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EXTENSION METHODS FOR NULLNORMS ON BOUNDED LATTICES

Merve Yeşilyurt and Ümit Ertuğrul

After nullnorms were defined on bounded lattices by Karaçal et al. [16], construction methods for nullnorms on bounded lattices have been widely studied in which the existence of t-norms (t-conorms) on sublattices of the bounded lattice L has generally been exploited. Extension methods of nullnorms are important as they also play a significant role for ordinal sum construction of nullnorms on bounded lattices. In this paper, we introduce extension construction methods for nullnorms on a bounded lattice L by exploiting the existence of a nullnorm V on a sublattice of L. Then, we demonstrate that our new construction methods are also different from the existing construction methods in the literature. Additionally, some illustrative examples are provided. Finally, we also give modified versions of our construction method by induction.

Keywords: nullnorm, extension method, bounded lattice, sublattice

Classification: 03E72, 03B52

1. INTRODUCTION

Nullnorms were introduced on the unit interval $[0, 1]$ by Calvo et al. in [2]. These operators play an important role in extensive areas like fuzzy logic, fuzzy quantifiers, decision making, expert systems, neural networks and so on [3, 18, 20]. Nullnorms with a zero element k are generalization of triangular norms and triangular conorms, where k be anywhere in the unit interval. In particular, a nullnorm is a t-norm (resp. t-conorm) when $k = 0$ (resp. $k = 1$).

Karaçal et al. [16] proved the existence of nullnorms on an arbitrary bounded lattice L with a zero element k, where k is an arbitrary element of $L\backslash\{0, 1\}$, by using the fact that a t-norm (t-conorm) on an arbitrary bounded lattice L always exists. Afterward, Ertuğrul [13] generalized the methods in [16] and obtained two general methods for constructing nullnorms on a general bounded lattice L. Xie and Ji [22] proposed new methods for obtaining nullnorms on L by using t-subnorm and t-subconorm operations under some additional constraints. Çaylı and Karaçal [5] proved that an idempotent nullnorm may not always exist on every bounded lattice. Dan et al. [11] presented a new method for constructing a nullnorm with a zero element k on a bounded lattice L by means of a closure operator g on $[0, k]$ and an interior operator h on $[k, 1]$. Çaylı and Karaçal [6]

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gave two methods to obtain nullnorms on bounded lattices with a zero element by using the given nullnorm and t-norm (t-conorm) under some constraints. Nullnorms have also been studied from different aspects in the literature [4, 6, 7, 8, 9, 10, 21].

This study aims to provide construction methods of nullnorms on a bounded lattice, based on a nullnorm, a t-norm and a t-conorm defined on sublattices of the bounded lattice. In this paper, we give two main construction methods for nullnorms on the bounded lattices. The first method extends a nullnorm V on $[a, b] \subseteq L$ with zero element $k \in (a, b)$ to a nullnorm F on L with the zero element k considering a t-conorm S on $[0, a]$ and a t-norm T on $[b, 1]$, where $V(x, y) = k$ for $(x, y) \in I_k^{a,b} \times [a, b] \cup [a, b] \times I_k^{a,b}$. By slightly modifying this method, we express it in a different way in Theorem 3.21. Then, when $I_a = I_b = \emptyset$, we give another construction method for nullnorms via a nullnorm V on $[a, b] \subseteq L$ with a zero element $k \in (a, b) \subseteq L$, a t-conorm S on $[0, a]$ and a t-norm T on $[b, 1]$, where the obtained nullnorm F_η on L and the priori nullnorm V have the same zero element $k \in (a, b) \subseteq L$. It is obvious that the methods in Theorems 3.8 and 3.24 are different from each other and also we show that they are different from those methods in the literature. It should be pointed out that the methods in Theorems 3.8 and 3.24 are nullnorms' extension methods to a bounded lattice L from its sublattice $[a, b]$ by choosing $T = T_\wedge$ and $S = S_\vee$ (see Corollaries 3.16 and 3.27). We also elaborate the construction method in detail by examining it on some important special lattices. Finally, we modify the methods by increasing the number of t-norms and t-conorms on the subintervals.

The paper is structured as follows. In Section 2, we recall the notions of a bounded lattice, t-norms, t-conorms and nullnorms on bounded lattices. Section 3 contains the main results: considering the existence of a nullnorm on $[a, b]$ satisfying $V(x, y) = k$ when $(x, y) \in I_k^{a,b} \times [a, b] \cup [a, b] \times I_k^{a,b}$, we introduce a new extension method for nullnorms on a bounded lattice L. Afterward, we introduce another construction method for nullnorms on a bounded lattice L where $I_a = I_b = \emptyset$. Then, we give some illustrative examples to demonstrate how we extend a nullnorm on a subinterval of the lattice to the whole lattice. Furthermore, some examples are provided to illustrate that our new extension methods are different from the existing methods in the literature. We give modified versions of our construction methods by induction.

2. NOTATIONS, DEFINITIONS AND A REVIEW OF PREVIOUS RESULTS

In this section, we recall some basic notions and results.

Definition 2.1. (Birkhoff [1]) A lattice $(L, \leq)$ is a bounded lattice if L has the top element 1 and the bottom element 0, that is, there exist two elements $0, 1 \in L$ such that $0 \leq x \leq 1$ for all $x \in L$.

Definition 2.2. (Birkhoff [1]) Let $(L, \leq, 0, 1)$ be a bounded lattice. The elements x and y are called comparable if $x \leq y$ or $y \leq x$. Otherwise, x and y are called incomparable and the notation $x||y$ is used for such elements. In the following, I_a denotes the family of all incomparable elements with a, i. e., $I_a = \{x \in L |\ x||a\}$.

We denote by I_k^a for the set of elements which are incomparable with k but comparable with a, i. e., $I_k^a = \{x \in L |\ \ x||k \text{ and } x \nparallel a\}$. Similarly, we denote by I_a^k for the set of elements which are incomparable with a but comparable with k, i.e., $I_a^k = \{x \in L |\ x||a \text{ and } x \nparallel k\}$. By $I_{a,k}$, we denote the set of elements which are incomparable with k and a, i.e., $I_{a,k} = \{x \in L|\ x||k \text{ and } x||a\}$.

Definition 2.3. (Birkhoff [1]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $a, b \in L$ with $a \leq b$. The sublattice $[a, b]$ is defined as

$$[a, b] = \{x \in L| \quad a \leq x \leq b\}.$$

Similarly, $(a, b] = \{x \in L| \quad a < x \leq b\}$, $[a, b) = \{x \in L| \quad a \leq x < b\}$ and $(a, b) = \{x \in L| \quad a < x < b\}$ can be defined.

Definition 2.4. (Klement et al. [17]) Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation T (S) on a bounded lattice L is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to the both variables and has a neutral element 1 (0).

Example 2.5. Let $(L, \leq, 0, 1)$ be a bounded lattice. The smallest t-norm T_W and the greatest t-norm $T_\wedge$ on bounded lattice L are given respectively as:

$$T_W(x, y) = \begin{cases} y & \text{if } x = 1, \\ x & \text{if } y = 1, \\ 0 & \text{otherwise} \end{cases}$$

$$T_\wedge(x, y) = x \wedge y.$$

The smallest t-conorm $S_\vee$ and the greatest t-conorm S_W on bounded lattice L are given respectively as:

$$S_\vee(x, y) = x \vee y$$

$$S_W(x, y) = \begin{cases} y & \text{if } x = 0, \\ x & \text{if } y = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Definition 2.6. (Grabisch et al. [15]) Let $(L, \leq, 0, 1)$ be a bounded lattice, A and B be two aggregation functions on L. Then, A is called smaller than B if for any elements $x, y \in L$, $A(x, y) \leq B(x, y)$.

Definition 2.7. (Karaçal et al. [16]) Let $(L, \leq, 0, 1)$ be a bounded lattice. A commutative, associative, non-decreasing in each variable function $F : L^2 \to L$ is called a nullnorm if there is an element $k \in L$ such that $F(x, 0) = x$ for all $x \in [0, k]$, $F(x, 1) = x$ for all $x \in [k, 1]$.

The element $k \in L$ is an annihilator (sometimes called zero element) for F since $F(x, k) = k$ for all $x \in L$. We use the notation D_k to denote the set $D_k = [0, k] \times [k, 1] \cup [k, 1] \times [0, k]$ for $k \in L \setminus \{0, 1\}$.

Proposition 2.8. (Drygaś [12], Karaçal et al. [16]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $k \in L \setminus \{0, 1\}$ and V be a nullnorm on L with the zero element k. The following properties hold:

(i) $V(x, y) = k$ for all $(x, y) \in D_k$.

(ii) $k \leq V(x, y)$ for all $(x, y) \in [k, 1]^2 \cup [k, 1] \times I_k \cup I_k \times [k, 1]$.

(iii) $V(x, y) \leq k$ for all $(x, y) \in [0, k]^2 \cup [0, k] \times I_k \cup I_k \times [0, k]$.

(iv) $V(x, y) \leq y$ for all $(x, y) \in L \times [k, 1]$.

(v) $V(x, y) \leq x$ for all $(x, y) \in [k, 1] \times L$.

(vi) $x \leq V(x, y)$ for all $(x, y) \in [0, k] \times L$.

(vii) $y \leq V(x, y)$ for all $(x, y) \in L \times [0, k]$.

$(viii)$ $x \vee y \leq V(x, y)$ for all $(x, y) \in [0, k]^2$.

(ix) $V(x, y) \leq x \wedge y$ for all $(x, y) \in [k, 1]^2$.

(x) $(x \wedge k) \vee (y \wedge k) \leq V(x, y)$ for all $(x, y) \in [0, k] \times I_k \cup I_k \times [0, k] \cup I_k \times I_k$.

(xi) $V(x, y) \leq (x \vee k) \wedge (y \vee k)$ for all $(x, y) \in [k, 1] \times I_k \cup I_k \times [k, 1] \cup I_k \times I_k$.

3. CONSTRUCTION OF NULLNORMS ON BOUNDED LATTICES

In this section, we recall the existing construction methods of nullnorms in the literature at first. Next, we propose a construction method for nullnorms on a bounded lattice L in Theorem 3.8 by exploiting the existence of a nullnorm V on a subinterval $[a, b]$ of L, where $V(x, y) = k$ when $(x, y) \in I_k^{a,b} \times [a, b] \cup [a, b] \times I_k^{a,b}$. Then, we propose another construction method in Theorem 3.24 for nullnorms on a bounded lattice L, where $I_a = I_b = \emptyset$ for $a, b \in L$. Since these methods can be seen as extension methods by $T = T_\wedge$ and $S = S_\vee$, we compare our methods with those construction methods in the literature. And also, we emphasize the differences between our construction methods and those in the literature. Some illustrative examples for our extension methods for nullnorms on a bounded lattice are provided.

Theorem 3.1. (Karaçal et al. [16]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $k \in L \setminus \{0, 1\}$, S be a t-conorm on $[0, k]$ and T be a t-norm on $[k, 1]$.Then the functions $V_k^S, V_k^T : L^2 \to L$ defined as follows

$$V_k^S(x, y) = \begin{cases} S(x, y) & (x, y) \in [0, k]^2, \\ k & (x, y) \in [k, 1)^2 \cup [k, 1] \times I_k \cup I_k \times [k, 1] \cup D_k, \\ S(x \wedge k, y \wedge k) & (x, y) \in [0, k] \times I_k \cup I_k \times [0, k] \cup I_k \times I_k, \\ x \wedge y & \text{otherwise} \end{cases} \tag{1}$$

and

$$V_k^T(x,y) = \begin{cases} T(x,y) & (x,y) \in [k,1]^2, \\ k & (x,y) \in [0,k)^2 \cup [0,k] \times I_k \cup I_k \times [0,k] \cup D_k, \\ T(x \vee k, y \vee k) & (x,y) \in [k,1] \times I_k \cup I_k \times [k,1] \cup I_k \times I_k, \\ x \vee y & \text{otherwise} \end{cases} \tag{2}$$

are nullnorms on L with zero element k.

Theorem 3.2. (Çaylı and Karaçal [6]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, k \in L \setminus \{0,1\}$, $[0,a]$ be a sublattice of L and $k \in [0,a]$ such that $x \nparallel k$ for all $x \in [0,a]$. If $x \geq a$ for all $x \in L \setminus [0,a]$, V^* is a nullnorm on $[0,a]$ with the zero element k and T is a t-norm on $[a,1]$, then the following operation $V_1 : L^2 \to L$ is a nullnorm with the zero element k, where

$$V_1(x,y) = \begin{cases} V^*(x,y) & (x,y) \in [0,a]^2, \\ k & (x,y) \in [0,k] \times [a,1] \cup [a,1] \times [0,k], \\ T(x,y) & (x,y) \in [a,1]^2, \\ x \wedge y & \text{otherwise.} \end{cases} \tag{3}$$

Theorem 3.3. (Çaylı and Karaçal [6]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, k \in L \setminus \{0,1\}$, $[a,1]$ be a sublattice of L and $k \in [a,1]$ such that $x \nparallel k$ for all $x \in [a,1]$. If $x \leq a$ for all $x \in L \setminus [a,1]$, V_* is a nullnorm on $[a,1]$ with the zero element k and S is a t-conorm on $[0,a]$, then the following operation $V_2 : L^2 \to L$ is a nullnorm with the zero element k, where

$$V_2(x,y) = \begin{cases} V_*(x,y) & (x,y) \in [a,1]^2, \\ k & (x,y) \in [k,1] \times [0,a] \cup [0,a] \times [k,1], \\ S(x,y) & (x,y) \in [0,a]^2, \\ x \vee y & \text{otherwise.} \end{cases} \tag{4}$$

Theorem 3.4. (Xie and Ji [22]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $k \in L \setminus \{0,1\}$. Let $T : [k,1]^2 \to [k,1]$ be a t-norm on $[k,1]$, $S : [0,k]^2 \to [0,k]$ be a t-conorm on $[0,k]$ and $R : [0,k]^2 \to [0,k]$ be a t-subconorm on $[0,k]$. If $S \leq R$ and $S(x, R(y,z)) = R(R(x,y),z) = R(S(x,y),z)$ for all $x, y, z \in [0,k]$, then $V_T^{S,R} : L^2 \to L$ is a nullnorm on L with the zero element k, where

$$V_T^{S,R}(x,y) = \begin{cases} S(x,y) & (x,y) \in [0,k]^2, \\ T(x,y) & (x,y) \in [k,1]^2, \\ R(x \wedge k, y \wedge k) & (x,y) \in [0,k] \times I_k \cup I_k \times [0,k] \cup I_k \times I_k, \\ k & \text{otherwise.} \end{cases} \tag{5}$$

Theorem 3.5. (Çaylı [7]) Let $k \in L \setminus \{0,1\}$ such that $d \nparallel e$ for all $d, e \in I_k$ and $T : [k,1]^2 \to [k,1]$ be a t-norm. The function $F_T : L^2 \to L$ defined by

$$F_T(x,y)=\begin{cases} x\vee y & (x,y)\in[0,k]^2,\\ T(x,y) & (x,y)\in[k,1]^2,\\ k & (x,y)\in D_k,\\ x\vee(y\wedge k) & (x,y)\in[0,k]\times I_k,\\ y\vee(x\wedge k) & (x,y)\in I_k\times[0,k],\\ x & (x,y)\in[k,1]\times I_k,\\ y & (x,y)\in I_k\times[k,1],\\ (x\wedge y)\vee(x\wedge k)\vee(y\wedge k) & (x,y)\in I_k\times I_k \end{cases} \tag{6}$$

is a nullnorm on L having the annihilator k iff $b||c$ for all $b\in I_k$ and $c\in[k,1[$.

Theorem 3.6. (Çaylı [7]) Let $k\in L\setminus\{0,1\}$ such that $d\nparallel e$ for all $d,e\in I_k$ and $S:[0,k]^2\to[0,k]$ be a t-conorm. The function $F_S:L^2\to L$ defined by

$$F_S(x,y)=\begin{cases} S(x,y) & (x,y)\in[0,k]^2,\\ x\wedge y & (x,y)\in[k,1]^2,\\ k & (x,y)\in D_k,\\ x & (x,y)\in[0,k]\times I_k,\\ y & (x,y)\in I_k\times[0,k],\\ x\wedge(y\vee k) & (x,y)\in[k,1]\times I_k,\\ y\wedge(x\vee k) & (x,y)\in I_k\times[k,1],\\ (x\vee y)\wedge(x\vee k)\wedge(y\vee k) & (x,y)\in I_k\times I_k \end{cases} \tag{7}$$

is a nullnorm on L having the annihilator k iff $b||c$ for all $b\in I_k$ and $c\in]0,k]$.

Theorem 3.7. (Ertuğrul [13]) Let $(L,\leq,0,1)$ be a bounded lattice, $k\in L\setminus\{0,1\}$, S be a t-conorm on $[0,k]$ and T be a t-norm on $[k,1]$. Then, the functions $V_T^S,V_S^T:L^2\to L$ can be defined as:

$$V_T^S=\begin{cases} S(x,y) & (x,y)\in[0,k]^2,\\ T(x,y) & (x,y)\in[k,1]^2,\\ S(x\wedge k,y\wedge k) & (x,y)\in[0,k]\times I_k\cup I_k\times[0,k]\cup I_k\times I_k,\\ k & \text{otherwise} \end{cases} \tag{8}$$

$$V_S^T=\begin{cases} S(x,y) & (x,y)\in[0,k]^2,\\ T(x,y) & (x,y)\in[k,1]^2,\\ T(x\vee k,y\vee k) & (x,y)\in[k,1]\times I_k\cup I_k\times[k,1]\cup I_k\times I_k,\\ k & \text{otherwise} \end{cases} \tag{9}$$

and they are nullnorms on L with zero element k.

In the following theorem, based on a nullnorm V on $[a,b]$ satisfying $V(x,y)=k$ when $(x,y)\in I_k^{a,b}\times[a,b]\cup[a,b]\times I_k^{a,b}$, we give a construction method for nullnorms on a bounded lattice at first.

Theorem 3.8. Let $(L,\leq,0,1)$ be a bounded lattice, $a,b,k\in L\setminus\{0,1\}$ with $a<k<b$, S be a t-conorm on $[0,a]$, T be a t-norm $[b,1]$, V be a nullnorm on $[a,b]$ with the zero

element k and $V(x,y)=k$ when $(x,y)\in I_k^{a,b}\times[a,b]\cup[a,b]\times I_k^{a,b}$. Then the following function $F: L^2\to L$ is a nullnorm with the zero element k on L, where

$$F(x,y)=\begin{cases} S(x,y) & (x,y)\in[0,a]^2,\\ T(x,y) & (x,y)\in[b,1]^2,\\ V(x\vee a,y\vee a) & (x,y)\in[a,b]^2\cup((0,a)\cup[a,k))\times I_a^{k,b}\cup\\ & I_a^{k,b}\times((0,a)\cup[a,k)\cup I_a^{k,b}),\\ V(x\wedge b,y\wedge b) & (x,y)\in((k,b]\cup(b,1))\times I_b^{a,k}\cup I_b^{a,k}\times\\ & ((k,b]\cup(b,1)\cup I_b^{a,k}),\\ x\vee y & (x,y)\in[0,a)\times[a,k)\cup[a,k)\times[0,a)\cup I_a^{k,b}\\ & \times\{0\}\cup\{0\}\times I_a^{k,b},\\ x\wedge y & (x,y)\in(k,b]\times(b,1]\cup(b,1]\times(k,b]\cup I_b^{a,k}\\ & \times\{1\}\cup\{1\}\times I_b^{a,k},\\ k & \text{otherwise.}\end{cases}\qquad(10)$$

Proof. *Monotonicity*: Let us show that for any elements $x,y\in L$ with $x\le y$, $F(x,z)\le F(y,z)$ for all $z\in L$. If $x=0$ or $x=1$, the inequality is satisfied. If x and y are both elements of $(0,a)$ or $[a,k]$ or $(k,b]$ or $(b,1)$ or $I_k^*=I_k^{a,b}\cup I_{a,k}^b\cup I_{k,b}^a\cup I_{a,k,b}$ or $I_a^{k,b}$ or $I_b^{a,k}$, $F(x,z)\le F(y,z)$ is always satisfied for all $z\in L$ since $x\le y$. It is clear that $F(x,z)=k=F(y,z)$, when $z\in I_k^*$ or $z=k$. Therefore, the mentioned cases are omitted and the rest of the proof is then split into all the remaining possible cases as follows.

1. Let $x\in(0,a)$. If $z\in(0,a)$, since $F(x,z)=S(x,z)$ and $F(y,z)$ equals to k or $y\vee z$ or $V(y\vee a,z\vee a)$, $F(x,z)\le F(y,z)$ for all $y\in L$. If $z\in[a,k)$, since $F(x,z)=x\vee z=z$ and $F(y,z)$ equals to k for all $y\in[k,1]\cup I_k^*$, $F(x,z)\le F(y,z)$. Therefore, we omit the case $z\in(0,a)\cup[a,k)$ and the other all possible cases are listed as follows.

1.1. $y\in[a,k)$,

1.1.1. If $z\in[a,k)$, then $F(x,z)=x\vee z\le y\vee z\le V(y,z)=F(y,z)$.

1.1.2. If $z\in(k,b]\cup(b,1)\cup I_b^{a,k}$, then $F(x,z)=k=F(y,z)$.

1.1.3. If $z\in I_a^{k,b}$, then $F(x,z)=V(x\vee a,z\vee a)\le V(y\vee a,z\vee a)=F(y,z)$.

1.2. $y=k$,

1.2.1. If $z\in(k,b]\cup(b,1)\cup I_b^{a,k}$, then $F(x,z)=k=F(y,z)$.

1.2.2. If $z\in I_a^{k,b}$, then $F(x,z)=V(x\vee a,z\vee a)=V(a,z\vee a)=z\vee a\le k=F(y,z)$.

1.3. $y\in(k,b]$,

1.3.1. If $z\in(k,b]\cup I_b^{a,k}$, then $F(x,z)=k=V(k,z\wedge b)\le V(y\wedge b,z\wedge b)=F(y,z)$.

1.3.2. If $z\in(b,1)$, then $F(x,z)=k\le y=y\wedge z=F(y,z)$.

1.3.3. If $z\in I_a^{k,b}$, then $F(x,z)=V(x\vee a,z\vee a)=V(a,z\vee a)=z\vee a\le k=F(y,z)$.

1.4. $y\in(b,1)$,

1.4.1. If $z\in(k,b]$, then $F(x,z)=k\le z=y\wedge z=F(y,z)$.

1.4.2. If $z\in(b,1)$, then $F(x,z)=k\le b\le T(y,z)=F(y,z)$.

1.4.3. If $z\in I_a^{k,b}$, then $F(x,z)=V(x\vee a,z\vee a)=V(a,z\vee a)=z\vee a\le k=F(y,z)$.

1.4.4. If $z\in I_b^{a,k}$, then $F(x,z)=k=V(k,z\wedge b)\le V(y\wedge b,z\wedge b)=F(y,z)$.

1.5. $y\in I_k^*$,

1.5.1. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,z) = k = F(y,z)$.
1.5.2. If $z \in I_a^{k,b}$, then $F(x,z) = V(x \vee a, z \vee a) = V(a, z \vee a) = z \vee a \leq k = F(y,z)$.
1.6. $y \in I_a^{k,b}$,
1.6.1. If $z \in [a,k)$, then $F(x,z) = x \vee z \leq y \vee z \vee a \leq V(y \vee a, z \vee a) = F(y,z)$.
1.6.2. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,z) = k = F(y,z)$.
1.6.3. If $z \in I_a^{k,b}$, then $F(x,z) = V(x \vee a, z \vee a) \leq V(y \vee a, z \vee a) = F(y,z)$.
1.7. $y \in I_b^{a,k}$,
1.7.1. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,z) = k = V(k, z \wedge b) \leq V(y \wedge b, z \wedge b) = F(y,z)$.
1.7.2. If $z \in I_a^{k,b}$, then $F(x,z) = V(x \vee a, z \vee a) = V(a, z \vee a) = z \vee a \leq k = F(y,z)$.

2. Let $x \in [a,k)$. If $z \in (0,a)$, since $F(x,z) = x \vee z = x$ and $F(y,z)$ equals to k for all $y \in I_a^{k,b} \cup (k,b] \cup (b,1) \cup I_k^{a,b} \cup I_{k,b}^{a} \cup I_b^{a,k}$, $F(x,z) \leq F(y,z)$. Therefore, we omit the case $z \in (0,a)$ and the other all possible cases are listed as follows.
2.1. $y = k$,
2.1.1. If $z \in [a,k) \cup I_a^{k,b}$, then $F(x,z) = V(x \vee a, z \vee a) \leq V(k, z \vee a) = k = F(y,z)$.
2.1.2. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,z) = k = F(y,z)$.
2.2. $y \in (k,b]$,
2.2.1. If $z \in [a,k) \cup (k,b]$, then $F(x,z) = V(x,z) \leq V(y,z) = F(y,z)$.
2.2.2. If $z \in (b,1)$, then $F(x,z) = k \leq y = y \wedge z = F(y,z)$.
2.2.3. If $z \in I_a^{k,b}$, then $F(x,z) = V(x \vee a, z \vee a) \leq V(y \vee a, k) = k = F(y,z)$.
2.2.4. If $z \in I_b^{a,k}$, then $F(x,z) = k = V(k, z \wedge b) \leq V(y \wedge b, z \wedge b) = F(y,z)$.
2.3. $y \in (b,1)$,
2.3.1. If $z \in [a,k)$, then $F(x,z) = V(x,z) \leq V(k,z) = k = F(y,z)$.
2.3.2. If $z \in (k,b]$, then $F(x,z) = V(x,z) = k \leq z = y \wedge z = F(y,z)$.
2.3.3. If $z \in (b,1)$, then $F(x,z) = k \leq b \leq T(y,z) = F(y,z)$.
2.3.4. If $z \in I_a^{k,b}$, then $F(x,z) = V(x \vee a, z \vee a) \leq V(y \vee a, k) = k = F(y,z)$.
2.3.5. If $z \in I_b^{a,k}$, then $F(x,z) = k \leq z \wedge b = V(b, z \wedge b) = V(y \wedge b, z \wedge b) = F(y,z)$.
2.4. $y \in I_k^{a,b} \cup I_{k,b}^{a}$,
2.4.1. If $z \in [a,k) \cup I_a^{k,b}$, then $F(x,z) = V(x \vee a, z \vee a) \leq k = F(y,z)$.
2.4.2. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,z) = k = F(y,z)$.
2.5. $y \in I_b^{a,k}$,
2.5.1. If $z \in [a,k) \cup I_a^{k,b}$, then $F(x,z) = V(x \vee a, z \vee a) \leq k = F(y,z)$.
2.5.2. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,z) = k = V(k, z \wedge b) \leq V(y \wedge b, z \wedge b) = F(y,z)$.

3. Let $x \in (k,b]$. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, since $F(x,z) = k$ and $F(y,z)$ equals to k for all $y \in (b,1) \cup I_b^{a,k}$, $F(x,z) \leq F(y,z)$. Therefore, we omit the case $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$ and the other all possible cases are listed as follows.
3.1. $y \in (b,1)$,
3.1.1. If $z \in (k,b]$, then $F(x,z) = V(x,z) \leq V(b,z) = z = y \wedge z = F(y,z)$.
3.1.2. If $z \in (b,1)$, then $F(x,z) = x \wedge z = x \leq b \leq T(y,z) = F(y,z)$.
3.1.3. If $z \in I_b^{a,k}$, then $F(x,z) = V(x \wedge b, z \wedge b) \leq V(y \wedge b, z \wedge b) = F(y,z)$.
3.2. $y \in I_b^{a,k}$,

3.2.1. If $z \in (k, b] \cup I_b^{a,k}$, then $F(x, z) = V(x \wedge b, z \wedge b) \leq V(y \wedge b, z \wedge b) = F(y, z)$.

3.2.2. If $z \in (b, 1)$, then $F(x, z) = x \wedge z = x \leq y \wedge b = V(y \wedge b, b) = V(y \wedge b, z \wedge b) = F(y, z)$.

4. Let $x \in I_k^{a,b}$.

If $z \in (0, a) \cup [a, k) \cup I_a^{k,b}$, since $F(x, z) = k$ and $F(y, z)$ equals to k for all $y \in (k, b] \cup (b, 1) \cup I_b^{a,k}$, $F(x, z) \leq F(y, z)$. Therefore, we omit the case $z \in (0, a) \cup [a, k) \cup I_a^{k,b}$ and the other all possible cases are listed as follows.

4.1. $y \in (k, b]$,

4.1.1. If $z \in (k, b] \cup I_b^{a,k}$, then $F(x, z) = k = V(k, z \wedge b) \leq V(y \wedge b, z \wedge b) = F(y, z)$.

4.1.2. If $z \in (b, 1)$, then $F(x, z) = k \leq y = y \wedge z = F(y, z)$.

4.2. $y \in (b, 1)$,

4.2.1. If $z \in (k, b]$, then $F(x, z) = k \leq z = y \wedge z = F(y, z)$.

4.2.2. If $z \in (b, 1)$, then $F(x, z) = k \leq b \leq T(y, z) = F(y, z)$.

4.2.3. If $z \in I_b^{a,k}$, then $F(x, z) = k = V(k, z \wedge b) \leq V(y \wedge b, z \wedge b) = F(y, z)$.

4.3. $y \in I_b^{a,k}$,

4.3.1. If $z \in (k, b] \cup (b, 1) \cup I_b^{a,k}$, then $F(x, z) = k = V(k, z \wedge b) \leq V(y \wedge b, z \wedge b) = F(y, z)$.

4.4. $y \in I_{k,b}^{a}$,

4.4.1. If $z \in L$, then $F(x, z) = k = F(y, z)$.

5. Let $x \in I_a^{k,b}$.

5.1. $y \in [a, k]$,

5.1.1. If $z \in (0, a)$, then $F(x, z) = V(x \vee a, z \vee a) = V(x \vee a, a) = x \vee a \leq y = y \vee z = F(y, z)$.

5.1.2. If $z \in [a, k) \cup I_a^{k,b}$, then $F(x, z) = V(x \vee a, z \vee a) \leq V(y \vee a, z \vee a) = F(y, z)$.

5.1.3. If $z \in (k, b] \cup (b, 1) \cup I_b^{a,k}$, then $F(x, z) = k = F(y, z)$.

5.2. $y \in (k, b]$,

5.2.1. If $z \in (0, a)$, then $F(x, z) = V(x \vee a, z \vee a) = V(x \vee a, a) = x \vee a \leq k = F(y, z)$.

5.2.2. If $z \in [a, k] \cup I_a^{k,b}$, then $F(x, z) = V(x \vee a, z \vee a) \leq V(y \vee a, k) = k = F(y, z)$.

5.2.3. If $z \in (k, b] \cup I_b^{a,k}$, then $F(x, z) = k = V(k, z \wedge b) \leq V(y \wedge b, z \wedge b) = F(y, z)$.

5.2.4. If $z \in (b, 1)$, then $F(x, z) = k \leq y = y \wedge z = F(y, z)$.

5.3. $y \in (b, 1)$,

5.3.1. If $z \in (0, a) \cup [a, k) \cup I_a^{k,b}$, then $F(x, z) = V(x \vee a, z \vee a) \leq V(y \vee a, k) = k = F(y, z)$.

5.3.2. If $z \in (k, b]$, then $F(x, z) = k \leq z = y \wedge z = F(y, z)$.

5.3.3. If $z \in (b, 1)$, then $F(x, z) = k \leq b \leq T(y, z) = F(y, z)$.

5.3.4. If $z \in I_b^{a,k}$, then $F(x, z) = k = V(k, z \wedge b) \leq V(y \wedge b, z \wedge b) = F(y, z)$.

5.4. $y \in I_k^*$,

5.4.1. If $z \in (0, a) \cup [a, k) \cup I_a^{k,b}$, then $F(x, z) = V(x \vee a, z \vee a) \leq V(y \vee a, k) = k = F(y, z)$.

5.4.2. If $z \in (k, b] \cup (b, 1) \cup I_b^{a,k}$, then $F(x, z) = k = F(y, z)$.

5.5. $y \in I_b^{a,k}$,

5.5.1. If $z \in (0, a)$, then $F(x, z) = V(x \vee a, z \vee a) \leq V(x \vee a, a) = x \vee a \leq k = F(y, z)$.

5.5.2. If $z \in [a,k) \cup I_a^{k,b}$, then $F(x,z) = V(x \vee a, z \vee a) \leq V(y \vee a, k) = k = F(y,z)$.
5.5.3. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,z) = k = V(y \wedge b, k) \leq V(y \wedge b, z \wedge b) = F(y,z)$.

6. Let $x \in I_b^{a,k}$.
6.1. $y \in (b,1)$,
6.1.1. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, then $F(x,z) = k = F(y,z)$.
6.1.2. If $z \in (k,b]$, then $F(x,z) = V(x \wedge b, z \wedge b) \leq V(b, z \wedge b) = z \wedge b = z = y \wedge z = F(y,z)$.
6.1.3. If $z \in (b,1)$, then $F(x,z) = V(x \wedge b, z \wedge b) \leq b \leq T(y,z) = F(y,z)$.
6.1.4. If $z \in I_b^{a,k}$, then $F(x,z) = V(x \wedge b, z \wedge b) \leq V(y \wedge b, z \wedge b) = F(y,z)$.

7. Let $x \in I_{a,k,b} \cup I_{k,b}^a$.
7.1. $y \in (b,1)$,
7.1.1. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, then $F(x,z) = k = F(y,z)$.
7.1.2. If $z \in (k,b]$, then $F(x,z) = k \leq z = y \wedge z = F(y,z)$.
7.1.3. If $z \in (b,1)$, then $F(x,z) = k \leq b \leq T(y,z) = F(y,z)$.
7.1.4. If $z \in I_b^{a,k}$, then $F(x,z) = k \leq z \wedge b = V(b, z \wedge b) = V(y \wedge b, z \wedge b) = F(y,z)$.
7.2. $y \in I_b^{a,k}$,
7.2.1. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, then $F(x,z) = k = F(y,z)$.
7.2.2. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,z) = k = V(k, z \wedge b) \leq V(y \wedge b, z \wedge b) = F(y,z)$.

8. Let $x \in I_{a,k}^b$. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, since $F(x,z) = k$ and $F(y,z)$ equals to k for all $y \in (k,b] \cup (b,1) \cup I_b^{a,k}$, $F(x,z) \leq F(y,z)$. Therefore, we omit the case $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$ and the other all possible cases are listed as follows.
8.1. $y \in (k,b]$,
8.1.1. If $z \in (k,b] \cup I_b^{a,k}$, then $F(x,z) = k = V(k, z \wedge b) \leq V(y \wedge b, z \wedge b) = F(y,z)$.
8.1.2. If $z \in (b,1)$, then $F(x,z) = k \leq y = y \wedge z = F(y,z)$.
8.2. $y \in (b,1)$,
8.2.1. If $z \in (k,b]$, then $F(x,z) = k \leq z = y \wedge z = F(y,z)$.
8.2.2. If $z \in (b,1)$, then $F(x,z) = k \leq b \leq T(y,z) = F(y,z)$.
8.2.3. If $z \in I_b^{a,k}$, then $F(x,z) = k \leq z \wedge b = V(b, z \wedge b) = V(y \wedge b, z \wedge b) = F(y,z)$.
8.3. $y \in I_k^{a,b} \cup I_{k,b}^a \cup I_{a,k,b}$,
8.3.1. If $z \in L$, then $F(x,z) = k = F(y,z)$.
8.4. $y \in I_b^{a,k}$,
8.4.1. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,z) = k = V(k, z \wedge b) \leq V(y \wedge b, z \wedge b) = F(y,z)$.

(ii) *Associativity*: We demonstrate that $F(x, F(y,z)) = F(F(x,y), z)$ for all $x,y,z \in L$. If one of the elements x, y and z is equal to k or element of I_k^*, it is clear that the equality is always satisfied. If $x = 0$ or $x = 1$, the equality is satisfied, therefore, we omit. Then, the proof is split into all remaining possible cases by considering the relationships between the elements x, y, z, a, b and k as follows.

1. Let $x \in (0,a)$. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, since $F(x,F(y,z)) = F(x,k) = k$ and $F(F(x,y),z) = F(k,z) = k$ for all $y \in (k,b] \cup (b,1) \cup I_b^{a,k}$, $F(x,F(y,z)) = F(F(x,y),z)$. Therefore, we omit the case $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$ and the other all possible cases are listed as follows.

1.1. $y \in (0,a)$,

1.1.1. If $z \in (0,a)$, then $F(x,F(y,z)) = F(x,S(y,z)) = S(x,S(y,z)) = S(S(x,y),z) = F(S(x,y),z) = F(F(x,y),z)$.

1.1.2. If $z \in [a,k)$, then $F(x,F(y,z)) = F(x,y \vee z) = F(x,z) = x \vee z = z = S(x,y) \vee z = F(S(x,y),z) = F(F(x,y),z)$.

1.1.3. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,k) = k = F(S(x,y),z) = F(F(x,y),z)$.

1.1.4. If $z \in I_a^{k,b}$, then $F(x,F(y,z)) = F(x,V(y \vee a, z \vee a)) = F(x,V(a,z \vee a)) = F(x,z \vee a) = x \vee z \vee a = z \vee a = V(a,z \vee a) = V(S(x,y) \vee a, z \vee a) = F(S(x,y),z) = F(F(x,y),z)$.

1.2. $y \in [a,k)$,

1.2.1. If $z \in (0,a)$, then $F(x,F(y,z)) = F(x,y \vee z) = F(x,y) = x \vee y = y = y \vee z = F(y,z) = F(x \vee y,z) = F(F(x,y),z)$.

1.2.2. If $z \in [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,V(y \vee a, z \vee a)) = x \vee V(y \vee a, z \vee a) = V(y \vee a, z \vee a) = F(y,z) = F(x \vee y,z) = F(F(x,y),z)$.

1.2.3. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,k) = k = F(y,z) = F(x \vee y,z) = F(F(x,y),z)$.

1.3. $y \in (k,b]$,

1.3.1. If $z \in (k,b] \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,V(y \wedge b, z \wedge b)) = k = F(k,z) = F(F(x,y),z)$.

1.3.2. If $z \in (b,1)$, then $F(x,F(y,z)) = F(x,y \wedge z) = F(x,y) = k = F(k,z) = F(F(x,y),z)$.

1.4. $y \in (b,1)$,

1.4.1. If $z \in (k,b]$, then $F(x,F(y,z)) = F(x,y \wedge z) = F(x,z) = k = F(k,z) = F(F(x,y),z)$.

1.4.2. If $z \in (b,1)$, then $F(x,F(y,z)) = F(x,T(y,z)) = k = F(k,z) = F(F(x,y),z)$.

1.4.3. If $z \in I_b^{a,k}$, then $F(x,F(y,z)) = F(x,V(y \wedge b, z \wedge b) = k = F(k,z) = F(F(x,y),z)$.

1.5. $y \in I_a^{k,b}$,

1.5.1. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,V(y \vee a, z \vee a)) = x \vee V(y \vee a, z \vee a) = V(y \vee a, z \vee a) = F(y \vee a, z) = F(V(a, y \vee a), z) = F(V(x \vee a, y \vee a), z) = F(F(x,y),z)$.

1.5.2. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,k) = k = F(y \vee a, z) = F(V(a, y \vee a), z) = F(V(x \vee a, y \vee a), z) = F(F(x,y),z)$.

1.6. $y \in I_b^{a,k}$,

1.6.1. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,V(y \wedge b, z \wedge b)) = k = F(k,z) = F(F(x,y),z)$.

2. Let $x \in [a,k)$. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, since $F(x,F(y,z)) = F(x,k) = k$ and $F(F(x,y),z) = F(k,z) = k$ for all $y \in (k,b] \cup (b,1) \cup I_b^{a,k}$, $F(x,F(y,z)) = F(F(x,y),z)$.

Therefore, we omit the case $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$ and the other all possible cases are listed as follows.
2.1. $y \in (0,a)$,
2.1.1. If $z \in (0,a)$, then $F(x,F(y,z)) = F(x,S(y,z)) = x \vee S(y,z) = x = x \vee z = F(x,z) = F(x \vee y, z) = F(F(x,y),z)$.
2.1.2. If $z \in [a,k)$, then $F(x,F(y,z)) = F(x,y \vee z) = F(x,z) = F(x \vee y,z) = F(F(x,y),z)$.
2.1.3. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,k) = k = F(x,z) = F(x \vee y,z) = F(F(x,y),z)$.
2.1.4. If $z \in I_a^{k,b}$, then $F(x,F(y,z)) = F(x,V(y \vee a, z \vee a)) = F(x,V(a,z \vee a)) = F(x,z \vee a) = V(x,z \vee a) = F(x,z) = F(x \vee y,z) = F(F(x,y),z)$.
2.2. $y \in [a,k)$,
2.2.1. If $z \in (0,a)$, then $F(x,F(y,z)) = F(x,y \vee z) = F(x,y) = V(x,y) = V(x,y) \vee z = F(V(x,y),z) = F(F(x,y),z)$.
2.2.2. If $z \in [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,V(y \vee a,z \vee a)) = V(x,V(y \vee a,z \vee a)) = V(V(x,y),z \vee a) = F(V(x,y),z) = F(F(x,y),z)$.
2.2.3. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,k) = k = F(V(x,y),z) = F(F(x,y),z)$.
2.3. $y \in (k,b]$,
2.3.1. If $z \in (k,b] \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,V(y \wedge b,z \wedge b)) = k = F(k,z) = F(V(x,y),z) = F(F(x,y),z)$.
2.3.2. If $z \in (b,1)$, then $F(x,F(y,z)) = F(x,y \wedge z) = F(x,y) = V(x,y) = k = F(k,z) = F(V(x,y),z) = F(F(x,y),z)$.
2.4. $y \in (b,1)$,
2.4.1. If $z \in (k,b]$, then $F(x,F(y,z)) = F(x,y \wedge z) = F(x,z) = V(x,z) = k = F(k,z) = F(F(x,y),z)$.
2.4.2. If $z \in (b,1)$, then $F(x,F(y,z)) = F(x,T(y,z)) = k = F(k,z) = F(F(x,y),z)$.
2.4.3. If $z \in I_b^{a,k}$, then $F(x,F(y,z)) = F(x,V(y \wedge b,z \wedge b)) = k = F(k,z) = F(F(x,y),z)$.
2.5. $y \in I_a^{k,b}$,
2.5.1. If $z \in (0,a)$, then $F(x,F(y,z)) = F(x,V(y \vee a,z \vee a)) = F(x,V(y \vee a,a)) = F(x,y \vee a) = V(x \vee a,y \vee a) = V(x \vee a,y \vee a) \vee z = F(V(x \vee a,y \vee a),z) = F(F(x,y),z)$.
2.5.2. If $z \in [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,V(y \vee a,z \vee a)) = V(x,V(y \vee a,z \vee a)) = V(V(x \vee a,y \vee a),z \vee a) = F(V(x \vee a,y \vee a),z) = F(F(x,y),z)$.
2.5.3. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,k) = k = F(V(x \vee a,y \vee a),z) = F(F(x,y),z)$.
2.6. $y \in I_b^{a,k}$,
2.6.1. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,V(y \wedge b,z \wedge b)) = k = F(k,z) = F(F(x,y),z)$.

3. Let $x \in (k,b]$.
3.1. $y \in (0,a)$,
3.1.1. If $z \in (0,a)$, then $F(x,F(y,z)) = F(x,S(y,z)) = k = F(k,z) = F(F(x,y),z)$.
3.1.2. If $z \in [a,k)$, then $F(x,F(y,z)) = F(x,y \vee z) = F(x,z) = V(x,z) = k = F(k,z) = F(F(x,y),z)$.

3.1.3. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,k) = k = F(k,z) = F(F(x,y),z)$.

3.1.4. If $z \in I_a^{k,b}$, then $F(x,F(y,z)) = F(x,V(y \vee a, z \vee a)) = k = F(k,z) = F(F(x,y),z)$.

3.2. $y \in [a,k)$,

3.2.1. If $z \in (0,a)$, then $F(x,F(y,z)) = F(x,y \vee z) = F(x,y) = V(x,y) = k = F(k,z) = F(V(x,y),z) = F(F(x,y),z)$.

3.2.2. If $z \in [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,V(y,z)) = V(x,V(y,z)) = k = F(k,z) = F(V(x,y),z) = F(F(x,y),z)$.

3.2.3. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,V(y,z)) = F(x,k) = k = F(k,z) = F(V(x,y),z) = F(F(x,y),z)$.

3.3. $y \in (k,b]$,

3.3.1. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,k) = k = F(V(x,y),z) = F(F(x,y),z)$.

3.3.2. If $z \in (k,b] \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,V(y \wedge b, z \wedge b)) = V(x,V(y \wedge b, z \wedge b)) = V(V(x,y), z \wedge b) = F(V(x,y),z) = F(F(x,y),z)$.

3.3.3. If $z \in (b,1)$, then $F(x,F(y,z)) = F(x,y \wedge z) = F(x,y) = V(x,y) = V(x,y) \wedge z = F(V(x,y),z) = F(F(x,y),z)$.

3.4. $y \in (b,1)$,

3.4.1. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,k) = k = F(x,z) = F(x \wedge y,z) = F(F(x,y),z)$.

3.4.2. If $z \in (k,b]$, then $F(x,F(y,z)) = F(x,y \wedge z) = F(x,z) = F(x \wedge y,z) = F(F(x,y),z)$.

3.4.3. If $z \in (b,1)$, then $F(x,F(y,z)) = F(x,T(y,z)) = x \wedge T(y,z) = x = x \wedge z = F(x,z) = F(x \wedge y,z) = F(F(x,y),z)$.

3.4.4. If $z \in I_b^{a,k}$, then $F(x,F(y,z)) = F(x,V(y \wedge b, z \wedge b)) = F(x,V(b, z \wedge b)) = F(x, z \wedge b) = V(x, z \wedge b) = F(x,z) = F(x \wedge y,z) = F(F(x,y),z)$.

3.5. $y \in I_a^{k,b}$,

3.5.1. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,V(y \vee a, z \vee a)) = k = F(k,z) = F(F(x,y),z)$.

3.5.2. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,k) = k = F(k,z) = F(F(x,y),z)$.

3.6. $y \in I_b^{a,k}$,

3.6.1. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,k) = k = F(V(x \wedge b, y \wedge b),z) = F(F(x,y),z)$.

3.6.2. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,V(y \wedge b, z \wedge b)) = V(x,V(y \wedge b, z \wedge b)) = V(V(x \wedge b, y \wedge b), z \wedge b) = F(V(x \wedge b, y \wedge b),z) = F(F(x,y),z)$.

4. Let $x \in (b,1)$.

4.1. $y \in (0,a)$,

4.1.1. If $z \in (0,a)$, then $F(x,F(y,z)) = F(x,S(y,z)) = k = F(k,z) = F(F(x,y),z)$.

4.1.2. If $z \in [a,k)$, then $F(x,F(y,z)) = F(x,y \vee z) = F(x,z) = k = F(k,z) = F(F(x,y),z)$.

4.1.3. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,k) = k = F(k,z) = F(F(x,y),z)$.

4.1.4. If $z \in I_a^{k,b}$, then $F(x,F(y,z)) = F(x,V(y \vee a, z \vee a)) = k = F(k,z) = F(F(x,y),z)$.
4.2. $y \in [a,k)$,
4.2.1. If $z \in (0,a)$, then $F(x,F(y,z)) = F(x,y \vee z) = F(x,y) = k = F(k,z) = F(F(x,y),z)$.
4.2.2. If $z \in [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,V(y \vee a, z \vee a)) = k = F(k,z) = F(F(x,y),z)$.
4.2.3. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,k) = k = F(k,z) = F(F(x,y),z)$.
4.3. $y \in (k,b]$,
4.3.1. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,k) = k = F(y,z) = F(x \wedge y, z) = F(F(x,y),z)$.
4.3.2. If $z \in (k,b] \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,V(y \wedge b, z \wedge b)) = x \wedge V(y \wedge b, z \wedge b) = V(y \wedge b, z \wedge b) = F(y,z) = F(x \wedge y, z) = F(F(x,y),z)$.
4.3.3. If $z \in (b,1)$, then $F(x,F(y,z)) = F(x,y \wedge z) = F(x,y) = x \wedge y = y = y \wedge z = F(y,z) = F(x \wedge y, z) = F(F(x,y),z)$.
4.4. $y \in (b,1)$,
4.4.1. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,k) = k = F(T(x,y),z) = F(F(x,y),z)$.
4.4.2. If $z \in (k,b]$, then $F(x,F(y,z)) = F(x,y \wedge z) = F(x,z) = x \wedge z = z = T(x,y) \wedge z = F(T(x,y),z) = F(F(x,y),z)$.
4.4.3. If $z \in (b,1)$, then $F(x,F(y,z)) = F(x,T(y,z)) = T(x,T(y,z)) = T(T(x,y),z) = F(T(x,y),z) = F(F(x,y),z)$.
4.4.4. If $z \in I_b^{a,k}$, then $F(x,F(y,z)) = F(x,V(y \wedge b, z \wedge b)) = x \wedge V(y \wedge b, z \wedge b) = V(y \wedge b, z \wedge b) = V(b, z \wedge b) = z \wedge b = V(b, z \wedge b) = V(T(x,y) \wedge b, z \wedge b) = F(T(x,y),z) = F(F(x,y),z)$.
4.5. $y \in I_a^{k,b}$,
4.5.1. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,V(y \vee a, z \vee a)) = k = F(k,z) = F(F(x,y),z)$.
4.5.2. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,k) = k = F(k,z) = F(F(x,y),z)$.
4.6. $y \in I_b^{a,k}$,
4.6.1. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,k) = k = F(V(x \wedge b, y \wedge b),z) = F(F(x,y),z)$.
4.6.2. If $z \in (k,b] \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,V(y \wedge b, z \wedge b)) = x \wedge V(y \wedge b, z \wedge b) = V(y \wedge b, z \wedge b) = F(y \wedge b, z) = F(V(b, y \wedge b), z) = F(V(x \wedge b, y \wedge b), z) = F(F(x,y),z)$.
4.6.3. If $z \in (b,1)$, then $F(x,F(y,z)) = F(x,V(y \wedge b, z \wedge b)) = x \wedge V(y \wedge b, z \wedge b) = V(y \wedge b, z \wedge b) = F(y \wedge b, b) = y \wedge b = y \wedge b \wedge z = F(y \wedge b, z) = F(V(b, y \wedge b), z) = F(V(x \wedge b, y \wedge b), z) = F(F(x,y),z)$.
5. Let $x \in I_a^{k,b}$. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, since $F(x,F(y,z)) = F(x,k) = k$ and $F(F(x,y),z) = F(k,z) = k$ for all $y \in (k,b] \cup (b,1) \cup I_b^{a,k}$, $F(x,F(y,z)) = F(F(x,y),z)$. Therefore, we omit the case $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$ and the other all possible cases are listed as follows.

5.1. $y \in (0,a)$,

5.1.1. If $z \in (0,a)$, then $F(x,F(y,z)) = F(x,S(y,z)) = V(x \vee a, S(y,z) \vee a) = V(x \vee a, a) = x \vee a = x \vee a \vee z = F(x \vee a, z) = F(V(x \vee a, a), z) = F(V(x \vee a, y \vee a), z) = F(F(x,y),z)$.

5.1.2. If $z \in [a,k)$, then $F(x,F(y,z)) = F(x, y \vee z) = F(x,z) = V(x \vee a, z \vee a) = F(x \vee a, z) = F(V(x \vee a, a), z) = F(V(x \vee a, y \vee a), z) = F(F(x,y),z)$.

5.1.3. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,k) = k = F(x \vee a, z) = F(V(x \vee a, a), z) = F(V(x \vee a, y \vee a), z) = F(F(x,y),z)$.

5.1.4. If $z \in I_a^{k,b}$, then $F(x,F(y,z)) = F(x, V(y \vee a, z \vee a)) = F(x, V(a, z \vee a)) = F(x, z \vee a) = V(x \vee a, z \vee a) = F(x \vee a, z) = F(V(x \vee a, a), z) = F(V(x \vee a, y \vee a), z) = F(F(x,y),z)$.

5.2. $y \in [a,k)$,

5.2.1. If $z \in (0,a)$, then $F(x,F(y,z)) = F(x, y \vee z) = F(x,y) = V(x \vee a, y \vee a) = V(x \vee a, y \vee a) \vee z = F(V(x \vee a, y \vee a), z) = F(F(x,y),z)$.

5.2.2. If $z \in [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x, V(y \vee a, z \vee a)) = V(x \vee a, V(y \vee a, z \vee a)) = V(V(x \vee a, y \vee a), z \vee a) = F(V(x \vee a, y \vee a), z) = F(F(x,y),z)$.

5.2.3. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,k) = k = F(V(x \vee a, y \vee a), z) = F(F(x,y),z)$.

5.3. $y \in (k,b]$,

5.3.1. If $z \in (k,b] \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x, V(y \wedge b, z \wedge b)) = k = F(k,z) = F(F(x,y),z)$.

5.3.2. If $z \in (b,1)$, then $F(x,F(y,z)) = F(x, y \wedge z) = F(x,y) = k = F(k,z) = F(F(x,y),z)$.

5.4. $y \in (b,1)$,

5.4.1. If $z \in (k,b]$, then $F(x,F(y,z)) = F(x, y \wedge z) = F(x,z) = k = F(k,z) = F(F(x,y),z)$.

5.4.2. If $z \in (b,1)$, then $F(x,F(y,z)) = F(x,T(y,z)) = k = F(k,z) = F(F(x,y),z)$.

5.4.3. If $z \in I_b^{a,k}$, then $F(x,F(y,z)) = F(x, V(y \wedge b, z \wedge b)) = k = F(k,z) = F(F(x,y),z)$.

5.5. $y \in I_a^{k,b}$,

5.5.1. If $z \in (0,a)$, then $F(x,F(y,z)) = F(x, V(y \vee a, z \vee a)) = F(x, V(y \vee a, a)) = F(x, y \vee a) = V(x \vee a, y \vee a) \vee z = F(V(x \vee a, y \vee a), z) = F(F(x,y),z)$.

5.5.2. If $z \in [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x, V(y \vee a, z \vee a)) = V(x \vee a, V(y \vee a, z \vee a)) = V(V(x \vee a, y \vee a), z \vee a) = F(V(x \vee a, y \vee a), z) = F(F(x,y),z)$.

5.5.3. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,k) = k = F(V(x \vee a, y \vee a), z) = F(F(x,y),z)$.

5.6. $y \in I_b^{a,k}$,

5.6.1. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x, V(y \wedge b, z \wedge b)) = k = F(k,z) = F(F(x,y),z)$.

6. Let $x \in I_b^{a,k}$.

6.1. $y \in (0,a)$,

6.1.1. If $z \in [0,a)$, then $F(x,F(y,z)) = F(x,S(y,z)) = k = F(k,z) = F(F(x,y),z)$.

6.1.2. If $z \in [a,k)$, then $F(x,F(y,z)) = F(x, y \vee z) = F(x,z) = k = F(k,z) = F(F(x,y),z)$.

6.1.3. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,k) = k = F(k,z) = F(F(x,y),z)$.

6.1.4. If $z \in I_a^{k,b}$, then $F(x,F(y,z)) = F(x,V(y \vee a, z \vee a)) = F(x,V(a,z \vee a)) = F(x,z \vee a) = k = F(k,z) = F(F(x,y),z)$.

6.2. $y \in [a,k)$,

6.2.1. If $z \in (0,a)$, then $F(x,F(y,z)) = F(x,y \vee z) = F(x,y) = k = F(k,z) = F(F(x,y),z)$.

6.2.2. If $z \in [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,V(y \vee a, z \vee a)) = k = F(k,z) = F(F(x,y),z)$.

6.2.3. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,k) = k = F(k,z) = F(F(x,y),z)$.

6.3. $y \in (k,b]$,

6.3.1. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,k) = k = F(V(x \wedge b, y \wedge b),z) = F(F(x,y),z)$.

6.3.2. If $z \in (k,b] \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,V(y \wedge b, z \wedge b)) = V(x \wedge b, V(y, z \wedge b)) = V(V(x \wedge b, y \wedge b), z \wedge b) = F(V(x \wedge b, y \wedge b),z) = F(F(x,y),z)$.

6.3.3. If $z \in (b,1)$, then $F(x,F(y,z)) = F(x,y \wedge z) = F(x,y) = V(x \wedge b, y) = V(x \wedge b, y) \wedge z = F(V(x \wedge b, y \wedge b),z) = F(F(x,y),z)$.

6.4. $y \in (b,1)$,

6.4.1. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,k) = k = F(x \wedge b, z) = F(V(x \wedge b, b),z) = F(V(x \wedge b, y \wedge b),z) = F(F(x,y),z)$.

6.4.2. If $z \in (k,b]$, then $F(x,F(y,z)) = F(x,y \wedge z) = F(x,z) = V(x \wedge b, z \wedge b) = F(x \wedge b, z) = F(V(x \wedge b, b),z) = F(V(x \wedge b, y \wedge b),z) = F(F(x,y),z)$.

6.4.3. If $z \in (b,1)$, then $F(x,F(y,z)) = F(x,T(y,z)) = V(x \wedge b, T(y,z) \wedge b) = V(x \wedge b, b) = x \wedge b = x \wedge b \wedge z = F(x \wedge b, z) = F(V(x \wedge b, b),z) = F(V(x \wedge b, y \wedge b),z) = F(F(x,y),z)$.

6.4.4. If $z \in I_b^{a,k}$, then $F(x,F(y,z)) = F(x,V(y \wedge b, z \wedge b)) = F(x,V(b,z \wedge b)) = F(x,z \wedge b) = V(x \wedge b, z \wedge b) = F(x \wedge b, z) = F(V(x \wedge b, b),z) = F(V(x \wedge b, y \wedge b),z) = F(F(x,y),z)$.

6.5. $y \in I_a^{k,b}$,

6.5.1. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,V(y \vee a, z \vee a)) = k = F(k,z) = F(F(x,y),z)$.

6.5.2. If $z \in (k,b] \cup (b,1) \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,k) = k = F(k,z) = F(F(x,y),z)$.

6.6. $y \in I_b^{a,k}$,

6.6.1. If $z \in (0,a) \cup [a,k) \cup I_a^{k,b}$, then $F(x,F(y,z)) = F(x,k) = k = F(V(x \wedge b, y \wedge b),z) = F(F(x,y),z)$.

6.6.2. If $z \in (k,b] \cup I_b^{a,k}$, then $F(x,F(y,z)) = F(x,V(y \wedge b, z \wedge b)) = V(x \wedge b, V(y \wedge b, z \wedge b)) = V(V(x \wedge b, y \wedge b), z \wedge b) = F(V(x \wedge b, y \wedge b),z) = F(F(x,y),z)$.

6.6.3. If $z \in (b,1)$, then $F(x,F(y,z)) = F(x,V(y \wedge b, z \wedge b)) = F(x,V(y \wedge b, b)) = F(x,y \wedge b) = V(x \wedge b, y \wedge b) = V(x \wedge b, y \wedge b) \wedge z = F(V(x \wedge b, y \wedge b),z) = F(F(x,y),z)$.

We have $F(x,0) = S(x,0) = x$ for all $x \in [0,a]$, $F(x,0) = x \vee 0 = x$ for all $x \in [a,k]$ and $F(t,1) = t \wedge 1 = t =$ for all $t \in [k,b]$, $F(t,1) = T(t,1) = t$ for all $t \in [b,1]$. The commutativity of F is obvious from the definition of F. Therefore, F is a nullnorm on L with the zero element k. □

The structure of the nullnorm F given in the formula (10) is summarized in Figure 1.

$I_b^{a,k}$	k	k	$V(x\wedge b, y\wedge b)$	$V(x\wedge b, y\wedge b)$	k	k	$V(x\wedge b, y\wedge b)$
$I_a^{k,b}$	$V(x\vee a, y\vee a)$	$V(x\vee a, y\vee a)$	k	k	k	$V(x\vee a, y\vee a)$	k
I_k^*	k	k	k	k	k	k	k
1	k	k	$x\wedge y$	$T(x,y)$	k	k	$V(x\wedge b, y\wedge b)$
b	k	$V(x,y)$	$V(x,y)$	$x\wedge y$	k	k	$V(x\wedge b, y\wedge b)$
k	$x\vee y$	$V(x,y)$	$V(x,y)$	k	k	$V(x\vee a, y\vee a)$	k
a	$S(x,y)$	$x\vee y$	k	k	k	$V(x\vee a, y\vee a)$	k
0	a	k	b	1	I_k^*	$I_a^{k,b}$	$I_b^{a,k}$

Fig. 1. The structure of the nullnorm F.

Remark 3.9. In Theorem 3.2 $I_a = I_k = \emptyset$. If we put the same constraints ($a = 0$ and $I_b^{a,k} = I_k^{a,b} = I_{k,b}^a = \emptyset$) in the formula (10), it coincides with the formula (3). (Note that the element b in the formula (10) corresponds the element a in the formula (3)). Therefore, Theorem 3.8 is more general than Theorem 3.2.

Similarly, $I_a = I_k = \emptyset$ in Theorem 3.3. If we put the same constraints ($b = 1$ and $I_a^{k,b} = I_k^{a,b} = I_{a,k}^b = \emptyset$) in the formula (10), it coincides with the formula (4). Therefore, Theorem 3.8 is more general than Theorem 3.3.

With the following examples, we show the differences among the method we have proposed and the ones in the literature.

Example 3.10. Consider the bounded lattice ($L_1 = \{0, s, t, u, w, x, y, z, 1\}, \leq, 0, 1$) characterized by the Hasse diagram in Figure 2 and the nullnorm V on $[s, y]$ with zero element u as in Table 1.

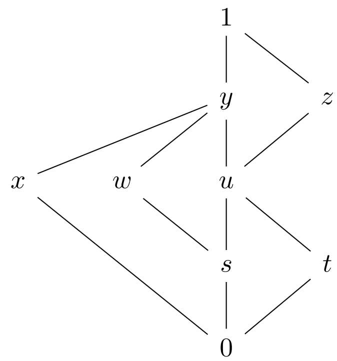

Fig. 2. Lattice diagram of L_1.

V	s	u	w	y
s	s	u	u	u
u	u	u	u	u
w	u	u	u	u
y	u	u	u	y

Tab. 1. The nullnorm V on $[s, y]$.

By applying the construction method in Theorem 3.8 and putting the nullnorm V on $[s, y]$ as in Table 1, when $T_\wedge = T$ and $S_\vee = S$ the corresponding nullnorm F on L_1 is obtained as in Table 2.

F	0	s	t	u	w	x	y	z	1
0	0	s	u	u	u	u	u	u	u
s	s	s	u	u	u	u	u	u	u
t	u	u	u	u	u	u	u	u	u
u	u	u	u	u	u	u	u	u	u
w	u	u	u	u	u	u	u	u	u
x	u	u	u	u	u	u	u	u	u
y	u	u	u	u	u	u	y	u	y
z	u	u	u	u	u	u	u	u	u
1	u	u	u	u	u	u	y	u	1

Tab. 2. The nullnorm F induced by the formula (10) in Theorem 3.8.

By applying construction methods in Theorems 3.1, 3.4 and 3.7, when $T_\wedge = T$ and $S_\vee = S$, we obtain

– $F(x,w) = u \neq s = 0 \vee s = S(x \wedge u, w \wedge u) = V_k^S(x,w)$;

– $F(x,w) = u \neq y = y \wedge y = T(y,y) = T(x \vee u, w \vee u) = V_k^T(x,w)$;

– $F(s,F(x,x)) = F(s,u) = V(s,u) = u \neq s = s \vee 0 = S(s,0) = V_T^{S,R}(s,0) = V_T^{S,R}(s,R(0,0)) = V_T^{S,R}(s,R(x \wedge u, x \wedge u)) = V_T^{S,R}(s,V_T^{S,R}(x,x))$;

– $F(x,w) = u \neq s = 0 \vee s = S(0,s) = S(x \wedge u, w \wedge u) = V_T^S(x,w)$;

– $F(x,w) = u \neq y = y \wedge y = T(y,y) = T(x \vee u, w \vee u) = V_S^T(x,w)$.

Therefore, it is clear that F is different from the nullnorms V_k^S, V_k^T, $V_T^{S,R}$, V_T^S and V_S^T obtained by the formulas (1), (2), (5), (8) and (9), respectively.

Example 3.11. Consider the bounded lattice $(L_1 = \{0,s,t,u,w,x,y,z,1\}, \leq, 0, 1)$ characterized by the Hasse diagram in Figure 2, the nullnorm V on $[0,u]$ with zero element s as in Table 3 and $T = T_\wedge$ on $[u,1]$. Then, we obtain nullnorm F on L_1 as in Table 4 by considering Theorem 3.8.

V	0	s	t	u
0	0	s	s	s
s	s	s	s	s
t	s	s	s	s
u	s	s	s	u

Tab. 3. The nullnorm V on $[0,u]$.

F	0	s	t	u	w	x	y	z	1
0	0	s	s	s	s	s	s	s	s
s	s	s	s	s	s	s	s	s	s
t	s	s	s	s	s	s	s	s	s
u	s	s	s	u	s	s	u	u	u
w	s	s	s	s	s	s	s	s	s
x	s	s	s	s	s	s	s	s	s
y	s	s	s	u	s	s	y	u	y
z	s	s	s	u	s	s	u	z	z
1	s	s	s	u	s	s	y	z	1

Tab. 4. The nullnorm F induced by the formula (10) in Theorem 3.8.

Note that Example 3.11 demonstrates that the method in Theorem 3.8 is much more applicable than the method in Theorem 3.2 since Theorem 3.2 does not produce a nullnorm on L_1 due to the fact that $\{t\} = I_s^{u,y} \neq \emptyset$ (or $\{w\} = I_u^{s,y} \neq \emptyset$), whereas the method in Theorem 3.8 produces.

Example 3.12. Consider the bounded lattice $(L_1 = \{0, s, t, u, w, x, y, z, 1\}, \leq, 0, 1)$ characterized by the Hasse diagram in Figure 2, the nullnorm V on $[u, 1]$ with zero element y as in Table 5 and $S = S_\vee$ on $[0, u]$. Then, we obtain nullnorm F on L_1 as in Table 6.

V	u	y	z	1
u	u	y	y	y
y	y	y	y	y
z	y	y	y	y
1	y	y	y	1

Tab. 5. The nullnorm V on $[u, 1]$.

F	0	s	t	u	w	x	y	z	1
0	0	s	t	u	w	y	y	y	y
s	s	s	u	u	y	y	y	y	y
t	t	u	t	u	y	y	y	y	y
u	u	u	u	u	y	y	y	y	y
w	w	y	y	y	y	y	y	y	y
x	y	y	y	y	y	y	y	y	y
y	y	y	y	y	y	y	y	y	y
z	y	y	y	y	y	y	y	y	y
1	y	y	y	y	y	y	y	y	1

Tab. 6. The nullnorm F induced by the formula (10) in Theorem 3.8.

It is worth noting that Example 3.12 demonstrates that the method in Theorem 3.8 is significantly more applicable than the method in Theorem 3.3, since Theorem 3.3 is not applicable to L_1 due to the fact that $\{f\} = I_y^{s,u} \neq \emptyset$ (or $\{t\} = I_u^{s,y} \neq \emptyset$), whereas the method in Theorem 3.8 constructs a nullnorm.

Example 3.13. Consider the bounded lattice $(L_2 = \{0, c, p, s, t, u, y, 1\}, \leq, 0, 1)$ characterized by the Hasse diagram in Figure 3, the nullnorm V on $[s, y]$ with zero element u, $T_\wedge = T$ and $S_\vee = S$.

Considering the method in Theorem 3.5, we obtain $F(s, c) = u \neq s = s \vee 0 = s \vee (c \wedge u) = F_T(s, c)$.

Therefore, it is clear that F is different from the nullnorm F_T obtained by the formula (6).

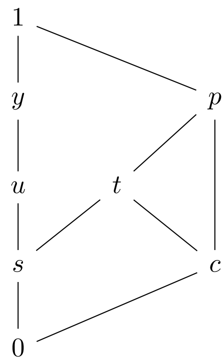

Fig. 3. Lattice diagram of L_2.

Example 3.14. Consider the bounded lattice $(L_3 = \{0, p, s, t, u, w, y, 1\}, \leq, 0, 1)$ characterized by the Hasse diagram in Figure 4, the nullnorm V on $[0, y]$ with zero element p, $T_\wedge = T$ and $S_\vee = S$.

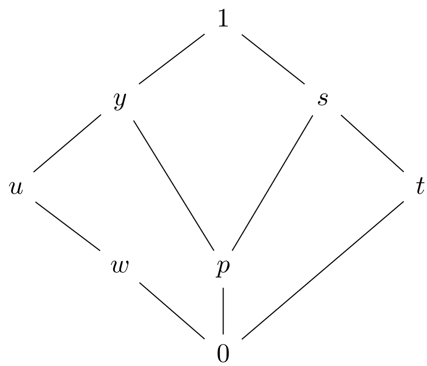

Fig. 4. Lattice diagram of L_3.

Let us apply the method given in Theorem 3.6. It follows that $F(u, w) = p \neq u = u \wedge y \wedge y = (u \vee w) \wedge (u \vee p) \wedge (w \vee p) = F_S(u, w)$.

Therefore, it is clear that F is different from the nullnorm F_S obtained by the formula (7).

Remark 3.15. If we put $T_\wedge = T$ and $S_\vee = S$ in Theorem 3.8 in the formula (10), the following construction method is obtained. The following method is an extension method of a nullnorm V on a subinterval $[a, b]$ of a given bounded lattice L to the whole lattice.

Corollary 3.16. Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b, k \in L \setminus \{0, 1\}$ with $a < k < b$, V be a nullnorm on $[a, b]$ with the zero element k and $V(x, y) = k$ when $(x, y) \in$

$I_k^{a,b} \times [a,b] \cup [a,b] \times I_k^{a,b}$. Then the following function $F^{a,b} : L^2 \to L$ is a nullnorm with the zero element k on L, where

$$F^{a,b}(x,y) = \begin{cases} V(x \vee a, y \vee a) & (x,y) \in [a,b]^2 \cup I_a^{k,b} \times ((0,a) \times [a,k)) \cup \\ & ((0,a) \times [a,k)) \times I_a^{k,b} \cup I_a^{k,b} \times I_a^{k,b}, \\ V(x \wedge b, y \wedge b) & (x,y) \in I_b^{a,k} \times ((k,b] \times (b,1]) \cup \\ & ((k,b] \times (b,1]) \times I_b^{a,k} \cup I_b^{a,k} \times I_b^{a,k}, \\ x \vee y & (x,y) \in [0,a) \times ([0,a) \cup [a,k)) \cup [a,k) \times \\ & [0,a) \cup I_a^{k,b} \times \{0\} \cup \{0\} \times I_a^{k,b}, \\ x \wedge y & (x,y) \in (k,b] \times (b,1] \cup (b,1] \times ((k,b] \cup \\ & (b,1]) \cup I_b^{a,k} \times \{1\} \cup \{1\} \times I_b^{a,k}, \\ k & \text{otherwise.} \end{cases} \tag{11}$$

In the following Corollaries 3.17, 3.18, 3.19 and 3.20 we apply our the formula (10) to some specific lattices.

Corollary 3.17. Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b, k \in L \setminus \{0,1\}$ with $a < k < b$ and V be a nullnorm on $[a,b]$ with the zero element k.

(i) If x is comparable with b for all $x \in L$ in Theorem 3.8, then the nullnorm $F_a : L^2 \to L$ with the zero element k on L is defined as follows:

$$F_a(x,y) = \begin{cases} S(x,y) & (x,y) \in [0,a]^2, \\ T(x,y) & (x,y) \in [b,1]^2, \\ V(x \vee a, y \vee a) & (x,y) \in [a,b]^2 \cup ((0,a) \cup [a,k)) \times I_a^{k,b} \cup \\ & I_a^{k,b} \times ((0,a) \cup [a,k) \cup I_a^{k,b}), \\ x \vee y & (x,y) \in [0,a) \times [a,k) \cup [a,k) \times [0,a) \cup \\ & I_a^{k,b} \times \{0\} \cup \{0\} \times I_a^{k,b}, \\ x \wedge y & (x,y) \in (k,b] \times (b,1] \cup (b,1] \times (k,b], \\ k & \text{otherwise.} \end{cases} \tag{12}$$

(ii) If x is comparable with a for all $x \in L$ in Theorem 3.8, then the nullnorm $F_b : L^2 \to L$ with the zero element k on L is defined as follows:

$$F_b(x,y) = \begin{cases} S(x,y) & (x,y) \in [0,a]^2, \\ T(x,y) & (x,y) \in [b,1]^2, \\ V(x,y) & (x,y) \in [a,b]^2, \\ V(x \wedge b, y \wedge b) & (x,y) \in ((k,b] \cup (b,1)) \times I_b^{a,k} \cup I_b^{a,k} \times \\ & ((k,b] \cup (b,1) \cup I_b^{a,k}), \\ x \vee y & (x,y) \in [0,a) \times [a,k) \cup [a,k) \times [0,a), \\ x \wedge y & (x,y) \in (k,b] \times (b,1] \cup (b,1] \times (k,b] \cup I_b^{a,k} \\ & \times \{1\} \cup \{1\} \times I_b^{a,k}, \\ k & \text{otherwise.} \end{cases} \tag{13}$$

(iii) If x is comparable with a and b for all $x \in L$ in Theorem 3.8, then the nullnorm $F_{a,b} : L^2 \to L$ with the zero element k on L is defined as follows:

$$F_{a,b}(x,y)=\begin{cases} S(x,y) & (x,y)\in[0,a]^2,\\ T(x,y) & (x,y)\in[b,1]^2,\\ V(x,y) & (x,y)\in[a,b]^2,\\ x\vee y & (x,y)\in[0,a)\times[a,k)\cup[a,k)\times[0,a),\\ x\wedge y & (x,y)\in(k,b]\times(b,1]\cup(b,1]\times(k,b],\\ k & \text{otherwise.}\end{cases} \tag{14}$$

Corollary 3.18. Let $(L,\leq,0,1)$ be a bounded lattice, $a,b,k\in L\setminus\{0,1\}$ with $a<k<b$ and V be a nullnorm on $[a,b]$ with the zero element k.

(i) If $a\in L$ is the smallest element of $L\setminus\{0\}$, the nullnorm $F_{a,1}$ obtained as follows by Theorem 3.8:

$$F_{a,1}(x,y)=\begin{cases} T(x,y) & (x,y)\in[b,1]^2,\\ V(x\wedge b,y\wedge b) & (x,y)\in[a,b]^2\cup((k,b]\cup(b,1])\times I_b^k\cup\\ & I_b^k\times((k,b]\cup(b,1]\cup I_b^k),\\ x\vee y & (x,y)\in\{0\}\times(\{0\}\cup[a,k))\cup[a,k)\times\\ & \{0\},\\ x\wedge y & (x,y)\in(k,b]\times(b,1]\cup(b,1]\times(k,b]\cup I_b^k\\ & \times\{1\}\cup\{1\}\times I_b^k,\\ k & \text{otherwise.}\end{cases} \tag{15}$$

(ii) If $b\in L$ is the greatest element of $L\setminus\{1\}$, the nullnorm $F_{b,1}$ obtained as follows by Theorem 3.8:

$$F_{b,1}(x,y)=\begin{cases} S(x,y) & (x,y)\in[0,a]^2,\\ V(x\vee a,y\vee a) & (x,y)\in[a,b]^2\cup((0,a)\cup[a,k))\times I_a^k\cup\\ & I_a^k\times((0,a)\cup[a,k)\cup I_a^k),\\ x\vee y & (x,y)\in[0,a)\times[a,k)\cup[a,k)\times[0,a)\\ & \cup I_a^k\times\{0\}\cup\{0\}\times I_a^k,\\ x\wedge y & (x,y)\in(k,b]\times\{1\}\cup\{1\}\\ & \times((k,b]\cup\{1\}),\\ k & \text{otherwise.}\end{cases} \tag{16}$$

(iii) If $a\in L$ is the smallest element of $L\setminus\{0\}$ and $b\in L$ is the greatest element of $L\setminus\{1\}$, the nullnorm $F_{a,b,1}$ obtained as follows by Theorem 3.8:

$$F_{a,b,1}(x,y)=\begin{cases} V(x,y) & (x,y)\in[a,b]^2,\\ x\vee y & (x,y)\in\{0\}\times(\{0\}\cup[a,k))\cup[a,k)\times\{0\},\\ x\wedge y & (x,y)\in(k,b]\times\{1\}\cup\{1\}\times((k,b]\cup\{1\}),\\ k & \text{otherwise.}\end{cases} \tag{17}$$

If $b = 1$ in the formula (10), we obtain the following nullnorm F^*.

Corollary 3.19. Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, k \in L \setminus \{0, 1\}$ with $a < k$, V be a nullnorm on $[a, 1]$ with the zero element k and satisfying $V(x, y) = k$ when $(x, y) \in I_k^{a,b} \times [a, b] \cup [a, b] \times I_k^{a,b}$. Then the following function $F^* : L^2 \to L$ is a nullnorm with the zero element k on L, where

$$F^*(x,y) = \begin{cases} S(x,y) & (x,y) \in [0,a]^2, \\ V(x \vee a, y \vee a) & (x,y) \in [a,1]^2 \cup ((0,a) \cup [a,k)) \times I_a^k \cup I_a^k \\ & \times((0,a) \cup [a,k) \cup I_a^k), \\ x \vee y & (x,y) \in [0,a) \times [a,k) \cup [a,k) \times [0,a) \cup I_a^k \\ & \times\{0\} \cup \{0\} \times I_a^k, \\ k & \text{otherwise.} \end{cases} \tag{18}$$

If $a = 0$ in the formula (10), we obtain the following nulnorm F_*.

Corollary 3.20. Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, k \in L \setminus \{0, 1\}$ with $k < b$, V be a nullnorm on $[0, b]$ with the zero element k and satisfying $V(x, y) = k$ when $(x, y) \in I_k^{a,b} \times [a, b] \cup [a, b] \times I_k^{a,b}$. Then the following function $F^* : L^2 \to L$ is a nullnorm with the zero element k on L, where

$$F_*(x,y) = \begin{cases} T(x,y) & (x,y) \in [b,1]^2, \\ V(x \wedge b, y \wedge b) & (x,y) \in [0,b]^2 \cup ((k,b] \cup (b,1]) \times I_b^k \cup I_b^k \\ & \times((k,b] \cup (b,1] \cup I_b^k), \\ x \wedge y & (x,y) \in (k,b] \times (b,1] \cup (b,1] \times (k,b] \cup I_b^k \\ & \times\{1\} \cup \{1\} \times I_b^k, \\ k & \text{otherwise.} \end{cases} \tag{19}$$

The method in Theorem 3.8 can be modified as in Theorem 3.21. It is applicable on any bounded lattice.

Theorem 3.21. Let $(L, \leq, 0, 1)$ be a bounded lattice, $a_1, a_2, \ldots, a_n, k, b_1, b_2, \ldots, b_n$ be a finite chain in L such that $0 = a_n < a_{n-1} < \ldots < a_1 < k < b_1 < \ldots < b_{n-1} < b_n = 1$ $(n \in \mathbf{N})$, S_1, S_2,…,S_{n-1} be t-conorms on subintervals $[a_2, a_1]$, $[a_3, a_2]$,…, $[a_n, a_{n-1}]$, T_1, T_2,…,T_{n-1} be t-norms on subintervals $[b_1, b_2]$, $[b_2, b_3]$,…,$[b_{n-1}, b_n]$, respectively, V^* be a nullnorm on $[a_1, b_1]$ with the zero element k satisfying $V^*(x, y) = k$ when $(x, y) \in I_k^{a_1,b_1} \times [a_1, b_1] \cup [a_1, b_1] \times I_k^{a_1,b_1}$. Then the operation $V = V_n : L^2 \to L$ defined recursively as follows is a nullnorm with the zero element k, where $V^* = V_1$ and for $i \in \{2, 3, \ldots, n\}$, the nullnorm $V_i : [a_i, b_i]^2 \to [a_i, b_i]$ with the zero element k is given by

$$V_i(x,y)=\begin{cases} S_{i-1}(x,y) & (x,y)\in[a_i,a_{i-1}]^2,\\ T_{i-1}(x,y) & (x,y)\in[b_{i-1},b_i]^2,\\ V_{i-1}(x\vee a_{i-1},y\vee a_{i-1}) & (x,y)\in[a_{i-1},b_{i-1}]^2\cup((a_i,a_{i-1})\cup\\ & [a_{i-1},k))\times I_{a_{i-1}}^{k,b_{i-1}}\cup I_{a_{i-1}}^{k,b_{i-1}}\times\\ & ((a_i,a_{i-1})\cup[a_{i-1},k)\cup I_{a_{i-1}}^{k,b_{i-1}}),\\ V_{i-1}(x\wedge b_{i-1},y\wedge b_{i-1}) & (x,y)\in((k,b_{i-1}]\cup(b_{i-1},b_i))\times\\ & I_{b_{i-1}}^{a_{i-1},k}\cup I_{b_{i-1}}^{a_{i-1},k}\times((k,b_{i-1}]\\ & \cup(b_{i-1},b_i)\cup I_{b_{i-1}}^{a_{i-1},k}),\\ x\vee y & (x,y)\in[a_i,a_{i-1})\times[a_{i-1},k)\cup\\ & [a_{i-1},k)\times[a_i,a_{i-1})\cup I_{a_{i-1}}^{k,b_{i-1}}\\ & \times\{a_i\}\cup\{a_i\}\times I_{a_{i-1}}^{k,b_{i-1}},\\ x\wedge y & (x,y)\in(k,b_{i-1}]\times(b_{i-1},b_i]\cup\\ & (b_{i-1},b_i]\times(k,b_{i-1}]\cup I_{b_{i-1}}^{a_{i-1},k}\times\\ & \{b_i\}\cup\{b_i\}\times I_{b_{i-1}}^{a_{i-1},k},\\ k & \text{otherwise.} \end{cases}\tag{20}$$

P r o o f. The proof follows easily from Theorem 3.8 by induction and therefore it is omitted. □

In the following Corollary 3.22, we give another construction method for nullnorms on a bounded lattice L when $I_k^{a,b}=\emptyset$. And also, we exemplify it with Example 3.25.

Corollary 3.22. Let $(L,\leq,0,1)$ be a bounded lattice, $a,b,k\in L\setminus\{0,1\}$ with $a<k<b$, S be t-conorm on $[0,a]$, T be a t-norm $[b,1]$, V be a nullnorm on $[a,b]$ with the zero element k and $I_k^{a,b}=\emptyset$ in L. Then the following function $F_k:L^2\to L$ is a nullnorm with the zero element k on L, where

$$F_k(x,y)=\begin{cases} S(x,y) & (x,y)\in[0,a]^2,\\ T(x,y) & (x,y)\in[b,1]^2,\\ V(x\vee a,y\vee a) & (x,y)\in[a,b]^2\cup((0,a)\cup[a,k))\times I_a^{k,b}\cup\\ & I_a^{k,b}\times((0,a)\cup[a,k)\cup I_a^{k,b}),\\ V(x\wedge b,y\wedge b) & (x,y)\in((k,b]\cup(b,1))\times I_b^{a,k}\cup I_b^{a,k}\times\\ & ((k,b]\cup(b,1)\cup I_b^{a,k}),\\ x\vee y & (x,y)\in[0,a)\times[a,k)\cup[a,k)\times[0,a)\cup\\ & I_a^{k,b}\times\{0\}\cup\{0\}\times I_a^{k,b},\\ x\wedge y & (x,y)\in(k,b]\times(b,1]\cup(b,1]\times(k,b]\cup I_b^{a,k}\\ & \times\{1\}\cup\{1\}\times I_b^{a,k},\\ k & \text{otherwise.} \end{cases}\tag{21}$$

P r o o f. The proof follows easily from Theorem 3.8 and therefore it is omitted. □

Remark 3.23. Consider the bounded lattice $(L_1 = \{0, s, t, u, w, x, y, z, 1\}, \leq, 0, 1)$ characterized by the Hasse diagram in Figure 2 and the nullnorm V on $[s, y]$ with zero element u.

It should be noticed that while Corollary 3.22 does not implement on the lattice L_1 since $\{w\} = I_u^{s,y} \neq \emptyset$, the method in Theorem 3.8 produces a nullnorm.

With the following Theorem 3.24, we give another construction method for nullnorms on a bounded lattice L when $I_a = I_b = \emptyset$.

Theorem 3.24. Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b, k \in L \setminus \{0, 1\}$ with $a < k < b$ such that $I_a = I_b = \emptyset$, S be t-conorm on $[0, a]$, T be a t-norm $[b, 1]$, V be a nullnorm on $[a, b]$ with the zero element k. Then the following function $F_\eta : L^2 \to L$ is a nullnorm with the zero element k on L, where

$$F_\eta(x,y) = \begin{cases} S(x,y) & (x,y) \in [0,a]^2, \\ T(x,y) & (x,y) \in [b,1]^2, \\ V(x,y) & (x,y) \in [a,b]^2, \\ V(a,y) & (x,y) \in [0,a) \times I_k^{a,b}, \\ V(x,a) & (x,y) \in I_k^{a,b} \times [0,a), \\ V(b,y) & (x,y) \in (b,1] \times I_k^{a,b}, \\ V(x,b) & (x,y) \in I_k^{a,b} \times (b,1], \\ x \vee y & (x,y) \in [0,a) \times [a,k] \cup [a,k] \times [0,a), \\ x \wedge y & (x,y) \in (k,b] \times (b,1] \cup (b,1] \times (k,b], \\ k & \text{otherwise.} \end{cases} \tag{22}$$

Proof.
(i) *Monotonicity*: Let us show that for any elements $x, y \in L$ with $x \leq y$, $F_\eta(x,z) \leq F_\eta(y,z)$ for all $z \in L$. If x and y are both elements of $[0,a)$ or $[a,k]$ or $(k,b]$ or $(b,1]$ or $I_k^{a,b}$, $F_\eta(x,z) \leq F_\eta(y,z)$ is always satisfied for all $z \in L$ since $x \leq y$. It is clear that $F_\eta(x,z) = k = F_\eta(y,z)$, when $z = k$. The proof is then split into all the remaining possible cases as follows.
1. Let $x \in [0,a)$. If $z \in (0,a)$, since $F(x,z) = S(x,z)$ and for all $y \in L$, $F(y,z)$ equals k or $y \vee z$ or $V(y,a)$, $F(x,z) \leq F(y,z)$. If $z \in [a,k)$, since $F(x,z) = x \vee z = z$ and for all $y \in [k,1] \cup I_k^*$, $F(y,z)$ equals k, $F(x,z) \leq F(y,z)$. Therefore, we omit the case $z \in (0,a) \cup [a,k)$ and the other all possible cases are listed as follows. Moreover, when $y \in [a,k) \cup (k,b] \cup I_k^{a,b}$ and $z \in I_k^{a,b}$, $F_\eta(x,z) = V(a,z) \leq V(y,z) = F_\eta(y,z)$.
1.1. $y \in [a,k)$,
  1.1.1. If $z \in [a,k)$, then $F_\eta(x,z) = x \vee z \leq y \vee z \leq V(y,z) = F_\eta(y,z)$.
  1.1.2. If $z \in (k,b] \cup (b,1]$, then $F_\eta(x,z) = k = F_\eta(y,z)$.
1.2. $y = k$,
  1.2.1. If $z \in [a,k)$, then $F_\eta(x,z) = x \vee z = z \leq k = F_\eta(y,z)$.
  1.2.2. If $z \in (k,b] \cup (b,1]$, then $F_\eta(x,z) = k = F_\eta(y,z)$.
  1.2.3. If $z \in I_k^{a,b}$, then $F_\eta(x,z) = V(a,z) \leq V(k,z) = k = F_\eta(y,z)$.
1.3. $y \in (k,b]$,
  1.3.1. If $z \in [a,k)$, then $F_\eta(x,z) = x \vee z = z \leq k = V(y,z) = F_\eta(y,z)$.
  1.3.2. If $z \in (k,b]$, then $F_\eta(x,z) = k = V(k,z) \leq V(y,z) = F_\eta(y,z)$.

1.3.3. If $z \in (b, 1]$, then $F_\eta(x, z) = k \le y = y \wedge z = F_\eta(y, z)$.
1.4. $y \in (b, 1]$,
1.4.1. If $z \in [a, k)$, then $F_\eta(x, z) = x \vee z = z \le k = F_\eta(y, z)$.
1.4.2. If $z \in (k, b]$, then $F_\eta(x, z) = k \le z = y \wedge z = F_\eta(y, z)$.
1.4.3. If $z \in (b, 1]$, then $F_\eta(x, z) = k \le b \le T(y, z) = F_\eta(y, z)$.
1.4.4. If $z \in I_k^{a,b}$, then $F_\eta(x, z) = V(a, z) \le V(b, z) = F_\eta(y, z)$.
1.5. $y \in I_k^{a,b}$,
1.5.1. If $z \in [a, k)$, then $F_\eta(x, z) = x \vee z = z = V(a, z) \le V(y, z) = F_\eta(y, z)$.
1.5.2. If $z \in (k, b]$, then $F_\eta(x, z) = k = V(y, k) \le V(y, z) = F_\eta(y, z)$.
1.5.3. If $z \in (b, 1]$, then $F_\eta(x, z) = k = V(y, k) \le V(y, b) = F_\eta(y, z)$.

2. Let $x \in [a, k)$. Clearly, when $y \in \{k\} \cup (k, b] \cup (b, 1]$ and $z \in [0, a)$, $F_\eta(x, z) = x \vee z = x \le k = F_\eta(y, z)$.
2.1. $y = k$,
2.1.1. If $z \in [a, k) \cup I_k^{a,b}$, then $F_\eta(x, z) = V(x, z) \le V(k, z) = k = F_\eta(y, z)$.
2.1.2. If $z \in (k, b] \cup (b, 1]$, then $F_\eta(x, z) = k = F_\eta(y, z)$.
2.2. $y \in (k, b]$,
2.2.1. If $z \in [a, k) \cup (k, b] \cup I_k^{a,b}$, then $F_\eta(x, z) = V(x, z) \le V(y, z) = F_\eta(y, z)$.
2.2.1. If $z \in (b, 1]$, then $F_\eta(x, z) = k \le y = y \wedge z = F_\eta(y, z)$.
2.3. $y \in (b, 1]$,
2.3.1. If $z \in [a, k)$, then $F_\eta(x, z) = V(x, z) \le V(k, z) = k = F_\eta(y, z)$.
2.3.2. If $z \in (k, b]$, then $F_\eta(x, z) = V(x, z) = k \le z = y \wedge z = F_\eta(y, z)$.
2.3.3. If $z \in (b, 1]$, then $F_\eta(x, z) = k \le b \le T(y, z) = F_\eta(y, z)$.
2.3.4. If $z \in I_k^{a,b}$, then $F_\eta(x, z) = V(x, z) \le V(b, z) = F_\eta(y, z)$.
2.4. $y \in I_k^{a,b}$,
2.4.1. If $z \in [0, a)$, then $F_\eta(x, z) = x \vee z = x \le y \wedge k = V(y \wedge k, a) \le V(y, a) = F_\eta(y, z)$.
2.4.2. If $z \in [a, k) \cup I_k^{a,b}$, then $F_\eta(x, z) = V(x, z) \le V(y, z) = F_\eta(y, z)$.
2.4.3. If $z \in (k, b]$, then $F_\eta(x, z) = k = V(y, k) \le V(y, z) = F_\eta(y, z)$.
2.4.4. If $z \in (b, 1]$, then $F_\eta(x, z) = k \le V(y, b) = F_\eta(y, z)$.

3. Let $x \in (k, b]$.
3.1. $y \in (b, 1]$,
3.1.1. If $z \in [0, a) \cup [a, k)$, then $F_\eta(x, z) = k = F_\eta(y, z)$.
3.1.2. If $z \in (k, b]$, then $F_\eta(x, z) = V(x, z) \le V(b, z) = z = y \wedge z = F_\eta(y, z)$.
3.1.3. If $z \in (b, 1]$, then $F_\eta(x, z) = x \wedge z = x \le b \le T(y, z) = F_\eta(y, z)$.
3.1.4. If $z \in I_k^{a,b}$, then $F_\eta(x, z) = V(x, z) \le V(b, z) = F_\eta(y, z)$.

4. Let $x \in I_k^{a,b}$. Clearly, when $y \in (k, b] \cup (b, 1]$ and $z \in [0, a)$, $F_\eta(x, z) = V(x, a) \le V(y, k) = k = F_\eta(y, z)$.
4.1. $y \in (k, b]$,
4.1.1. If $z \in [a, k) \cup (k, b] \cup I_k^{a,b}$, then $F_\eta(x, z) = V(x, z) \le V(y, z) = F_\eta(y, z)$.
4.1.2. If $z \in (b, 1]$, then $F_\eta(x, z) = V(x, b) \le V(y, b) = y = y \wedge z = F_\eta(y, z)$.
4.2. $y \in (b, 1]$,
4.2.1. If $z \in [a, k)$, then $F_\eta(x, z) = V(x, z) \le V(y, k) = k = F_\eta(y, z)$.

4.2.2. If $z \in I_k^{a,b}$, then $F_\eta(x,z) = V(x,z) \leq V(b,z) = F_\eta(y,z)$.

4.2.3. If $z \in (k,b]$, then $F_\eta(x,z) = V(x,z) \leq V(b,z) = z = y \wedge z = F_\eta(y,z)$.

4.2.4. If $z \in (b,1]$, then $F_\eta(x,z) = V(x,b) \leq b \leq T(y,z) = F_\eta(y,z)$.

(ii) *Associativity*: We demonstrate that $F_\eta(x, F_\eta(y,z)) = F_\eta(F_\eta(x,y),z)$ for all $x,y,z \in L$. If one of the elements x,y and z is equal to k, it is clear that the equality is always satisfied. Then, the proof is split into all remaining possible cases by considering the relationships between the elements x,y,z,a,b and k as follows.

1. Let $x \in [0,a)$.

1.1. $y \in [0,a)$,

1.1.1. If $z \in [0,a)$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,S(y,z)) = S(x,S(y,z)) = S(S(x,y),z) = F_\eta(S(x,y),z) = F_\eta(F_\eta(x,y),z)$.

1.1.2. If $z \in [a,k)$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,y\vee z) = F_\eta(x,z) = x \vee z = z = S(x,y)\vee z = F_\eta(S(x,y),z) = F_\eta(F_\eta(x,y),z)$.

1.1.3. If $z \in (k,b] \cup (b,1]$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(S(x,y),z) = F_\eta(F_\eta(x,y),z)$.

1.1.4. If $z \in I_k^{a,b}$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,V(a,z)) = V(a,z) = F_\eta(S(x,y),z) = F_\eta(F_\eta(x,y),z)$.

1.2. $y \in [a,k)$,

1.2.1. If $z \in [0,a)$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,y\vee z) = F_\eta(x,y) = x\vee y = y = y\vee z = F_\eta(y,z) = F_\eta(x\vee y,z) = F_\eta(F_\eta(x,y),z)$.

1.2.2. If $z \in [a,k) \cup I_k^{a,b}$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,V(y,z)) = x \vee V(y,z) = V(y,z) = F_\eta(y,z) = F_\eta(x\vee y,z) = F_\eta(F_\eta(x,y),z)$.

1.2.3. If $z \in (k,b] \cup (b,1]$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(y,z) = F_\eta(x\vee y,z) = F_\eta(F_\eta(x,y),z)$.

1.3. $y \in (k,b]$,

1.3.1. If $z \in [0,a) \cup [a,k)$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.

1.3.2. If $z \in (k,b] \cup I_k^{a,b}$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,V(y,z)) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.

1.3.3. If $z \in (b,1]$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,y\wedge z) = F_\eta(x,y) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.

1.4. $y \in (b,1]$,

1.4.1. If $z \in [0,a) \cup [a,k)$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.

1.4.2. If $z \in (k,b]$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,y\wedge z) = F_\eta(x,z) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.

1.4.3. If $z \in (b,1]$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,T(y,z)) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.

1.4.4. If $z \in I_k^{a,b}$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,V(b,z)) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.

1.5. $y \in I_k^{a,b}$,

1.5.1. If $z \in [0,a)$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,V(y,a)) = x \vee V(y,a) = V(y,a) = V(a,y)\vee z = F_\eta(V(a,y),z) = F_\eta(F_\eta(x,y),z)$.

1.5.2. (i) If $z \in [a,k)\cup I_k^{a,b}$ and $V(y,z) \in [a,k]$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,V(y,z)) = x\vee V(y,z) = V(y,z) = V(a,V(y,z)) = V(V(a,y),z) = F_\eta(V(a,y),z) = F_\eta(F_\eta(x,y),z)$.

1.5.2. *(ii)* If $z \in I_k^{a,b}$ and $V(y,z) \in [k,b]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,z)) = k = V(a, V(y,z)) = V(V(a,y),z) = F_\eta(V(a,y),z) = F_\eta(F_\eta(x,y),z)$.

1.5.2. *(iii)* If $z \in I_k^{a,b}$ and $V(y,z) \in I_k^{a,b}$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,z)) = V(a, V(y,z)) = V(V(a,y),z) = F_\eta(V(a,y),z) = F_\eta(F_\eta(x,y),z)$.

1.5.3. If $z \in (k,b]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,z)) = k = V(a, V(y,z)) = F_\eta(V(a,y),z) = F_\eta(F_\eta(x,y),z)$.

1.5.4. If $z \in (b,1]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,b)) = k = F_\eta(V(a,y),z) = F_\eta(F_\eta(x,y),z)$.

2. Let $x \in [a,k)$. Clearly, when $y \in [a,k) \cup I_k^{a,b}$ and $z \in [a,k) \cup I_k^{a,b}$, $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,z)) = V(x, V(y,z)) = V(V(x,y),z) = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.

2.1. $y \in [0,a)$,

2.1.1. If $z \in [0,a)$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, S(y,z)) = x \vee S(y,z) = x = x \vee z = F_\eta(x,z) = F_\eta(x \vee y, z) = F_\eta(F_\eta(x,y),z)$.

2.1.2. If $z \in [a,k)$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, y \vee z) = F_\eta(x,z) = F_\eta(x \vee y, z) = F_\eta(F_\eta(x,y),z)$.

2.1.3. If $z \in (k,b] \cup (b,1]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(x,z) = F_\eta(x \vee y, z) = F_\eta(F_\eta(x,y),z)$.

2.1.4. If $z \in I_k^{a,b}$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(a,z)) = V(x, V(a,z)) = V(V(x,a),z) = V(x,z) = F_\eta(x,z) = F_\eta(x \vee y, z) = F_\eta(F_\eta(x,y),z)$.

2.2. $y \in [a,k)$,

2.2.1. If $z \in [0,a)$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, y \vee z) = F_\eta(x,y) = V(x,y) = V(x,y) \vee z = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.

2.2.2. If $z \in (k,b] \cup (b,1]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.

2.3. $y \in (k,b]$,

2.3.1. If $z \in [0,a) \cup [a,k)$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(k,z) = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.

2.3.2. If $z \in (k,b] \cup I_k^{a,b}$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,z)) = V(x, V(y,z)) = k = F_\eta(k,z) = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.

2.3.3. If $z \in (b,1]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, y \wedge z) = F_\eta(x,y) = V(x,y) = k = F_\eta(k,z) = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.

2.4. $y \in (b,1]$,

2.4.1. If $z \in [0,a) \cup [a,k)$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.

2.4.2. If $z \in (k,b]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, y \wedge z) = F_\eta(x,z) = V(x,z) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.

2.4.3. If $z \in (b,1]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, T(y,z)) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.

2.4.4. If $z \in I_k^{a,b}$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(b,z)) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.

2.5. $y \in I_k^{a,b}$,

2.5.1. If $z \in [0,a)$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,a)) = V(x, V(y,a)) = V(V(x,y),a) = V(x,y) \vee z = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.

2.5.2. If $z \in (b,1]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,b)) = k = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.

3. Let $x \in (k,b]$. Clearly, when $y \in [a,k) \cup (k,b] \cup I_k^{a,b}$ and $z \in (k,b] \cup I_k^{a,b}$, $F_\eta(x,F_\eta(y,z)) = F_\eta(x,V(y,z)) = V(x,V(y,z)) = k = F_\eta(k,z) = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.
3.1. $y \in [0,a)$,
3.1.1. If $z \in [0,a)$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,S(y,z)) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.
3.1.2. If $z \in [a,k)$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,y \vee z) = F_\eta(x,z) = V(x,z) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.
3.1.3. If $z \in (k,b] \cup (b,1]$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.
3.1.4. If $z \in I_k^{a,b}$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,V(a,z)) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$
3.2. $y \in [a,k)$,
3.2.1. If $z \in [0,a)$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,y \vee z) = F_\eta(x,y) = V(x,y) = k = F_\eta(k,z) = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.
3.2.2. If $z \in [a,k)$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,V(y,z)) = V(x,V(y,z)) = k = F_\eta(k,z) = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.
3.2.3. If $z \in (b,1]$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(k,z) = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.
3.3. $y \in (k,b]$,
3.3.1. If $z \in [0,a) \cup [a,k)$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.
3.3.2. If $z \in (b,1]$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,y \wedge z) = F_\eta(x,y) = V(x,y) = V(x,y) \wedge z = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.
3.4. $y \in (b,1]$,
3.4.1. If $z \in [0,a) \cup [a,k)$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(x,z) = F_\eta(x \wedge y,z) = F_\eta(F_\eta(x,y),z)$.
3.4.2. If $z \in (k,b]$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,y \wedge z) = F_\eta(x,z) = F_\eta(x \wedge y,z) = F_\eta(F_\eta(x,y),z)$.
3.4.3. If $z \in (b,1]$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,T(y,z)) = x \wedge T(y,z) = x = x \wedge z = F_\eta(x,z) = F_\eta(x \wedge y,z) = F_\eta(F_\eta(x,y),z)$.
3.4.4. If $z \in I_k^{a,b}$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,V(b,z)) = V(V(x,b),z) = V(x,z) = F_\eta(x,z) = F_\eta(x \wedge y,z) = F_\eta(F_\eta(x,y),z)$.
3.5. $y \in I_k^{a,b}$,
3.5.1. If $z \in [0,a)$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,V(y,a)) = k = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.
3.5.2. If $z \in [a,k)$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,V(y,z)) = V(x,V(y,z)) = V(V(x,y),z) = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.
3.5.3. If $z \in (b,1]$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,V(y,b)) = V(x,V(y,b)) = V(V(x,y),b) = V(x,y) = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.

4. Let $x \in (b,1]$. Clearly, when $y \in [0,a) \cup [a,k)$ and $z \in (k,b] \cup (b,1]$, $F_\eta(x,F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.
4.1. $y \in [0,a)$,
4.1.1. If $z \in [0,a)$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,S(y,z)) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.
4.1.2. If $z \in [a,k)$, then $F_\eta(x,F_\eta(y,z)) = F_\eta(x,y \vee z) = F_\eta(x,z) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.

4.1.3. If $z \in I_k^{a,b}$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(a,z)) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.

4.2. $y \in [a,k)$,

4.2.1. If $z \in [0,a)$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, y \vee z) = F_\eta(x,y) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.

4.2.2. If $z \in [a,k) \cup I_k^{a,b}$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,z)) = k = F_\eta(k,z) = F_\eta(F_\eta(x,y),z)$.

4.3. $y \in (k,b]$,

4.3.1. If $z \in [0,a) \cup [a,k)$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(y,z) = F_\eta(x \wedge y, z) = F_\eta(F_\eta(x,y),z)$.

4.3.2. If $z \in (k,b] \cup I_k^{a,b}$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,z)) = x \wedge V(y,z) = V(y,z) = F_\eta(y,z) = F_\eta(x \wedge y, z) = F_\eta(F_\eta(x,y),z)$.

4.3.3. If $z \in (b,1]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, y \wedge z) = F_\eta(x,y) = y = y \wedge z = F_\eta(y,z) = F_\eta(x \wedge y, z) = F_\eta(F_\eta(x,y),z)$.

4.4. $y \in (b,1]$,

4.4.1. If $z \in [0,a) \cup [a,k)$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(T(x,y),z) = F_\eta(F_\eta(x,y),z)$.

4.4.2. If $z \in (k,b]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, y \wedge z) = F_\eta(x,z) = x \wedge z = z = T(x,y) \wedge z = F_\eta(T(x,y),z) = F_\eta(F_\eta(x,y),z)$.

4.4.3. If $z \in (b,1]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, T(y,z)) = T(x, T(y,z)) = T(T(x,y),z)$ $= F_\eta(T(x,y),z) = F_\eta(F_\eta(x,y),z)$.

4.4.4. If $z \in I_k^{a,b}$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(b,z)) = V(b,z) = F_\eta(T(x,y),z) = F_\eta(F_\eta(x,y),z)$.

4.5. $y \in I_k^{a,b}$,

4.5.1. If $z \in [0,a)$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,a)) = k = F_\eta(V(b,y),z) = F_\eta(F_\eta(x,y),z)$.

4.5.2. *(i)* If $z \in [a,k) \cup I_k^{a,b}$ and $V(y,z) \in [a,k]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,z)) = k = V(b, V(y,z)) = V(V(b,y),z) = F_\eta(V(b,y),z) = F_\eta(F_\eta(x,y),z)$.

4.5.2. *(ii)* If $z \in I_k^{a,b}$ and $V(y,z) \in [k,b]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,z)) = x \wedge V(y,z) = V(y,z) = V(b, V(y,z)) = V(V(b,y),z) = F_\eta(V(b,y),z) = F_\eta(F_\eta(x,y),z)$.

4.5.2. *(iii)* If $z \in I_k^{a,b}$ and $V(y,z) \in I_k^{a,b}$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,z)) = V(b, V(y,z)) = V(V(b,y),z) = F_\eta(V(b,y),z) = F_\eta(F_\eta(x,y),z)$.

4.5.3. If $z \in (k,b]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,z)) = V(y,z) = V(b, V(y,z)) = V(V(b,y),z) = F_\eta(V(b,y),z) = F_\eta(F_\eta(x,y),z)$.

4.5.4. If $z \in (b,1]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,b)) = x \wedge V(y,b) = V(y,b) = V(b,y) = V(b,y) \wedge z = F_\eta(V(b,y),z) = F_\eta(F_\eta(x,y),z)$.

5. Let $x \in I_k^{a,b}$. Clearly, when $y \in [a,k) \cup I_k^{a,b}$ and $z \in [a,k) \cup I_k^{a,b}$, $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,z)) = V(x, V(y,z)) = V(V(x,y),z) = F_\eta(V(x,y),z) = F_\eta(F_\eta(x,y),z)$.

5.1. $y \in [0,a)$,

5.1.1. If $z \in [0,a)$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, S(y,z)) = V(x,a) = F_\eta(V(x,a),z) = F_\eta(F_\eta(x,y),z)$.

5.1.2. If $z \in [a,k)$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, y \vee z) = F_\eta(x,z) = V(x,z) = V(x, V(a,z)) = F_\eta(V(x,a),z) = F_\eta(F_\eta(x,y),z)$.

5.1.3. If $z \in (k,b] \cup (b,1]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(V(x,a),z) = F_\eta(F_\eta(x,y),z)$.

5.1.4. If $z \in I_k^{a,b}$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(a,z)) = V(x, V(a,z)) = V(V(x,a), z)$ $= F_\eta(V(x,a), z) = F_\eta(F_\eta(x,y), z)$.

5.2. $y \in [a,k)$,

5.2.1. If $z \in [0,a)$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, y \vee z) = F_\eta(x,y) = V(x,y) = F_\eta(V(x,y), z) = F_\eta(F_\eta(x,y), z)$.

5.2.2. If $z \in (k,b] \cup (b,1]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(V(x,y), z) = F_\eta(F_\eta(x,y), z)$.

5.3. $y \in (k,b]$,

5.3.1. If $z \in [0,a) \cup [a,k)$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(V(x,y), z) = F_\eta(F_\eta(x,y), z)$.

5.3.2. If $z \in (k,b] \cup I_k^{a,b}$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,z)) = V(x, V(y,z)) = V(V(x,y), z) = F_\eta(V(x,y), z) = F_\eta(F_\eta(x,y), z)$.

5.3.3. If $z \in (b,1]$, then $F_\eta(x, F_\eta(y,z))) = F_\eta(x, y \wedge z) = F_\eta(x,y) = V(x,y) = F_\eta(V(x,y), z) = F_\eta(F_\eta(x,y), z)$.

5.4. $y \in (b,1]$,

5.4.1. If $z \in [0,a) \cup [a,k)$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x,k) = k = F_\eta(V(x,b), z) = F_\eta(F_\eta(x,y), z)$.

5.4.2. If $z \in (k,b]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, y \wedge z) = F_\eta(x,z) = V(x,z) = F_\eta(V(x,b), z) = F_\eta(F_\eta(x,y), z)$.

5.4.3. If $z \in (b,1]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, T(y,z)) = V(x,b) = F_\eta(V(x,b), z) = F_\eta(F_\eta(x,y), z)$.

5.4.4. If $z \in I_k^{a,b}$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(b,z)) = V(x, V(b,z)) = V(V(x,b), z)$ $= F_\eta(V(x,b), z) = F_\eta(F_\eta(x,y), z)$.

5.5. $y \in I_k^{a,b}$,

5.5.1. *(i)* If $z \in [0,a]$ and $V(x,y) \in [a,k]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,a)) = V(x, V(y,a)) = V(V(x,y), a) = V(x,y) = V(x,y) \vee z = F_\eta(V(x,y), z) = F_\eta(F_\eta(x,y), z)$.

5.5.1. *(ii)* If $z \in [0,a]$ and $V(x,y) \in [k,b]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,a)) = V(x, V(y,a)) = V(V(x,y), a) = k = V(V(x,y), a) = F_\eta(V(x,y), z) = F_\eta(F_\eta(x,y), z)$.

5.5.1. *(iii)* If $z \in [0,a]$ and $V(x,y) \in I_k^{a,b}$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,a)) = V(x, V(y,a)) = V(V(x,y), a) = F_\eta(V(x,y), z) = F_\eta(F_\eta(x,y), z)$.

5.5.2. If $z \in (k,b]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,z)) = V(x, V(y,z)) = V(V(x,y), z)$ $= F_\eta(V(x,y), z) = F_\eta(F_\eta(x,y), z)$.

5.5.3. If $z \in (b,1]$, then $F_\eta(x, F_\eta(y,z)) = F_\eta(x, V(y,b)) = V(x, V(y,b)) = V(V(x,y), b)$ $= F_\eta(V(x,y), z) = F_\eta(F_\eta(x,y), z)$.

We have $F_\eta(x,0) = x \vee 0 = x$ for all $x \in [0,k]$ and $F_\eta(t,1) = t \wedge 1 = t$ for all $t \in [k,1]$. The commutativity of F_η is obvious from the definition of F_η. Therefore, F_η is a nullnorm on L with the zero element k. □

The structure of the nullnorm F_η given in formula (22) is summarized in Figure 5.

$I_k^{a,b}$	$V(a,y)$	$V(x,y)$	$V(x,y)$	$V(b,y)$	$V(x,y)$
1	k	k	$x \wedge y$	$T(x,y)$	$V(x,b)$
b	k	$V(x,y)$	$V(x,y)$	$x \wedge y$	$V(x,y)$
k	$x \vee y$	$V(x,y)$	$V(x,y)$	k	$V(x,y)$
a	$S(x,y)$	$x \vee y$	k	k	$V(x,a)$
0	a	k	b	1	$I_k^{a,b}$

Fig. 5. The structure of the nullnorm F_η.

Example 3.25. Consider the bounded lattice $(L_4 = \{0, c, d, p, q, s, t, u, x, y, 1\}, \leq, 0, 1)$ characterized by the Hasse diagram in Figure 6 and the nullnorm V on $[s, y]$ with zero element u as in Table 7.

V	c	d	p	s	u	x	y
c	s	s	u	s	u	x	u
d	s	s	u	s	u	x	u
p	u	u	u	u	u	u	p
s	s	s	u	s	u	x	u
u	u	u	u	u	u	u	u
x	x	x	u	x	u	y	u
y	u	u	p	u	u	u	y

Tab. 7. The nullnorm V on $[s, y]$.

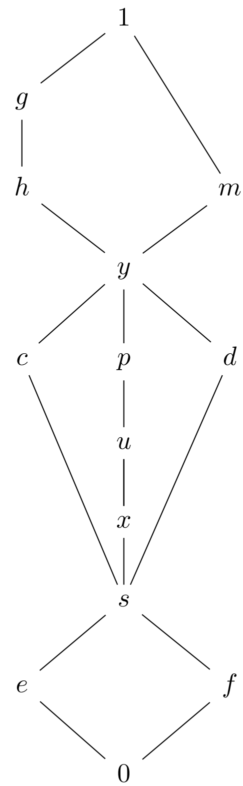

Fig. 6. Lattice diagram of L_4.

By applying the construction method in Theorem 3.24 and putting the nullnorm V on $[s, y]$ as in Table 7, the corresponding nullnorm F_η on L_4 is obtained as in Table 8.

By applying construction methods in Theorems 3.1, 3.4 and 3.7, when $T_W = T$ and $S_W = S$, we obtain

– $F\eta(e, x) = x \neq s = S(e, x) = V_k^S(e, x)$;

– $F\eta(p, g) = p \neq y = T(p, g) = V_k^T(p, g)$;

– $F\eta(f, x) = x \neq s = S(f, x) = V_T^{S,R}(f, x)$;

– $F\eta(e, x) = x \neq s = S(e, x) = V_T^S(e, x)$;

– $F\eta(p, h) = p \neq y = T(p, h) = V_S^T(p, h)$.

Therefore, it is clear that F_η is different from the nullnorms V_k^S, V_k^T, $V_T^{S,R}$, V_T^S and V_S^T obtained by the formulas (1), (2), (5), (8) and (9), respectively.

– It should be noticed that the restrictions of the Theorems 3.2 and 3.3 are not satisfied on L_4, i. e., $\{c\} = I_u^{s,y} \neq \emptyset$ and $\{d\} = I_u^{s,y} \neq \emptyset$ whereas the method in Theorem 3.24 produces.

F_η	0	c	d	e	f	g	h	m	p	s	u	x	y	1
0	0	s	s	e	f	u	u	u	u	s	u	x	u	u
c	s	s	s	s	s	u	u	u	u	s	u	x	u	u
d	s	s	s	s	s	u	u	u	u	s	u	x	u	u
e	e	s	s	s	s	u	u	u	u	s	u	x	u	u
f	f	c	d	s	s	u	u	u	u	s	u	x	u	u
g	u	u	u	u	u	y	y	y	p	u	u	u	y	g
h	u	u	u	u	u	y	y	y	p	u	u	u	y	h
m	u	u	u	u	u	y	y	y	p	u	u	u	y	m
p	u	u	u	u	u	p	p	p	p	u	u	u	p	p
s	s	s	s	s	s	u	u	u	u	s	u	x	u	u
u	u	u	u	u	u	u	u	u	u	u	u	u	u	u
x	x	x	x	x	x	u	u	u	u	x	u	y	u	u
y	u	u	u	u	u	y	y	y	p	u	u	u	y	y
1	u	u	u	u	u	g	h	m	p	u	u	u	y	1

Tab. 8. The nullnorm F_η induced by the formula (22) in Theorem 3.24.

Remark 3.26. If we put $T_\wedge = T$ and $S_\vee = S$ in Theorem 3.24 in the formula (22), the following construction method is obtained. The following method is an extension method of a nullnorm V on a subinterval $[a,b]$ of a given bounded lattice L to the whole lattice.

Corollary 3.27. Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b, k \in L\setminus\{0,1\}$ with $a < k < b$ such that $I_a = I_b = \emptyset$, V be a nullnorm on $[a,b]$ with the zero element k. Then the following function $F_\beta : L^2 \to L$ is a nullnorm with the zero element k on L, where

$$F_\beta(x,y) = \begin{cases} V(x,y) & (x,y) \in [a,b]^2, \\ V(a,y) & (x,y) \in [0,a) \times I_k^{a,b}, \\ V(x,a) & (x,y) \in I_k^{a,b} \times [0,a), \\ V(b,y) & (x,y) \in (b,1] \times I_k^{a,b}, \\ V(x,b) & (x,y) \in I_k^{a,b} \times (b,1], \\ x \vee y & (x,y) \in [0,a]^2 \cup [0,a) \times [a,k] \cup [a,k] \times [0,a), \\ x \wedge y & (x,y) \in [b,1]^2 \cup (k,b] \times (b,1] \cup (b,1] \times (k,b], \\ k & \text{otherwise}. \end{cases} \tag{23}$$

Remark 3.28. Note that the formula (22) in Theorem 3.24 coincides with the formula (14) in Corollary 3.17 under same constraints. It shows that Theorems 3.8 and 3.24 produce same nullnorm on L if L satisfies both constraints in Theorems 3.8 and 3.24.

Theorem 3.29. Let $(L, \leq, 0, 1)$ be a bounded lattice, $a_1, a_2, \ldots, a_m, k, b_1, b_2, \ldots, b_m$ be a finite chain in L such that $0 = a_m < a_{m-1} < \ldots < a_1 < k < b_1 < \ldots < b_{m-1} < b_m = 1$ ($m \in \mathbf{N}$), $I_{a_j} = I_{b_j} = \emptyset$ ($j \in \{1, 2, \ldots, n-1\}$), S_1, S_2,…,S_{m-1} be t-conorms on subintervals $[a_2, a_1]$, $[a_3, a_2]$,...,$[a_m, a_{m-1}]$, T_1, T_2,...,T_{m-1} be t-norms on subintervals $[b_1, b_2]$, $[b_2, b_3]$,...,$[b_{m-1}, b_m]$, respectively, V_* be a nullnorm on $[a_1, b_1]$ with the zero element k. Then the operation $V = V_m : L^2 \to L$ defined recursively as follows is a nullnorm with the zero element k, where $V_* = V_1$ and for $j \in \{2, 3, \dots, m\}$, the nullnorm $V_j : [a_j, b_j]^2 \to [a_j, b_j]$ with the zero element k is given by

$$V_j(x,y) = \begin{cases} S_{j-1}(x,y) & (x,y) \in [a_j, a_{j-1}]^2, \\ T_{j-1}(x,y) & (x,y) \in [b_{j-1}, b_j]^2, \\ V_{j-1}(x,y) & (x,y) \in [a_{j-1}, b_{j-1}]^2, \\ V_{j-1}(a_{j-1}, y) & (x,y) \in [a_j, a_{j-1}) \times I_k^{a_{j-1}, b_{j-1}}, \\ V_{j-1}(x, a_{j-1}) & (x,y) \in I_k^{a_{j-1}, b_{j-1}} \times [a_j, a_{j-1}), \\ V_{j-1}(b_{j-1}, y) & (x,y) \in (b_{j-1}, b_j] \times I_k^{a_{j-1}, b_{j-1}}, \\ V_{j-1}(x, b_{j-1}) & (x,y) \in I_k^{a_{j-1}, b_{j-1}} \times (b_{j-1}, b_j], \\ x \vee y & (x,y) \in [a_j, a_{j-1}) \times [a_{j-1}, k] \cup [a_{j-1}, k] \\ & \times [a_j, a_{j-1}), \\ x \wedge y & (x,y) \in (k, b_{j-1}] \times (b_{j-1}, b_j] \cup (b_{j-1}, b_j] \\ & \times (k, b_{j-1}], \\ k & \text{otherwise.} \end{cases} \tag{24}$$

P r o o f. The proof follows easily from Theorem 3.24 by induction and therefore it is omitted. □

4. CONCLUDING REMARKS

Nullnorms on bounded lattices, particularly the construction of nullnorms on related algebraic structures, are active research areas. There are several known construction methods on bounded lattices, in which the authors use the existence of t-norms, t-conorms etc. The idea of extending a nullnorm on a subinterval of a bounded lattice to the whole lattice is much more general, and also it serves the purpose of constructing a nullnorm. In this way, the ordinal sums of nullnorms can also be obtained. In this paper, we investigate how to extend a given nullnorm on a subinterval of a bounded lattice L to the whole lattice L. We propose two main construction methods for nullnorms on bounded lattices under totally different assumptions. We also apply these methods to some special lattices. We also highlight the differences between our extension methods and the existing construction methods. Moreover, we have generalized our construction methods so that they can be applied to any proper bounded lattice. We believe that our extension methods provide the inspiration for other aggregation functions.

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