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Mathematica Bohemica, Vol. 151 (2026), No. 1, 29–55

Persistent URL: <http://dml.cz/dmlcz/153385>

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ON REPDIGITS AS PRODUCT OF k -FIBONACCI
AND k -LUCAS NUMBERS

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Received March 25, 2024. Published online January 27, 2025.

Communicated by Carlos Alexis Gómez Ruíz

Abstract. For an integer $k \geq 2$, let $(F_n^{(k)})_{n \geq -(k-2)}$, $(L_n^{(k)})_{n \geq -(k-2)}$ be k -Fibonacci and k -Lucas sequences, respectively. For these sequences the first k terms are $0, \dots, 0, 1$ and $0, \dots, 0, 2, 1$, respectively, and each term afterwards is the sum of the preceding k terms. In this paper, we determine all possibilities such that $F_n^{(k)} L_m^{(k)}$ can represent a repdigit.

Keywords: k -Fibonacci numbers; k -Lucas numbers; repdigits; linear form in logarithms; reduction method

MSC 2020: 11B39, 11J86

1. INTRODUCTION

For an integer $k \geq 2$, let $(F_n^{(k)})_{n \geq -(k-2)}$ be the k -generalized Fibonacci sequence defined as

$$(1.1) \quad F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \dots + F_{n-k}^{(k)} \quad \forall n \geq 2$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \dots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. Let $(L_n^{(k)})_{n \geq -(k-2)}$ be the k -generalized Lucas sequence following the same recursive pattern as the k -Fibonacci sequence but with the initial conditions $L_{-(k-2)}^{(k)} = L_{-(k-3)}^{(k)} = \dots = L_{-1}^{(k)} = 0$, $L_0^{(k)} = 2$, and $L_1^{(k)} = 1$.

Recall that a positive integer is called a repdigit if it has only one distinct digit in its decimal expansion. In particular, such number has the form $a(10^l - 1)/9$ for some $l \geq 1$ and $1 \leq a \leq 9$. Several problems involving generalized Fibonacci sequences or generalized Lucas sequences and repdigits have been of interest to mathematicians. For instance, Luca [16] and Marques [17] proved that 55 and 44

are the largest repdigits in the sequences $F^{(2)}$ and $F^{(3)}$, respectively. Moreover, Marques conjectured that there are no repdigits with at least two digits, belonging to $F^{(k)}$ for $k > 3$. This conjecture was confirmed by Bravo and Luca in [7]. Also in [16], Luca showed that 11 is the largest repdigit in the sequence $L^{(2)}$. This result was generalized by Bravo and Luca in [8]. For more results see [1], [3], [4], [6], [9], [11], [19], [22], [24] and references therein. In [15], Erduvan and Keskin studied repdigits as product of two Fibonacci and two Lucas numbers. Motivated by these results, we study the solutions of the Diophantine equation

$$(1.2) \quad F_n^{(k)} L_m^{(k)} = \frac{a(10^l - 1)}{9}.$$

We will prove the following results.

Theorem 1.1. *All the solutions of the Diophantine equation (1.2) in positive integers n, m, l, k , and a with $0 \leq m < n$, $k \geq 2$, $l \geq 2$, and $1 \leq a \leq 9$, are*

$$(a, k, l, m, n) \in \{(4, 3, 2, 1, 8), (5, 2, 2, 1, 10), (8, 2, 2, 5, 6), (8, 3, 2, 0, 8)\}.$$

Theorem 1.2. *All the solutions of the Diophantine equation (1.2) in positive integers n, m, l, k , and a with $1 \leq n \leq m$, $k \geq 2$, $l \geq 2$, and $1 \leq a \leq 9$, are*

$$(a, k, l, m, n) \in \{(1, 2, 2, 5, 1), (1, 2, 2, 5, 2), (2, 4, 2, 5, 1), (2, 4, 2, 5, 2), \\ (2, 2, 2, 5, 3), (3, 2, 2, 5, 4), (4, 4, 2, 5, 3), (5, 2, 2, 5, 5), (8, 4, 2, 5, 4)\}.$$

In order to prove our theorems, we use linear forms in logarithms of algebraic numbers and the reduction method due to Dujella-Pethő. We start by introducing necessary results and definitions which are used in the rest of the work.

2. THE TOOLS

2.1. Linear forms in logarithms. For any nonzero algebraic number η of degree d over \mathbb{Q} , whose minimal polynomial over \mathbb{Z} is $a \prod_{j=1}^d (X - \eta^{(j)})$, we denote by

$$h(\eta) = \frac{1}{d} \left(\log |a| + \sum_{j=1}^d \log \max(1, |\eta^{(j)}|) \right)$$

the usual absolute logarithmic height of η . In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log \max\{|p|, q\}$. The following

properties of the function logarithmic height $h(\cdot)$, which will be used in the next sections without special reference, are also known:

$$(2.1) \quad h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2,$$

$$(2.2) \quad h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma),$$

$$(2.3) \quad h(\eta^s) = |s|h(\eta), \quad s \in \mathbb{Z},$$

where η, γ are algebraic numbers. With this notation, Matveev proved the following theorem (see [18]). Here, we use a version due to Bugeaud, Mignotte and Siksek [10], Theorem 9.4.

Theorem 2.1. *Let η_1, \dots, η_s be real algebraic numbers and let b_1, \dots, b_s be nonzero integers. Let $d_{\mathbb{K}}$ be the degree of the number field $\mathbb{Q}(\eta_1, \dots, \eta_s)$ over \mathbb{Q} and let A_j be a positive real number satisfying*

$$A_j \geq \max\{d_{\mathbb{K}}h(\eta_j), |\log \eta_j|, 0.16\} \quad \text{for } j = 1, \dots, s.$$

Assume that

$$B \geq \max\{|b_1|, \dots, |b_s|\}.$$

If $\eta_1^{b_1} \dots \eta_s^{b_s} - 1 \neq 0$, then

$$|\eta_1^{b_1} \dots \eta_s^{b_s} - 1| \geq \exp(-1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_{\mathbb{K}}^2 (1 + \log d_{\mathbb{K}})(1 + \log B)A_1 \dots A_s).$$

2.2. The reduction algorithm. Our second tool is a version of the reduction method of Baker and Davenport [2]. We use a slight variant of the version given by Dujella and Pethő [14].

Lemma 2.2. *Let M be a positive integer and let A, B, μ, γ be given real numbers with $A > 0$ and $B > 1$. Assume that p/q is a convergent of the continued fraction of γ such that $q > 6M$. Let*

$$\varepsilon = \|\mu q\| - M \cdot \|\gamma q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

in positive integers u, v and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

2.3. Properties of k -generalized Fibonacci sequence and k -generalized Lucas sequence. In this subsection, we recall some important facts and properties of the k -Fibonacci and k -Lucas sequences which are used later. The characteristic polynomial of these sequences is

$$\Psi_k(x) = x^k - x^{k-1} - \dots - x - 1.$$

The polynomial $\Psi_k(x)$ is irreducible over $\mathbb{Q}[x]$ and has just one root $\alpha(k)$ outside the unit circle (see, for example, [20], [21] and [25]). The other roots are strictly inside the unit circle. Furthermore, in [25], Wolfram showed that

$$(2.4) \quad 2(1 - 2^{-k}) < \alpha(k) < 2 \quad \text{for all } k \geq 2.$$

To simplify the notation, in general, we omit the dependence on k of α . For $s \geq 2$, let

$$(2.5) \quad f_s(x) := \frac{x - 1}{2 + (s + 1)(x - 2)}.$$

In [5], Bravo, Gómez, and Luca proved the inequalities

$$(2.6) \quad 1/2 < f_k(\alpha) < 3/4 \quad \text{and} \quad |f_k(\alpha^{(i)})| < 1, \quad 2 \leq i \leq k,$$

where $\alpha := \alpha^{(1)}, \dots, \alpha^{(k)}$ are all the zeros of $\Psi_k(x)$. In addition, they proved that the logarithmic height of f satisfies

$$(2.7) \quad h(f_k(\alpha)) < 3 \log k \quad \text{for all } k \geq 2.$$

With the above notation, Dresden and Du [13] showed that

$$(2.8) \quad F_n^{(k)} = \sum_{i=1}^k f_k(\alpha^{(i)}) \alpha^{(i)n-1}$$

and

$$(2.9) \quad |e_k(n)| < \frac{1}{2}, \quad \text{where } e_k(n) = F_n^{(k)} - f_k(\alpha) \alpha^{n-1}$$

for all $n \geq 2 - k$ and $k \geq 2$. Furthermore, for $n \geq 1$ and $k \geq 2$, it was shown in [7] that

$$(2.10) \quad \alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}.$$

Note that the first $k + 1$ nonzero terms in $F^{(k)}$ are powers of two, namely

$$(2.11) \quad F_1^{(k)} = 1, \quad F_2^{(k)} = 1, \quad F_3^{(k)} = 2, \quad F_4^{(k)} = 4, \dots, F_{k+1}^{(k)} = 2^{k-1}.$$

For the k -Lucas sequence, in [8] Bravo and Luca proved that

$$(2.12) \quad L_n^{(k)} = \sum_{i=1}^k (2\alpha^{(i)} - 1) f_k(\alpha^{(i)}) \alpha^{(i)n-1}$$

and

$$(2.13) \quad |e'_k(n)| < \frac{3}{2}, \quad \text{where } e'_k(n) = L_n^{(k)} - (2\alpha - 1) f_k(\alpha) \alpha^{n-1}$$

for all $n \geq 2 - k$ and $k \geq 2$. In addition, for $n \geq 1$ and $k \geq 2$, it was proved in the same paper that

$$(2.14) \quad \alpha^{n-1} \leq L_n^{(k)} \leq 2\alpha^n.$$

Furthermore, note that if $2 \leq n \leq k$, then $L_n^{(k)} = 3 \cdot 2^{n-2}$.

2.4. Other tools. We conclude this section by recalling the following lemmas that we will need later.

Lemma 2.3 ([12], Lemma 2.2). *Let $d, x \in \mathbb{R}$ and $0 < d < 1$. If $|x| < d$, then*

$$|\log(1+x)| < \frac{-\log(1-d)}{d} |x|.$$

Lemma 2.4 ([6], Lemma 3). *If $n < 2^{k/2}$, then*

$$(2.15) \quad F_n^{(k)} = 2^{n-2}(1 + \zeta_1), \quad \text{where } |\zeta_1| < \frac{2}{2^{k/2}}.$$

Lemma 2.5 ([23], Lemma 2.6). *For $k \geq 2$ and $n \geq k + 1$, we have*

$$(2.16) \quad L_n^{(k)} = 3 \times 2^{n-2}(1 + \zeta_2), \quad \text{where } |\zeta_2| < \frac{1}{2^{k/2}}.$$

3. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1 in four steps. Let us start with the first step.

3.1. An upper bound for l in terms of n . We begin our analysis of (1.2) for $2 \leq m < n \leq k + 1$. In this case, we have $F_n^{(k)} = 2^{n-2}$ and $L_m^{(k)} = 3 \cdot 2^{m-2}$, so the equation (1.2) becomes

$$(3.1) \quad 3 \cdot 2^{n+m-4} = \frac{a(10^l - 1)}{9}.$$

For any rational number x , let $\nu_2(x)$ denote the 2-adic valuation of x . Since $\nu_2(a(10^l - 1)/9) \leq 3$, then by comparing the 2-adic valuation on both sides of (3.1), one gets $2 \leq m < n \leq 7$. In this range, equation (3.1) has no solutions. So, from now, we assume that $n \geq k + 2 \geq 4$ and $m \geq 2$.

We next comment on the size of l versus n . By inequalities (2.10), (2.14), and $10^{l-1} < a(10^l - 1)/9$, we obtain

$$10^{l-1} < \frac{a(10^l - 1)}{9} = F_n^{(k)} L_m^{(k)} < \alpha^{n-1} 2\alpha^m < 2\alpha^{n+m-1}.$$

Then, we deduce that

$$l < \frac{\log 2}{\log 10} + (2n - 1) \left(\frac{\log \alpha}{\log 10} \right) + 1 = n \left(\frac{2 \log \alpha}{\log 10} \right) - \left(\frac{\log \alpha}{\log 10} \right) + \frac{\log 2}{\log 10} + 1.$$

Moreover, by using the fact that $3/2 = 2(1 - 2^{-2}) < \alpha < 2$ for $k \geq 2$ (see (2.4)), we get

$$(3.2) \quad l < n.$$

3.2. An inequality for n versus k . Now, we show the following lemma that gives an upper bound for n in terms of k .

Lemma 3.1. *If (a, k, l, m, n) is a solution in positive integers of equation (1.2) with $k \geq 2$ and $n \geq k + 2$, then we have the inequality*

$$(3.3) \quad n < 1.9 \times 10^{30} k^8 \log^5 k.$$

Proof. Using estimates (2.8) and (2.12), equation (1.2) can be rearranged as

$$(3.4) \quad (f_k(\alpha)\alpha^{n-1} + e_k(n))((2\alpha - 1)f_k(\alpha)\alpha^{m-1} + e'_k(m)) = a \left(\frac{10^l - 1}{9} \right),$$

i.e.,

$$(2\alpha - 1)f_k^2(\alpha)\alpha^{n+m-2} - \frac{a10^l}{9} = -e'_k(m)f_k(\alpha)\alpha^{n-1} - e_k(n)(2\alpha - 1)f_k(\alpha)\alpha^{m-1} \\ - e_k(n)e'_k(m) - \frac{a}{9}.$$

Thus, we obtain

$$\left| (2\alpha - 1)f_k^2(\alpha)\alpha^{n+m-2} - \frac{a10^l}{9} \right| \leq \frac{3f_k(\alpha)}{2}\alpha^{n-1} + \frac{(2\alpha - 1)f_k(\alpha)}{2}\alpha^{m-1} + \frac{7}{4}.$$

If we divide both sides by $(2\alpha - 1)f_k^2(\alpha)\alpha^{n+m-2}$ and use the fact that $f_k(\alpha) > 1/2$ and $\alpha > 1.5$, we arrive at

$$(3.5) \quad |\Gamma_1| \leq \frac{3}{2\alpha^{m-1}} + \frac{1}{\alpha^{n-1}} + \frac{7}{2\alpha^{n+m-2}} < \frac{6\alpha}{\alpha^m} < \frac{12}{\alpha^m},$$

where

$$(3.6) \quad \Gamma_1 := \frac{a}{9(2\alpha - 1)f_k^2(\alpha)} \cdot \alpha^{-(n+m-2)} \cdot 10^l - 1.$$

We have $\Gamma_1 \neq 0$, otherwise we would get

$$\frac{a10^l}{9} = (2\alpha - 1)f_k^2(\alpha)\alpha^{n+m-2}.$$

Conjugating the above relation by some automorphism of the Galois group of the decomposition field of $\Psi(x)$ over \mathbb{Q} and then taking absolute values, we get that for any $i \geq 2$, we have

$$\frac{100}{9} \leq \frac{a10^l}{9} = |(2\alpha_i - 1)| \cdot |f_k(\alpha_i)|^2 \cdot |\alpha_i|^{n+m-2} < 3,$$

which leads to a contradiction. Let us apply Theorem 2.1 to Γ_1 given by (3.6). To this end, we take as parameters

$$(\eta_1, b_1) := \left(\frac{a}{9(2\alpha - 1)f_k^2(\alpha)}, 1 \right), \quad (\eta_2, b_2) := (\alpha, -(n + m - 2)), \quad (\eta_3, b_3) := (10, l).$$

The algebraic numbers η_1, η_2, η_3 are elements of the number field $\mathbb{K} := \mathbb{Q}(\alpha)$ and $d_{\mathbb{K}} = k$. Since $h(\eta_2) = (\log \alpha)/k < (\log 2)/k$ and $h(\eta_3) = \log 10$, then we can choose

$$A_2 := \log 2 = \max\{kh(\eta_2), |\log \eta_2|, 0.16\}$$

and

$$A_3 := k \log 10 = \max\{kh(\eta_3), |\log \eta_3|, 0.16\}.$$

Next, we compute A_1 . Using estimate (2.7) and properties (2.2), (2.3), it follows that for all $k \geq 2$

$$h(\eta_1) \leq h\left(\frac{a}{9}\right) + h(2\alpha - 1) + 2h(f_k(\alpha)) < \log 9 + \log 3 + 6 \log k < 12 \log k.$$

Hence, we get

$$A_1 := 12k \log k > \max\{kh(\eta_1), |\log \eta_1|, 0.16\}.$$

Finally, the fact that $m < n$ and inequality (3.2) imply that we can take $B := 2n$. Therefore, according to Theorem 2.1, we have

$$\begin{aligned} |\Gamma_1| &> \exp(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot k^2(1 + \log k)(1 + \log 2n)(12k \log k)(\log 2)(k \log 10)) \\ &> \exp(-2.8 \cdot 10^{12}(1 + \log k)(k^4 \log k)(1 + \log 2n)). \end{aligned}$$

Using the fact $1 + \log k < 2.5 \log k$, which holds for $k \geq 2$, we obtain

$$(3.7) \quad |\Gamma_1| > \exp(-7.1 \cdot 10^{12} k^4 \log^2 k (1 + \log 2n)).$$

Comparing the obtained bounds of $|\Gamma_1|$ gives us

$$(3.8) \quad m \log \alpha < 7.2 \cdot 10^{12} k^4 \log^2 k (1 + \log 2n).$$

We return to equation (1.2) and we rewrite it as

$$(f_k(\alpha)\alpha^{n-1} + e_k(n))L_m^{(k)} = \frac{a(10^l - 1)}{9},$$

i.e.,

$$(3.9) \quad f_k(\alpha)\alpha^{n-1} - \frac{a10^l}{9L_m^{(k)}} = -\frac{a}{9L_m^{(k)}} - e_k(n).$$

So, we get

$$\left| f_k(\alpha)\alpha^{n-1} - \frac{a10^l}{9L_m^{(k)}} \right| \leq \frac{a}{9L_m^{(k)}} + \frac{1}{2} \leq \frac{3}{2}.$$

Dividing by $f_k(\alpha)\alpha^{n-1}$, we get

$$(3.10) \quad |\Gamma_2| \leq \frac{3}{2f_k(\alpha)\alpha^{n-1}} < \frac{3\alpha}{\alpha^n} < \frac{6}{\alpha^n},$$

where

$$(3.11) \quad \Gamma_2 := \frac{a}{9L_m^{(k)} f_k(\alpha)} \cdot \alpha^{-(n-1)} \cdot 10^l - 1.$$

We can prove that $\Gamma_2 \neq 0$ by a similar method used to show that $\Gamma_1 \neq 0$. Now, we apply Theorem 2.1 to Γ_2 by taking

$$(\eta_1, b_1) := \left(\frac{a}{9L_m^{(k)} f_k(\alpha)}, 1 \right), \quad (\eta_2, b_2) := (\alpha, -(n-1)), \quad (\eta_3, b_3) := (10, l).$$

It is clear that $\mathbb{K} := \mathbb{Q}(\alpha)$ contains η_1, η_2, η_3 and has degree $d_{\mathbb{K}} = k$. As calculated before, we take

$$A_2 := \log 2, \quad A_3 := k \log 10, \quad \text{and} \quad B := 2n.$$

We need to compute A_1 . Estimates (2.7), (3.8), and properties (2.1)–(2.3) imply that, for all $k \geq 2$, we have

$$\begin{aligned} h(\eta_1) &\leq h\left(\frac{a}{9}\right) + h(L_m^{(k)}) + h(f_k(\alpha)) < \log 9 + \log 2 + m \log \alpha + 3 \log k \\ &< 7.3 \cdot 10^{12} k^4 \log^2 k (1 + \log 2n). \end{aligned}$$

On the other hand, since

$$\eta_1 := \frac{a}{9L_m^{(k)} f_k(\alpha)} < 2 \quad \text{and} \quad \eta_1^{-1} = \frac{9L_m^{(k)} f_k(\alpha)}{a} < \frac{27\alpha^m}{2},$$

then, by (3.8), we get

$$|\log \eta_1| < m \log \alpha + \log 13.5 < 7.3 \cdot 10^{12} k^4 \log^2 k (1 + \log 2n).$$

Thus, we conclude that

$$\max\{kh(\eta_1), |\log \eta_1|, 0.16\} < 7.3 \cdot 10^{12} k^5 \log^2 k (1 + \log 2n) := A_1.$$

Applying Theorem 2.1 and comparing the resulting inequality with (3.10), we obtain

$$n \log \alpha < 2.21 \cdot 10^{25} k^8 \log^3 k \log^2 n,$$

where we have used the facts $1 + \log k < 2.5 \log k$ and $1 + \log 2n < 2.3 \log n$ which hold for $k \geq 2$ and $n \geq 4$. So, we deduce that

$$(3.12) \quad \frac{n}{\log^2 n} < 5.5 \cdot 10^{25} k^8 \log^3 k.$$

It is easy to check that the inequality

$$(3.13) \quad \frac{x}{\log^2 x} < A \text{ implies } x < 4A \log^2 A \text{ holds, whenever } A \geq 100.$$

Thus, putting $A := 5.5 \cdot 10^{25} k^8 \log^3 k$ in inequality (3.13) and using $59.3 + 8 \log k + 3 \log \log k < 92 \log k$, which holds for all $k \geq 2$, we obtain

$$\begin{aligned} n &< 4(5.5 \cdot 10^{25} k^8 \log^3 k)(\log(5.5 \cdot 10^{25} k^8 \log^3 k))^2 \\ &< (2.21 \cdot 10^{26} k^8 \log^3 k)(59.3 + 8 \log k + 3 \log \log k)^2 \\ &< 1.9 \cdot 10^{30} k^8 \log^5 k. \end{aligned}$$

This gives (3.3) and completes the proof of Lemma 3.1. \square

3.3. The case $2 \leq k \leq 440$. For this subsection, we consider $k \in [2, 440]$. Put

$$(3.14) \quad \Lambda_1 := \log(\Gamma_1 + 1) = l \log 10 - (n + m - 2) \log \alpha + \log(a/(9(2\alpha - 1)f_k^2(\alpha))).$$

Suppose that $m \geq 10$, so by estimate (3.5) and the fact that $\alpha > 1.5$, we have $|\Gamma_1| < 0.21$. Taking $d = 0.21$ in Lemma 2.3, we obtain

$$(3.15) \quad |\Lambda_1| < \frac{-\log 0.79}{0.21} \cdot |\Gamma_1| < 13.5 \cdot \alpha^{-m}.$$

So, we get

$$(3.16) \quad \left| l \cdot \frac{\log 10}{\log \alpha} - (n + m - 2) + \frac{\log(a/(9(2\alpha - 1)f_k^2(\alpha)))}{\log \alpha} \right| < 33.3 \cdot \alpha^{-m}.$$

For all $a \in \{1, \dots, 9\}$, we apply Lemma 2.2 to Λ_1 with the parameters

$$\gamma := \frac{\log 10}{\log \alpha}, \quad \mu := \frac{\log(a/(9(2\alpha - 1)f_k^2(\alpha)))}{\log \alpha}, \quad \text{and} \quad (A, B) := (33.3, \alpha).$$

For each $k \in [2, 440]$ and $a \in \{1, \dots, 9\}$, we find a good approximation of γ and a convergent p_l/q_l of the continued fraction of γ such that $q_l > 6M_k$ and $\varepsilon = \varepsilon(k) = \|\mu q\| - M_k \|\gamma q\| > 0$, where $M_k := \lfloor 1.9 \times 10^{30} k^8 \log^5 k \rfloor$, which is an upper bound of l from Lemma 3.1. Using Mathematica, we see that q_{121} satisfies the assumptions of Lemma 2.2. After doing this, we apply Lemma 2.2 to inequality (3.16). A computer program with Mathematica shows for $k = 417$ and $a = 7$ that $\varepsilon > 0.000064$ and the maximum value of $\log(Aq/\varepsilon)/\log B$, for $k \in [2, 440]$ and $a \in \{1, \dots, 9\}$, is 204.861, which is an upper bound of m by Lemma 2.2.

For $2 \leq m < 205$, we consider

$$(3.17) \quad \Lambda_2 := \log(\Gamma_2 + 1) = l \log 10 - (n - 1) \log \alpha + \log(a/(9L_m^{(k)} f_k(\alpha))).$$

Suppose that $n \geq 10$, so by estimate (3.10) and the fact that $\alpha > 1.5$, we have $|\Gamma_2| < 0.11$. Taking $d = 0.11$ in Lemma 2.3, we obtain

$$|\Lambda_2| < \frac{-\log 0.89}{0.11} \cdot |\Gamma_2| < 6.4 \cdot \alpha^{-n}.$$

So, we get

$$(3.18) \quad \left| l \cdot \frac{\log 10}{\log \alpha} - (n - 1) - \frac{\log(a/(9L_m^{(k)} f_k(\alpha)))}{\log \alpha} \right| < 15.8 \cdot \alpha^{-n}.$$

For all $a \in \{1, \dots, 9\}$ and $2 \leq m \leq 204$, we apply Lemma 2.2 to Λ_2 by fixing

$$\gamma := \frac{\log 10}{\log \alpha}, \quad \mu := \frac{\log(a/(9L_m^{(k)} f_k(\alpha)))}{\log \alpha}, \quad \text{and} \quad (A, B) := (15.8, \alpha).$$

Again, for each $(k, m) \in [2, 440] \times [2, 204]$ and $a \in \{1, \dots, 9\}$, we find a good approximation of γ and a convergent p_l/q_l of the continued fraction of γ such that $q_l > 6M_k$ and $\varepsilon = \varepsilon(k) = \|\mu q\| - M_k \|\gamma q\| > 0$, where $M_k := \lfloor 1.9 \times 10^{30} k^8 \log^5 k \rfloor$, which is an upper bound of l from Lemma 3.1. With the help of Mathematica, we see again that q_{121} satisfies the assumptions of Lemma 2.2. After doing this, we apply Lemma 2.2 to inequality (3.18). A computer program with Mathematica revealed for $k = 417$, $a = 7$ and for all $2 \leq m \leq 204$ that $\varepsilon > 0.000064$ and the maximum value of $\log(Aq/\varepsilon)/\log B$, for $(k, m) \in [2, 440] \times [2, 204]$ and $a \in \{1, \dots, 9\}$, is 203.786, which is an upper bound of n by Lemma 2.2.

Hence, we deduce that the possible solutions (a, k, l, m, n) of equation (1.2), for which $k \in [2, 440]$ and $a \in \{1, \dots, 9\}$, satisfy $m < n \leq 203$. Therefore, we use inequality (3.2) to obtain $l \leq 202$.

Finally, we use Mathematica to compare $F_n^{(k)} L_m^{(k)}$ and $a(10^l - 1)/9$, for the ranges $k + 2 \leq n \leq 203$, $m < n$, $1 \leq a \leq 9$ and $2 \leq l \leq 202$ with $l < n$, and check that the only solutions of equation (1.2) are those listed in Theorem 1.1.

3.4. The case $k > 440$. In this subsection, we analyze the case $k > 440$. For such k , it is easy to check that

$$m \leq n < 1.9 \cdot 10^{30} k^8 \log^5 k < 2^{k/2}.$$

Thus, by Lemmas 2.4 and 2.5, $F_n^{(k)}$ and $L_m^{(k)}$ can be rewritten as

$$(3.19) \quad F_n^{(k)} = 2^{n-2}(1 + \zeta_1), \quad \text{where } |\zeta_1| < \frac{2}{2^{k/2}}$$

and

$$(3.20) \quad L_m^{(k)} = 3 \cdot 2^{m-2}(1 + \zeta_2), \quad \text{where } |\zeta_2| < \frac{1}{2^{k/2}},$$

respectively. Substituting (3.19) and (3.20) in (1.2), we obtain

$$3 \cdot 2^{n+m-4}(1 + \zeta_1)(1 + \zeta_2) = \frac{a(10^l - 1)}{9}.$$

Hence, we get

$$\left| 3 \cdot 2^{n+m-4} - \frac{a10^l}{9} \right| \leq \frac{9 \cdot 2^{n+m-4}}{2^{k/2}} + \frac{6 \cdot 2^{n+m-4}}{2^k} + 1.$$

Consequently, the above inequality and the fact that $n \geq 4$ give

$$(3.21) \quad \left| 1 - \frac{a}{27} \cdot 10^l \cdot 2^{-(n+m+4)} \right| < \frac{3}{2^{k/2}} + \frac{2}{2^k} + \frac{1}{3 \cdot 2^m} < \frac{5.5}{2^{\min\{k/2, m\}}}.$$

Put

$$\Gamma_3 = \frac{a}{27} \cdot 2^{-(n+m-4)} \cdot 10^l - 1.$$

We have $\Gamma_3 \neq 0$, because if $\Gamma_3 = 0$ we get $a \cdot 10^l = 27 \cdot 2^{n+m-4}$. This implies that 5 divides $27 \cdot 2^{n+m-4}$, which is impossible. Now, we put

$$(\eta_1, b_1) := \left(\frac{a}{27}, 1 \right), \quad (\eta_2, b_2) := (2, -(n+m-4)), \quad (\eta_3, b_3) := (10, l).$$

Then, we obtain

$$h(\eta_1) = \log 27, \quad h(\eta_2) = \log 2, \quad \text{and} \quad h(\eta_3) = \log 10.$$

Note that $\eta_1, \eta_2, \eta_3 \in \mathbb{K} := \mathbb{Q}$. Thus $d_{\mathbb{K}} = 1$. So, we take

$$A_1 := \log 27, \quad A_2 := \log 2, \quad A_3 := \log 10, \quad \text{and} \quad B := 2n.$$

Thus, applying Theorem 2.1, we get

$$|\Gamma_3| > \exp(-1.74 \cdot 10^{12} \log n),$$

where we have used the fact that $1 + \log 2n < 2.3 \log n$ for all $n \geq 4$. By comparing the resulting inequality with (3.21), we obtain

$$\min\{k/2, m\} < 2.52 \cdot 10^{12} \log n.$$

By Lemma 3.1 and using the fact that $69.72 + 8 \log k + 5 \log \log k < 21 \log k$ for $k > 440$, we get

$$\begin{aligned} \min\{k/2, m\} &< 2.52 \cdot 10^{12} \log(1.9 \cdot 10^{30} k^8 \log^5 k) \\ &< 2.52 \cdot 10^{12} (69.72 + 8 \log k + 5 \log \log k) \\ &< 5.3 \cdot 10^{13} \log k. \end{aligned}$$

If $\min\{k/2, m\} = k/2$, then we get $k < 1.1 \cdot 10^{14} \log k$. Solving this inequality and using Lemma 3.1 we conclude that

$$(3.22) \quad k < 4 \cdot 10^{15} \quad \text{and} \quad n < 7.5 \cdot 10^{162}.$$

If $\min\{k/2, m\} = m$, we obtain in this case that

$$(3.23) \quad m < 5.3 \cdot 10^{13} \log k.$$

Now, we go back to (3.9) and we rewrite it as

$$2^{n-2} - \frac{a10^l}{9L_m^{(k)}} = \frac{-a}{9L_m^{(k)}} - 2^{n-2} \zeta_1.$$

Thus, we obtain

$$\left| 2^{n-2} - \frac{a10^l}{9L_m^{(k)}} \right| \leq \frac{2^{n-1}}{2^{k/2}} + 1.$$

Dividing by 2^{n-2} and using the fact that $n \geq k + 2$, we get

$$(3.24) \quad |\Gamma_4| \leq \frac{1}{2^{n-2}} + \frac{2}{2^{k/2}} < \frac{1}{2^k} + \frac{2}{2^{k/2}} < \frac{3}{2^{k/2}},$$

where

$$(3.25) \quad \Gamma_4 := \frac{a}{9L_m^{(k)}} \cdot 2^{-(n-2)} \cdot 10^l - 1.$$

Note that $\Gamma_4 \neq 0$, since otherwise we would get $a10^l/(9L_m^{(k)}) = 2^{n-2}$. If $a \in \{1, \dots, 8\}$ then it is obvious that the left side cannot be an integer. If $a = 9$, then we have $10^l/L_m^{(k)} = 2^{n-2}$ or in this case $m < k/2$ implies that $m < k$ and so $L_m^{(k)} = 3 \cdot 2^{m-2}$.

Then we would get $10^l/(3 \cdot 2^{m-2}) = 2^{n-2}$ and this leads to a contradiction. Therefore, $\Gamma_4 \neq 0$. Now, we apply Theorem 2.1 by fixing

$$(\eta_1, b_1) := \left(\frac{a}{9L_m^{(k)}}, 1 \right), \quad (\eta_2, b_2) := (2, -(n-2)), \quad (\eta_3, b_3) := (10, l).$$

As calculated before, we take $A_2 := \log 2$ and $A_3 := \log 10$. We take $B := n$. Next, we estimate $h(\eta_1)$. By the fact that $L_m^{(k)} < 2\alpha^m$ and inequality (3.23), we obtain

$$h(\eta_1) \leq h\left(\frac{a}{9}\right) + h(L_m^{(k)}) < \log 9 + \log 2 + m \log \alpha < 3.7 \cdot 10^{13} \log k.$$

So, we choose $A_1 := 3.7 \cdot 10^{13} \log k$. Therefore, Theorem 2.1 gives

$$(3.26) \quad |\Gamma_4| > \exp(-1.53 \cdot 10^{25} \log k \log n),$$

where we used the fact that $1 + \log n < 1.8 \log n$, which holds for $n \geq 4$. From (3.24) and (3.26), it results in

$$k < 4.42 \cdot 10^{25} \log k \log n.$$

By Lemma 3.1 and using the fact that $69.72 + 8 \log k + 5 \log \log k < 21 \log k$ for all $k > 440$, we get

$$\begin{aligned} k &< 4.42 \cdot 10^{25} \log k (\log(1.9 \cdot 10^{30} k^8 \log^5 k)) \\ &< 4.42 \cdot 10^{25} \log k (69.72 + 8 \log k + 5 \log \log k) \\ &< 9.3 \cdot 10^{26} \log^2 k. \end{aligned}$$

Solving this inequality and using Lemma 3.1, we obtain

$$(3.27) \quad k < 4.64 \cdot 10^{30} \quad \text{and} \quad n < 7.2 \cdot 10^{284}.$$

Comparing (3.22) and (3.27), we conclude that inequalities (3.27) always hold. The obtained bounds are very large, next we will reduce them. Put

$$(3.28) \quad \Lambda_3 := \log(\Gamma_3 + 1) = l \log 10 - (n + m - 4) \log 2 + \log\left(\frac{a}{27}\right).$$

Assume that $m \geq 10$, then we get $|\Gamma_3| < 0.02$. Taking $d = 0.02$ in Lemma 2.3, we obtain

$$|\Lambda_3| < \frac{-\log 0.98}{0.02} \cdot |\Gamma_3| < 6 \cdot 2^{-\min\{k/2, m\}}.$$

So, we deduce that

$$(3.29) \quad \left| l \cdot \frac{\log 10}{\log 2} - (n + m - 4) + \frac{\log(a/27)}{\log 2} \right| < 9 \cdot 2^{-\min\{k/2, m\}}.$$

For $a \in \{1, \dots, 9\}$, we apply Lemma 2.2 to Λ_3 with the data

$$\gamma := \frac{\log 10}{\log 2}, \quad \mu := \frac{\log(a/27)}{\log 2}, \quad \text{and} \quad (A, B) := (9, 2).$$

We now want to reduce our bound, which is too large, by using Lemma 2.2. We take $M := 7.2 \cdot 10^{284}$, which is an upper bound on l by (3.2) and (3.27). Then, we apply Lemma 2.2 to inequality (3.29) in order to obtain an upper bound on k . A computer search with Maple shows that q_{575} satisfies the assumptions of Lemma 2.2, for all $a \in \{1, \dots, 9\}$. Thus, the application of Lemma 2.2 yields to the different results presented in the following table.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geq$	0.37	0.37	0.25	0.37	0.37	0.25	0.34	0.37	0.13
$\min\{k/2, m\} \leq$	956	956	956	956	956	956	956	956	957

According to the obtained results, we find that $\min\{k/2, m\} < 958$ which holds in all cases.

Case 1: $\min\{k/2, m\} = k/2$. In this case, we get

$$(3.30) \quad k < 1916.$$

Case 2: $\min\{k/2, m\} = m$. In this case, we obtain that $m \leq 957$. Let $2 \leq m \leq 957$ and

$$\Lambda_4 := \log(\Gamma_4 + 1) = l \log 10 - (n - 2) \log 2 + \log\left(\frac{a}{9L_m^{(k)}}\right).$$

Since $k > 440$, then from (3.24), we have $|\Gamma_4| < 0.01$. Hence by Lemma 2.3, we obtain

$$(3.31) \quad |\Lambda_4| < -\frac{\log(0.99)}{0.01} \cdot |\Gamma_4| < 3.02 \cdot 2^{-k/2}.$$

So, we get

$$(3.32) \quad \left| l \cdot \frac{\log 10}{\log 2} - (n - 2) + \frac{\log(a/(9L_m^{(k)}))}{\log 2} \right| < 4.4 \cdot 2^{-k/2}.$$

For all $a \in \{1, \dots, 9\}$ and $2 \leq m \leq 957$, we apply Lemma 2.2 to Λ_4 with the parameters

$$\gamma := \frac{\log 10}{\log 2}, \quad \mu := \frac{\log(a/(9L_m^{(k)}))}{\log 2}, \quad M := 7.2 \cdot 10^{284}, \quad \text{and} \quad (A, B) := (4.4, 2).$$

Note that $m < k/2$. This implies that $m < k$, which holds for $k \geq 2$ and we can replace $L_m^{(k)}$ by $3 \cdot 2^{m-2}$ in our calculations. Using again Maple, one can see that q_{575} satisfies the assumptions of Lemma 2.2 for all $a \in \{1, \dots, 9\}$ and $2 \leq m \leq 957$. Thus, applying Lemma 2.2 gives us the following different results which hold for all $2 \leq m \leq 957$.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geq$	0.37	0.37	0.25	0.37	0.37	0.25	0.34	0.37	0.13
$k/2 \leq$	955	955	955	955	955	955	955	955	956

From the obtained results, we observe that in all cases $k < 1913$. So, in both cases, we have $k < 1916$.

With this new bound, we get $n < 9 \cdot 10^{60}$. Again, we apply Lemma 2.2 with the same above data but this time we take $M := 9 \cdot 10^{60}$. With the help of Maple we see that q_{131} satisfies the assumptions of Lemma 2.2 for all $a \in \{1, \dots, 9\}$. By this application, we obtain the following results.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geq$	0.05	0.05	0.37	0.05	0.05	0.37	0.34	0.05	0.32
$\min\{k/2, m\} \leq$	217	217	214	217	217	214	214	217	214

After this, we can see that $\min\{k/2, m\} < 218$ which holds in all cases.

As we have done above, we obtain for the first case that $k < 436$ and for the second case, we use again q_{131} which fulfills the assumptions of Lemma 2.2 for all $a \in \{1, \dots, 9\}$ and $2 \leq m \leq 217$. Then, we obtain the following results which hold for all $2 \leq m \leq 217$ and give us that $k < 433$ in all cases.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geq$	0.05	0.05	0.37	0.05	0.05	0.37	0.34	0.05	0.32
$k/2 \leq$	216	216	213	216	216	213	213	216	213

Thus, for both cases it holds that $k < 436$, which contradicts our assumption that $k > 440$. This completes the proof of Theorem 1.1. \square

4. PROOF OF THEOREM 1.2

This section is devoted to proving Theorem 1.2. This proof is similar to that of Theorem 1.1. For the sake of completeness, we give most of the details.

4.1. An upper bound for l in terms of m . We begin our analysis of (1.2) for $2 \leq n \leq m \leq k$. In this case, we have $F_n^{(k)} = 2^{n-2}$ and $L_m^{(k)} = 3 \cdot 2^{m-2}$, so equation (1.2) turns to

$$(4.1) \quad 3 \cdot 2^{m+n-4} = \frac{a(10^l - 1)}{9}.$$

For any rational number x , let $\nu_2(x)$ denote the 2-adic valuation of x . Since $\nu_2(a(10^l - 1)/9) \leq 3$, then by comparing the 2-adic valuations on both sides of (4.1) one gets $2 \leq n \leq m \leq 7$. In this range, equation (4.1) has no solutions. So, from now, we assume that $m \geq k + 1$.

Now, by inequalities (2.10), (2.14), and $10^l - 1 \leq a(10^l - 1)/9$, we obtain

$$(4.2) \quad l < m.$$

4.2. An inequality for m versus k . Here, we give an inequality for m in terms of k by showing the following lemma.

Lemma 4.1. *If (a, k, l, n, m) is a solution of equation (1.2) in positive integers with $k \geq 2$ and $m \geq k + 1$, then we have the inequality*

$$(4.3) \quad m < 2.6 \times 10^{30} k^8 \log^5 k.$$

Proof. Using estimates (2.8) and (2.12), equation (1.2) can be rewritten as

$$(4.4) \quad (f_k(\alpha)\alpha^{n-1} + e_k(n))((2\alpha - 1)f_k(\alpha)\alpha^{m-1} + e'_k(m)) = a\left(\frac{10^l - 1}{9}\right),$$

i.e.,

$$(2\alpha - 1)f_k^2(\alpha)\alpha^{m+n-2} - \frac{a10^l}{9} = -e'_k(m)f_k(\alpha)\alpha^{n-1} - e_k(n)(2\alpha - 1)f_k(\alpha)\alpha^{m-1} \\ - e'_k(m)e_k(n) - \frac{a}{9}.$$

Thus, by taking the absolute value and dividing both sides by $(2\alpha - 1)f_k^2(\alpha)\alpha^{n+m-2}$, we obtain

$$(4.5) \quad |\Gamma'_1| \leq \frac{3}{2\alpha^{m-1}} + \frac{1}{\alpha^{n-1}} + \frac{7}{2\alpha^{n+m-2}} < \frac{6\alpha}{\alpha^n} < \frac{12}{\alpha^n},$$

where

$$(4.6) \quad \Gamma'_1 := \frac{a}{9(2\alpha - 1)f_k^2(\alpha)} \cdot \alpha^{-(n+m-2)} \cdot 10^l - 1.$$

As seen previously, we have $\Gamma'_1 = \Gamma_1 \neq 0$. To apply Theorem 2.1 to Γ'_1 , we take

$$(\eta_1, b_1) := \left(\frac{a}{9(2\alpha - 1)f_k^2(\alpha)}, 1 \right), \quad (\eta_2, b_2) := (\alpha, -(n + m - 2)), \quad (\eta_3, b_3) := (10, l).$$

The algebraic numbers η_1, η_2, η_3 are elements of the number field $\mathbb{K} := \mathbb{Q}(\alpha)$ and $d_{\mathbb{K}} = k$. So, as in the previous section, we take

$$A_1 := 12k \log k, \quad A_2 := \log 2, \quad A_3 := k \log 10, \quad \text{and} \quad B := 2m.$$

Now, we apply Theorem 2.1 to Γ'_1 and compare the resulting inequality with (4.5) to obtain

$$(4.7) \quad n \log \alpha < 7.2 \cdot 10^{12} k^4 \log^2 k (1 + \log 2m).$$

We return to equation (1.2) and rewrite it as

$$F_n^{(k)}((2\alpha - 1)f_k(\alpha)\alpha^{m-1} + e'_k(m)) = \frac{a(10^l - 1)}{9},$$

i.e.,

$$(4.8) \quad (2\alpha - 1)f_k(\alpha)\alpha^{m-1} - \frac{a10^l}{9F_n^{(k)}} = \frac{-a}{9F_n^{(k)}} - e'_k(m).$$

So, we get

$$\left| (2\alpha - 1)f_k(\alpha)\alpha^{m-1} - \frac{a10^l}{9F_n^{(k)}} \right| \leq \frac{a}{9F_n^{(k)}} + \frac{3}{2} \leq \frac{5}{2}.$$

Dividing by $(2\alpha - 1)f_k(\alpha)\alpha^{m-1}$, we get

$$(4.9) \quad |\Gamma'_2| \leq \frac{5}{2(2\alpha - 1)f_k(\alpha)\alpha^{m-1}} < \frac{5\alpha}{2\alpha^m} < \frac{5}{\alpha^m},$$

where

$$(4.10) \quad \Gamma'_2 := \frac{a}{9(2\alpha - 1)F_n^{(k)}f_k(\alpha)} \cdot \alpha^{-(m-1)} \cdot 10^l - 1.$$

We have $\Gamma'_2 \neq 0$. Now, we apply Theorem 2.1 to Γ'_2 by fixing

$$(\eta_1, b_1) := \left(\frac{a}{9(2\alpha - 1)F_n^{(k)} f_k(\alpha)}, 1 \right), \quad (\eta_2, b_2) := (\alpha, -(m - 1)), \quad (\eta_3, b_3) := (10, l).$$

It is obvious that $\mathbb{K} := \mathbb{Q}(\alpha)$ contains η_1, η_2, η_3 and $d_{\mathbb{K}} = k$. As calculated before, we take

$$A_2 := \log 2, \quad A_3 := k \log 10, \quad \text{and} \quad B := 2m.$$

We need to compute A_1 . Estimates (2.7), (4.7), and the properties (2.1)–(2.3) imply that

$$\begin{aligned} h(\eta_1) &\leq h\left(\frac{a}{9}\right) + h(2\alpha - 1) + h(F_n^{(k)}) + h(f_k(\alpha)) \\ &< \log 9 + \log 3 + (n - 1) \log \alpha + \log k \\ &< 7.3 \cdot 10^{12} k^4 \log^2 k (1 + \log 2m) \end{aligned}$$

for $k \geq 2$. On the other hand, since

$$\eta_1 := \frac{a}{9(2\alpha - 1)F_n^{(k)} f_k(\alpha)} < 1 \quad \text{and} \quad \eta_1^{-1} = \frac{9(2\alpha - 1)F_n^{(k)} f_k(\alpha)}{a} < \frac{81\alpha^{(n-1)}}{4},$$

then, by (4.7), we get

$$|\log \eta_1| < (n - 1) \log \alpha + \log 20.3 < 7.3 \cdot 10^{12} k^4 \log^2 k (1 + \log 2m).$$

Thus, we conclude that

$$\max\{kh(\eta_1), |\log \eta_1|, 0.16\} < 7.3 \cdot 10^{12} k^5 \log^2 k (1 + \log 2m) := A_1.$$

Applying Theorem 2.1 and comparing the resulting inequality with (4.9) give

$$m \log \alpha < 2.9 \cdot 10^{25} k^8 \log^3 k \log^2 m,$$

where we used the facts $1 + \log k < 2.5 \log k$ and $1 + \log 2m < 2.6 \log m$, which hold for $k \geq 2$ and $m \geq 3$. So, we obtain

$$(4.11) \quad \frac{m}{\log^2 m} < 7.2 \times 10^{25} k^8 \log^3 k.$$

We use (3.13) by putting $A := 7.2 \times 10^{25} k^8 \log^3 k$ and the fact that $59.6 + 8 \log k + 3 \log \log k < 93 \log k$, which holds for all $k \geq 2$, to obtain

$$\begin{aligned} m &< 4(7.2 \times 10^{25} k^8 \log^3 k)(\log(7.2 \times 10^{25} k^8 \log^3 k))^2 \\ &< (3 \times 10^{26} k^8 \log^3 k)(59.6 + 8 \log k + 3 \log \log k)^2 \\ &< 2.6 \times 10^{30} k^8 \log^5 k. \end{aligned}$$

This gives (4.3) and completes the proof of Lemma 4.1. □

4.3. The case $2 \leq k \leq 430$. For this subsection, we consider $k \in [2, 430]$. Put

$$(4.12) \quad \Lambda'_1 := \log(\Gamma'_1 + 1) = l \log 10 - (n + m - 2) \log \alpha + \log\left(\frac{a}{9(2\alpha - 1)f_k^2(\alpha)}\right).$$

Suppose that $n \geq 10$, so by estimate (4.5) and the fact that $\alpha > 1.5$, we have $|\Gamma'_1| < 0.21$. Taking $d = 0.21$ in Lemma 2.3, we obtain

$$(4.13) \quad |\Lambda'_1| < \frac{-\log 0.79}{0.21} \cdot |\Gamma'_1| < 13.5 \cdot \alpha^{-n}.$$

So, we get

$$(4.14) \quad \left| l \cdot \frac{\log 10}{\log \alpha} - (n + m - 2) + \frac{\log(a/(9(2\alpha - 1)f_k^2(\alpha)))}{\log \alpha} \right| < 33.3 \cdot \alpha^{-n}.$$

For $a \in \{1, \dots, 9\}$ and $k \in [2, 430]$, we apply Lemma 2.2 to Λ'_1 . For this application, we put

$$\gamma := \frac{\log 10}{\log \alpha}, \quad \mu := \frac{\log(a/(9(2\alpha - 1)f_k^2(\alpha)))}{\log \alpha}, \quad \text{and} \quad (A, B) := (33.3, \alpha).$$

For each $k \in [2, 430]$ and $a \in \{1, \dots, 9\}$, we find a good approximation of γ and a convergent p_l/q_l of the continued fraction of γ such that $q_l > 6M_k$ and $\varepsilon = \varepsilon(k) = \|\mu q\| - M_k \|\gamma q\| > 0$, where $M_k := \lfloor 2.6 \times 10^{30} k^8 \log^5 k \rfloor$, which is an upper bound of l from Lemma 4.1. After doing this, we apply Lemma 2.2 to inequality (4.14). Using Mathematica, we find that q_{121} fulfills the assumptions of Lemma 2.2. A computer program with Mathematica reveals for $k = 402$ and $a = 7$ that $\varepsilon > 0.0011$ and the maximum value of $\log(Aq/\varepsilon)/\log B$, for $k \in [2, 430]$ and $a \in \{1, \dots, 9\}$, is 200.757. This is an upper bound of n by Lemma 2.2.

For $2 \leq n < 201$, we consider

$$(4.15) \quad \Lambda'_2 := \log(\Gamma'_2 + 1) = l \log 10 - (m - 1) \log \alpha + \log\left(\frac{a}{9(2\alpha - 1)F_n^{(k)} f_k(\alpha)}\right).$$

Suppose that $m \geq 10$, then by estimate (4.9) and the fact that $\alpha > 1.5$, we have $|\Gamma'_2| < 0.09$. Taking $d = 0.09$ in Lemma 2.3, we obtain

$$|\Lambda'_2| < \frac{-\log 0.91}{0.09} \cdot |\Gamma'_2| < 5.3 \cdot \alpha^{-m}.$$

So, we get

$$(4.16) \quad \left| l \cdot \frac{\log 10}{\log \alpha} - (m - 1) - \frac{\log(a/(9(2\alpha - 1)F_n^{(k)} f_k(\alpha)))}{\log \alpha} \right| < 13.1 \cdot \alpha^{-m}.$$

For $a \in \{1, \dots, 9\}$ and $(k, n) \in [2, 430] \times [2, 200]$, we apply Lemma 2.2 to Λ'_2 by taking

$$\gamma := \frac{\log 10}{\log \alpha}, \quad \mu := \frac{\log(a/(9(2\alpha - 1)F_n^{(k)}f_k(\alpha)))}{\log \alpha}, \quad \text{and} \quad (A, B) := (13.1, \alpha).$$

Again, for each $(k, n) \in [2, 430] \times [2, 200]$ and $a \in \{1, \dots, 9\}$, we find a good approximation of γ and a convergent p_l/q_l of the continued fraction of γ such that $q_l > 6M_k$ and $\varepsilon = \varepsilon(k) = \|\mu q\| - M_k \|\gamma q\| > 0$, where $M_k := \lfloor 2.6 \times 10^{30} k^8 \log^5 k \rfloor$, which is an upper bound of l from Lemma 4.1. After doing this, we apply Lemma 2.2 to inequality (4.16). Using Mathematica again, we see that q_{123} confirms the assumptions of Lemma 2.2. A computer program with Mathematica proves, for $k = 427$, $a = 1$ and $n = 58$, that $\varepsilon > 0.000019$ and the maximum value of $\log(Aq/\varepsilon)/\log B$, for $(k, n) \in [2, 430] \times [2, 200]$ and $a \in \{1, \dots, 9\}$, is 208.792. This is also an upper bound of m by Lemma 2.2.

Hence, we deduce that the possible solutions (a, k, l, m, n) of equation (1.2), for which $k \in [2, 430]$ and $a \in \{1, \dots, 9\}$, satisfy $n \leq m \leq 208$. Therefore, we use inequality (3.2) to obtain $l \leq 207$.

Finally, we use Mathematica to compare $F_n^{(k)}L_m^{(k)}$ and $a(10^l - 1)/9$ for the ranges $k + 1 \leq m \leq 208$, $n \leq m$, $a \in \{1, \dots, 9\}$ and $2 \leq l \leq 207$, with $l < m$, and check that the only solutions of equation (1.2) are those given in Theorem 1.2.

4.4. The case $k > 430$. For $k > 430$, it is easy to check that

$$n \leq m < 2.6 \cdot 10^{30} k^8 \log^5 k < 2^{k/2}.$$

Thus, by Lemmas 2.4 and 2.5, $F_n^{(k)}$ and $L_m^{(k)}$ can be rewritten as

$$(4.17) \quad F_n^{(k)} = 2^{n-2}(1 + \zeta_1), \quad \text{where } |\zeta_1| < \frac{2}{2^{k/2}}$$

and

$$(4.18) \quad L_m^{(k)} = 3 \cdot 2^{m-2}(1 + \zeta_2), \quad \text{where } |\zeta_2| < \frac{1}{2^{k/2}},$$

respectively. Substituting (4.17) and (4.18) in (1.2) and using the fact that $m \geq 3$, we obtain

$$(4.19) \quad |\Gamma'_3| < \frac{5.5}{2^{\min\{k/2, n\}}},$$

where

$$\Gamma'_3 = \frac{a}{27} \cdot 2^{-(n+m-4)} \cdot 10^l - 1.$$

One can check that $\Gamma'_3 \neq 0$. Thus, we apply Theorem 2.1 by taking

$$(\eta_1, b_1) := \left(\frac{a}{27}, 1\right), \quad (\eta_2, b_2) := (2, -(n+m-4)), \quad (\eta_3, b_3) := (10, l).$$

We put $d_{\mathbb{K}} = 1$, $A_1 := \log 27$, $A_2 := \log 2$, $A_3 := \log 10$, and $B := 2m$. Thus, applying Theorem 2.1 and comparing the resulting inequality with (4.19), we obtain

$$\min\{k/2, n\} < 3 \cdot 10^{12} \log m.$$

By Lemma 4.1 and using the fact that $70.1 + 8 \log k + 5 \log \log k < 21 \log k$ for all $k > 430$, we get

$$\min\{k/2, n\} < 6.3 \cdot 10^{13} \log k.$$

If $\min\{k/2, n\} = k/2$, then we get $k < 1.3 \cdot 10^{14} \log k$. Solving this inequality and using Lemma 4.1, we conclude that

$$(4.20) \quad k < 4.7 \cdot 10^{15} \quad \text{and} \quad m < 3.8 \cdot 10^{163}.$$

If $\min\{k/2, n\} = n$, then we obtain

$$(4.21) \quad n < 6.3 \cdot 10^{13} \log k.$$

Now, we go back to (4.8) and we rewrite it as

$$3 \cdot 2^{m-2} - \frac{a10^l}{9F_n^{(k)}} = \frac{-a}{9F_n^{(k)}} - 3 \cdot 2^{m-2} \zeta_2.$$

Taking the absolute value and dividing by $3 \cdot 2^{m-2}$, we use the fact that $m \geq k+1$ to get

$$(4.22) \quad |\Gamma'_4| \leq \frac{1}{3 \cdot 2^{m-2}} + \frac{1}{2^{k/2}} < \frac{1}{3 \cdot 2^{k-2}} + \frac{1}{2^{k/2}} < \frac{2}{2^{k/2}},$$

where

$$(4.23) \quad \Gamma'_4 := \frac{a}{27F_n^{(k)}} \cdot 2^{-(m-2)} \cdot 10^l - 1.$$

As above, we show that $\Gamma'_4 \neq 0$, since if not, we would obtain $a10^l/(27F_n^{(k)}) = 2^{m-2}$ or in this case $F_n^{(k)} = 2^{n-2}$ and so we would obtain $a10^l/(27 \cdot 2^{n-2}) = 2^{m-2}$. This is impossible. Now, we apply Theorem 2.1 by fixing

$$(\eta_1, b_1) := \left(\frac{a}{27F_n^{(k)}}, 1\right), \quad (\eta_2, b_2) := (2, -(m-2)), \quad (\eta_3, b_3) := (10, l).$$

As calculated before, we take $A_2 := \log 2$, $A_3 := \log 10$, and $B := m$. Next, we estimate $h(\eta_1)$. As $F_n^{(k)} < \alpha^{n-1}$ and using inequality (4.21), we obtain

$$h(\eta_1) \leq h\left(\frac{a}{27}\right) + h(F_n^{(k)}) < \log 27 + (n-1) \log \alpha < 4.4 \cdot 10^{13} \log k.$$

So, we choose $A_1 := 4.4 \cdot 10^{13} \log k$. Therefore, Theorem 2.1 gives

$$(4.24) \quad |\Gamma'_4| > \exp(-2.1 \cdot 10^{25} \log k \log m),$$

where we used the fact that $1 + \log m < 2 \log m$, which holds for all $m \geq 3$. From (4.22) and (4.24), we deduce that

$$k < 6.1 \cdot 10^{25} \log k \log m.$$

By Lemma 4.1 and using the fact that $70.1 + 8 \log k + 5 \log \log k < 21 \log k$ for all $k > 430$, we get

$$\begin{aligned} k &< 6.1 \cdot 10^{25} \log k (\log(2.6 \cdot 10^{30} k^8 \log^5 k)) \\ &< 6.1 \cdot 10^{25} \log k (70.1 + 8 \log k + 5 \log \log k) \\ &< 1.3 \cdot 10^{27} \log^2 k. \end{aligned}$$

Solving this inequality and using Lemma 4.1, we obtain

$$(4.25) \quad k < 6.55 \cdot 10^{30} \quad \text{and} \quad m < 1.6 \cdot 10^{286}.$$

From (4.20) and (4.25), we conclude that inequalities (4.25) always hold. The obtained bounds are very large, next we will reduce them. Put

$$(4.26) \quad \Lambda'_3 := \log(\Gamma'_3 + 1) = l \log 10 - (n + m - 4) \log 2 + \log\left(\frac{a}{27}\right).$$

Assume that $n \geq 10$, we get then $|\Gamma'_3| < 0.02$. Taking $d = 0.02$ in Lemma 2.3, we obtain

$$|\Lambda'_3| < \frac{-\log 0.98}{0.02} \cdot |\Gamma'_3| < 6 \cdot 2^{-\min\{k/2, n\}}.$$

So, we get

$$(4.27) \quad \left| l \cdot \frac{\log 10}{\log 2} - (n + m - 4) + \frac{\log(a/27)}{\log 2} \right| < 9 \cdot 2^{-\min\{k/2, n\}}.$$

For $a \in \{1, \dots, 9\}$, we apply Lemma 2.2 to Λ'_3 with the data

$$\gamma := \frac{\log 10}{\log 2}, \quad \mu := \frac{\log(a/27)}{\log 2}, \quad \text{and} \quad (A, B) := (9, 2).$$

We now want to reduce our bound of k , which is too large, by using Lemma 2.2. We take $M := 1.6 \cdot 10^{286}$, which is an upper bound of l by (4.2) and (4.25). After that, we apply Lemma 2.2 to (4.27) in order to obtain an upper bound of k . A computer search with Maple shows that q_{580} satisfies the assumptions of Lemma 2.2 for all $a \in \{1, \dots, 9\}$. Thus, the application of Lemma 2.2 leads to the different results presented in the following table.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geq$	0.45	0.45	0.03	0.45	0.45	0.03	0.44	0.45	0.48
$\min\{k/2, n\} \leq$	961	961	965	961	961	965	961	961	961

According to the above results, we see that $\min\{k/2, n\} < 966$ across all cases.

Case 1: $\min\{k/2, n\} = k/2$. In this case, we get

$$(4.28) \quad k < 1932.$$

Case 2: $\min\{k/2, n\} = n$. In this case, we obtain $n \leq 965$. For $2 \leq n \leq 965$, let

$$\Lambda'_4 := \log(\Gamma'_4 + 1) = l \log 10 - (m - 2) \log 2 + \log\left(\frac{a}{27F_n^{(k)}}\right).$$

Since $k > 430$, then from (4.22), we have $|\Gamma'_4| < 0.01$. Hence by Lemma 2.3, we obtain

$$(4.29) \quad |\Lambda'_4| < -\frac{\log(0.99)}{0.01} \cdot |\Gamma'_4| < 2.02 \cdot 2^{-k/2}.$$

So, we get

$$(4.30) \quad \left| l \cdot \frac{\log 10}{\log 2} - (m - 2) + \frac{\log(a/(27F_n^{(k)}))}{\log 2} \right| < 3 \cdot 2^{-k/2}.$$

Again, for $a \in \{1, \dots, 9\}$ and $2 \leq n \leq 965$, we apply Lemma 2.2 to Λ'_4 with the parameters

$$\gamma := \frac{\log 10}{\log 2}, \quad \mu := \frac{\log(a/(27F_n^{(k)}))}{\log 2}, \quad M := 1.6 \cdot 10^{286}, \quad \text{and} \quad (A, B) := (3, 2).$$

Note that $n < k/2$. This implies that $n \leq k + 1$, which holds for $k \geq 2$ and we can replace $F_n^{(k)}$ by 2^{n-2} in our calculations. We use again Maple to see that q_{580} satisfies the assumptions of Lemma 2.2 for all $a \in \{1, \dots, 9\}$ and $2 \leq n \leq 965$. Hence, applying Lemma 2.2 leads us to the following varied results which hold for all $2 \leq n \leq 965$.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geq$	0.45	0.45	0.03	0.45	0.45	0.03	0.44	0.45	0.48
$k/2 \leq$	959	959	963	959	959	963	959	959	959

The obtained results reveal across all instances that $k \leq 1926$. As a result, in both cases, we have $k < 1932$.

With this new bound, we get $m < 1.25 \cdot 10^{61}$. So, we apply again Lemma 2.2 with the same above data but this time we take $M := 1.25 \cdot 10^{61}$. With the help of Maple, we see that q_{140} satisfies the assumptions of Lemma 2.2 for all $a \in \{1, \dots, 9\}$. In the context of this application which yields different results presented in the table below, we find that $\min\{k/2, n\} < 235$.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geq$	0.33	0.33	0.46	0.33	0.33	0.46	0.24	0.33	0.23
$\min\{k/2, n\} \leq$	233	233	233	233	233	233	233	233	234

For the first case, we obtain that $k < 470$ and for the second case we use again q_{140} , which satisfies the assumptions of Lemma 2.2 for all $a \in \{1, \dots, 9\}$ and $2 \leq n \leq 234$, to obtain the following results which hold for all $2 \leq n \leq 234$ and reveals that $k \leq 464$.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geq$	0.33	0.33	0.46	0.33	0.33	0.46	0.24	0.33	0.23
$k/2 \leq$	231	231	231	231	231	231	232	231	232

So, in both cases we have $k < 470$. With this new bound, we get $m < 5.5 \cdot 10^{55}$. So, we apply again Lemma 2.2 to the same above data but this time we take $M := 5.5 \cdot 10^{55}$. In this case, q_{128} satisfies the assumptions of Lemma 2.2 for all $a \in \{1, \dots, 9\}$. The third application of Lemma 2.2 leads to the following results from which we observe that $\min\{k/2, n\} < 209$, which holds in all cases.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geq$	0.27	0.27	0.18	0.27	0.27	0.18	0.24	0.27	0.09
$\min\{k/2, n\} \leq$	207	207	207	207	207	207	207	207	208

The first case gives us that $k < 418$ and for the second case we use again q_{128} which satisfies the assumptions of Lemma 2.2 for all $a \in \{1, \dots, 9\}$ and $2 \leq n \leq 208$. This application yields the following results which hold for all $2 \leq n \leq 208$ and tells us that $k < 415$.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geq$	0.27	0.27	0.18	0.27	0.27	0.18	0.24	0.27	0.09
$k/2 \leq$	205	205	206	205	205	206	205	205	207

Consequently, we deduce that in both cases we have $k < 418$, which contradicts our assumption that $k > 430$. This completes the proof of Theorem 1.2. \square

A c k n o w l e d g e m e n t . The authors would like to express their gratitude to the reviewer for the careful reading of this paper and the remarks which have qualitatively improved the work. This paper was written when the third author was visiting the Max-Planck-Institut für Mathematik and then when the first author was visiting the Purdue University Northwest. They thank the institutions for the great working environment, hospitality and support.

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