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ON A GENERALIZED DENSITY POINT DEFINED BY FAMILIES
OF SEQUENCES INVOLVING IDEALS

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Abstract. We introduce the notion of $\mathcal{I}_{(s)}$-density point corresponding to the family of unbounded and $\mathcal{I}$-monotonic increasing positive real sequences, where $\mathcal{I}$ is the ideal of subsets of the set of natural numbers. We study the corresponding topology in the space of reals and investigate several properties of this topology. Also we present a characterization of equality between the classical density topology and $\mathcal{I}_{(s)}$-density topology.

Keywords: density topology; ideal; $\mathcal{I}$-density topology

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1. Introduction

A series of important developments in density topology was evolved from the foundational result of Goffman et al. [11] to the most remarkable work of Filipczak and Hejduk in [10] where they defined the density point by families of sequences. Density topologies were studied extensively in several spaces like the space of real numbers [21], Euclidean n-space [25], metric spaces [18] etc. In the recent past the notion of classical Lebesgue density point has been generalized by weakening the assumptions on the sequences of intervals and consequently several notions like $\langle s \rangle$-density point by Filipczak and Hejduk [10], $\mathcal{J}$-density point by Hejduk and Wiertelak [16], $\mathcal{S}$-density point by Strobin and Wiertelak [24] were obtained. A significant volume of work in this area was carried out by distinguished researchers in the last few decades

[1], [6], [9], [13], [15], [27]. In recent time Banerjee and Debnath have found a new way to generalize density topology using ideals in [2].

The usual notion of convergence does not always capture the properties of a vast class of nonconvergent sequences in fine details. In order to include more sequences under purview the idea of convergence of real sequences was generalized to the notion of statistical convergence [8], [23] followed by the idea of ideal convergence [17].

$\langle s \rangle$-Density topology [10] is the object of our interest and plays a central role in our study. The prime objective of this paper is to investigate a generalized density point defined by families of sequences. In this paper we try to generalize the $\langle s \rangle$-density point by involving the notion of ideal $\mathcal{I}$ of subsets of naturals. We have given the notion of $\mathcal{I}_{(s)}$-density and induced $\mathcal{I}_{(s)}$-density topology in the space of reals. We have shown that an $\mathcal{I}_{(s)}$-density point is dependent on the nature of the sequence (s). Some natural properties of this topology have been studied. Also we have given a characterization of equality between this topology and classical density topology.

2. Preliminaries

Let us recall the definition of asymptotic density. Here $\mathbb{N}$ stands for the set of natural numbers and for $K \subset \mathbb{N}$ we denote by $K(n)$ the set $\{k \in K \colon k \leqslant n\}$, $|K(n)|$ being the cardinality of $K(n)$. The asymptotic density of K is defined by

$$d(K) = \lim_{n \to \infty} \frac{|K(n)|}{n},$$

provided the limit exists. The notion of asymptotic density was used to define the idea of statistical convergence by Fast [8] generalizing the idea of usual convergence of real sequences. A sequence $\{x_n\}_{n \in \mathbb{N}}$ of real numbers is said to be statistically convergent to x_0 if for any given $\varepsilon > 0$ the set $K(\varepsilon) = \{k \in \mathbb{N} \colon |x_k - x_0| \geqslant \varepsilon\}$ has asymptotic density zero.

After this pioneering work, the theory of statistical convergence of real sequences was generalized to the idea of $\mathcal{I}$-convergence of real sequences by Kostyrko et al. [17] using the notion of ideal $\mathcal{I}$ of subsets of $\mathbb{N}$. We use the symbol $2^{\mathbb{N}}$ to denote the power set of $\mathbb{N}$.

Definition 2.1 ([17])**.** A nonvoid class $\mathcal{I} \subset 2^{\mathbb{N}}$ is called an ideal if $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$. Clearly $\{\emptyset\}$ and $2^{\mathbb{N}}$ are ideals on $\mathbb{N}$ which are called trivial ideals. An ideal is called nontrivial if it is not trivial.

$\mathcal{I}$ is admissible if it contains all singletons. By Fin we denote the ideal of all finite subsets of $\mathbb{N}$. It is easy to verify that the family $\mathcal{J} = \{A \subset \mathbb{N} \colon d(A) = 0\}$ forms a nontrivial admissible ideal of subsets of $\mathbb{N}$. If $\mathcal{I}$ is a nontrivial ideal, then the family

of sets $\{M \subset \mathbb{N} \colon \mathbb{N} \setminus M \in \mathcal{I}\}$ denoted by $\mathcal{F}(\mathcal{I})$ is a filter on $\mathbb{N}$ and it is called the filter associated with the ideal $\mathcal{I}$ on $\mathbb{N}$.

Definition 2.2 ([17])**.** A sequence $\{x_n\}_{n \in \mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-convergent to x_0 if the set $K(\varepsilon) = \{k \in \mathbb{N} \colon |x_k - x_0| \geqslant \varepsilon\}$ belongs to $\mathcal{I}$ for any $\varepsilon > 0$.

If $\mathcal{I} = \text{Fin}$ then Fin-convergence coincides with the usual notion of convergence of real numbers. If $\mathcal{I}$ is an admissible ideal then for a sequence of real numbers, the usual convergence implies $\mathcal{I}$-convergence with the same limit. Several works were carried out in this direction by many authors [3], [4], [19]. Throughout the paper the ideal $\mathcal{I}$ always stands for a nontrivial admissible ideal of subsets on $\mathbb{N}$.

Now let us introduce the following notations. Throughout, $\mathbb{R}$ stands for the set of all real numbers. We use the notation $\mathcal{L}$ for the σ-algebra of Lebesgue measurable sets on $\mathbb{R}$ and λ for the Lebesgue measure on $\mathbb{R}$ [12]. Wherever we write $\mathbb{R}$ it means that $\mathbb{R}$ is equipped with the natural topology unless otherwise stated. We use the symbol $2^{\mathbb{R}}$ to denote the power set of $\mathbb{R}$. By the "Euclidean F_σ and Euclidean G_δ set" we mean F_σ and G_δ sets in $\mathbb{R}$ equipped with the natural topology. The symmetric difference of two sets A and B is $(A \setminus B) \cup (B \setminus A)$ and it is denoted by $A \triangle B$. For any two sets $A, B \in \mathcal{L}$ the fact that $\lambda(A \triangle B) = 0$ is denoted by $A \sim B$. By "a sequence of intervals $\{J_n\}_{n \in \mathbb{N}}$ about a point p" we mean $p \in \bigcap_{n \in \mathbb{N}} J_n$. The length of the interval J_n is denoted by $|J_n|$.

Definition 2.3 ([26])**.** For $E \in \mathcal{L}$ and a point $p \in \mathbb{R}$ we say that the point p is a classical density point of E if and only if

$$\lim_{h \to 0+} \frac{\lambda(E \cap [p-h, p+h])}{2h} = 1.$$

Equivalently, we can say that the point $p \in \mathbb{R}$ is a classical density point of E if and only if

$$\lim_{h \to 0+} \frac{\lambda((\mathbb{R} \setminus E) \cap [p-h, p+h])}{2h} = 0.$$

The set of all classical density points of E is denoted by $\Phi(E)$. The collection

$$\mathcal{T}_d = \{E \in \mathcal{L} \colon E \subseteq \Phi(E)\}$$

is a topology in the real line [26] and is called the classical density topology.

Theorem 2.4 ([20])**.** *For any set $H \in \mathcal{L}$,*

$$\lambda(H \triangle \Phi(H)) = 0.$$

The above theorem is known as the Lebesgue Density Theorem.

Definition 2.5 ([14]). We say that an operator $\Phi\colon \mathcal{L} \to \mathcal{L}$ is a lower density operator if the following conditions are satisfied:

(1) $\Phi(\emptyset) = \emptyset$, $\Phi(\mathbb{R}) = \mathbb{R}$;
(2) for all $A, B \in \mathcal{L}$, $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$;
(3) for all $A, B \in \mathcal{L}$, $A \sim B \Longrightarrow \Phi(A) = \Phi(B)$;
(4) for all $A \in \mathcal{L}$, $A \sim \Phi(A)$.

Definition 2.6 ([14]). We say that an operator $\Psi\colon \mathcal{L} \to 2^{\mathbb{R}}$ is an almost density operator if the following conditions are satisfied:

(1) $\Psi(\emptyset) = \emptyset$, $\Psi(\mathbb{R}) = \mathbb{R}$;
(2) for all $A, B \in \mathcal{L}$, $\Psi(A \cap B) = \Psi(A) \cap \Psi(B)$;
(3) for all $A, B \in \mathcal{L}$, $A \sim B \Longrightarrow \Psi(A) = \Psi(B)$;
(4) for all $A \in \mathcal{L}$, $\lambda(\Psi(A) \setminus A) = 0$.

R e m a r k 2.7. A lower density operator is an almost density operator but not conversely. For an example of almost density operator that is not a lower density operator, see [14].

Theorem 2.8 ([14]). *Let $\Psi\colon \mathcal{L} \to 2^{\mathbb{R}}$ be an almost density operator. Then the family $\mathcal{T}_{\Psi} = \{B \in \mathcal{L}\colon B \subseteq \Psi(B)\}$ forms a topology on $\mathbb{R}$.*

In [10] Filipczak and Hejduk introduced the notion of $\langle s \rangle$-density as follows. Let $\mathcal{S}$ be the family of all unbounded and nondecreasing sequences of positive reals. Every sequence $\{s_n\} \in \mathcal{S}$ is denoted by $\langle s \rangle$. Then a new kind of density point is defined.

Definition 2.9 ([10]). Let $\langle s \rangle \in \mathcal{S}$. We say that $x \in \mathbb{R}$ is a density point of a set $A \in \mathcal{L}$ with respect to a sequence $\langle s \rangle \in \mathcal{S}$ or an $\langle s \rangle$-density point of A if

$$\lim_{n \to \infty} \frac{\lambda(A \cap [x - 1/s_n, x + 1/s_n])}{2/s_n} = 1.$$

The point x is an $\langle s \rangle$-dispersion point of A if x is an $\langle s \rangle$-density point of $\mathbb{R} \setminus A$.

Proposition 2.10 ([5]). *Let $A \in \mathcal{L}$ and $x \in \mathbb{R}$. Then*

$$\lim_{h \to 0+} \frac{\lambda(A \cap [x - h, x + h])}{2h} = 1$$

if and only if

$$\lim_{n \to \infty} \frac{\lambda(A \cap [x - 1/n, x + 1/n])}{2/n} = 1.$$

So if we choose, in particular, $s_n = n$ for all $n \in \mathbb{N}$ in Definition 2.9 then we obtain the notion of classical density point.

For any sequence $\langle s \rangle \in \mathcal{S}$ and set $A \in \mathcal{L}$ let

$$\Phi_{\langle s \rangle}(A) = \{x \in \mathbb{R} \colon x \text{ is } \langle s \rangle\text{-density point of A}\}.$$

Proposition 2.11 ([10]). *For every pair of Lebesgue measurable sets $A, B \in \mathcal{L}$ and a sequence $\langle s \rangle \in \mathcal{S}$ we have*

(1) $\Phi_{\langle s \rangle}(\emptyset) = \emptyset$, $\Phi_{\langle s \rangle}(\mathbb{R}) = \mathbb{R}$;

(2) $\Phi_{\langle s \rangle}(A \cap B) = \Phi_{\langle s \rangle}(A) \cap \Phi_{\langle s \rangle}(B)$;

(3) $A \sim B \Longrightarrow \Phi_{\langle s \rangle}(A) = \Phi_{\langle s \rangle}(B)$;

(4) $\Phi_{\langle s \rangle}(A) \sim A$;

(5) $\Phi(A) \subseteq \Phi_{\langle s \rangle}(A)$.

Corollary 2.12 ([10]). *The operator $\Phi_{\langle s \rangle} \colon \mathcal{L} \to \mathcal{L}$ is a lower density operator.*

Theorem 2.13 ([10]). *For every sequence $\langle s \rangle \in \mathcal{S}$ the family*

$$\mathcal{T}_{\langle s \rangle} = \{A \in \mathcal{L} \colon A \subseteq \Phi_{\langle s \rangle}(A)\}$$

forms a topology such that $\mathcal{T}_d \subseteq \mathcal{T}_{\langle s \rangle}$.

In [10] a characterization of the equality $\mathcal{T}_d = \mathcal{T}_{\langle s \rangle}$ was formulated.

Theorem 2.14 ([10]). *Let $\langle s \rangle \in \mathcal{S}$ be a sequence, then $\mathcal{T}_d = \mathcal{T}_{\langle s \rangle}$ if and only if* $\liminf\limits_{n \to \infty} s_n / s_{n+1} > 0$.

3. $\mathcal{I}_{(s)}$-DENSITY

Throughout we consider the ideal $\mathcal{I}$ to be a nontrivial admissible ideal.

Definition 3.1 ([22]). A real valued sequence $x = \{x_n\}_{n \in \mathbb{N}}$ is said to be $\mathcal{I}$-monotonic increasing (or $\mathcal{I}$-monotonic decreasing), if there is a set $\{k_1 < k_2 < \ldots\} \in \mathcal{F}(\mathcal{I})$ such that $x_{k_i} \leqslant x_{k_{i+1}}$ (or $x_{k_i} \geqslant x_{k_{i+1}}$) for every $i \in \mathbb{N}$.

The family of all unbounded, $\mathcal{I}$-monotonic increasing positive real sequences is denoted by $\Sigma_{\mathcal{I}}$. If a sequence $\{s_n\}_{n \in \mathbb{N}}$ is chosen from the family $\Sigma_{\mathcal{I}}$ it is denoted by (s).

Definition 3.2. Now for $A \in \mathcal{L}$, $p \in \mathbb{R}$ and $(s) \in \Sigma_{\mathcal{I}}$, let us take a collection of closed intervals about p like $J_n = [p - 1/s_n, p + 1/s_n]$ for $n \in \mathbb{N}$.

Now let us take $x_n = \lambda(A \cap J_n)/|J_n|$. Then clearly $x = \{x_n\}_{n \in \mathbb{N}}$ is a sequence of non-negative real numbers. If $\mathcal{I} - \lim_n x_n$ exists then we denote its value by $\mathcal{I}_{(s)} - d(p, A)$ which we call the $\mathcal{I}_{(s)}$-density of A at the point p and clearly $\mathcal{I}_{(s)} - d(p, A) = \mathcal{I} - \lim x_n$.

A point $p_0 \in \mathbb{R}$ is an $\mathcal{I}_{(s)}$-density point of $A \in \mathcal{L}$ if $\mathcal{I}_{(s)} - d(p_0, A) = 1$.

If a point $p_0 \in \mathbb{R}$ is an $\mathcal{I}_{(s)}$-density point of the set $\mathbb{R} \setminus A$, then p_0 is an $\mathcal{I}_{(s)}$-dispersion point of A.

If in the above definition we take $J_n = [p, p+1/s_n]$ for $n \in \mathbb{N}$ so that $\mathcal{I} - \lim_n x_n = 1$ then the point $p \in \mathbb{R}$ is a right $\mathcal{I}_{(s)}$-density point of A and if we take $J_n = [p - 1/s_n, p]$ for $n \in \mathbb{N}$ so that $\mathcal{I} - \lim_n x_n = 1$ then the point $p \in \mathbb{R}$ is a left $\mathcal{I}_{(s)}$-density point of A. We note that

$$\lambda\left(A \cap \left[p - \frac{1}{s_n}, p + \frac{1}{s_n}\right]\right) = \lambda\left(A \cap \left[p - \frac{1}{s_n}, p\right]\right) + \lambda\left(A \cap \left[p, p + \frac{1}{s_n}\right]\right).$$

It can be easily proved that if a point $p \in \mathbb{R}$ is both left and right $\mathcal{I}_{(s)}$-density point of A, then the point p is an $\mathcal{I}_{(s)}$-density point of A.

N o t e 3.3. The following three kinds of density points may be distinguished in this context.

(1) If we take the sequence $s_n = n$ for all $n \in \mathbb{N}$ then by Proposition 2.10 we obtain the notion of classical density point.
(2) If $\{s_n\} \in \mathcal{S}$ then the notion of $\langle s \rangle$-density point is obtained.
(3) For $\{s_n\} \in \Sigma_{\mathcal{I}}$ we introduced the notion of $\mathcal{I}_{(s)}$-density point.

E x a m p l e 3.4. Let us consider the ideal $\mathcal{J}$ of a subcollection of $2^{\mathbb{N}}$ where $\mathcal{J}$ consists of all those subsets of $\mathbb{N}$ whose asymptotic density is zero. Let us take the set A as the open interval $(-1, 1)$ and the point p to be 0. For any positive real sequence $\{s_n\}_{n \in \mathbb{N}}$ let us consider a collection of closed bounded intervals $J_n = [-1/s_n, 1/s_n]$ for all $n \in \mathbb{N}$. We make a choice of the sequence $\{s_n\}_{n \in \mathbb{N}}$ as follows:

$$s_n = \begin{cases} n! & \text{if } n \neq m^2 \quad \text{where } m \in \mathbb{N}, \\ n^{-1} & \text{if } n = m^2 \quad \text{where } m \in \mathbb{N}. \end{cases}$$

Then $s_2 < s_3 < s_5 < s_6 < s_7 < \ldots$ and also the set of positive integers $\{2, 3, 5, 6, 7, 8, 10, \ldots\} = \{n \in \mathbb{N}: n \neq m^2 \text{ where } m \in \mathbb{N}\} \in \mathcal{F}(\mathcal{J})$, since $\{n \in \mathbb{N}: n = m^2 \text{ where } m \in \mathbb{N}\}$ has natural density zero. So, $\{s_n\}_{n \in \mathbb{N}}$ is $\mathcal{J}$-monotonic increasing and unbounded positive real sequence, i.e., $\{s_n\} \in \Sigma_{\mathcal{J}}$.

Now the sequence $x_n = \lambda(A \cap J_n)/|J_n|$ for $n \in \mathbb{N}$ becomes

$$x_n = \begin{cases} 1 & \text{if } n \neq m^2 \quad \text{where } m \in \mathbb{N}, \\ n^{-1} & \text{if } n = m^2 \quad \text{where } m \in \mathbb{N}. \end{cases}$$

Therefore, since the subsequence $\{x_n\}_{n=m^2}$ converges to 0 and the subsequence $\{x_n\}_{n \neq m^2}$ converges to 1, $\lim_n x_n$ does not exist. Since for any $\varepsilon > 0$, $\{n \in \mathbb{N}: |x_n - 1| \geqslant \varepsilon\} \subseteq \{n \in \mathbb{N}: \text{there exists } m \in \mathbb{N} \text{ such that } n = m^2\}$ and $\{n \in \mathbb{N}: \text{there exists } m \in \mathbb{N} \text{ such that } n = m^2\} \in \mathcal{J}$, so $\{n \in \mathbb{N}: |x_n - 1| \geqslant \varepsilon\} \in \mathcal{J}$. Thus, $\mathcal{J} - \lim_n x_n = 1$. Consequently, 0 is a $\mathcal{J}_{(s)}$-density point of A.

In order to establish that an $\mathcal{I}_{(s)}$-density point is indeed a generalization of $\langle s \rangle$-density point we need the following proposition and its corollary.

Proposition 3.5. *Let $A \in \mathcal{L}$, $x \in \mathbb{R}$ and $\langle s \rangle = \{s_n\}_{n \in \mathbb{N}} \in \mathcal{S}$. If $\{r_n\}_{n \in \mathbb{N}}$ be any real sequence such that there exists $n_0 \in \mathbb{N}$, for which $s_n = r_n$ for $n \geqslant n_0$, then x is $\langle s \rangle$-density point of A if and only if x is $\langle r_n \rangle$-density point of A.*

The result is obvious, so the proof is omitted.

Corollary 3.6. *In Definition 2.9 if we choose $\{s_n\}$ to be any unbounded positive real sequence and there exists $n_0 \in \mathbb{N}$ for which $\{s_n\}$ for $n \geqslant n_0$ is nondecreasing, then the definition remains valid as well and we can write without any loss of generality that x is $\langle s \rangle$-density point of A.*

N o t e 3.7.   In particular in Definition 3.2, if we take $\mathcal{I} = \mathrm{Fin}$, then the collection Σ_{Fin} contains unbounded and Fin-monotonic increasing positive real sequences.

Theorem 3.8. *Let $A \in \mathcal{L}$, $x \in \mathbb{R}$ and $(s) = \{s_n\}_{n \in \mathbb{N}} \in \Sigma_{\mathrm{Fin}}$. If x is a $\mathrm{Fin}_{(s)}$-density point of A then it is $\langle s \rangle$-density point of A.*

P r o o f.   Let $(s) \in \Sigma_{\mathrm{Fin}}$. So, (s) is a Fin-monotonic increasing positive real sequence. Thus, there exists $\{k_1 < k_2 < k_3 < \ldots\} \in \mathcal{F}(\mathrm{Fin})$ such that $s_{k_i} \leqslant s_{k_{i+1}}$ for every $i \in \mathbb{N}$. Now if $\mathbb{N} \setminus \{k_1 < k_2 < k_3 < \ldots\} = \{l_1, l_2, \ldots, l_N\}$, then there exists some $r_0 \in \mathbb{N}$ such that $l_N < k_{r_0}$. So, $\{s_n\}_{n \geqslant k_{r_0}}$ is nondecreasing. We claim that for $t \in \mathbb{N}$,

(3.1) $$k_{r_0 + (t+1)} = k_{(r_0 + t)} + 1.$$

Since $\{k_1 < k_2 < k_3 < \ldots\} \cup \{l_1, l_2, \ldots, l_N\} = \mathbb{N}$ and $l_N < k_{r_0}$, so $k_{r_0 + i} = k_{r_0} + i$ for $i \in \mathbb{N}$. Thus we get consecutive natural numbers from k_{r_0} onwards in the set $\{k_1 < k_2 < k_3 < \ldots\}$.

Now let x be a $\mathrm{Fin}_{(s)}$-density point of A. So for any given $\varepsilon > 0$,

$$\left\{ n \in \mathbb{N} \colon 1 - \varepsilon < \frac{\lambda(A \cap [x - 1/s_n, x + 1/s_n])}{2/s_n} < 1 + \varepsilon \right\} \in \mathcal{F}(\mathrm{Fin}).$$

Thus,

$$\mathcal{B} = \left\{ n \in \mathbb{N} \colon 1 - \varepsilon < \frac{\lambda(A \cap [x - 1/s_n, x + 1/s_n])}{2/s_n} < 1 + \varepsilon \right\}$$
$$\cap \{ k_1 < k_2 < k_3 < \ldots \} \in \mathcal{F}(\mathrm{Fin}).$$

As a result, $\mathcal{B} \subset \{ k_1 < k_2 < k_3 < \ldots \}$ and so $\{ k_1 < k_2 < k_3 < \ldots \} = \mathcal{B} \cup \mathcal{C}$ for some set $\mathcal{C}$. Clearly,

$$\mathcal{C} = \{ k_1 < k_2 < k_3 < \ldots \} \setminus \mathcal{B} = \{ k_1 < k_2 < k_3 < \ldots \} \cap \mathcal{B}^c.$$

Thus, $\mathcal{C} \subset \mathcal{B}^c$ and $\mathcal{B} \in \mathcal{F}(\mathrm{Fin})$ which implies $\mathcal{C} \in \mathrm{Fin}$. So, $\mathcal{C}$ is a finite set. Consequently,

$$\mathcal{B} = \{ k_1 < k_2 < k_3 < \ldots \} \setminus \mathcal{C}.$$

Clearly, for any given $\varepsilon > 0$,

$$\left\{ n \in \mathbb{N} \colon 1 - \varepsilon < \frac{\lambda(A \cap [x - 1/s_n, x + 1/s_n])}{2/s_n} < 1 + \varepsilon \right\}$$
$$\supset \{ k_1 < k_2 < k_3 < \ldots \} \setminus \mathcal{C} \text{ where } \mathcal{C} \text{ is a finite set.}$$

So there exists $q_0 \in \mathbb{N}$ such that

$$\left\{ n \in \mathbb{N} \colon 1 - \varepsilon < \frac{\lambda(A \cap [x - 1/s_n, x + 1/s_n])}{2/s_n} < 1 + \varepsilon \right\} \supset \{ k_{q_0} < k_{q_0+1} < k_{q_0+2} < \ldots \}.$$

In particular, if we choose $k_{m_0} = \max\{ k_{r_0}, k_{q_0} \}$, then $s_{k_{m_0}} < s_{k_{m_0+1}} < s_{k_{m_0+2}} < \ldots$ and we note that by (3.1), $k_{m_0 + (t+1)} = k_{m_0+t} + 1$ for $t \in \mathbb{N} \cup \{ 0 \}$.

Consequently, for all $n \geqslant k_{m_0}$ the set

$$\left\{ n \in \mathbb{N} \colon 1 - \varepsilon < \frac{\lambda(A \cap [x - 1/s_n, x + 1/s_n])}{2/s_n} < 1 + \varepsilon \right\}$$

contains all consecutive natural numbers on and from k_{m_0}, i.e.,

$$1 - \varepsilon < \frac{\lambda(A \cap [x - 1/s_n, x + 1/s_n])}{2/s_n} < 1 + \varepsilon \quad \text{for all } n \geqslant k_{m_0}.$$

Thus,

$$\lim_{n \to \infty} \frac{\lambda(A \cap [x - 1/s_n, x + 1/s_n])}{2/s_n} = 1.$$

Now, by Corollary 3.6 we can conclude that x is $\langle s \rangle$-density point of A. As a result our definition of $\mathcal{I}_{(s)}$-density coincides with the definition of $\langle s \rangle$-density when $\mathcal{I} = \mathrm{Fin}$.

$\square$

In the next example we investigate the role played by sequences in Definition 3.2. For any given set $A \subset \mathbb{R}$ we consider the set $-A = \{-x \colon x \in A\}$.

E x a m p l e 3.9. This example gives some insight to the role a sequence plays in the above case.

For the ideal $\mathcal{J}$ as in Example 3.4 there exists a set $A \in \mathcal{L}$ and sequences (s) and $(c) \in \Sigma_{\mathcal{J}}$ such that 0 is a $\mathcal{J}_{(c)}$-density point of $-A \cup A$ and 0 is a $\mathcal{J}_{(s)}$-dispersion point of $-A \cup A$. Now if we make a choice of the sequence $\{s_n\}_{n \in \mathbb{N}}$ as

$$
s_n = \begin{cases} n! & \text{if } n \neq m^2 \quad \text{where } m \in \mathbb{N}, \\ n^{-1} & \text{if } n = m^2 \quad \text{where } m \in \mathbb{N}, \end{cases}
$$

then we clearly from Example 3.4 have $\{s_n\} \in \Sigma_{\mathcal{J}}$ and it is denoted by (s). Now let us take a set

$$
A = \bigcup_{n=1}^{\infty} \left[\frac{1}{(n+1)!}, \frac{1}{n!\sqrt{n+1}} \right].
$$

Then A is Lebesgue measurable as it is a countable union of closed intervals. Since A is a subset of $[0, 1]$, so $\lambda(A) = \omega \leqslant 1$ for some positive real number ω. We put $x_n = s_n \lambda(A \cap [0, 1/s_n])$ for $n \in \mathbb{N}$.

Now if $n \neq m^2$, then

$$
x_n = s_n \lambda \left(A \cap \left[0, \frac{1}{s_n} \right] \right) = n! \lambda \left(A \cap \left[0, \frac{1}{n!} \right] \right) \leqslant \frac{n!}{n!\sqrt{n+1}} = \frac{1}{\sqrt{n+1}}.
$$

If $n = m^2$, then

$$
x_n = s_n \lambda \left(A \cap \left[0, \frac{1}{s_n} \right] \right) = \frac{\lambda(A \cap [0, n])}{n} = \frac{\lambda(A)}{n} = \frac{\omega}{n} \leqslant \frac{1}{n}.
$$

Since $\{x_n\}$ is a sequence of non-negative real numbers, so $\lim_n x_n = 0$. Hence, $\mathcal{J} - \lim_n x_n = 0$. So, 0 is a right $\mathcal{J}_{(s)}$-dispersion point of A for (s) in $\Sigma_{\mathcal{J}}$. Now if we take

$$
-A = \bigcup_{n=1}^{\infty} \left[-\frac{1}{(n+1)!}, -\frac{1}{n!\sqrt{n+1}} \right],
$$

then by similar calculation it can be shown that 0 is a left $\mathcal{J}_{(s)}$-dispersion point of $-A$ for $(s) \in \Sigma_{\mathcal{J}}$. We observe that $-A \cup A$ is symmetric about origin. Clearly, 0 is both right and left $\mathcal{J}_{(s)}$-dispersion point of $-A \cup A$ for (s) in $\Sigma_{\mathcal{J}}$. Consequently, 0 is a $\mathcal{J}_{(s)}$-dispersion point of $-A \cup A$ for (s) in $\Sigma_{\mathcal{J}}$.

Now, instead of taking the sequence $\{s_n\}_{n \in \mathbb{N}}$ we make some other choice of the sequence $\{c_n\}_{n \in \mathbb{N}}$ where

$$c_n = \begin{cases} n!\sqrt{n+1} & \text{if } n \neq m^2 \quad \text{where } m \in \mathbb{N}, \\ n^{-1} & \text{if } n = m^2 \quad \text{where } m \in \mathbb{N}. \end{cases}$$

Then, $c_2 < c_3 < c_5 < c_6 < c_7 < \ldots$ and so the set of positive integers $\{2, 3, 5, 6, 7, 8, 10, \ldots\} = \{n \in \mathbb{N}: n \neq m^2 \text{ where } m \in \mathbb{N}\} \in \mathcal{F}(\mathcal{J})$, i.e., $\{c_n\}_{n \in \mathbb{N}}$ is $\mathcal{J}$-monotonic increasing and unbounded positive real sequence. So, $\{c_n\} \in \Sigma_{\mathcal{J}}$ and we denote $\{c_n\}$ as (c). We put $y_n = c_n \lambda(A \cap [0, 1/c_n])$ for all $n \in \mathbb{N}$. Thus, for $n \neq m^2$,

$$\begin{aligned} y_n &= n!\sqrt{n+1}\lambda\left(A \cap \left[0, \frac{1}{n!\sqrt{n+1}}\right]\right) \\ &\geqslant n!\sqrt{n+1}\lambda\left(A \cap \left[\frac{1}{(n+1)!}, \frac{1}{n!\sqrt{n+1}}\right]\right) \\ &= 1 - \frac{1}{\sqrt{n+1}}. \end{aligned}$$

So, clearly,

$$1 - \frac{1}{\sqrt{n+1}} \leqslant y_n \leqslant 1 + \frac{1}{\sqrt{n+1}} \quad \text{for all } n \neq m^2 \quad \text{for } m \in \mathbb{N}.$$

Thus, $\{n \in \mathbb{N}: |y_n - 1| \leqslant 1/\sqrt{n+1}\} \supseteq \{n \in \mathbb{N}: n \neq m^2 \text{ where } m \in \mathbb{N}\}$. For arbitrary small $\varepsilon > 0$, choose n large enough, say there exists $n_0 \in \mathbb{N}$ so that for $n > n_0$ we have $1/\sqrt{n+1} < \varepsilon$. Then,

$$\{n \in \mathbb{N}: |y_n - 1| < \varepsilon\} \supseteq \{n \in \mathbb{N}: n \neq m^2 \text{ where } m \in \mathbb{N}\} \setminus \{1, 2, \ldots, n_0\}.$$

Since $\{n \in \mathbb{N}: n \neq m^2 \text{ for } m \in \mathbb{N}\} \setminus \{1, 2, \ldots, n_0\} \in \mathcal{F}(\mathcal{J})$, so $\{n \in \mathbb{N}: |y_n - 1| < \varepsilon\} \in \mathcal{F}(\mathcal{J})$ and thus $\mathcal{J} - \lim_{n \to \infty} y_n = 1$. Consequently, 0 is a right $\mathcal{J}_{(c)}$-density point of A. By a similar calculation it can be shown that 0 is a left $\mathcal{J}_{(c)}$-density point of $-A$ for (c) in $\Sigma_{\mathcal{J}}$. Clearly, 0 is both right and left $\mathcal{J}_{(c)}$-density point of $-A \cup A$ for (c) in $\Sigma_{\mathcal{J}}$. Consequently, 0 is a $\mathcal{J}_{(c)}$-density point of $-A \cup A$ for (c) in $\Sigma_{\mathcal{J}}$. Although 0 is a $\mathcal{J}_{(s)}$-dispersion point of $-A \cup A$ for (s) in $\Sigma_{\mathcal{J}}$.

R e m a r k 3.10. Thus in general, for any given set $A \subset \mathbb{R}$, the notion of $\mathcal{I}_{(s)}$-density point of A with respect to the sequence $(s) \in \Sigma_{\mathcal{I}}$ is dependent on the nature of the sequence (s). It may vary from sequence to sequence.

Proposition 3.11. *For any two disjoint sets $A, B \in \mathcal{L}$ and $x \in \mathbb{R}$,*

$$\mathcal{I}_{(s)} - d(x, A \cup B) = \mathcal{I}_{(s)} - d(x, A) + \mathcal{I}_{(s)} - d(x, B) \quad \text{where } (s) \in \Sigma_{\mathcal{I}}.$$

P r o o f. The result follows from the fact that $\lambda(A \cup B) = \lambda(A) + \lambda(B)$. So the detailed proof is omitted. □

N o t e 3.12. If in Proposition 3.11 we replace $\mathcal{I}_{(s)}$-density by right (or left) $\mathcal{I}_{(s)}$-density then the result holds good as well.

4. $\mathcal{I}_{(s)}$-DENSITY TOPOLOGY

For any real sequence $(s) \in \Sigma_{\mathcal{I}}$ and any set $A \in \mathcal{L}$ let us consider the collection

$$\Phi^{\mathcal{I}}_{(s)}(A) = \{x \in \mathbb{R} \colon x \text{ is } \mathcal{I}_{(s)}\text{-density point of } A\}.$$

N o t e 4.1. If $\mathcal{I} = \text{Fin}$, then by Theorem 3.8

$$\begin{aligned}
\Phi^{\text{Fin}}_{(s)}(A) &= \{x \in \mathbb{R} \colon x \text{ is } \text{Fin}_{(s)}\text{-density point of } A\} \\
&= \{x \in \mathbb{R} \colon x \text{ is } \langle s \rangle\text{-density point of } A\} \\
&= \Phi_{\langle s \rangle}(A).
\end{aligned}$$

Definition 4.2 ([7])**.** We recall that the countable union of closed sets is called F_σ sets. Countable intersection of F_σ sets is called $F_{\sigma\delta}$. Thus, $F_{\sigma\delta} = (F_\sigma)_\delta$.

Lemma 4.3. *If $H, G \in \mathcal{L}$, then $|\lambda(H) - \lambda(G)| \leqslant \lambda(H \triangle G)$.*

The proof is omitted.

Proposition 4.4. *For any set $A \in \mathcal{L}$ and a sequence $(s) \in \Sigma_{\mathcal{I}}$ the set $\Phi^{\mathcal{I}}_{(s)}(A)$ is a $F_{\sigma\delta}$ set.*

P r o o f. For $A \in \mathcal{L}$ and $(s) \in \Sigma_{\mathcal{I}}$, let us consider the function $G(p, n) \colon \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ defined as

$$G(p, n) = \lambda\left(A \cap \left[p - \frac{1}{s_n}, p + \frac{1}{s_n} \right] \right).$$

Now for $p, q \in \mathbb{R}$ and fixed $n \in \mathbb{N}$ we get by Lemma 4.3,

$$
\begin{aligned}
|G(p,n) - G(q,n)| &= \left| \lambda \left(A \cap \left[p - \frac{1}{s_n}, p + \frac{1}{s_n} \right] \right) - \lambda \left(A \cap \left[q - \frac{1}{s_n}, q + \frac{1}{s_n} \right] \right) \right| \\
&\leqslant \lambda \left(\left(A \cap \left[p - \frac{1}{s_n}, p + \frac{1}{s_n} \right] \right) \triangle \left(A \cap \left[q - \frac{1}{s_n}, q + \frac{1}{s_n} \right] \right) \right) \\
&= \lambda \left(A \cap \left(\left[p - \frac{1}{s_n}, p + \frac{1}{s_n} \right] \triangle \left[q - \frac{1}{s_n}, q + \frac{1}{s_n} \right] \right) \right) \\
&\leqslant \left| \left[p - \frac{1}{s_n}, p + \frac{1}{s_n} \right] \triangle \left[q - \frac{1}{s_n}, q + \frac{1}{s_n} \right] \right| \\
&\leqslant 2|p - q|.
\end{aligned}
$$

Hence $G(\cdot, n)$ for fixed n satisfies the Lipschitz condition, thus it is continuous. So for fixed n the function $\frac{1}{2} s_n G(p, n)$ is continuous with respect to p. Now, $p \in \Phi^{\mathcal{I}}_{(s)}(A)$ if and only if for any $F_k = \{k_1 < k_2 < \ldots\} \in \mathcal{F}(\mathcal{I})$ such that $s_{k_i} \leqslant s_{k_{i+1}}$ for all $i \in \mathbb{N}$ we have for every $r \in \mathbb{N}$ that there exists $m \in \mathbb{N}$ such that for every $n > m$ and $n \in F_k$,

$$
\frac{s_n}{2} G(p, n) \geqslant 1 - \frac{1}{r}.
$$

Hence,

$$
\Phi^{\mathcal{I}}_{(s)}(A) = \bigcap_{r=1}^{\infty} \bigcup_{m \in \mathbb{N}} \bigcap_{n > m, n \in F_k} \left\{ p \in \mathbb{R} \colon \frac{s_n}{2} G(p, n) \geqslant 1 - \frac{1}{r} \right\}.
$$

Since $\frac{1}{2} s_n G(p, n)$ is continuous with respect to p so $\{ p \in \mathbb{R} \colon \frac{1}{2} s_n G(p, n) \geqslant 1 - 1/r \}$ is a closed set. Therefore, $\Phi^{\mathcal{I}}_{(s)}(A) \in F_{\sigma\delta}$. In particular $\Phi^{\mathcal{I}}_{(s)}(A) \in \mathcal{L}$. $\qquad \square$

Proposition 4.5. *For every pair of sets $A, B \in \mathcal{L}$ and a sequence $(s) \in \Sigma_{\mathcal{I}}$ we have*

(1) $\Phi^{\mathcal{I}}_{(s)}(\emptyset) = \emptyset$, $\Phi^{\mathcal{I}}_{(s)}(\mathbb{R}) = \mathbb{R}$;
(2) $\Phi^{\mathcal{I}}_{(s)}(A \cap B) = \Phi^{\mathcal{I}}_{(s)}(A) \cap \Phi^{\mathcal{I}}_{(s)}(B)$;
(3) $A \sim B \Longrightarrow \Phi^{\mathcal{I}}_{(s)}(A) = \Phi^{\mathcal{I}}_{(s)}(B)$;
(4) $\Phi(A) \subseteq \Phi_{\langle s \rangle}(A) \subseteq \Phi^{\mathcal{I}}_{(s)}(A)$;
(5) $\Phi^{\mathcal{I}}_{(s)}(A) \sim A$.

P r o o f. (1) $\Phi^{\mathcal{I}}_{(s)}(\emptyset) = \emptyset$ by voidness since an empty set has no points so it has no $\mathcal{I}_{(s)}$-density points.

Clearly, $\Phi^{\mathcal{I}}_{(s)}(\mathbb{R}) \subseteq \mathbb{R}$. Now, for any $x \in \mathbb{R}$ let $J_n = [x - 1/s_n, x + 1/s_n]$ for all $n \in \mathbb{N}$. Then

$$
\frac{\lambda(\mathbb{R} \cap J_n)}{|J_n|} = \frac{\lambda(J_n)}{|J_n|} = \frac{|J_n|}{|J_n|} = 1 \quad \text{for all } n \in \mathbb{N}.
$$

So for any given $\varepsilon > 0$,

$$\left\{ n \in \mathbb{N} \colon \left| \frac{\lambda(\mathbb{R} \cap J_n)}{|J_n|} - 1 \right| < \varepsilon \right\} = \mathbb{N} \in \mathcal{F}(\mathcal{I}).$$

Thus

$$\mathcal{I} - \lim_{n \to \infty} \frac{\lambda(\mathbb{R} \cap J_n)}{|J_n|} = 1.$$

So, $x \in \Phi^{\mathcal{I}}_{(s)}(\mathbb{R})$. Hence, $\Phi^{\mathcal{I}}_{(s)}(\mathbb{R}) = \mathbb{R}$.

(2) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, so $\Phi^{\mathcal{I}}_{(s)}(A \cap B) \subseteq \Phi^{\mathcal{I}}_{(s)}(A)$ and $\Phi^{\mathcal{I}}_{(s)}(A \cap B) \subseteq \Phi^{\mathcal{I}}_{(s)}(B)$. Consequently, $\Phi^{\mathcal{I}}_{(s)}(A \cap B) \subseteq \Phi^{\mathcal{I}}_{(s)}(A) \cap \Phi^{\mathcal{I}}_{(s)}(B)$. Now we are to prove $\Phi^{\mathcal{I}}_{(s)}(A) \cap \Phi^{\mathcal{I}}_{(s)}(B) \subseteq \Phi^{\mathcal{I}}_{(s)}(A \cap B)$. Let $x \in \Phi^{\mathcal{I}}_{(s)}(A) \cap \Phi^{\mathcal{I}}_{(s)}(B)$. Thus $x \in \Phi^{\mathcal{I}}_{(s)}(A)$ and $x \in \Phi^{\mathcal{I}}_{(s)}(B)$. For $J_n = [x - 1/s_n, x + 1/s_n]$ for all $n \in \mathbb{N}$ and for any given $\varepsilon > 0$ we have

$$A_\varepsilon = \left\{ n \colon \frac{\lambda(A \cap J_n)}{|J_n|} > 1 - \varepsilon \right\} \in \mathcal{F}(\mathcal{I}) \text{ and } B_\varepsilon = \left\{ n \colon \frac{\lambda(B \cap J_n)}{|J_n|} > 1 - \varepsilon \right\} \in \mathcal{F}(\mathcal{I}).$$

Now, since

$$\lambda(A \cap J_n) + \lambda(B \cap J_n) - \lambda(A \cap B \cap J_n) \leqslant |J_n|$$

so for any $\{k_1 < k_2 < \ldots\} \in \mathcal{F}(\mathcal{I})$ such that $s_{k_i} \leqslant s_{k_{i+1}}$ for all $i \in \mathbb{N}$ we have for $n \in \{k_1 < k_2 < \ldots\}$,

$$(4.1) \qquad \frac{\lambda(A \cap J_n)}{|J_n|} + \frac{\lambda(B \cap J_n)}{|J_n|} \leqslant 1 + \frac{\lambda((A \cap B) \cap J_n)}{|J_n|}.$$

So for $n \in \{k_1 < k_2 < \ldots\} \cap A_\varepsilon \cap B_\varepsilon$ from the equation (4.1) we have

$$\frac{\lambda((A \cap B) \cap J_n)}{|J_n|} \geqslant \frac{\lambda(A \cap J_n)}{|J_n|} + \frac{\lambda(B \cap J_n)}{|J_n|} - 1 > 1 - 2\varepsilon.$$

Thus,

$$\left\{ n \colon \frac{\lambda((A \cap B) \cap J_n)}{|J_n|} > 1 - 2\varepsilon \right\} \supseteq \{k_1 < k_2 < \ldots\} \cap A_\varepsilon \cap B_\varepsilon$$

and $\{k_1 < k_2 < \ldots\} \cap A_\varepsilon \cap B_\varepsilon \in \mathcal{F}(\mathcal{I})$. So,

$$\mathcal{I} - \lim_n \frac{\lambda((A \cap B) \cap J_n)}{|J_n|} = 1.$$

Therefore, $x \in \Phi^{\mathcal{I}}_{(s)}(A \cap B)$. So we are done. As a corollary to this we can conclude $\Phi^{\mathcal{I}}_{(s)}(A) \subseteq \Phi^{\mathcal{I}}_{(s)}(B)$ for $A \subseteq B$, i.e., $\Phi^{\mathcal{I}}_{(s)}(\cdot)$ is monotonic.

(3) Let $\{J_n\}_{n\in\mathbb{N}}$ be any sequence of closed intervals in $\mathbb{R}$. If $\lambda(A\triangle B) = 0$, then we claim that $\lambda(A\cap J_n) = \lambda(B\cap J_n)$ for every interval $J_n \subset \mathbb{R}$. Now

$$A = A\cap(B\cup B^c) = (A\cap B)\cup(A\cap B^c) = (A\cap B)\cup(A\setminus B) \subset B\cup(A\triangle B).$$

So, for any $n\in\mathbb{N}$ we have

$$\lambda(A\cap J_n) \leqslant \lambda((B\cup(A\triangle B))\cap J_n) \leqslant \lambda(B\cap J_n) + \lambda((A\triangle B)\cap J_n) = \lambda(B\cap J_n)$$

since

$$\lambda((A\triangle B)\cap J_n) \leqslant \lambda(A\triangle B) = 0.$$

Similarly, $\lambda(B\cap J_n) \leqslant \lambda(A\cap J_n)$ for all $n\in\mathbb{N}$. So, we have $\lambda(A\cap J_n) = \lambda(B\cap J_n)$ for all $n\in\mathbb{N}$. For $(s)\in\Sigma_{\mathcal{I}}$ let $J_n = [x-1/s_n, x+1/s_n]$ for all $n\in\mathbb{N}$. Then,

$$x\in\Phi^{\mathcal{I}}_{(s)}(A) \Leftrightarrow \mathcal{I}-\lim_n \frac{\lambda(A\cap J_n)}{|J_n|} = 1$$

$$\Leftrightarrow \mathcal{I}-\lim_n \frac{\lambda(B\cap J_n)}{|J_n|} = 1$$

$$\Leftrightarrow x\in\Phi^{\mathcal{I}}_{(s)}(B).$$

Consequently, $\Phi^{\mathcal{I}}_{(s)}(A) = \Phi^{\mathcal{I}}_{(s)}(B)$.

(4) By Proposition 2 from [10] we have $\Phi(A) \subseteq \Phi_{\langle s\rangle}(A)$. Now we claim that $\Phi_{\langle s\rangle}(A) \subseteq \Phi^{\mathcal{I}}_{(s)}(A)$. We notice that if $\mathcal{I}$ is an admissible ideal then Fin $\subset \mathcal{I}$. For any $x\in\mathbb{R}$ let $x\in\Phi_{\langle s\rangle}(A)$. Then by Note 4.1, $x\in\Phi^{\text{Fin}}_{(s)}(A)$. Thus for any given $\varepsilon > 0$,

$$\left\{n\in\mathbb{N}: \left|\frac{\lambda(A\cap[x-1/s_n, x+1/s_n])}{2/s_n} - 1\right| \geqslant \varepsilon\right\} \in \text{Fin}.$$

Thus,

$$\left\{n\in\mathbb{N}: \left|\frac{\lambda(A\cap[x-1/s_n, x+1/s_n])}{2/s_n} - 1\right| \geqslant \varepsilon\right\} \in \mathcal{I}$$

since Fin $\subseteq \mathcal{I}$. So, $x\in\Phi^{\mathcal{I}}_{(s)}(A)$. Consequently, $\Phi_{\langle s\rangle}(A) \subset \Phi^{\mathcal{I}}_{(s)}(A)$.

(5) We are to show $\lambda(\Phi^{\mathcal{I}}_{(s)}(A)\triangle A) = 0$. Now, $\Phi^{\mathcal{I}}_{(s)}(A)\triangle A = (A\setminus\Phi^{\mathcal{I}}_{(s)}(A))\cup(\Phi^{\mathcal{I}}_{(s)}(A)\setminus A)$. Since $\Phi(A)\subseteq\Phi^{\mathcal{I}}_{(s)}(A)$ so $A\setminus\Phi^{\mathcal{I}}_{(s)}(A)\subseteq A\setminus\Phi(A)$. By the Lebesgue Density Theorem 2.4, $\lambda(A\setminus\Phi(A)) = 0$. So, $\lambda(A\setminus\Phi^{\mathcal{I}}_{(s)}(A)) = 0$. Now we are to show $\lambda(\Phi^{\mathcal{I}}_{(s)}(A)\setminus A) = 0$. We note that $\Phi^{\mathcal{I}}_{(s)}(A)\cap\Phi^{\mathcal{I}}_{(s)}(\mathbb{R}\setminus A) = \Phi^{\mathcal{I}}_{(s)}(A\cap(\mathbb{R}\setminus A)) = \Phi^{\mathcal{I}}_{(s)}(\emptyset) = \emptyset$. Hence $\Phi^{\mathcal{I}}_{(s)}(A)\subseteq\mathbb{R}\setminus\Phi^{\mathcal{I}}_{(s)}(\mathbb{R}\setminus A)$. So,

$$\Phi^{\mathcal{I}}_{(s)}(A)\setminus A \subseteq (\mathbb{R}\setminus A)\setminus\Phi^{\mathcal{I}}_{(s)}(\mathbb{R}\setminus A) \subseteq (\mathbb{R}\setminus A)\setminus\Phi(\mathbb{R}\setminus A).$$

Since $\mathbb{R}\setminus A\in\mathcal{L}$, so by the Lebesgue Density Theorem 2.4, $\lambda((\mathbb{R}\setminus A)\setminus\Phi(\mathbb{R}\setminus A)) = 0$. Therefore, $\lambda(\Phi^{\mathcal{I}}_{(s)}(A)\setminus A) = 0$ since λ is complete measure. Hence, $\lambda(\Phi^{\mathcal{I}}_{(s)}(A)\triangle A) = 0$.

$\square$

Corollary 4.6. *The operator* $\Phi^{\mathcal{I}}_{(s)} \colon \mathcal{L} \to \mathcal{L}$ *is a lower density operator.*

As a consequence of Remark 2.7, Theorem 2.8 and Corollary 4.6 we have the following theorem.

Theorem 4.7. *For every sequence* $(s) \in \Sigma_{\mathcal{I}}$ *the family* $\mathcal{T}^{\mathcal{I}}_{(s)} = \{A \in \mathcal{L} \colon A \subseteq \Phi^{\mathcal{I}}_{(s)}(A)\}$ *forms a topology.*

N o t e 4.8. We call $\mathcal{T}^{\mathcal{I}}_{(s)}$ to be the $\mathcal{I}_{(s)}$-density topology on the space of reals and by Proposition 4.5 (4), since $\Phi(A) \subseteq \Phi_{\langle s \rangle}(A) \subseteq \Phi^{\mathcal{I}}_{(s)}(A)$ we can conclude that $\mathcal{T}_d \subseteq \mathcal{T}_{\langle s \rangle} \subseteq \mathcal{T}^{\mathcal{I}}_{(s)}$.

R e m a r k 4.9. As we have introduced the $\mathcal{I}_{(s)}$-density for $(s) \in \Sigma_{\mathcal{I}}$ and for $\mathcal{I} = \mathrm{Fin}$, the $\langle s \rangle$-density coincides with $\mathcal{I}_{(s)}$-density, so $\mathcal{T}_{\langle s \rangle} = \mathcal{T}^{\mathcal{I}}_{(s)}$ if $\mathcal{I} = \mathrm{Fin}$.

In the following theorem the natural properties of $\mathcal{T}^{\mathcal{I}}_{(s)}$-topologies are listed.

Theorem 4.10. *For any* $(s) \in \Sigma_{\mathcal{I}}$ *and* $A \in \mathcal{T}^{\mathcal{I}}_{(s)}$ *we have*

(1) $A + x \in \mathcal{T}^{\mathcal{I}}_{(s)}$ *for all* $x \in \mathbb{R}$ *where* $A + x = \{a + x \colon a \in A\}$,
(2) $-A \in \mathcal{T}^{\mathcal{I}}_{(s)}$ *where* $-A = \{-a \colon a \in A\}$.

The proof is omitted.

5. Characterization of equality for $\mathcal{T}_d$ and $\mathcal{T}^{\mathcal{I}}_{(s)}$

The characterization of equality between $\mathcal{T}_d$ and $\mathcal{T}_{\langle s \rangle}$ as mentioned in Theorem 2.14 uncovers several important consequences which can be seen in [10]. So in this section we will concentrate on the problem of characterization given below.

Problem.   Is there any characterization of equality for $\mathcal{T}_d$ and $\mathcal{T}^{\mathcal{I}}_{(s)}$ as given in [10] for $\mathcal{T}_d$ and $\mathcal{T}_{\langle s \rangle}$?

The next theorem answers our question affirmatively. Here we have formulated a weaker condition for the sequence $(s) \in \Sigma_{\mathcal{I}}$ so that the classical density topology coincides with the $\mathcal{I}_{(s)}$-density topology.

Theorem 5.1. *Let* $(s) \in \Sigma_{\mathcal{I}}$ *be a real sequence. Then* $\mathcal{T}_d = \mathcal{T}^{\mathcal{I}}_{(s)}$ *if and only if for any* $\{n_1 < n_2 < \ldots < n_k < \ldots\} \in \mathcal{F}(\mathcal{I})$ *such that* $s_{n_k} \leqslant s_{n_{k+1}}$ *for all* $k \in \mathbb{N}$, *the condition* $\liminf s_{n_k}/s_{n_{(k+1)}} > 0$ *holds.*

P r o o f.   First suppose that the condition holds. It is sufficient to show that for any $A \in \mathcal{L}$, $\Phi(A) = \Phi^{\mathcal{I}}_{(s)}(A)$, when (s) satisfies the condition given in the statement. By Proposition 4.5 (4) we have $\Phi(A) \subseteq \Phi^{\mathcal{I}}_{(s)}(A)$. Now, we need to show $\Phi^{\mathcal{I}}_{(s)}(A) \subseteq \Phi(A)$, i.e., if $x \in \mathbb{R}$ is an $\mathcal{I}_{(s)}$-density point of A, then x is a classical density point of A. Since $\liminf s_{n_k}/s_{n_{(k+1)}} > 0$ so there exists a subsequence of $\{s_{n_k}\}_{k \in \mathbb{N}}$, say $\{s_{n_{k_l}}\}_{l \in \mathbb{N}}$, such that

$$\lim_{l \to \infty} \frac{s_{n_{k_l}}}{s_{n_{(k_l+1)}}} = \sigma > 0.$$

Thus there exists $l_0 \in \mathbb{N}$ such that for any $l \geqslant l_0$ we have

$$\frac{\sigma}{2} < \frac{s_{n_{k_l}}}{s_{n_{(k_l+1)}}} < \frac{3\sigma}{2}.$$

Since x is an $\mathcal{I}_{(s)}$-density point of A, so clearly

$$\mathcal{I} - \lim_{n \to \infty} \frac{\lambda(A^c \cap [x - 1/s_n, x + 1/s_n])}{2/s_n} = 0,$$

where A^c denotes $\mathbb{R} \setminus A$. Thus, for any given $\varepsilon > 0$ the set

$$C_\varepsilon = \left\{ n \in \mathbb{N} \colon \frac{s_n}{2} \lambda \left(A^c \cap \left[x - \frac{1}{s_n}, x + \frac{1}{s_n} \right] \right) < \frac{\varepsilon \sigma}{2} \right\} \in \mathcal{F}(\mathcal{I}).$$

Since $C_\varepsilon \in \mathcal{F}(\mathcal{I})$ and $\{n_1 < n_2 < \ldots < n_k < \ldots\} \in \mathcal{F}(\mathcal{I})$, so $C_\varepsilon \cap \{n_1 < n_2 < \ldots < n_k < \ldots\} \neq \emptyset$ and moreover it is an infinite set. Thus, there exists a natural number $p_0 > l_0$ such that for some $p \in \mathbb{N}$ we have

$$n_{k_p} \in \{n_1 < n_2 < \ldots < n_k < \ldots\} \cap C_\varepsilon$$

and for $p \geqslant p_0$ we get

$$\frac{s_{n_{k_p}}}{2} \lambda \left(A^c \cap \left[x - \frac{1}{s_{n_{k_p}}}, x + \frac{1}{s_{n_{k_p}}} \right] \right) < \frac{\varepsilon \sigma}{2}.$$

We fix $t \in \mathbb{R}$ such that $0 < t < 1/s_{n_{k_{p_0}}}$. So, there exists $p \geqslant p_0$ for which $n_{k_p} \in \{n_1 < n_2 < \ldots < n_k < \ldots\} \cap C_\varepsilon$ such that

$$\frac{1}{s_{n_{(k_p+1)}}} \leqslant t < \frac{1}{s_{n_{k_p}}}.$$

Hence, we have

$$\frac{\lambda(A^c \cap [x-t, x+t])}{2t} \leqslant \frac{\lambda(A^c \cap [x-1/s_{n_{k_p}}, x+1/s_{n_{k_p}}])}{2/s_{n_{(k_p+1)}}}$$

$$= \frac{\lambda(A^c \cap [x-1/s_{n_{k_p}}, x+1/s_{n_{k_p}}])}{2/s_{n_{k_p}}} \cdot \frac{s_{n_{(k_p+1)}}}{s_{n_{k_p}}}$$

$$< \frac{\varepsilon\sigma}{2} \cdot \frac{2}{\sigma} = \varepsilon.$$

Therefore, x is a classical density point of A.

Conversely, let $\mathcal{T}_d = \mathcal{T}^{\mathcal{I}}_{(s)}$. We show for any $\{l_1 < l_2 < \ldots < l_n < \ldots\} \in \mathcal{F}(\mathcal{I})$ such that $s_{l_k} \leqslant s_{l_{k+1}}$ for all $k \in \mathbb{N}$, that the condition $\liminf s_{l_k}/s_{l_{(k+1)}} > 0$ holds. Suppose if possible there exists $\{n_1 < n_2 < \ldots < n_k < \ldots\} \in \mathcal{F}(\mathcal{I})$ such that $s_{n_k} \leqslant s_{n_{(k+1)}}$ for all $k \in \mathbb{N}$ and $\liminf s_{n_k}/s_{n_{(k+1)}} = 0$. Then there exists a subsequence $\{s_{n_{k_l}}\}_{l\in\mathbb{N}}$ of $\{s_{n_k}\}_{k\in\mathbb{N}}$ such that $\lim_{l\to\infty} s_{n_{k_l}}/s_{n_{(k_l+1)}} = 0$. Thus for any given $\varepsilon > 0$ there exists $l_0 \in \mathbb{N}$ such that for $l \geqslant l_0$,

$$(5.1) \qquad\qquad \sqrt{\frac{s_{n_{k_l}}}{s_{n_{(k_l+1)}}}} < \varepsilon.$$

Now let us define a set

$$\mathbb{A} = \bigcup_{i=1}^{\infty} \left[\frac{1}{s_{n_{(k_i+1)}}}, \frac{1}{\sqrt{s_{n_{k_i}} s_{n_{(k_i+1)}}}} \right].$$

We are to show that 0 is a right $\mathcal{I}_{(s)}$-dispersion point of $\mathbb{A}$. It is enough to show for any given $\varepsilon > 0$ that $\{n \in \mathbb{N}\colon s_n\lambda(\mathbb{A} \cap [0, 1/s_n]) < \varepsilon\} \in \mathcal{F}(\mathcal{I})$. Now let us take $\alpha_j = \min\{r \in \mathbb{N}\colon s_{n_{k_r}} \geqslant s_{n_j}\}$ for $j \in \mathbb{N}$. Thus, $s_{n_{k_{\alpha_j}}} \geqslant s_{n_j}$ for all $j \in \mathbb{N}$. We claim that for any $j \in \mathbb{N}$,

$$\mathbb{A} \cap \left[0, \frac{1}{s_{n_j}}\right] = \bigcup_{i=\alpha_j}^{\infty} \left[\frac{1}{s_{n_{(k_i+1)}}}, \frac{1}{\sqrt{s_{n_{k_i}} s_{n_{(k_i+1)}}}} \right] \cap \left[0, \frac{1}{s_{n_j}}\right].$$

Now we consider the following two cases.

Case (i): Let $s_{n_j} = s_{n_{k_{\alpha_j}}}$ for all $j \in \mathbb{N}$. Since $k_{\alpha_j} < k_{\alpha_j} + 1$ for all $j \in \mathbb{N}$, so $n_{k_{\alpha_j}} < n_{k_{\alpha_j}+1}$ for all $j \in \mathbb{N}$. Thus, $s_{n_{k_{\alpha_j}}} \leqslant s_{n_{(k_{\alpha_j}+1)}}$. Therefore,

$$\frac{1}{s_{n_{(k_{\alpha_j}+1)}}} \leqslant \frac{1}{s_{n_{k_{\alpha_j}}}}.$$

So,

$$(5.2) \qquad \frac{1}{s_{n_{(k_{\alpha_j}+1)}}} \leqslant \frac{1}{\sqrt{s_{n_{k_{\alpha_j}}} s_{n_{(k_{\alpha_j}+1)}}}} \leqslant \frac{1}{s_{n_{k_{\alpha_j}}}}.$$

Also, since $k_i + 1 \leqslant k_{i+1}$ for all $i \in \mathbb{N}$, so $k_{\alpha_j-1} + 1 \leqslant k_{\alpha_j}$ for all j. Thus, $n_{k_{\alpha_j-1}+1} \leqslant n_{k_{\alpha_j}}$ for all $j \in \mathbb{N}$. Therefore, $s_{n_{(k_{\alpha_j}-1+1)}} \leqslant s_{n_{k_{\alpha_j}}}$ for all $j \in \mathbb{N}$. Consequently,

$$\frac{1}{s_{n_{k_{\alpha_j}}}} \leqslant \frac{1}{s_{n_{(k_{\alpha_j}-1+1)}}} \qquad \text{for all } j \in \mathbb{N}.$$

So we can conclude

$$\mathbb{A} \cap \left[0, \frac{1}{s_{n_j}}\right] = \bigcup_{i=\alpha_j}^{\infty} \left[\frac{1}{s_{n_{(k_i+1)}}}, \frac{1}{\sqrt{s_{n_{k_i}} s_{n_{(k_i+1)}}}}\right] \cap \left[0, \frac{1}{s_{n_j}}\right].$$

Case (ii): Let $s_{n_j} < s_{n_{k_{\alpha_j}}}$ for all $j \in \mathbb{N}$. From the choice of α_j it is clear that $s_{n_{k_{(\alpha_j-1)}}} < s_{n_j} < s_{n_{k_{\alpha_j}}}$ for all $j \in \mathbb{N}$. We claim that

$$\frac{1}{s_{n_j}} \leqslant \frac{1}{s_{n_{(k_{\alpha_j}-1+1)}}} \qquad \text{for all } j \in \mathbb{N}.$$

If possible, let

$$\frac{1}{s_{n_{(k_{\alpha_j}-1+1)}}} < \frac{1}{s_{n_j}} \qquad \text{for all } j \in \mathbb{N},$$

i.e., $s_{n_j} < s_{n_{(k_{\alpha_j}-1+1)}}$. So $j < k_{(\alpha_j-1)} + 1$ which implies $j + 1 \leqslant k_{(\alpha_j-1)} + 1$ for all $j \in \mathbb{N}$. Hence, $j \leqslant k_{(\alpha_j-1)}$ and so $n_j \leqslant n_{k_{(\alpha_j-1)}}$ for all $j \in \mathbb{N}$. Thus $s_{n_j} \leqslant s_{n_{k_{(\alpha_j-1)}}}$. This is a contradiction. So,

$$(5.3) \qquad \frac{1}{s_{n_{k_{\alpha_j}}}} < \frac{1}{s_{n_j}} \leqslant \frac{1}{s_{n_{k_{(\alpha_j-1)}+1}}}.$$

From the equations (5.2) and (5.3) we observe that

$$\frac{1}{s_{n_{(k_{\alpha_j}+1)}}} \leqslant \frac{1}{\sqrt{s_{n_{k_{\alpha_j}}} s_{n_{(k_{\alpha_j}+1)}}}} \leqslant \frac{1}{s_{n_{k_{\alpha_j}}}} < \frac{1}{s_{n_j}} \qquad \text{for all } j \in \mathbb{N}.$$

So we can conclude

$$\mathbb{A} \cap \left[0, \frac{1}{s_{n_j}}\right] = \bigcup_{i=\alpha_j}^{\infty} \left[\frac{1}{s_{n_{(k_i+1)}}}, \frac{1}{\sqrt{s_{n_{k_i}} s_{n_{(k_i+1)}}}}\right] \cap \left[0, \frac{1}{s_{n_j}}\right].$$

For $\alpha_j \geqslant l_0$,

$$s_{n_j}\lambda\left(\mathbb{A} \cap \left[0, \frac{1}{s_{n_j}}\right]\right) = s_{n_j}\lambda\left(\bigcup_{i=1}^{\infty}\left[\frac{1}{s_{n_{(k_i+1)}}}, \frac{1}{\sqrt{s_{n_{k_i}}s_{n_{(k_i+1)}}}}\right] \cap \left[0, \frac{1}{s_{n_j}}\right]\right)$$

$$= s_{n_j}\lambda\left(\bigcup_{i=\alpha_j}^{\infty}\left[\frac{1}{s_{n_{(k_i+1)}}}, \frac{1}{\sqrt{s_{n_{k_i}}s_{n_{(k_i+1)}}}}\right] \cap \left[0, \frac{1}{s_{n_j}}\right]\right)$$

$$< s_{n_{k_{\alpha_j}}} \cdot \frac{1}{\sqrt{s_{n_{k_{\alpha_j}}}s_{n_{(k_{\alpha_j}+1)}}}}$$

$$= \sqrt{\frac{s_{n_{k_{\alpha_j}}}}{s_{n_{k_{\alpha_j}}+1}}} < \varepsilon \quad \text{by the equation (5.1).}$$

Now take $j_0 = \min\{j \in \mathbb{N}: \alpha_j \geqslant l_0\}$. Then $s_{n_{j_0}}\lambda(\mathbb{A} \cap [0, 1/s_{n_{j_0}}]) < \varepsilon$. So,

$$\left\{n \in \mathbb{N}: s_n\lambda\left(\mathbb{A} \cap \left[0, \frac{1}{s_n}\right]\right) < \varepsilon\right\} \supset \{n_{j_0}, n_{j_0+1}, n_{j_0+2}, \ldots\}.$$

Now, $\{n_{j_0}, n_{j_0+1}, n_{j_0+2}, \ldots\} = \{n_1, n_2, n_3, \ldots, n_k, \ldots\} \setminus \{n_1, n_2, \ldots, n_{j_0-1}\} \in \mathcal{F}(\mathcal{I})$. Consequently, we have $\{n \in \mathbb{N}: s_n\lambda(\mathbb{A} \cap [0, 1/s_n]) < \varepsilon\} \in \mathcal{F}(\mathcal{I})$. So 0 is a right $\mathcal{I}_{(s)}$-dispersion point of $\mathbb{A}$. Clearly, 0 is an $\mathcal{I}_{(s)}$-dispersion point of $X = (-\mathbb{A}) \cup \mathbb{A}$. Now let us take

$$\mathbb{B} = \bigcup_{j=1}^{\infty}\left(\left(\frac{1}{\sqrt{s_{n_{k_j}}s_{n_{(k_j+1)}}}}, \frac{1}{s_{n_{k_j}}}\right) \cup \left(\frac{1}{s_{n_{k_{(j+1)}}}}, \frac{1}{s_{n_{(k_j+1)}}}\right)\right) = \bigcup_{j=1}^{\infty}(\mathbb{C}_j \cup \mathbb{D}_j),$$

where

$$\mathbb{C}_j = \left(\frac{1}{\sqrt{s_{n_{k_j}}s_{n_{(k_j+1)}}}}, \frac{1}{s_{n_{k_j}}}\right) \quad \text{and} \quad \mathbb{D}_j = \left(\frac{1}{s_{n_{k_{(j+1)}}}}, \frac{1}{s_{n_{(k_j+1)}}}\right).$$

Then, $\mathbb{A} \cap \mathbb{B} = \emptyset$ and $\mathbb{A} \cup \mathbb{B} = (0, 1/s_{n_{k_1}})$. It can be easily seen that 0 is a right $\mathcal{I}_{(s)}$-density point of $\mathbb{A} \cup \mathbb{B}$.

Thus, by Proposition 3.11 we can say that 0 is a right $\mathcal{I}_{(s)}$-density point of $\mathbb{B}$. Now if we take $Y = -\mathbb{B} \cup \mathbb{B} \cup \{0\}$, then 0 is an $\mathcal{I}_{(s)}$-density point of Y. Also since $\mathbb{B} \subseteq \Phi(\mathbb{B}) \subseteq \Phi_{(s)}^{\mathcal{I}}(\mathbb{B})$ by Proposition 4.5 (4) so we can conclude that $Y \subseteq \Phi_{(s)}^{\mathcal{I}}(Y)$. Therefore, $Y \in \mathcal{T}_{(s)}^{\mathcal{I}}$. Further we show that $Y \notin \mathcal{T}_d$.

First we see that

$$\sqrt{s_{n_{k_l}} s_{n_{(k_l+1)}}} \lambda\left(\bigcup_{j=1}^{\infty} \mathbb{C}_j \cap \left[0, \frac{1}{\sqrt{s_{n_{k_l}} s_{n_{(k_l+1)}}}}\right]\right)$$

$$= \sqrt{s_{n_{k_l}} s_{n_{(k_l+1)}}} \lambda\left(\bigcup_{j=l+1}^{\infty} \mathbb{C}_j \cap \left[0, \frac{1}{\sqrt{s_{n_{k_l}} s_{n_{(k_l+1)}}}}\right]\right)$$

$$\leqslant \sqrt{s_{n_{k_l}} s_{n_{(k_l+1)}}} \cdot \frac{1}{s_{n_{(k_l+1)}}}$$

$$= \sqrt{\frac{s_{n_{k_l}}}{s_{n_{(k_l+1)}}}} \to 0 \text{ as } l \to \infty,$$

since

$$\frac{1}{s_{n_{k_{(l+1)}}}} \leqslant \frac{1}{s_{n_{(k_l+1)}}}.$$

So 0 is not a right density point of $\bigcup_{j=1}^{\infty} \mathbb{C}_j$. Next we see that

$$(5.4) \qquad \sqrt{s_{n_{k_{(l+1)}}} s_{n_{k_{(l+1)}+1}}} \lambda\left(\bigcup_{j=1}^{\infty} \mathbb{D}_j \cap \left[0, \frac{1}{\sqrt{s_{n_{k_{(l+1)}}} s_{n_{k_{(l+1)}+1}}}}\right]\right)$$

$$= \sqrt{s_{n_{k_{(l+1)}}} s_{n_{k_{(l+1)}+1}}} \lambda\left(\bigcup_{j=l+1}^{\infty} \mathbb{D}_j \cap \left[0, \frac{1}{\sqrt{s_{n_{k_{(l+1)}}} s_{n_{k_{(l+1)}+1}}}}\right]\right)$$

$$\leqslant \sqrt{s_{n_{k_{(l+1)}}} s_{n_{k_{(l+1)}+1}}} \cdot \frac{1}{s_{n_{k_{(l+1)}+1}}}$$

$$= \sqrt{\frac{s_{n_{k_{(l+1)}}}}{s_{n_{k_{(l+1)}+1}}}}.$$

In the equation (5.4), since

$$\sqrt{\frac{s_{n_{k_{(l+1)}}}}{s_{n_{k_{(l+1)}+1}}}} \to 0 \text{ as } l \to \infty, \text{ so } 0$$

is not a right density point of $\bigcup_{j=1}^{\infty} \mathbb{D}_j$. Hence we can conclude that 0 is not a right density point of $\mathbb{B}$. Clearly 0 is not a density point of Y. So, $Y \not\subset \Phi(Y)$. Therefore, $Y \notin \mathcal{T}_d$. This is a contradiction since $\mathcal{T}_d = \mathcal{T}_{(s)}^{\mathcal{I}}$. Thus our assumption was wrong. Thus for any $\{n_1 < n_2 < \ldots < n_k < \ldots\} \in \mathcal{F}(\mathcal{I})$ such that $s_{n_k} \leqslant s_{n_{k+1}}$ for all $k \in \mathbb{N}$, the condition $\liminf s_{n_k}/s_{n_{(k+1)}} > 0$ must hold. This completes the proof. $\qquad\square$

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