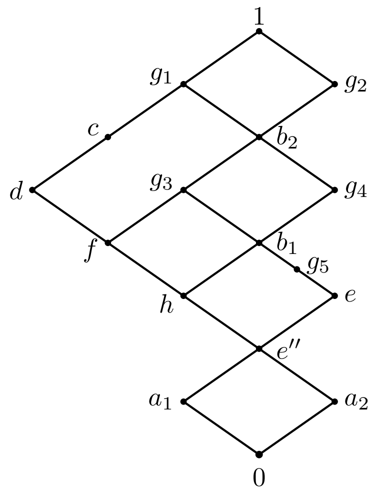

Fig. 14. The lattice L_{12} in Example 4.10 (v)

S_1 and S_2 are t-conorms on $[e, b_1]$ and $[b_1, b_2]$ respectively; while T is a t-norm on $[0, e]$ such that $T(e'', z) = z$ for all $z \in [0, e[$. From Figure 14 we know that $I_e \nsubseteq]e, 1]^l$ and other conditions in Theorem 4.1 are satisfied, where $B_0 = \{c\}, B_1 = \{d\}, B_2 = \{f\}$ and $B_3 = \{h\}$. The function U defined by the formula (14) is not a uninorm on L_{12}, because $U(U(d, g_3), g_1) = U(d \vee g_3, g_1) = U(g_1, g_1) = 1$, while $U(d, U(g_3, g_1)) = U(d, g_1) = g_1$.

(vi) The condition "B_i is closed w.r.t the operation $\wedge$ for each $i \in \{1, 2, \ldots, n-1\}$" in Theorem 4.1 cannot be removed.

Consider the lattice L_{13} drawn in Figure 15. Assume that T is a t-norm on $[0, e]$ such that $T(e'', z) = z$ for all $z \in [0, e[$, and $S = \{< e, b, S_1 >, < b, 1, S_2 >\}$ is a t-conorm on $[e, 1]$, where S_1 and S_2 are t-conorms on $[e, b]$ and $[b, 1]$ respectively. From Figure 15 we know that $f_1 \wedge f_2 \notin B_1$ for $f_1, f_2 \in B_1$ and other conditions in Theorem 4.1 are satisfied, where $B_0 = \{c\}, B_1 = \{f_1, f_2\}$ and $B_2 = \{d\}$. The function U defined by the formula (14) is not a uninorm on L_{13}, because $U(U(f_1, f_2), g_1) = U(f_1 \wedge f_2, g_1) = U(d, g_1) = d$, while $U(f_1, U(f_2, g_1)) = U(f_1, g_1) = g_1$.

(vii) The condition "B_0 is closed w.r.t the operation $\vee$" in Theorem 4.1 cannot be omitted.

Consider the lattice L_{14} drawn in Figure 16. Assume that T is a t-norm on $[0, e]$ such that $T(e'', z) = z$ for all $z \in [0, e[$ and the ordinal sum $S = \{< e, b, S_1 >, < b, 1, S_2 >\}$ is a t-conorm on $[e, 1]$, where S_1 and S_2 are t-conorms on $[e, b]$ and $[b, 1]$ respectively, and S_1 is defined by

$$S_1(x, y) = \begin{cases} x \vee y & \text{if } e \in \{x, y\}, \\ b & \text{otherwise.} \end{cases} \tag{23}$$

From Figure 16 we know that $d_1 \vee d_2 \notin B_0$ for $d_1, d_2 \in B_0$ and the other conditions in Theorem 4.1 are satisfied, where $B_0 = \{d_1, d_2, d_3\}, B_1 = \{f\}$ and $B_2 = \{c\}$. The function U defined by the formula (14) is not a uninorm on L_{14}, because $U(U(d_1, d_2), g_3) = U(d_1 \vee d_2, g_3) = U(g_3, g_3) = b$, while $U(d_1, U(d_2, g_3)) = U(d_1, g_3) = g_3$.

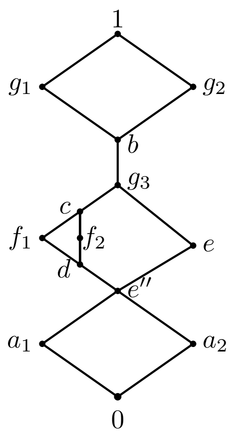

Fig. 15. The lattice L_{13} in Example 4.10 (vi)

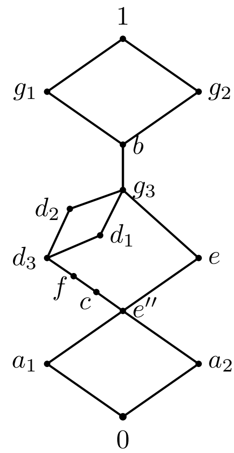

Fig. 16 The lattice L_{14} in Example 4.10 (vii)

5. CONCLUSION

In this article, by considering the underlying t-norms or t-conorms with ordinal sum structure, new construction methods of uninorms on bounded lattices were proposed. We also showed that our methods can cover several existing methods in literature as their particular cases. Furthermore, some illustrative examples for the new constructions of uninorms on bounded lattices were provided. This study opens up a new approach for further exploring the complex structure of uninorms on bounded lattices.

Around the subject of this research, there are still several other aspects that need further exploration. A study on more general cases will be our future research focus rather than just limited to $[0, e]$ or $[e, 1]$ as a chain of subintervals. On the other hand, the uninorms obtained from our new construction methods belong to $\mathcal{U}_\wedge \setminus \mathcal{U}_{\min}$ or $\mathcal{U}_\vee \setminus \mathcal{U}_{\max}$. Currently, there are many methods in the literature that have constructed uninorms in $\mathcal{U}_\wedge \setminus \mathcal{U}_{\min}$ or $\mathcal{U}_\vee \setminus \mathcal{U}_{\max}$ (such as [5, 6, 30], etc.), but the classes $\mathcal{U}_\wedge$ and $\mathcal{U}_\vee$ have not been characterized yet. Therefore, this is also a problem that needs to be solved in the future.

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