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SEMIMULTIPLICATIVE GENERALIZED ARITHMETICAL FUNCTIONS

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Abstract. By a generalized arithmetical function we mean a function from the set of positive integers to a ring with identity, and we say that a generalized arithmetical function f is semimultiplicative if $f(n) = c_f f_M(n/a_f)$, where c_f is a unit in the ring, a_f is a positive integer and f_M is a multiplicative generalized arithmetical function. We study basic properties of these functions, connections to Selberg multiplicative functions and to the Dirichlet convolution. Particular attention is paid to the commutativity and noncommutativity of the function values.

Keywords: generalized arithmetical function; semimultiplicative function; Selberg multiplicative function; Dirichlet convolution

MSC 2020: 11A25

1. Introduction

An arithmetical function is a complex-valued function on the set of positive integers. In this paper we consider functions from the set of positive integers to a ring with identity. We refer to these functions as generalized arithmetical functions. It is sometimes convenient to interpret that the domain of a generalized arithmetical function is the set of positive real numbers but the function value is zero if the argument is not a positive integer.

Bouzeffour, Jedidi and Garayev [4] investigate generalized arithmetical functions whose range is a unital associative algebra. Our approach is, in a sense, similar to that of [4] but we do not consider scalar multiplication. Therefore we confine ourselves to rings with identity. Some authors have studied generalized arithmetical functions whose range is a commutative ring with identity, see, e.g., Chawdhury [6], Elliott [8],

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Ferrero [9], Lu [16], and Popken [17], or an integral domain with identity, see, e.g., Alkan et al. [1], [2]. Many authors have studied integer-valued arithmetical functions, see, e.g., Delange [7]. Of course, the usual complex-valued arithmetical functions are examples of generalized arithmetical functions [3], [20]. Rings, in general, are not commutative. Therefore it is a crucial question in this paper to analyze the role of commutativity (and noncommutativity).

In the study of the arithmetical functions, multiplicative functions are very important. Bouzeffour, Jedidi and Garayev [4] pay much attention to these functions. Multiplicativity, however, can be easily destroyed, e.g., by multiplying the function value by a constant or by dividing the argument by a constant. The concept of semimultiplicativity has been developed to avoid these problems. The concept of a semimultiplicative usual arithmetical function originates with Rearick [18], [19]. The concept of a Selberg multiplicative usual arithmetical function is the same as the concept of a semimultiplicative usual arithmetical function, see [21]. Semimultiplicative and Selberg multiplicative usual arithmetical functions have been studied e.g., in [5], [10], [12], [13], [14], [15], [22].

The object of this paper is to introduce semimultiplicative generalized arithmetical functions (for definition, see Section 3.1). We study basic properties of semimultiplicative generalized arithmetical functions (see Section 3.1), connections between semimultiplicative and Selberg multiplicative generalized arithmetical functions (see Section 3.2) and connections between semimultiplicative generalized arithmetical functions and the Dirichlet convolution (see Section 3.3).

2. Preliminaries

Let R be an arbitrary but fixed ring with identity $\mathbf{1}$. We use $\mathbf{0}$ to denote the zero of R and assume that $\mathbf{0} \neq \mathbf{1}$. We say that $r \in R$ is a unit if it possesses a multiplicative inverse r^{-1}. The group of units of R is denoted as $R^{\times}$. Throughout this paper we consider functions from the set $\mathbb{Z}^{+}$ of positive integers to R which we refer to as generalized arithmetical functions. We denote by $\mathcal{A}$ the class of all generalized arithmetical functions. All functions in this paper are generalized arithmetical functions unless otherwise stated.

2.1. The Dirichlet convolution. The Dirichlet convolution of generalized arithmetical functions f and g is the generalized arithmetical function $f * g$ defined as

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d). \tag{2.1}$$

Considering a generalized arithmetical function as a function of a real variable which is zero if the argument is not a positive integer, we can write

$$(f * g)(n) = \sum_{k=1}^{\infty} f(k)g(n/k). \tag{2.2}$$

It is easy to see that the Dirichlet convolution is assosiative but not commutative in general. The class $\mathcal{A}$ forms a semigroup under the Dirichlet convolution. Therefore the right and left identities are identical and the right and left inverses are identical if they exist.

The generalized arithmetical function δ defined as

$$\delta(1) = \mathbf{1}, \tag{2.3}$$

$$\delta(n) = \mathbf{0} \quad \text{whenever } n \neq 1, \tag{2.4}$$

serves as the identity under the Dirichlet convolution so that

$$\delta * f = f * \delta = f \tag{2.5}$$

for all generalized arithmetical functions f. For a generalized arithmetical function f such that $f(1)$ is a unit, the generalized arithmetical function f^{*-1} is the Dirichlet inverse of f if

$$f * f^{*-1} = f^{*-1} * f = \delta. \tag{2.6}$$

The next theorem can be proved in a similar way as for the usual arithmetical functions (see [3], pp. 30–31). We therefore omit the proof.

Theorem 2.1. *Let f be a generalized arithmetical function such that $f(1)$ is a unit. Then f^{*-1} is uniquely determined and*

$$f^{*-1}(1) = f(1)^{-1}, \tag{2.7}$$

$$f^{*-1}(n) = -f(1)^{-1} \sum_{\substack{d|n \\ d<n}} f(n/d) f^{*-1}(d) \tag{2.8}$$

whenever $n > 1$.

2.2. Multiplicative functions. A generalized arithmetical function f is said to be multiplicative if $f(1) = \mathbf{1}$ and

$$f(mn) = f(m)f(n) \tag{2.9}$$

whenever $(m, n) = 1$.

The next theorems can be proved essentially in the same way as for the usual arithmetical functions (see [3], pp. 35–36). We therefore omit the proofs.

Theorem 2.2. *If f and g are multiplicative and if $f(m)$ and $g(n)$ commute whenever $(m, n) = 1$, then the Dirichlet convolution $f * g$ is multiplicative.*

Theorem 2.3. *If g and $f * g$ are multiplicative and if $f(m)$ and $g(n)$ commute whenever $(m, n) = 1$, then f is multiplicative.*

Theorem 2.4. *If f and $f * g$ are multiplicative and if $f(m)$ and $g(n)$ commute whenever $(m, n) = 1$, then g is multiplicative.*

2.3. Commutative properties of the Dirichlet inverse. One of the difficulties in the study of generalized arithmetical functions is that the multiplication in the ring R is not necessarily commutative. In this section we present some theorems concerning commutative properties of the values of the Dirichlet inverse of generalized arithmetical functions. For this purpose we give the following definition.

Definition 2.1. Let A be a subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$. We say that A is a D-set if A satisfies:

(i) $\langle 1, n\rangle \in A$ for all $n \in \mathbb{Z}^+$,
(ii) if $\langle m, n\rangle \in A$, then $\langle d, n\rangle \in A$ for all $d \mid m$.

Example 2.1. For example, $\mathbb{Z}^+ \times \mathbb{Z}^+$, $\{\langle m, n\rangle \mid (m, n) = 1\}$ and $\{\langle n, m\rangle \mid (m, n) = 1\}$ are D-sets, where (m, n) is the gcd of m and n.

Theorem 2.5. *Let f and g be generalized arithmetical functions such that $f(1)$ is a unit. Let A be a* D-set. *Then*

$$f(m)g(n) = g(n)f(m) \tag{2.10}$$

whenever $\langle m, n\rangle \in A$ if and only if

$$f^{*-1}(m)g(n) = g(n)f^{*-1}(m) \tag{2.11}$$

whenever $\langle m, n\rangle \in A$.

P r o o f. Suppose that (2.10) holds. We prove (2.11) by induction on m. For $m=1$ we have $f^{*-1}(m)=f(1)^{-1}$. In addition, by (2.10), $f(1)g(n)=g(n)f(1)$, and thus (2.11) can be easily seen to be true.

Now, suppose that (2.11) holds whenever $m<k$ ($k\geqslant 2$) and $\langle m,n\rangle\in A$. We prove that (2.11) holds whenever $m=k$ and $\langle k,n\rangle\in A$. By Theorem 2.1, we have

$$g(n)f^{*-1}(k)=g(n)\Big(-f(1)^{-1}\sum_{\substack{d\mid k\\ d<k}}f(k/d)f^{*-1}(d)\Big). \tag{2.12}$$

By the definition of a D-set, $\langle 1,n\rangle\in A$ and, since $\langle k,n\rangle\in A$, we have $\langle k/d,n\rangle\in A$ for all $d\mid k$. Thus

$$g(n)f^{*-1}(k)=-f(1)^{-1}\sum_{\substack{d\mid k\\ d<k}}f(k/d)g(n)f^{*-1}(d). \tag{2.13}$$

Now, by the inductive assumption

$$\begin{aligned} g(n)f^{*-1}(k)&=\Big(-f(1)^{-1}\sum_{\substack{d\mid k\\ d<k}}f(k/d)f^{*-1}(d)\Big)g(n)\\ &=f^{*-1}(k)g(n). \end{aligned} \tag{2.14}$$

Thus, we have proved (2.11).

The converse follows easily by substituting f for f^{*-1} in the first part. This can be carried out since $f^{*-1}(1)$ is a unit and $(f^{*-1})^{*-1}=f$. Now, the proof is complete. □

Theorem 2.6. *If f is multiplicative, then*

$$f(m)f(n)=f(n)f(m), \tag{2.15}$$

$$f^{*-1}(m)f(n)=f(n)f^{*-1}(m), \tag{2.16}$$

$$f^{*-1}(m)f^{*-1}(n)=f^{*-1}(n)f^{*-1}(m) \tag{2.17}$$

whenever $(m,n)=1$.

P r o o f. By the definition of multiplicative functions, $f(m)f(n)=f(mn)=f(nm)=f(n)f(m)$ whenever $(m,n)=1$. Thus (2.15) holds. Equations (2.16) and (2.17) can be established by Theorem 2.5 and Example 2.1. □

E x a m p l e 2.2. Consider generalized arithmetical functions whose range is the ring of real 2×2 matrices. Let g be a usual additive arithmetical function with nonnegative integer values. (Additivity means that $g(mn) = g(m) + g(n)$ whenever $(m, n) = 1$.) Let A be a real 2×2 matrix. Let f be the generalized arithmetical function defined as $f(n) = A^{g(n)}$. Then for $(m, n) = 1$,

$$f(mn) = A^{g(mn)} = A^{g(m)+g(n)} = A^{g(m)}A^{g(n)} = f(m)f(n),$$

which implies that f is multiplicative. Let then A and B be two suitable noncommutative real 2×2 matrices. Let $\omega(n)$ denote the number of distinct prime divisors of n with $\omega(1) = 0$. Let h be the generalized arithmetical function defined as $h(n) = A^{\omega(n)}B^{\omega(n)}$. Then for example, $h(2)h(15) \neq h(15)h(2)$ (for suitable A and B), and therefore, on the basis of (2.15), h is not multiplicative.

We complete this section by a lemma that is not concerned with the Dirichlet inverse but will be used often in Section 3.2. The proof is easy.

Lemma 2.1. *Given generalized arithmetical functions f, g and positive integers m, n such that $f(m)$ is a unit, then*

$$f(m)g(n) = g(n)f(m) \tag{2.18}$$

if and only if

$$f(m)^{-1}g(n) = g(n)f(m)^{-1}. \tag{2.19}$$

3. Properties of semimultiplicative functions

3.1. Definition and basic properties.

Definition 3.1. A generalized arithmetical function f is semimultiplicative if

$$f(n) = c_f f_M(n/a_f), \tag{3.1}$$

where $c_f \in R^\times$, $a_f \in \mathbb{Z}^+$ and f_M is a multiplicative generalized arithmetical function. (As noted in the introduction, $f(x) = 0$ if x is not a positive integer.)

E x a m p l e 3.1. The constant function $f(n) = \mathbf{1}$ is multiplicative, thus it is semimultiplicative with $c_f = 1$, $a_f = 1$ and $f_M = f$. The constant function $g(n) = c$ with $c \neq \mathbf{0}$ is semimultiplicative with $c_g = c$, $a_g = 1$ and $g_M = f$. The function h defined as $h(n) = c$ with $c \neq \mathbf{0}$ for even n, and $h(n) = \mathbf{0}$ for odd n, is semimultiplicative with $c_h = c$, $a_h = 2$ and $h_M = f$.

E x a m p l e 3.2. If f is semimultiplicative, then the functions g and h defined as $g(n) = bf(n)$ and $h(n) = f(n/k)$, where $b \in R^\times$ and $k \in \mathbb{Z}^+$, are semimultiplicative with $c_g = bc_f$ and $a_h = ka_f$.

Theorem 3.1. *Suppose that f is semimultiplicative. Then*

(1) $f(a_f) = c_f$,

(2) *a_f is the smallest value of n such that $f(n) \neq \mathbf{0}$, and $f(n) = \mathbf{0}$ whenever $a_f \nmid n$,*

(3) *a_f is the smallest value of n such that $f(n)$ is a unit, and $f(n)$ is a nonunit whenever $a_f \nmid n$,*

(4) *c_f, a_f and f_M are unique.*

P r o o f. 1) By the definitions of semimultiplicative and multiplicative generalized arithmetical functions, $f(a_f) = c_f f_M(a_f/a_f) = c_f f_M(1) = c_f \mathbf{1} = c_f$.

2) Since $f_M(x) = \mathbf{0}$ if x is not a positive integer, we see that $f(n) = c_f f_M(n/a_f) = \mathbf{0}$ whenever $a_f \nmid n$ and $n = a_f$ is the smallest value of n such that $f(n) \neq \mathbf{0}$.

3) This follows from 1) and 2) and the property that c_f is a unit.

4) a_f is unique, since it is the smallest value of n such that $f(n) \neq \mathbf{0}$. Since $c_f = f(a_f)$, we see that c_f is unique, and therefore, finally, f_M is unique. □

Theorem 3.2. *A generalized arithmetical function f is semimultiplicative if and only if*

$$f(n) = f'_M(n/a'_f)c'_f, \tag{3.2}$$

where $c'_f \in R^\times$, $a'_f \in \mathbb{Z}^+$ and f'_M is a multiplicative generalized arithmetical function. Further, we have $c'_f = c_f$, $a'_f = a_f$ and $f'_M = c_f f_M (c_f)^{-1}$.

P r o o f. If f is semimultiplicative, then

$$f(n) = c_f(n/a_f) = c_f f(n/a_f)(c_f)^{-1}c_f. \tag{3.3}$$

Denoting $c'_f = c_f$, $a'_f = a_f$ and $f'_M = c_f f_M(c_f)^{-1}$ we obtain (3.2). Note that f'_M is multiplicative since $f'_M(mn) = c_f f_M(mn)(c_f)^{-1} = c_f f_M(m) f_M(n)(c_f)^{-1} = c_f f_M(m)(c_f)^{-1} c_f f_M(n)(c_f)^{-1} = f'_M(m) f'_M(n)$ whenever $(m,n) = 1$.

The converse part can be proved in a similar way. □

R e m a r k 3.1. If R is the ring of $m \times m$ complex matrices ($m \geqslant 2$), then the function values $f_M(n)$ and $f'_M(n)$ are similar matrices.

Theorem 3.3. *If f is semimultiplicative, r, $s \in R^\times$ and $k \in \mathbb{Z}^+$, then the generalized arithmetical function g, defined by $g(n) = rf(n/k)s$, is semimultiplicative with $c_g = rc_f s$, $a_g = ka_f$ and $g_M = s^{-1} f_M s$.*

P r o o f. If f is semimultiplicative, then

$$g(n) = rf(n/k)s = rc_f ss^{-1} f_M(n/(ka_f))s. \tag{3.4}$$

Since $s^{-1} f_M s$ is multiplicative, we obtain Theorem 3.3. □

Theorem 3.4. *A generalized arithmetical function f is multiplicative if and only if f is semimultiplicative with $c_f = \mathbf{1}$, $a_f = 1$ and $f_M = f$.*

P r o o f. We can write $f(n) = \mathbf{1} f(n/1)$. □

Theorem 3.5. *A generalized arithmetical function f is multiplicative if and only if f is semimultiplicative with $f(1) = \mathbf{1}$.*

P r o o f. If f is multiplicative, then $f(1) = \mathbf{1}$ according to the definition of multiplicative functions. Let f be semimultiplicative with $f(1) = \mathbf{1}$. Then $a_f = 1$ since a_f is the smallest value of n such that $f(n) \neq \mathbf{0}$. Further, $c_f = \mathbf{1}$, since $c_f = f(a_f) = f(1)$. Thus $f(n) = c_f f_M(n/a_f) = f_M(n)$ and therefore f is multiplicative. □

Theorem 3.6. *Suppose that f is semimultiplicative. Then*

$$f_M(n) = f(a_f)^{-1} f(a_f n) \tag{3.5}$$

and $f(a_f)^{-1} f(a_f x)$ is a multiplicative generalized arithmetical function in x. Analogously,

$$f'_M(n) = f(a_f n) f(a_f)^{-1} \tag{3.6}$$

and $f(a_f x) f(a_f)^{-1}$ is a multiplicative generalized arithmetical function in x.

Conversely, suppose that f is a generalized arithmetical function such that $f(n)$ is not always a nonunit. Let a be a positive integer such that $f(a)$ is a unit. If $f(a)^{-1} f(ax)$ (or $f(ax) f(a)^{-1}$) is a multiplicative generalized arithmetical function in x, then f is semimultiplicative.

P r o o f. Since $f(n) = c_f f_M(n/a_f)$, we have $f(a_f n) = c_f f_M(n)$. Further, $c_f = f(a_f)$ is a unit and thus we obtain $f_M(n) = f(a_f)^{-1} f(a_f n)$. It is known that $f(m) = \mathbf{0}$ whenever $a_f \nmid m$. Thus $f(a_f)^{-1} f(a_f x) = \mathbf{0}$ if x is not a positive integer, that is, $f(a_f)^{-1} f(a_f x)$ is a generalized arithmetical function in x, and since it equals f_M, it is multiplicative. The analogous result for f'_M follows in a similar manner.

To prove the converse part denote $g(x) = f(a)^{-1} f(ax)$. Then $f(n) = f(a) g(n/a)$ and thus f is semimultiplicative with $a_f = a$, $c_f = f(a)$ and $f_M = g$. □

Theorem 3.7. *Let f be a generalized arithmetical function such that $f(n)$ is not always a nonunit. Let a be the smallest value of n such that $f(n)$ is a unit. Then f is semimultiplicative if and only if*

$$f(amn) = f(am)f(a)^{-1}f(an) \quad \text{whenever } (m,n) = 1 \tag{3.7}$$

and

$$f(n) = \mathbf{0} \quad \text{if } a \nmid n. \tag{3.8}$$

P r o o f. If f is semimultiplicative, then according to Theorem 3.6, $f(a)^{-1}f(an)$ is multiplicative, and thus $f(a)^{-1}f(amn) = f(a)^{-1}f(am)f(a)^{-1}f(an)$ whenever $(m,n) = 1$. This shows (3.7). Further, (3.8) holds by Part 2 of Theorem 3.1.

Conversely, (3.7) and (3.8) show that $f(a)^{-1}f(ax)$ is a multiplicative arithmetical function in x and according to the converse part of Theorem 3.6, f is semimultiplicative. □

Corollary 3.1. *Let f be a complex-valued arithmetical function such that f is not identically zero. Let a be the smallest value of n such that $f(n) \neq 0$. Then f is semimultiplicative if and only if*

$$f(amn)f(a) = f(am)f(an) \quad \text{whenever } (m,n) = 1 \tag{3.9}$$

and

$$f(n) = 0 \quad \text{if } a \nmid n. \tag{3.10}$$

Theorem 3.8. *Let f be a generalized arithmetical function such that $f(n)$ is not always a nonunit. Let a be the smallest value of n such that $f(n)$ is a unit. Then f is semimultiplicative if and only if*

$$f(ap_1^{n_1}p_2^{n_2}\dots p_r^{n_r}) = f(ap_1^{n_1})f(a)^{-1}f(ap_2^{n_2})\dots f(a)^{-1}f(ap_r^{n_r}) \tag{3.11}$$

for all primes p_i and all nonnegative integers n_i, and

$$f(n) = \mathbf{0} \quad \text{if } a \nmid n. \tag{3.12}$$

P r o o f. Theorem 3.8 follows directly from Theorem 3.7. □

Corollary 3.2. *Let f be a complex-valued arithmetical function such that f is not identically zero. Let a be the smallest value of n for which $f(n) \neq 0$. Then f is semimultiplicative if and only if*

$$f(ap_1^{n_1}p_2^{n_2}\dots p_r^{n_r})f(a)^{r-1} = f(ap_1^{n_1})f(ap_2^{n_2})\dots f(ap_r^{n_r}) \tag{3.13}$$

for all primes p_i and all nonnegative integers n_i, and (3.10) *holds.*

In the study of semimultiplicative functions we must often check that $f(m)$ and $f(n)$ commute. To make this easier we present the next remark. The proof is straightforward.

R e m a r k 3.2. Suppose that f is semimultiplicative. Then $f(m)$ and $f(n)$ commute for all positive integers m and n if and only if $f(a_f p^\alpha)$ and $f(a_f q^\beta)$ commute for all primes p and q and all nonnegative integers α and β.

3.2. Semimultiplicative and Selberg multiplicative functions. We say that a generalized arithmetical function f is Selberg multiplicative if f can be written as

$$f(n) = \prod_{p\in\mathbb{P}} f_p(n(p)), \tag{3.14}$$

where $n = \prod_{p\in\mathbb{P}} p^{n(p)}$ is the canonical factorization of n and for each prime p, $f_p(n(p))$ is a generalized arithmetical function on $\mathbb{Z}^+ \cup \{0\}$.

E x a m p l e 3.3. If f is Selberg multiplicative, then the functions g and h defined as $g(n) = f(kn)$ and $h(n) = f([k,n])$, where $k \in \mathbb{Z}^+$ and $[k,n]$ is the lcm of k and n, are Selberg multiplicative with $g_p(n(p)) = f_p(k(p)+n(p))$ and $h_p(n(p)) = f_p(\max\{k(p)+n(p)\}$.

Theorem 3.8 can be written in the following way.

Theorem 3.9. *Let f be a generalized arithmetical function such that $f(n)$ is not always a nonunit. Let a be the smallest value of n such that $f(n)$ is a unit. Then f is semimultiplicative if and only if a is the smallest value of n such that $f(n)$ is nonzero and*

$$f(n) = \left(\prod_{p\in\mathbb{P}} f(ap^{n(p)-a(p)})f(a)^{-1}\right)f(a) \tag{3.15}$$

for all $n \in \mathbb{Z}^+$.

P r o o f. Let f be a semimultiplicative generalized arithmetical function. Suppose that $a \nmid n$. Then $f(n) = \mathbf{0}$. Further, $a \nmid ap^{n(p)-a(p)}$ for some $p \in \mathbb{P}$ and thus $f(ap^{n(p)-a(p)}) = \mathbf{0}$. Therefore also the right-hand side of (3.15) is equal to $\mathbf{0}$. Suppose now that $a \mid n$. Then by Theorem 3.8

$$\begin{aligned} f(n) = f(am) &= \Big(\prod_{p|m} f(ap^{m(p)})f(a)^{-1}\Big)f(a) \\ &= \Big(\prod_{p|(n/a)} f(ap^{n(p)-a(p)})f(a)^{-1}\Big)f(a). \end{aligned} \tag{3.16}$$

For $p \nmid (n/a)$ we have $f(ap^{n(p)-a(p)})f(a)^{-1} = f(a)f(a)^{-1} = \mathbf{1}$. Thus (3.16) can be written as (3.15). Since f is semimultiplicative, it follows from Part 2 of Theorem 3.1 that a is the smallest value of n such that $f(n)$ is nonzero.

Conversely, assume that (3.15) holds and a is the smallest value of n such that $f(n)$ is nonzero. If $a \nmid n$, then $ap^{n(p)-a(p)} < a$ for some $p \in \mathbb{P}$ and thus $f(n) = \mathbf{0}$. If $a \mid n$, then (3.15) becomes (3.11). Therefore f is semimultiplicative on the basis of Theorem 3.8. □

Corollary 3.3. *Let f be a generalized arithmetical function such that $f(n)$ is not always a nonunit. If f is semimultiplicative, then f is Selberg multiplicative.*

P r o o f. Let a be the smallest value of n such that $f(n)$ is a unit. Define $f_2(m) = f(a2^{m-a(2)})$ and for primes $p > 2$, $f_p(m) = f(a)^{-1}f(ap^{m-a(p)})$, where $m \in \mathbb{Z}^+ \cup \{0\}$. □

R e m a r k 3.3. For complex-valued arithmetical functions, Selberg multiplicative functions are exactly the same as semimultiplicative functions [11]. For generalized arithmetical functions, a Selberg multiplicative function is not necessarily semimultiplicative. This is shown in the next example.

E x a m p l e 3.4. Let f be a function from $\mathbb{Z}^+$ into the ring of 2×2 real matrices such that $f(2)$ and $f(3)$ do not commute, $f(p^m)$ is the identity matrix for all primes p and nonnegative integers m with $p^m \neq 2, 3$ and

$$f(p_1^{a_1}p_2^{a_2}\dots p_r^{a_r}) = f(p_1^{a_1})f(p_2^{a_2})\dots f(p_r^{a_r}) \tag{3.17}$$

for all primes p_i, nonnegative integers a_i and $r \geqslant 2$. Then f is Selberg multiplicative with $f_p(m) = f(p^m)$ for all primes p and nonnegative integers m. Now, since $f(1)$ is the identity matrix, $a = 1$ is the smallest value of n such that $f(n)$ is a unit. Therefore, on the basis of Theorem 3.7 if f were semimultiplicative, we should have $f(2)f(3) = f(3)f(2)$. But this does not hold and therefore f is not semimultiplicative.

3.3. Semimultiplicative functions and the Dirichlet convolution. In this section we present some properties of semimultiplicative generalized arithmetical functions with respect to the Dirichlet convolution.

Theorem 3.10. *Let two semimultiplicative generalized arithmetical functions* f *and* g *such that* $f(m)$ *and* $g(n)$ *commute whenever* $(a_f, a_g) \leqslant (m, n) \leqslant [a_f, a_g]$, *be given. Then* $f * g$ *is semimultiplicative with* $c_{f*g} = c_f c_g$, $(f * g)_M = f_M * g_M$ *and* $a_{f*g} = a_f a_g$.

R e m a r k 3.4. Rearick [19], p. 51, and Selberg [21], p. 233, noted for complex-valued arithmetical functions that the Dirichlet convolution of two semimultiplicative arithmetical functions is semimultiplicative. See also [11].

P r o o f. By the definition of semimultiplicative generalized arithmetical functions,

$$(3.18) \qquad (f * g)(n) = \sum_{k=1}^{\infty} f(k) g(n/k) = \sum_{k=1}^{\infty} f(k) g(a_g) g_M(n/(k a_g)).$$

For every k with $a_f \mid k$ we have

$$(3.19) \qquad (a_f, a_g) \leqslant (k, a_g) \leqslant [a_f, a_g]$$

and thus, by assumptions of this theorem, $f(k)$ and $g(a_g)$ commute. In addition, for all k with $a_f \nmid k$, $f(k) = \mathbf{0}$ and so $f(k)$ and $g(a_g)$ commute. Thus $f(k)$ and $g(a_g)$ commute for all $k \in \mathbb{Z}^+$. For this reason

$$(3.20) \qquad \begin{aligned} (f * g)(n) &= g(a_g) \sum_{k=1}^{\infty} f(k) g_M(n/(k a_g)) \\ &= g(a_g) \sum_{k=1}^{\infty} f(a_f) f_M(k/a_f) g_M(n/(k a_g)) \\ &= f(a_f) g(a_g) \sum_{k=1}^{\infty} f_M(k/a_f) g_M(n/(k a_g)). \end{aligned}$$

Furthermore, $f_M(k/a_f) = \mathbf{0}$ if $a_f \nmid k$. Therefore

$$(3.21) \qquad \begin{aligned} (f * g)(n) &= f(a_f) g(a_g) \sum_{k=1}^{\infty} f_M(k) g_M(n/(k a_f a_g)) \\ &= f(a_f) g(a_g) (f_M * g_M)(n/(a_f a_g)). \end{aligned}$$

Now, we show that $f_M * g_M$ is multiplicative. By Theorem 2.2, it is enough to show that $f_M(n)$ and $g_M(m)$ commute whenever $(m, n) = 1$. This holds since $f_M(n) = f(a_f)^{-1} f(a_f n)$, $g_M(m) = g(a_g)^{-1} g(a_g m)$ and

$$(a_f, a_g) \leqslant (a_f n, a_g m) \leqslant [a_f, a_g] \tag{3.22}$$

for $(m, n) = 1$. Thus $f_M * g_M$ is multiplicative. Since in addition, $f(a_f)g(a_g)$ is a unit, applying (3.21) and the definition of semimultiplicative generalized arithmetical functions we obtain Theorem 3.10. □

Lemma 3.1. *Assume that g and $f * g$ are semimultiplicative. Then $f(n)$ is not always a nonunit, $a_{f*g} = aa_g$ and $(f * g)(a_{f*g}) = f(a)g(a_g)$, where a is the smallest value of n such that $f(n)$ is a unit.*

Remark 3.5. The positive integer a in Lemma 3.1 is also the smallest value of n such that $f(n) \neq \mathbf{0}$.

Proof. Since $f * g$ is not identically zero, f is not identically zero. Let a be the smallest value of n such that $f(n) \neq \mathbf{0}$. (In fact, we will see later that $f(a)$ is a unit.) Then

$$(f * g)(n) = \sum_{k=1}^{\infty} f(k)g(n/k) = \sum_{k=a}^{\infty} f(k)g(n/k). \tag{3.23}$$

First, let $n < aa_g$. If $k \geqslant a$, then $n/k \leqslant n/a < a_g$ and thus $g(n/k) = \mathbf{0}$. Therefore by (3.23),

$$(f * g)(n) = \mathbf{0}. \tag{3.24}$$

Second, let $n = aa_g$. If $k > a$, then $aa_g/k < a_g$ and thus $g(aa_g/k) = \mathbf{0}$. Therefore by (3.23),

$$(f * g)(aa_g) = f(a)g(a_g). \tag{3.25}$$

Since $g(a_g)$ is a unit and $f(a) \neq \mathbf{0}$, we see that $f(a)g(a_g) \neq \mathbf{0}$, that is,

$$(f * g)(aa_g) \neq \mathbf{0}. \tag{3.26}$$

It follows that

$$a_{f*g} = aa_g, \tag{3.27}$$

$$(f * g)(a_{f*g}) = f(a)g(a_g). \tag{3.28}$$

Moreover, since $(f * g)(a_{f*g})$ and $g(a_g)$ are units, $f(a)$ is a unit. Thus $f(n)$ is not always a nonunit and a is the smallest value of n such that $f(n)$ is a unit. Now the proof is complete. □

Lemma 3.2. *Assume that g and $f * g$ are semimultiplicative, and let a be the smallest value of n such that $f(n)$ is a unit. Then*

$$f(n) = \mathbf{0} \quad \text{if } a \nmid n. \tag{3.29}$$

Proof. If $a = 1$, Lemma 3.2 is trivial. In what follows, assume that $a > 1$. We prove that for all $s \in \mathbb{Z}^+ \cup \{0\}$,

$$f(n) = \mathbf{0} \quad \text{whenever } sa < n < (s+1)a. \tag{3.30}$$

We proceed by induction on s. If $s = 0$, then by the remark to Lemma 3.1, equation (3.30) holds.

Assume that (3.30) holds for all $s < t$ ($t \in \mathbb{Z}^+$). We prove that (3.30) holds for $s = t$. Let m be a positive integer such that

$$ta_g a < m < (t+1)a_g a. \tag{3.31}$$

By Lemma 3.1, $a_g a = a_{f*g}$ and therefore

$$\mathbf{0} = (f * g)(m) = \sum_{k=1}^{\infty} f(k)g(m/k). \tag{3.32}$$

For $k \geqslant (t+1)a$ we have $m/k \leqslant m/((t+1)a) < a_g$ and consequently, $g(m/k) = \mathbf{0}$. Thus, by (3.32),

$$\sum_{k=1}^{(t+1)a-1} f(k)g(m/k) = \mathbf{0}. \tag{3.33}$$

By the inductive assumption,

$$\sum_{i=1}^{t-1} f(ia)g(m/(ia)) + \sum_{k=ta}^{(t+1)a-1} f(k)g(m/k) = \mathbf{0}. \tag{3.34}$$

From (3.31),

$$ta_g/i < m/(ia) < (t+1)a_g/i, \quad i = 1, 2, \ldots, t-1. \tag{3.35}$$

There does not exist any positive integer j such that $m/(ia) = ja_g$ for some $i = 1, 2, \ldots, t-1$. Namely, otherwise we would have $t < ij < t+1$. Thus $g(m/(ia)) = \mathbf{0}$ for all $i = 1, 2, \ldots, t-1$. Therefore, by (3.34),

$$\sum_{k=ta}^{ta+a-1} f(k)g(m/k) = \mathbf{0}. \tag{3.36}$$

Now, we take

$$m = (ta+u)a_g,\ \ u = 1, 2, \dots, a-1,$$

in (3.36). Then

$$\sum_{k=ta}^{ta+a-1} f(k)g((ta+u)a_g/k) = \mathbf{0},\ \ u = 1, 2, \dots, a-1. \tag{3.37}$$

In (3.37),

$$0 < (ta+u)a_g/k < 2a_g, \tag{3.38}$$

and $(ta+u)a_g/k = a_g$ if and only if $k = ta+u$. Thus, in (3.37),

$$g((ta+u)a_g/k) = \mathbf{0} \quad \text{if } k \neq ta+u.$$

We see that

$$f(ta+u)g(a_g) = \mathbf{0},\ \ u = 1, 2, \dots, a-1.$$

Since $g(a_g)$ is a unit, we have

$$f(ta+u) = \mathbf{0},\ \ u = 1, 2, \dots, a-1.$$

In other words,

$$f(n) = \mathbf{0}$$

for all n such that $ta < n < (t+1)a$. Now the induction is completed. □

Theorem 3.11. *Suppose that g and $f * g$ are semimultiplicative such that $f(m)$ and $g(n)$ commute whenever $(a_g, a_{f*g}/a_g) \leqslant (m,n) \leqslant [a_g, a_{f*g}/a_g]$. Then f is semimultiplicative with $c_f = c_{f*g}(c_g)^{-1}$, $f_M = (f*g)_M * (g_M)^{*-1}$ and $a_f = a_{f*g}/a_g$.*

Proof. By Lemma 3.1, there exists a positive integer a that is the smallest value of n such that $f(n)$ is a unit. Denote $h(n) = f(a)^{-1}f(an)$ for all positivve integers n. Then by Theorem 3.6,

$$(h * g_M)(n) = \sum_{k=1}^{\infty} h(k)g_M(n/k) = \sum_{k=1}^{\infty} f(a)^{-1}f(ak)g(a_g)^{-1}g(a_g n/k).$$

By Lemma 3.1,

$$a_{f*g} = aa_g \tag{3.39}$$

and therefore

$$(a_g, a_{f*g}/a_g) \leqslant (ak, a_g) \leqslant [a_g, a_{f*g}/a_g]$$

for all $k \in \mathbb{Z}^+$. On the basis of the assumptions of this theorem, we see that $f(ak)$ and $g(a_g)^{-1}$ commute for all $k \in \mathbb{Z}^+$. In addition, $f(a)^{-1}$ and $g(a_g)^{-1}$ commute. Thus

$$(h * g_M)(n) = (f(a)g(a_g))^{-1} \sum_{k=1}^{\infty} f(ak)g(a_g n/k).$$

Now, by Lemma 3.2,

$$(h * g_M)(n) = (f(a)g(a_g))^{-1} \sum_{k=1}^{\infty} f(k)g(a_g an/k) = (f(a)g(a_g))^{-1}(f * g)(a_g an).$$

Thus, by Lemma 3.1, $(f * g)(a_{f*g}) = f(a)g(a_g)$, and by Theorem 3.6,

$$(f * g)_M = h * g_M.$$

We show that $h(n) = f(a)^{-1}f(an)$ is multiplicative. Since $h * g_M$ and g_M are multiplicative, we aim to apply Theorem 2.3. It suffices to show that $h(n)$ and $g_M(m)$ commute whenever $(m, n) = 1$, that is, $f(a)^{-1}f(an)$ and $g(a_g)^{-1}g(a_g m)$ commute whenever $(m, n) = 1$. It is easy to see that

$$(a_g, a_{f*g}/a_g) = (a_g, a) \leqslant (an, a_g m) \leqslant [a_g, a] = [a_g, a_{f*g}/a_g]$$

whenever $(m, n) = 1$. Thus, on the basis of the assumptions of this theorem, $f(a)^{-1}f(an)$ and $g(a_g)^{-1}g(a_g m)$ commute whenever $(m, n) = 1$. We have now shown that $h(n) = f(a)^{-1}f(an)$ is multiplicative. Since in addition $f(a)^{-1}f(ax)$ is a generalized arithmetical function in x by Lemma 3.2, we can apply Theorem 3.6 to deduce that f is semimultiplicative. Now, certainly, $a_f = a$, that is, by (3.39), $a_f = a_{f*g}/a_g$. Further, we can apply Theorem 3.10 to obtain $c_{f*g} = c_f c_g$ and $(f * g)_M = f_M * g_M$, which completes the proof. □

In a similar way to Theorem 3.11 we obtain:

Theorem 3.12. *Suppose that f and $f * g$ are semimultiplicative such that $f(m)$ and $g(n)$ commute whenever $(a_f, a_{f*g}/a_f) \leqslant (m, n) \leqslant [a_f, a_{f*g}/a_g]$. Then g is semimultiplicative with $c_g = (c_f)^{-1}c_{f*g}$, $g_M = (f_M)^{*-1} * (f * g)_M$ and $a_g = a_{f*g}/a_f$.*

Corollary 3.4. *Suppose that f and g are complex-valued arithmetical functions. If f and $f * g$ are semimultiplicative, then g is semimultiplicative with $c_g = c_{f*g}/c_f$, $g_M = (f_M)^{*-1} * (f * g)_M$ and $a_g = a_{f*g}/a_f$.*

Theorem 3.13. *Suppose that f is semimultiplicative with $a_f = 1$. Further, suppose $f(m)$ and $f(n)$ commute whenever $(m,n) = 1$. Then f^{*-1} is semimultiplicative with $c_{f^{*-1}} = (c_f)^{-1}$, $(f^{*-1})_M = (f_M)^{*-1}$ and $a_{f^{*-1}} = 1$.*

Proof. We have

$$f * f^{*-1} = \delta,$$

where δ is semimultiplicative with $c_\delta = \mathbf{1}$, $\delta_M = \delta$ and $a_\delta = 1$. Thus, by Theorem 2.5, we can conclude that $f(m)$ and $f^{*-1}(n)$ commute whenever $(m,n) = 1$, that is, $(a_f, a_{f*f^{*-1}}/a_f) \leqslant (m,n) \leqslant [a_f, a_{f*f^{*-1}}]$. Now, by Theorem 3.12, we can see that Theorem 3.13 holds. □

Corollary 3.5. *Suppose that f is a complex-valued arithmetical function such that f is semimultiplicative with $a_f = 1$. Then f^{*-1} is semimultiplicative with $c_{f^{*-1}} = 1/c_f$, $(f^{*-1})_M = (f_M)^{*-1}$ and $a_{f^{*-1}} = 1$.*

Theorem 3.14. *If f is multiplicative, then f^{*-1} is multiplicative.*

Proof. By Theorems 2.6, 3.4 and 3.13, f^{*-1} is semimultiplicative with $c_{f^{*-1}} = \mathbf{1}$, $(f^{*-1})_M = f^{*-1}$ and $a_{f^{*-1}} = 1$. Thus, by Theorem 3.4, f^{*-1} is multiplicative. □

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