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# A REGULARITY CRITERION OF 3D MAGNETO-MICROPOLAR EQUATIONS WITH THE PRESSURE TERM

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*Abstract.* This work focuses on the 3D incompressible magneto-micropolar (MMP) equations with the mixed pressure-velocity-magnetic field in view of Lorentz spaces. Also, we generalize some known results to MMP equations in view of Besov spaces.

*Keywords*: 3D magneto-micropolar equation; regularity criterion; pressure function; Besov space

*MSC 2020*: 35Q35, 35B65, 76D05

## 1. Introduction

This paper is concerned with the decay rates in time of the weak solutions to the non-Newtonian magneto-micropolar fluid equations in $\mathbb{R}^3$, which are described by

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla P = -(\mu+\chi)\Delta u + \chi\nabla\times w + (b\cdot\nabla)b, \\ \partial_t b + (u\cdot\nabla)b = -\nu\Delta b + (b\cdot\nabla)u, \\ \nabla\cdot u(\cdot,t) = \nabla\cdot b(\cdot,t) = 0, \\ \partial_t w + (u\cdot\nabla)w = -\kappa\Delta w + \eta\nabla(\nabla\cdot w) + \chi\nabla\times u - 2\chi w, \end{cases} \tag{1.1}$$

where $u = u(x,t)$, $w = w(x,t)$, $b = b(x,t)$ and $P := \mathcal{P} + \frac{1}{2}|b|^2$ denote the fluid velocity, the micro-rotation velocity (angular velocity of the rotation of the fluid particles), the magnetic field and the pressure function, respectively. The positive constant $\kappa$ in (1.1) corresponds with the angular viscosity, $\nu$ is the inverse of the

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magnetic Reynolds number and $\chi$ is the micro-rotational viscosity. We consider the initial value problem of (1.1), which requires initial conditions

$$u(x,0)=u_0(x),\quad w(x,0)=w_0(x)\quad\text{and}\quad b(x,0)=b_0(x),\quad x\in\mathbb{R}^3, \tag{1.2}$$

and we also assume that $\operatorname{div} u_0 = \operatorname{div} b_0$. In this paper, we assume that $\gamma=\nu=\chi=\kappa=\mu=\eta=1$.

**Definition 1.1.** The vector-valued function $(u,b,w)$ is called a weak solution of equations (1.1)–(1.2) on $(0,T)\times\mathbb{R}^3$ if it satisfies the following conditions:

(A) $u,b,w\in L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^3))$;
(B) $\operatorname{div} u=\operatorname{div} b=0$ in the sense of distribution;
(C) $(u,b,w)$ verifies (1.1) in the sense of distribution;
(D) the energy inequality, that is,

$$\begin{aligned}\|(u,b,w)(\tau)\|_{L^2}^2&+2(\mu+\xi)\int_0^t\|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau\\&+2\nu\int_0^t\|b(\tau)\|_{L^2}^2\,\mathrm{d}\tau+2\gamma\int_0^t\|w(\tau)\|_{L^2}^2\,\mathrm{d}\tau\\&+2\kappa\int_0^t\|\operatorname{div} w(\tau)\|_{L^2}^2\,\mathrm{d}\tau+2\xi\int_0^t\|w(\tau)\|_{L^2}^2\,\mathrm{d}\tau\\&\leqslant\|(u,b,w)(0)\|_{L^2}^2.\end{aligned} \tag{1.3}$$

Let us now briefly mention some previous results for (1.1)–(1.2). These equations were introduced by Ahmadi and Shahinpoor (see [1]), who based on the theory of micropolar fluids proposed by Eringen (see [6]), studied stability of solutions in bounded domains. Rojas-Medar (see [21]) proved the existence and uniqueness of strong solutions in bounded domains in $\mathbb{R}^n$, by the spectral Galerkin method. After that, Ortega-Torres and Rojas-Medar showed local in time (or global in time for small initial data) the existence and uniqueness of strong solutions in [21], [19].

For the regularity criteria in Lorentz space, Li and Niu (see [16]) proved that a weak solution $(u,b,\ w)$ for the standard 3D MHD equations become regular under the scaling invariant conditions for the total pressure, in particular, so called Serrin's conditions, $\pi\in L^{q,\infty}(0,T;L^{p,\infty}(\mathbb{R}^3))$ with $3/p+2/q\leqslant 2$ and $p>\frac{3}{2}$ (compare to [4], [25], [24], [23] for Navier-Stokes equations).

Very recently, Kanamaru and Yamamoto [13], Gala, Ragusa, Wu [7], [8], [20] investigated logarithmically improved extension criteria involving the pressure for the Navier-Stokes equations in $\mathbb{R}^n$ with respect to $P$ or $\nabla P$ using a new tool developed by the authors in [11].

In this direction, our result is stated as follows.

**Theorem 1.2.** *Suppose that* $(u_0, b_0, w_0) \in H^1(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ *and* $(u, b, w)$ *is a (local in time) strong solution to equations* (1.1)–(1.2) *on an interval* $[0, T)$ *with* $0 < T < \infty$. *Assume that one of the following conditions is satisfied:*

$$\text{(A)} \qquad \int_0^T \frac{\|\mathcal{P}(\tau)\|^r_{\dot{B}^{-3(1/p-1/q)}_{q,\infty}(\mathbb{R}^3)}}{\log(\mathrm{e} + \|u(\tau)\|_{H^s})}\,\mathrm{d}\tau < \infty, \quad \textit{and} \quad \int_0^T \|b(\tau)\|^{2p}_{L^{2r}}\,\mathrm{d}\tau < \infty,$$

*for* $2/r + 3/p = 2$, $\frac{3}{2} < p < 3$, $p \leqslant q < 3p/(3-p)$,

$$\text{(B)} \qquad \int_0^T \frac{\|\nabla\mathcal{P}(\tau)\|^r_{\dot{B}^{-3(1/p-1/q)}_{q,\infty}(\mathbb{R}^3)}}{\log(\mathrm{e} + \|u(\tau)\|_{H^s})}\,\mathrm{d}\tau < \infty, \quad \textit{and} \quad \int_0^T \|b(\tau)\|^{3p}_{L^{3r}}\,\mathrm{d}\tau < \infty,$$

*for* $2/r + 3/p = 3$, $1 < p < \frac{3}{2}$, $p \leqslant q < 3p/(3-p)$. *Then* $(u, b, w)$ *can be extended beyond* $T > 0$.

Remark 1.3. In Theorem 1.2, a proof of Part (B) is easy to check by the lifting property for Besov space (see Lemma 2.4 below) and thus it is omitted.

Remark 1.4. In Theorem 1.2, the condition for $b$ is required additionally. In particular, even for the 3D MHD equations (namely $w = 0$ in (1.1)), it is not yet known to obtain the regularity criterion for the pressure function $\mathcal{P}$ only unlike Navier-Stokes regularity results for the pressure function. According to the arguments in [25], the norm for $b$ is replaced by a suitable norm for $\nabla b$ under some calculations for the mixed term $u$ and $b$.

**Theorem 1.5.** *Under the assumption for the initial data in Theorem* 1.2, *suppose that* $(u, b, w)$ *is a strong solution to equations* (1.1)–(1.2) *in the time interval* $[0, T)$ *for some* $0 < T < \infty$. *If the total pressure function* $P(x, t)$ *satisfies the condition*

$$P \in L^{2+r}(0, T; \dot{B}^r_{\infty,\infty}(\mathbb{R}^3)), \quad -1 \leqslant r \leqslant 1, \tag{1.4}$$

*then* $(u, b, w)$ *can be extended beyond* $T > 0$.

Remark 1.6. In [9], the authors proved that if the pressure satisfies the critical growth condition (1.4), a weak solution for 3D MHD equations (namely, $w = 0$ in the equations (1.1)) remains regular on $(0, T]$ based on a suitable function decomposition method together with Besov space techniques. Here, regular solutions mean $L^\infty$-solutions. This result also holds for our equations by the same arguments and thus its proof is left as an exercise to the readers.

Recently, Beirão da Veiga and Yang (see [2]) obtained generalized regular criteria for the mixed pressure-velocity problems in Lorentz spaces for Leray-Hopf weak solutions to 3D Navier-Stokes equations, that is, $w = 0 = b$ in equations (1.1) based on the relation $|P| \simeq |u|^2$. More rigorously, they showed that if

$$\frac{P}{(\mathrm{e}^{-|x|^2} + |u|)^\theta} \in L^p(0, T; L^{q,\infty}(\Omega)), \quad p, q < \infty,$$

and

$$\frac{2}{p} + \frac{3}{q} = 2 - \theta, \quad 0 \leqslant \theta \leqslant 1, \tag{1.5}$$

then $u$ is regular on $(0, T]$. In this respect, inspired by [2], our result is stated as follows.

**Theorem 1.7.** *Under the assumption for the initial data in Theorem* 1.2*, if the condition*

$$\frac{P}{(\mathrm{e}^{-|x|^2} + |u| + |b|)^\theta} \in L^p(0, T; L^{q,\infty}(\mathbb{R}^3)), \quad p, q < \infty, \tag{1.6}$$

*with relation* (1.5) *is satisfied, then* $(u, b, w)$ *is regular on* $(0, T] \times \mathbb{R}^3$.

Remark 1.8. In particular, if $b = 0$ in equations (1.1), namely, for the micropolar equations or Boussinesq equations, the results in [13] and [2] are also established in our way (see also [15]).

Remark 1.9. These results are shown as an extension of the manuscript [14] of the author.

## 2. Proof of Theorem 1.2

Next, we recall some definitions for Lorentz spaces. For $p, q \in [1, \infty]$, we define

$$\|f\|_{L^{p,q}(\mathbb{R}^3)} = \begin{cases} \left(p \int_0^\infty \alpha^q |\{x \in \mathbb{R}^3 \colon |f(x)| > \alpha\}|^{q/p} \dfrac{\mathrm{d}\alpha}{\alpha}\right)^{1/q}, & q < \infty, \\ \sup\limits_{\alpha > 0} \alpha |\{x \in \mathbb{R}^3 : |f(x)| > \alpha\}|^{1/p}, & q = \infty. \end{cases}$$

Furthermore,

$$L^{p,q}(\mathbb{R}^3) = \{f \colon f \text{ is a measurable function on } \mathbb{R}^3 \text{ and } \|f\|_{L^{p,q}(\mathbb{R}^3)} < \infty\}.$$

Following [3], Lorentz space $L^{p,q}(\mathbb{R}^3)$ may be also defined by real interpolation methods

$$L^{p,q}(\mathbb{R}^3) = (L^{p_1}(\mathbb{R}^3), L^{p_2}(\mathbb{R}^3))_{\alpha,q} \tag{2.1}$$

with

$$\frac{1}{p} = \frac{1-\alpha}{p_1} + \frac{\alpha}{p_2}, \quad 1 \leqslant p_1 < p < p_2 \leqslant \infty.$$

**Lemma 2.1** ([18]). *Assume* $1 \leqslant p_1, p_2 \leqslant \infty$, $1 \leqslant q_1, q_2 \leqslant \infty$ *and* $u \in L^{p_1,q_1}(\mathbb{R}^3)$, $v \in L^{p_2,q_2}(\mathbb{R}^3)$. *Then* $uv \in L^{p_3,q_3}(\mathbb{R}^3)$ *with* $1/p_3 = 1/p_1 + 1/p_2$ *and* $1/q_3 \leqslant 1/q_1 + 1/q_2$, *and the inequality*

$$\|uv\|_{L^{p_3,q_3}(\mathbb{R}^3)} \leqslant C\|u\|_{L^{p_1,q_1}(\mathbb{R}^3)}\|v\|_{L^{p_2,q_2}(\mathbb{R}^3)}$$

*is valid.*

**Lemma 2.2** ([17], Lemma 10). *Let* $1 < p < q < \infty$, $\alpha > 0$ *and* $s = \alpha(q/p-1) > 0$. *Then it holds that*

$$\|f\|_{L^q} \leqslant C(n,\alpha,p,q)\|f\|_{\dot{H}^s_p}^{p/q}\|f\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{p/q}$$

*for all* $f \in (\dot{H}^s_p \cap \dot{B}^{-\alpha}_{\infty,\infty})(\mathbb{R}^3)$.

**Lemma 2.3** ([11], Lemma 5). *Let* $\frac{3}{2} \leqslant p \leqslant 3$ *and* $p \leqslant q < 3p/(3-p)$. *Then there exists an absolute constant* $C > 0$ *such that the estimate*

$$\begin{aligned}\int_{\mathbb{R}^3} uvw\,\mathrm{d}x \leqslant C(&\|\nabla u\|_{L^2}\|v\|_{L^2}^{2-3/p}\|\nabla v\|_{L^2}^{3/p-1} + \|\nabla v\|_{L^2}\|u\|_{L^2}^{2-3/p}\|\nabla u\|_{L^2}^{3/p-1})\\ &\times \|w\|_{\dot{B}^{-3(1/p-1/q)}_{q,\infty}}\end{aligned}$$

*holds for every* $u, v \in (\dot{H}^1 \cap L^2)(\mathbb{R}^3)$ *and* $w \in \dot{B}^{-3(1/p-1/q)}_{q,\infty}(\mathbb{R}^3)$.

**Theorem 2.4** ([22], Theorem 3.17). *Let* $1 \leqslant p, q \leqslant \infty$ *and* $s, \alpha \in \mathbb{R}$. *Then*

$$(-\Delta)^{\alpha/2}\colon\ \dot{B}^s_{pq} \to \dot{B}^{s-\alpha}_{pq}$$

*is an isomorphism.*

**Lemma 2.5.** *Assume that* $(u, b, w)$ *is a smooth solution of system* (1.1)–(1.2). *Then*

$$\|P\|_{L^s(\mathbb{R}^3)} \lesssim (\|u\|_{L^{2s}(\mathbb{R}^3)}^2 + \|b\|_{L^{2s}(\mathbb{R}^3)}^2),$$

*and*

$$\|\nabla P\|_{L^s(\mathbb{R}^3)} \lesssim (\||u||\nabla u|\|_{L^s(\mathbb{R}^3)} + \||b||\nabla b|\|_{L^s(\mathbb{R}^3)}), \quad 1 < s < \infty.$$

Proof. Since

$$P = (-\Delta)^{-1} \sum_{i,j=1}^{3} \frac{\partial^2 (u_i u_j)}{\partial x_i \partial x_j}, \quad \nabla P = (-\Delta)^{-1} \sum_{i,j=1}^{3} \frac{\partial^2 (\nabla (u_i u_j))}{\partial x_i \partial x_j},$$

by the Calderón-Zygmund inequality we obtain:

$$\|\nabla^2 (-\Delta)^{-1} f\|_{L^p(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}, \quad 1 < p < \infty.$$

□

**Proposition 2.6** ([5] and [10]). *Assume that $(u_0, b_0) \in L^s_\sigma(\mathbb{R}^3)$ and $w_0 \in L^s(\mathbb{R}^3)$ with $s > 3$. Then there exists a constant $T > 0$ and a unique local strong solution $(u, b, w)$ of equations (1.1)–(1.2) such that $(u, b, w) \in BC([0, T); L^s(\mathbb{R}^3))$. Moreover, if $T^* > 0$ is the maximal time of the local smooth solution, then for any $t \in (0, T^*)$,*

$$\|(u, b, w)(t)\|_{L^s} \geqslant \frac{C}{(T^* - t)^{(s-3)/2s}},$$

*where the constant $C > 0$ is independent of $T^*$ and $s$.*

For this, according to the argument in [25] or [12], we can establish a Serrin's type regularity criterion on the gradient of pressure function $\mathcal{P}$. Multiplying both sides of $(1.1)_1$ by $u|u|^2$, $(1.1)_2$ by $b|b|^2$ and $(1.1)_3$ by $w|w|^2$, and then integrating them over $\mathbb{R}^3$, it follows that

$$\begin{aligned}
(2.2)\quad & \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|(u, b, w)\|_{L^4}^4 + \|\nabla(|u|^2, |b|^2, |w|^2)\|_{L^2}^2 + \||u||\nabla u|\|_{L^2}^2 \\
& + \||b||\nabla b|\|_{L^2}^2 + \||w||\nabla w|\|_{L^2}^2 + 2\chi\|w\|_{L^4}^4 + \kappa\||w|\operatorname{div} w\|_{L^2}^2 \\
& \lesssim \underbrace{\int_{\mathbb{R}^3} \nabla\mathcal{P} \cdot |u|^2 u \,\mathrm{d}x\,\mathrm{d}t}_{\mathcal{J}_1} + \underbrace{\int_{\mathbb{R}^3} (b \cdot \nabla) b \cdot |u|^2 u \,\mathrm{d}x\,\mathrm{d}t}_{\mathcal{J}_2} \\
& + \underbrace{\frac{1}{2}\int_{\mathbb{R}^3} \nabla(|b|^2) \cdot |u|^2 u \,\mathrm{d}x\,\mathrm{d}t}_{\mathcal{J}_3} + \underbrace{\int_{\mathbb{R}^3} (b \cdot \nabla) u \cdot |b|^2 b \,\mathrm{d}x\,\mathrm{d}t}_{\mathcal{J}_4} \\
& + \underbrace{\frac{\chi}{2}\int_{\mathbb{R}^3} |w|^2 w \cdot (\nabla \times u)\,\mathrm{d}x}_{\mathcal{J}_5} + \underbrace{\frac{\chi}{2}\int_{\mathbb{R}^3} |w|^2 w \cdot (\nabla \times u)\,\mathrm{d}x}_{\mathcal{J}_6} \\
& - \underbrace{\int_{\mathbb{R}^3} \operatorname{div} w (w \cdot \nabla |w|^2)\,\mathrm{d}x}_{\mathcal{J}_7}.
\end{aligned}$$

For $\mathcal{J}_1$, after the integration by parts, Hölder and Young's inequalities imply

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \nabla\mathcal{P}\cdot|u|^2u\,\mathrm{d}x\\
&\qquad\leqslant \int_{\mathbb{R}^3}|\mathcal{P}|^2|u|^2\,\mathrm{d}x+\frac{1}{8}\||u||\nabla u|\|_{L^2}^2\\
&\qquad\lesssim (\|\nabla\mathcal{P}\|_{L^2}\|u\|_{L^4}^{2(2-3/p)}\|\nabla|u|^2\|_{L^2}^{3/p-1}\\
&\qquad\quad+\|\nabla|u|^2\|_{L^2}\|\mathcal{P}\|_{L^2}^{2-3/p}\|\nabla\mathcal{P}\|_{L^2}^{3/p-1})\|\mathcal{P}\|_{\dot{B}_{q,\infty}^{-3(1/p-1/q)}}\\
&\qquad\lesssim ((\||u||\nabla u|+|b||\nabla b|)\|_{L^2}\|u\|_{L^4}^{2(2-3/p)}\|\nabla|u|^2\|_{L^2}^{3/p-1}+\|\nabla|u|^2\|_{L^2}\||u|^2\\
&\qquad\quad+|b|^2\|_{L^2}^{2-3/p}\|(|u||\nabla u|+|b||\nabla b|)\|_{L^2}^{3/p-1})\|\mathcal{P}\|_{\dot{B}_{q,\infty}^{-3(1/p-1/q)}}\\
&\qquad\lesssim \|\mathcal{P}\|_{\dot{B}_{q,\infty}^{-3(1/p-1/q)}}^{2p/(2p-3)}(\|u\|_{L^4}^4+\|b\|_{L^4}^4)\\
&\qquad\quad+\frac{1}{8}(\|\nabla|u|^2\|_{L^2}^2+\|\nabla|b|^2\|_{L^2}^2+\||u||\nabla u|\|_{L^2}^2+\||b||\nabla b|\|_{L^2}^2).
\end{aligned}
$$

Using the integration by parts, $\mathcal{J}_2$, $\mathcal{J}_3$ and $\mathcal{J}_4$ is bounded by

$$
\begin{aligned}
|J_2|+|J_3|+|J_4| &\leqslant C(\||b|^2u\|_{L^2}^2+\frac{1}{16}(\|\nabla|u|^2\|_{L^2}^2+\|\nabla|b|^2\|_{L^2}^2)\\
&\leqslant C\|b\|_{L^{a_1}}^{2a_1/(a_1-3)}(\||u|^2\|_{L^2}^2+\||b|^2\|_{L^2}^2)\\
&\quad+\frac{1}{16}(\|\nabla|u|^2\|_{L^2}^2+\|\nabla|b|^2\|_{L^2}^2).
\end{aligned}
$$

In a similar way, for $\mathcal{J}_5$ and $\mathcal{J}_6$, it shows that

$$
|J_5|+|J_6|\leqslant C\int_{\mathbb{R}^3}(|u|^4+|w|^4)+|\operatorname{div}w|^2\,\mathrm{d}x+\frac{1}{16}\int_{\mathbb{R}^3}(||w||\nabla w||^2+|\nabla|w|^2|^2)\,\mathrm{d}x.
$$

Plugging this into (2.2), we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|(u,b,w)\|_{L^4}^4+\|\nabla(|u|^2,|b|^2,|w|^2)\|_{L^2}^2+\||u||\nabla u|\|_{L^2}^2+\||b||\nabla b|\|_{L^2}^2+\||w||\nabla w|\|_{L^2}^2\\
&\qquad\lesssim \frac{\|\mathcal{P}\|_{\dot{B}_{q,\infty}^{-3(1/p-1/q)}}^{2p/(2p-3)}}{\log(\mathrm{e}+\|u\|_{L^s})}(\log(\mathrm{e}+\|u\|_{L^s}))\|(u,b,w)\|_{L^4}^4+\|b\|_{L^{a_1}}^{2a_1/(a_1-3)}\|(u,b,w)\|_{L^4}^4.
\end{aligned}
$$

According to the arguments in [13], pages 2868–2869, the desired result is obtained and the proof of Theorem 1.2 is complete. □

## 3. Proof of Theorem 1.7

In fact, this proof is similar to that in Theorem 1.7. Only differences occur in the term associated with the pressure function. Indeed, it is estimated by

$$\int_{\mathbb{R}^3} \nabla P\cdot |u|^2 u\,\mathrm{d}x \leqslant \int_{\mathbb{R}^3} |P|^2|u|^2\,\mathrm{d}x + \frac{1}{8}\||u||\nabla u|\|_{L^2}^2.$$

For the first term, namely $I$, above we let

$$V = \mathrm{e}^{-|x|^2} + |u|, \quad \Pi = \frac{P}{(\mathrm{e}^{-|x|^2} + u + b)^{\theta}}.$$

Due to the definition of $V$, we observe that

$$\|V^2\|_{L^2}^2 \lesssim (1 + \|u\|_{L^2}^2 + \||u|^2\|_{L^2}^2 + \|b\|_{L^2}^2 + \||b|^2\|_{L^2}^2),$$

and

$$\|\nabla V^2\|_{L^2}^2 \lesssim (1 + \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla |u|^2\|_{L^2} + \|b\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \|\nabla |b|^2\|_{L^2}).$$

By the interpolation and Sobolev inequalities in Lorentz spaces (2.1) we have

$$\begin{cases} \|V^2\|_{L^{(2-\alpha)r_1,2}} \lesssim \|V^2\|_{L^2}^{1-\delta_1}\|\nabla V^2\|_{L^2}^{\delta_1}, \\ \|V^2\|_{L^{\alpha r_2,2}} \lesssim \|V^2\|_{L^2}^{1-\delta_2}\|\nabla V^2\|_{L^2}^{\delta_2}, \qquad 0<\delta_1,\delta_2<1, \end{cases} \tag{3.1}$$

with

$$\frac{1}{(2-\alpha)r_1} = \frac{1-\delta_1}{2} + \frac{\delta_1}{6}, \quad \frac{1}{\alpha r_2} = \frac{1-\delta_2}{2} + \frac{\delta_2}{6}.$$

And thus, we get

$$\begin{aligned} I &= \int_{\mathbb{R}^3} \Big(\frac{P}{(\mathrm{e}^{-|x|^2} + |u|)^{\theta}}\Big)^{\alpha} |P|^{2-\alpha}(\mathrm{e}^{-|x|^2} + |u|)^{\alpha\theta}|u|^2\,\mathrm{d}x \\ &\lesssim \|\Pi\|_{L^{q,\infty}}^{\alpha}\|P\|_{L^{(2-\alpha)r_1,2}}^{2-\alpha}\|V^2\|_{L^{\alpha r_2,2}}^{\alpha}, \quad \frac{\alpha}{q} + \frac{1}{r_1} + \frac{1}{r_2} = 1 \\ &\lesssim \|\Pi\|_{L^{q,\infty}}^{\alpha}\||u|^2 + |b|^2\|_{L^{(2-\alpha)r_1,2}}^{2-\alpha}\|V^2\|_{L^{\alpha r_2,2}}^{\alpha} \\ &\lesssim \|\Pi\|_{L^{q,\infty}}^{\alpha}\|V^2\|_{L^{(2-\alpha)r_1,2}}^{2-\alpha}\|V^2\|_{L^{\alpha r_2,2}}^{\alpha} \\ &\lesssim \|\Pi\|_{L^{q,\infty}}^{\alpha}\|V^2\|_{L^2}^{(1-\delta_1)(2-\alpha)}\|\nabla V^2\|_{L^2}^{\delta_1(2-\alpha)}\|V^2\|_{L^2}^{(1-\delta_2)\alpha}\|\nabla V^2\|_{L^2}^{\delta_2\alpha} \\ &\lesssim \|\Pi\|_{L^{q,\infty}}^{2\alpha/(2-\delta_1(2-\alpha)-\delta_2\alpha)}\|V^2\|_{L^2}^2 + \frac{1}{4}\|\nabla V^2\|_{L^2}^2, \end{aligned} \tag{3.2}$$

where we use inequality (3.1).

Due to the definition of $V$, we know from (3.2) that

$$I \lesssim \|\Pi\|_{L^{q,\infty}}^{2\alpha/(2-\delta_1(2-\alpha)-\delta_2\alpha)}(1+\|u\|_{L^4}^4+\|b\|_{L^4}^4) + C(1+\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\frac{1}{4}(\|\nabla|u|^2\|_{L^2}^2+\|\nabla|b|^2\|_{L^2}^2). \tag{3.3}$$

Combining estimates $\mathcal{J}_2$–$\mathcal{J}_6$ in Theorem 1.2 and (3.3), we finally obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|(u,b,w)\|_{L^4}^4 &+ \|\nabla(|u|^2,|b|^2,|w|^2)\|_{L^2}^2+\||u||\nabla u|\|_{L^2}^2+\||b||\nabla b|\|_{L^2}^2\\ &+\||w||\nabla w|\|_{L^2}^2+2\|w\|_{L^4}^4+\||w|\operatorname{div} w\|_{L^2}^2\\ &\lesssim \|(u,b,w)\|_{L^4}^4+C(1+\|\nu\|_{L^2}^2)\\ &\quad+\|\Pi\|_{L^{q,\infty}}^{2\alpha/(2-\delta_1(2-\alpha)-\delta_2\alpha)}(1+\|(u,b,w)\|_{L^4}^4),\end{aligned}$$

where we use $u,b\in L^\infty(0,T;L^2(\mathbb{R}^3))$ from (1.3). By Grönwall's lemma, we obtain that $(u,b,w)$ is regular in $\mathbb{R}^3\times[0,T)$ provided that

$$\Pi\in L^{2\alpha/(2-\delta_1(2-\alpha)-\delta_2\alpha)}(0,T;L^{q,\infty}(\mathbb{R}^3)),$$

which completes the proof of Theorem 1.7. □

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