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Mathematica Bohemica, Vol. 151 (2026), No. 1, 89–103

Persistent URL: <http://dml.cz/dmlcz/153388>

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SYMMETRIC PARTIALLY DARBOUX FUNCTIONS
AND MULTIPLE POINTS MEAN VALUE RESULTS

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Received July 29, 2024. Published online February 19, 2025.
Communicated by Jiří Spurný

Abstract. We consider mean value results for symmetric multivariable functions whose partial functions satisfy the intermediate value property.

Keywords: Darboux function; symmetric function; mean value theorem

MSC 2020: 26B99, 26A15, 26A24

1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{N} = \{1, 2, \dots\}$ denote the set of natural numbers. The following question comes naturally in the circle of ideas around Lagrange's mean value theorem to similar questions asked in several journals (see [2], [4], [6]).

- (\diamond) Let a and b be real numbers such that $a < b$, and let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function that is differentiable on (a, b) . Then for any $n \in \mathbb{N}$, $n \geq 2$, there are pairwise distinct points c_1, \dots, c_n of (a, b) that satisfy

$$\left(\frac{f(b) - f(a)}{b - a} \right)^2 = \frac{1}{n(n-1)} \sum_{i \neq j} f'(c_i) f'(c_j).$$

It turns out that (\diamond) and the problems raised in the above quoted journals as well can be solved in an unified way via a theorem for multivariable partially Darboux functions (the subsequent Theorem 1), whose proof is given in Section 2.

Then in Section 3 we apply *ad-hoc* methods for other multiple points mean value problems. The fourth section of the paper is a revisited version of a few classical mean value results.

Hencefort, unless explicitly mentioned, an interval of \mathbb{R} is denoted by I and is supposed to be “nondegenerate”, i.e., it contains at least two points.

Given a set $S \subset \mathbb{R}$, a function $f: S \rightarrow \mathbb{R}$ is said to possess the *Darboux property* or that f is a *Darboux function* if for any interval $I \subset S$ the restriction $f|_I$ has the *intermediate value property*, that is, whenever x_1 and x_2 are points in I and y is any value between $f(x_1)$ and $f(x_2)$, there is a number x between x_1 and x_2 such that $f(x) = y$.

Let A be a subset of \mathbb{R}^n . We say that a function $\Phi: A \rightarrow \mathbb{R}$ is *partially Darboux* if for any $a \in A$ and index $j \in \{1, \dots, n\}$ the restriction of Φ to $A_j(a)$ possesses the *Darboux property*, where $A_j(a)$ is the slice of A cut by the parallel to the j th axis of \mathbb{R}^n through the point a ; we view $A_j(a)$ as a subset of \mathbb{R} .

For any $n \in \mathbb{N}$ the symmetric group S_n of all permutations of $\{1, 2, \dots, n\}$ with composition of permutations as multiplicative operation acts in a canonical way on the real euclidean space \mathbb{R}^n . In particular, we get a canonical definition of a symmetric subset of \mathbb{R}^n and then of a symmetric real-valued function defined on a symmetric set.

Lemma 1. *Let $f: I \rightarrow \mathbb{R}$ be a Darboux function. Then for any nonempty proper set $A \subset I$ such that $f(A)$ has empty interior it follows that $f(I) \subset \overline{f(I \setminus A)}$.*

Proof. Indeed, reasoning by contradiction, suppose that there exists a point $x_0 \in A$ such that $f(x_0) \notin \overline{f(I \setminus A)}$ and see what happens. First note that either $f(x_0)$ is interior to $f(I)$, or else $f(x_0)$ is an extremal value of $f(I)$.

In the first case, since f has the Darboux property, there exist two points $a, b \in I$ such that $f(a) \neq f(b)$ (*a fortiori* $a \neq b$), x_0 is between a and b , and the nonempty open interval with extremities $f(a)$ and $f(b)$ contains $f(x_0)$ and is disjoint with $f(I \setminus A)$, hence, it is contained in $f(A)$ implying that $f(A)$ has nonempty interior, contradicting the hypothesis.

Similarly, we deal with the case when $f(x_0)$ is an extremity of $f(I)$. The proof of the lemma follows. \square

Theorem 1. *Let $\Phi: I^n \rightarrow \mathbb{R}$ be a symmetric partially Darboux function, and let a_1, \dots, a_n be points of I such that $\Phi(a_1, \dots, a_n)$ is not an extremal value of Φ . Then there are pairwise distinct points c_1, \dots, c_n of I that satisfy*

$$\Phi(c_1, \dots, c_n) = \Phi(a_1, \dots, a_n).$$

Remark 1. Since $\Phi(I^n)$ is an interval, it follows that $\Phi(a_1, \dots, a_n)$ is not an extremum of Φ if and only if $\Phi(a_1, \dots, a_n)$ is interior to $\Phi(I^n)$. In practice this

condition is satisfied if either the points a_1, \dots, a_n are interior to I and the restriction of Φ to an open set of interior points of I^n has no local extrema, or there exists a curve $\Gamma \subset I^n$ passing through (a_1, \dots, a_n) such that $\Phi(a_1, \dots, a_n)$ is interior to $\Phi(\Gamma)$ in the relative topology. Also, the condition is satisfied if there exists a parametrization $\gamma: (-1, 1) \rightarrow \mathbb{R}^n$ of a curve Σ such that $\gamma(0) = (a_1, \dots, a_n)$ and the composition $\Phi \circ \gamma$ is injective and has the Darboux property, since then $\Phi(a_1, \dots, a_n)$ is the interior to $\Phi(\Sigma)$. This is true because any real-valued, injective, continuous function defined on an interval of \mathbb{R} is either increasing or decreasing. As a matter of fact, the following more general result holds true.

Lemma 2. *Let $S \subset \mathbb{R}$ be a nonempty set. A sufficient condition for a Darboux function $f: S \rightarrow \mathbb{R}$ to be continuous is that the fiber $f^{-1}(y)$ is closed for all points y in a dense subset of $f(S)$.*

In particular, finite-to-one (a fortiori injective) Darboux functions are continuous (and strictly monotone).

Proof. Reasoning by contradiction, assume that there is a point $a \in S$, where f is not continuous and see what happens. Then there exists $\varepsilon_0 > 0$ such that for any $n \in \mathbb{N}$ there is a point $x_n \in T_n = (a - 1/n, a + 1/n) \cap S$ with $|f(x_n) - f(a)| \geq \varepsilon_0$. Now, since f is a Darboux function, for any $n \in \mathbb{N}$, one of the two intervals $[f(a), f(a) + \varepsilon_0]$ or $[f(a) - \varepsilon_0, f(a)]$ is contained in $f(T_n)$. Hence, without loss of generality, assume that there is an increasing sequence $(\nu_k)_k$ of positive integers such that $[f(a), f(a) + \varepsilon_0] \subset f(T_{\nu_k})$ for all k . Thus, for any $y \in (f(a), f(a) + \varepsilon_0)$, the point a is a limit point of $f^{-1}(y)$, and since a does not belong to $f^{-1}(y)$, it follows that the set $f^{-1}(y)$ is not closed, contradicting the hypothesis. Hence, the function f is continuous, concluding the lemma. \square

Proposition 1. *Let $f: I \rightarrow \mathbb{R}$ be a nonconstant Darboux function, and let $\Phi: f(I)^n \rightarrow \mathbb{R}$ be a symmetric partially Darboux function. Then for any point $c \in I$ such that $\Phi(f(c), \dots, f(c))$ is not an extremal value of Φ there are pairwise distinct points c_1, \dots, c_n of I such that*

$$\Phi(f(c), \dots, f(c)) = \Phi(f(c_1), \dots, f(c_n)).$$

This proposition follows by Theorem 1 since the composed function

$$I^n \ni (x_1, \dots, x_n) \mapsto \Phi(f(x_1), \dots, f(x_n)) \in \mathbb{R}$$

is partially Darboux.

Remark 2. The condition on the point c holds true in each of the following cases: (i) either $f(c)$ is not an extremum for f and Φ restricted to the interior of I^n is an open map or has no local extrema or (ii) there exists a curve $\Gamma \subset I^n$ passing through $(f(c), \dots, f(c))$ such that $\Phi(f(c), \dots, f(c))$ is interior to $\Phi(\Gamma)$.

Proposition 1 can be applied to solve several problems published in mathematical journals; see [4] and the references therein as well as our Sections 3 and 4.

Proof of (\diamond) . Let $c \in (a, b)$ be such that $f'(c) = (f(b) - f(a))/(b - a)$. If $f'(c)$ is an extremal value of f' on (a, b) , then it is easily seen that f is an affine function and the proof is immediate. Otherwise, we apply Proposition 1 to the function

$$\Phi: \mathbb{R}^n \rightarrow \mathbb{R}, (x_1, \dots, x_n) \mapsto \sum_{i \neq j} x_i x_j,$$

which is of class \mathcal{C}^∞ and has no local extrema. □

2. PROOF OF THEOREM 1

Recall that for two points $\alpha, \beta \in \mathbb{R}^n$, the segment $[\alpha, \beta]$ is the set of all convex combinations of α and β , that is, $[\alpha, \beta] = \{(1 - t)\alpha + t\beta: t \in [0, 1]\}$. We term a segment $[\alpha, \beta]$ *simple* if either $\alpha = \beta$, or $\alpha \neq \beta$ and the vector $\beta - \alpha$ is parallel to one of the coordinate axes of \mathbb{R}^n .

A *manhattan path* (in short, M-path) between two points of \mathbb{R}^n is a polygonal path γ that joins these points and each segment of γ is simple.

A set S of \mathbb{R}^n is said to be *M-path connected* if any two points of S can be joined by an M-path contained entirely in S .

Since the topology of \mathbb{R}^n is generated by open cubes, it is easily seen that any connected open subset of \mathbb{R}^n is M-path connected.

Notice that if S is an M-path connected subset of \mathbb{R}^n , then for any partially Darboux function $\Phi: S \rightarrow \mathbb{R}$ the set $\Phi(S)$ is an interval.

Let a and b be real numbers such that $a < b$. In the sequel K denotes the set of all points $(x_1, \dots, x_n) \in \mathbb{R}^n$ with $a \leq x_1 \leq \dots \leq x_n \leq b$ and K° is the set of interior points of K , i.e., $K^\circ = \{(x_1, \dots, x_n) \in \mathbb{R}^n: a < x_1 < \dots < x_n < b\}$.

A key point in the proof is the following lemma.

Lemma 3. *For any partially Darboux function $\Phi: K \rightarrow \mathbb{R}$ it follows that $\Phi(K) \subseteq \overline{\Phi(K^\circ)}$. Besides, the sets $\Phi(K)$ and $\Phi(K^\circ)$ are intervals.*

Proof. First we check that K is M-path connected. For this we claim that any point $(x_1, x_2, \dots, x_n) \in K$ can be joined by an M-path in K to the “lower left” corner (a, a, \dots, a) of K .

To illustrate the intuitive idea behind the proof, for any $k \in \{0, \dots, n\}$ let α_k be the point of \mathbb{R}^n in which the first k components are equal to a and followed in order by x_{k+1}, \dots, x_n so that we have $\alpha_0 = (x_1, \dots, x_n)$ and $\alpha_n = (a, \dots, a)$.

Therefore, since the simple segments $[\alpha_n, \alpha_{n-1}]$, $[\alpha_{n-1}, \alpha_{n-2}]$, \dots , $[\alpha_1, \alpha_0]$ are contained in K , the claim follows.

On the other hand, notice that for any M-path γ in K its image through Φ is connected. Therefore, since K° and K are M-path connected, we obtain that $\Phi(K^\circ)$ and $\Phi(K)$ are connected, hence, these sets are intervals, whence the second part of the lemma.

The consistent part of the lemma is the inclusion $\Phi(K) \subset \overline{\Phi(K^\circ)}$, which we now verify. Define $\nu: K \rightarrow \{0, 1, \dots, n\}$, which assigns to a point $\omega = (x_1, \dots, x_n)$ of K the number $\nu(\omega)$ of equality signs in the sequence of inequalities

$$a \leq x_1 \leq \dots \leq x_n \leq b.$$

Claim. For any point ω of K there is a sequence $(\omega^{(j)})_j$ of points in K such that $\Phi(\omega^{(j)}) \rightarrow \Phi(\omega)$ and $\nu(\omega^{(j)}) < \nu(\omega)$ for all j .

Note that by Lemma 1 it follows that if $f: I \rightarrow \mathbb{R}$ is a function which possesses the Darboux property, then for any $x \in I$ there is a sequence $(x_j)_j$ of points in $I \setminus \{x\}$, which converges to x , and such that the sequence of general term $f(x_j)$ converges to $f(x)$.

Then, back to the proof of the claim, consider for simplicity the case $\nu(\omega) = n$ so that ω is either (a, \dots, a) or (b, \dots, b) . To make a choice, let $\omega = (a, \dots, a)$. Remark that the segment H with extremities (a, \dots, a, a) and (a, \dots, a, b) lies in K and $\nu(\omega_t) = n - 1$ for every $t \in (a, b)$, where $\omega_t = (a, \dots, a, t)$. On the other hand, $\Phi|_H$ has the Darboux property, hence by Lemma 1 we are done in this case.

In general, put $x_0 = a$, $x_{n+1} = b$, and select an index $j \in \{0, 1, \dots, n+1\}$ such that $x_j < x_{j+1}$.

Now, if $j = 0$, the segment determined by (x_1, x_2, \dots, x_n) and (a, x_2, \dots, x_n) lies in K and for every number $t \in (a, x_1)$ one has $\nu(\omega_t) = \nu(\omega) - 1$, where $\omega_t = (t, x_2, \dots, x_n)$ and continue as explained above.

Similarly, we deal with the case when $j \in \{1, \dots, n\}$, and we leave the details to the interested reader.

Eventually, since $K^\circ = \{\omega \in K: \nu(\omega) = 0\}$, the claim gives the desired inclusion, whence the proof of the lemma. \square

End of the proof of Theorem 1. Let D be the set of all points of I^n whose coordinates are pairwise distinct and let

$$\Omega = \{(x_1, \dots, x_n) \in I^n : x_1 < \dots < x_n\}.$$

Let \overline{D} and $\overline{\Omega}$ be the closures of D and Ω in I^n respectively. Since Φ is symmetric, it follows that D is the union of the sets $F_\sigma(\Omega)$ when the permutation σ runs throughout the set S_n , where $F_\sigma: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the function

$$\mathbb{R}^n \ni (x_1, \dots, x_n) \rightarrow (x_{\sigma(1)}, \dots, x_{\sigma(n)}) \in \mathbb{R}^n.$$

Note that $\overline{D} = I^n$, and by Lemma 3 it is easily seen that $\Phi(\overline{D}) \subseteq \overline{\Phi(D)}$, whence

$$\Phi(I^n) = \Phi(\overline{D}) \subseteq \overline{\Phi(D)} \subseteq \overline{\Phi(I^n)},$$

so that $\overline{\Phi(D)} = \overline{\Phi(I^n)}$. Note also that if A and B are subsets of \mathbb{R} such that A is an interval and $A \subseteq B \subseteq \overline{A}$, then B is an interval, too, and the sets A and B have the same interior set. Thus, since the sets $\Phi(D)$ and $\Phi(I^n)$ are intervals, by the above fact they have the same interior set, hence the point $\Phi(a_1, \dots, a_n)$, which lies in the interior of $\Phi(I^n)$, belongs to $\Phi(D)$, whence the theorem. \square

Remark 3. The symmetry assumption of the function Φ in Theorem 1 is essential. Consider the function $\Phi: [-1, 1]^2 \rightarrow \mathbb{R}$, $(x_1, x_2) \rightarrow x_1 - x_2$, for which $0 = \Phi(0, 0)$ is not an extremal value of Φ since $\Phi([-1, 1]^2) = [-2, 2]$, and Φ vanished precisely on the diagonal set of $[-1, 1]^2$.

However, in certain particular cases, via *ad-hoc* arguments we may show that the symmetric assumption can be replaced by asking that Φ is “nonseparating” in the sense that for any $\sigma \in S_n$ we have $\Phi(\Omega) \cap \Phi(F_\sigma(\Omega)) \neq \emptyset$.

Further, Lemma 3 does not hold if K is the closure of an arbitrary connected open set $\Omega \subset \mathbb{R}^n$ ($n \geq 2$). For instance, consider the open triangle

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x/2 < y < 2x, x + y < 1\},$$

and let $\Phi: K \rightarrow \{0, 1\}$ be defined by setting $\Phi(0, 0) = 0$ and 1 otherwise. Obviously, Φ is partially Darboux and $\Phi(K) = \{0, 1\}$, which is not an interval. Note that K is not M -path connected; cf. the following proposition.

In the same circle of ideas we observe that a more general statement than Lemma 3, which we do not need in this paper, holds true.

Proposition 2. *Let $W \subseteq \mathbb{R}^n$ be a nonempty M -path connected set such that its closure \overline{W} is M -path connected. Then for any partially Darboux function $\Phi: \overline{W} \rightarrow \mathbb{R}$ we have that $\Phi(W)$ and $\Phi(\overline{W})$ are intervals of \mathbb{R} , and*

$$\Phi(\overline{W}) \subseteq \overline{\Phi(W)}.$$

Remark 4. The origin of Lemma 3 when Φ is continuous, goes back to Pompeiu, who have proved Darboux theorem stating that derivatives enjoy the intermediate value property. Here is a brief presentation of Pompeiu's proof.

Let $\mathfrak{h}: I \rightarrow \mathbb{R}$ be a differentiable function, where I is an open interval. Then for any open interval $J \subseteq I$ the set $\mathfrak{h}'(J)$ is an interval.

Indeed, let Σ be the set of slopes of the graph of $\mathfrak{h}|_J$, namely

$$\Sigma = \{(\mathfrak{h}(x_1) - \mathfrak{h}(x_2))/(x_1 - x_2) : (x_1, x_2) \in J^2, x_1 \neq x_2\},$$

and observe that Σ is an interval since it is the image through the following continuous map of a connected set, namely

$$\{(x_1, x_2) \in J^2 : x_1 < x_2\} \ni (x_1, x_2) \mapsto (\mathfrak{h}(x_1) - \mathfrak{h}(x_2))/(x_1 - x_2).$$

Now, thanks to Lagrange and using the definition of the derivative we have the inclusions, respectively,

$$\Sigma \subseteq T \subseteq \overline{\Sigma},$$

hence T is also an interval, which is what we want.

Notice that both inclusions above can be strict. Take $\mathfrak{h}: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^3$, so that for $J = (-1, 1)$ we have $\Sigma = (0, 3)$ and $T = [0, 3)$.

We invite the reader to re-consider the solutions of Problems 1.2–1.6 as well as Theorems 2.2–2.5 from [4] using Proposition 1 with an appropriately selected function Φ as done in the proof of (\diamond) . In particular, all results discussed in [4] can be solved in this manner and, for some of them, when two functions are involved, one needs a reduction step in the proof similarly to that when we deduce Cauchy's mean value theorem from Lagrange's mean value by using the standard facts of the derivative of a composite function and of the inverse function.

One can invent similar problems by carefully selecting the function Φ in Proposition 1.

3. ON A MULTIPLE POINTS MEAN VALUE RESULT

The following proposition extends Problem 1.1 from [4], which was inspired by [6], and is recovered setting below $\nu_1 = \dots = \nu_k = 1$.

Proposition 3. *Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function that is differentiable on the open interval (a, b) . Then for any $\nu_1, \dots, \nu_k \in \mathbb{N}$ there are pairwise distinct*

points c_1, \dots, c_k in (a, b) such that setting $n = \nu_1 + \dots + \nu_k$ we have

$$\left(\frac{f(b) - f(a)}{b - a} \right)^n = \prod_{j=1}^k (f'(c_j))^{\nu_j}.$$

P r o o f. We provide two solutions (see (i) and (ii) from below) to the above question, each having its own flavour.

(i) Thanks to Lagrange, there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Case 1. If $f'(c)$ is a global extremum of f' on (a, b) , then f is an affine function and the proof is immediate. Also, if $f'(c) = 0$, then take $c_1 = c$ and c_2, \dots, c_n arbitrary pairwise distinct points of $(a, b) \setminus \{c\}$.

Case 2. Suppose that $f'(c) \in (\inf_{(a,b)} f', \sup_{(a,b)} f')$ and $f'(c) \neq 0$. Then there exist points $a_*, b_* \in (a, b)$ such that $f'(a_*) < f'(c) < f'(b_*)$ and the values $f'(a_*)$, $f'(b_*)$ have the same sign as $f'(c)$. We claim that we may find pairwise distinct points y_1, \dots, y_n in the interval $(f'(a_*), f'(b_*))$ such that

$$(f'(c))^n = y_1^{\nu_1} \dots y_n^{\nu_n},$$

so that the conclusion of the proposition results because f' possesses the intermediate value property.

Now, the claim is a routine consequence of the following lemma.

Lemma 4. *Let α and β be real numbers such that $\alpha < 0 < \beta$. Then the following assertions hold true.*

- (a) *For any positive numbers $\lambda_1, \dots, \lambda_n$ ($n \geq 2$) there exist pairwise distinct nonzero numbers $\gamma_1, \dots, \gamma_n \in (\alpha, \beta)$ such that $\lambda_1 \gamma_1 + \dots + \lambda_n \gamma_n = 0$.*
- (b) *For any real number λ except for $\lambda = 1$ there exist distinct numbers γ_1 and γ_2 in (α, β) such that $\gamma_1 = \lambda \gamma_2$.*

P r o o f. We check (a) by induction over n . For $n = 2$, take $\gamma_1 = -\varepsilon \lambda_2$ and $\gamma_2 = \varepsilon \lambda_1$ with $\varepsilon > 0$ small enough. Now, let us see how to pass from n to $n+1$. By the induction hypothesis there are pairwise distinct nonzero numbers $\gamma'_1, \dots, \gamma'_n \in (\alpha, \beta)$ such that $\lambda_1 \gamma'_1 + \dots + \lambda_n \gamma'_n = 0$. Then consider $\gamma_j = \gamma'_j - \varepsilon$ for $j \in \{1, \dots, n\}$ and set $\gamma_{n+1} = -(\lambda_1 \gamma_1 + \dots + \lambda_n \gamma_n) / \lambda_{n+1}$ for a real number ε . Then it is easily seen that if $\varepsilon \neq 0$ is small enough, then $\gamma_1, \dots, \gamma_n$, and γ_{n+1} fulfil the required conditions.

The assertion (b) follows by cases analysis. If $\lambda < 0$, then we conclude easily by assertion (a) for $n = 2$. The case $\lambda = 0$ is obvious, and for $\lambda > 0$, $\lambda \neq 1$, we take $\gamma_2 \in (0, \beta)$ small enough such that $\gamma_1 = \lambda \gamma_2 < \beta$. □

(ii) First, recall Ampère's chord theorem [1] (see also [3] and [13] for some comments and historical backgrounds), namely the following statement.

Proposition 4. *Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then for any $n \in \{2, 3, \dots\}$ there is an interval $[a', b'] \subset [a, b]$ of length $(b - a)/n$ such that*

$$\frac{f(b') - f(a')}{b' - a'} = \frac{f(b) - f(a)}{b - a}.$$

Besides, we may choose $[a', b'] \subset (a, b)$ provided that either $n \geq 3$, or $n = 2$ and $f((a + b)/2) \neq (f(a) + f(b))/2$.

Chord theorem is very useful, for instance, we have the following proof of Lagrange's mean value theorem due to Pompeiu [8] (see also [7], [9], [10], and [11]), which avoid the classical approach via Rolle's theorem.

The proof goes as follows. By Proposition 4 there exists a sequence $([a_n, b_n])_n$ of intervals of (a, b) with nonzero length which is decreasing with respect to inclusion and whose length tends to 0 such that

$$\frac{f(b) - f(a)}{b - a} = \frac{f(b_n) - f(a_n)}{b_n - a_n}.$$

Then by the nested interval property of \mathbb{R} the intersection of the above intervals is a singleton, say a point c of (a, b) , which is easily seen to be a Lagrange point for f because f is differentiable at c .

It is worth mentioning that in Pompeiu's proof the function f may possess infinite derivative at some points in (a, b) . (Think of $x^{1/3}$ over an interval $[a, b]$ such that $ab < 0$.)

Now, Ampère's chord theorem, Pompeiu's idea from above and the following result conclude Proposition 3. □

Proposition 5. *For any continuous function $f: [a, b] \rightarrow \mathbb{R}$ with $f(a) \neq f(b)$, and any $m, n \in \mathbb{N}$, there exists a point $c \in (a, b)$ that satisfies*

$$\left(\frac{f(c) - f(a)}{c - a} \right)^m \left(\frac{f(b) - f(c)}{b - c} \right)^n = \left(\frac{f(b) - f(a)}{b - a} \right)^{m+n}.$$

Proof. Clearly, without loss of generality, we may assume that $[a, b] = [0, 1]$, $f(0) = 0$ and $f(1) = 1$. Thus, we have to show that the equation

$$(f(x))^m (1 - f(x))^n = x^m (1 - x)^n$$

has at least a solution in the open interval $(0, 1)$. Discarding the case when f has a fix point in $(0, 1)$ and the proof results immediately, suppose that f has no fix point in $(0, 1)$. Now, observe that the function

$$h: [0, 1] \rightarrow \mathbb{R}, t \rightarrow t^m(1-t)^n$$

is strictly increasing on the interval $[0, \alpha]$ and is strictly decreasing on the interval $[\alpha, 1]$, where $\alpha = m/(m+n)$.

Further, consider the auxilliary function $g: [0, 1] \rightarrow \mathbb{R}$,

$$[0, 1] \ni x \rightarrow g(x) := h(f(x)) - h(x).$$

Then since f has no fix point in $(0, 1)$, we have that either $f(x) > x$ for all $x \in (0, 1)$, or $f(x) < x$ for all $x \in (0, 1)$. In each case we check that there are points $x_1 \in (0, \varepsilon)$ and $x_2 \in (1 - \varepsilon, 1)$ for some $\varepsilon > 0$ small enough such that the values $g(x_1)$ and $g(x_2)$ have opposite signs.

For instance, suppose that $f(x) < x$ for all $x \in (0, 1)$. Because f is continuous and $f(1) = 1$, there exist $x_2 \in (\alpha, 1)$ such that $\alpha < f(x_2)$. Thus $f(x_2)$ and x_2 belong to $(\alpha, 1)$, $f(x_2) < x_2$ and h is strictly decreasing on $[\alpha, 1]$, hence $g(x_2) > 0$.

On the other hand, if there exists $x_1 \in (0, 1)$ such that $f(x_1) = 0$, then $g(x_1) < 0$. Otherwise, let $f(x) > 0$ for all $x \in (0, 1)$, so that from the continuity of f and the condition $f(0) = 0$, there exists $x_1 \in (0, \alpha)$ that verifies $0 < f(x_1) < \alpha$. Thus, both $f(x_1)$ and x_1 belong to $(0, \alpha)$ and $f(x_1) < x_1$. Since h is strictly increasing on $[0, \alpha]$, we have that $g(x_1) < 0$.

Eventually, because g is continuous, there exists a number x_* between x_1 and x_2 such that $g(x_*) = 0$, concluding the proof in first case.

Similarly, we deal with the case when $f(x) > x$ for all $x \in (0, 1)$. □

By Lemma 4 we get the following results which extend Theorem 2.1 (a) in [4], and respectively the problem proposed in [5], which is recovered for $n = 4$.

Corollary 1. *Let $a, b \in \mathbb{R}$ such that $a < b$ and let $h: (a, b) \rightarrow \mathbb{R}$ be a Darboux function. Then for any $\alpha \in (\inf h, \sup h)$ and any positive numbers $\lambda_1, \dots, \lambda_n$ whose sum is 1, there are paarwise distinct points c_1, \dots, c_n of (a, b) that satisfy*

$$\alpha = \lambda_1 h(c_1) + \dots + \lambda_n h(c_n).$$

Corollary 2. *Let $f: [0, 1] \rightarrow \mathbb{R}$ be continuous on $[0, 1]$ and differentiable on $(0, 1)$ and assume that either $f(0) = f(1)$ or $f(1)$ is not an extremal value for f .*

- (a) Then for any real λ except $\lambda = 1$ there are distinct $c_1, c_2 \in (0, 1)$ such that $f'(c_1) = \lambda f'(c_2)$.
- (b) For any $s_1, s_2, \dots, s_n \in [0, \infty)$ there are pairwise distinct $c_1, c_2, \dots, c_n \in (0, 1)$ such that $s_1 f'(c_1) + s_2 f'(c_2) + \dots + s_n f'(c_n) = 0$.

Proof. Taking into account Lemma 4, the solution results by the following fact.

Let $f: [0, 1] \rightarrow \mathbb{R}$ be a continuous function which is differentiable on $(0, 1)$. Then $\inf_{(0,1)} f' < \sup_{(0,1)} f'$ if and only if f is not an affine function. Besides, if either $f(0) = f(1)$ and f is not a constant function, or $f(1)$ is not an extremal value, then $\inf_{(0,1)} f' < 0 < \sup_{(0,1)} f'$.

The first part is obvious, hence omitted. For the additional statement we see easily that f cannot be an affine function. Then note that if $f(1)$ is not an extremal value, then there exist two distinct points $a', b' \in [0, 1]$ such that $f(a') = f(b')$. Therefore, in both cases, by Rolle's theorem, there exists a point $c \in (0, 1)$ such that $f'(c) = 0$, hence $\inf_{(0,1)} f' < 0 < \sup_{(0,1)} f'$. \square

4. MORE MEAN VALUE MULTIPLE POINTS RESULTS

First, we improve a classical result.

Proposition 6. *Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function that is differentiable on the open interval (a, b) . Then for any $n \in \mathbb{N}$, $n \geq 2$, and positive numbers $\lambda_1, \dots, \lambda_n$ with sum 1, there exists a nontrivial arithmetic progression $\xi_1, \xi_2, \dots, \xi_n$ of points in (a, b) such that*

$$(1) \quad \frac{f(b) - f(a)}{b - a} = \lambda_1 f'(\xi_1) + f'(\xi_2) + \dots + \lambda_n f'(\xi_n).$$

Proof. Put $\mu = (f(b) - f(a))/(b - a)$. By Lagrange's mean value theorem there exists $\xi \in (a, b)$ such that $f'(\xi) = \mu$. Observe that $f'(\xi)$ is a global extremum for f' on (a, b) if and only if f is affine. (This results by routine arguments using the known ascending criterion which says that a real-valued, differentiable function defined on an open interval of \mathbb{R} is ascending provided that it has non-negative derivative.)

Now, discarding the case when f is an affine function and the proof of the proposition is obvious, let us assume that f is not an affine function. Hence, $f'(\xi)$ is an interior point of the image of (a, b) through f' . It follows that $f'(\xi)$ is not a global extremum for f' on (a, b) and that f' is not the constant function. Then there are

points $a_*, b_* \in (a, b)$ such that $a_* < \xi < b_*$ and $f'(\xi)$ lies in the open interval with extremities $f'(a_*)$ and $f'(b_*)$. Without any loss of generality, take $f'(a_*) < f'(b_*)$.

Let T be the rectangle in \mathbb{R}^2 consisting of all couples of points $(x, t) \in \mathbb{R}^2$ such that $x \in [a_*, b_*]$ and $a - b_* < (n - 1)t < b - a_*$. Then consider $F: T \rightarrow \mathbb{R}$ defined by

$$F(x, t) = \lambda_1 f'(x) + \lambda_2 f'(x + t) + \dots + \lambda_n f'(x + (n - 1)t), \quad (x, t) \in T.$$

Clearly, for any $x \in [a_*, b_*]$ the function $F(x, \cdot)$ is a derivative, hence it has the intermediate value property, hence by Lemma 1 there are points (a_*, t_1) and (b_*, t_2) of T with $t_1 > 0$ and $t_2 > 0$ that satisfy $F(a_*, t_1) < \mu < F(b_*, t_2)$.

Now, the function $g: [0, 1] \rightarrow \mathbb{R}$ defined for $s \in [0, 1]$ by setting

$$g(s) = F((1 - s)a_* + sb_*, (1 - s)t_1 + st_2)$$

is easily seen to be a derivative, and since μ lies between $g(0)$ and $g(1)$, by Darboux theorem, there exists $s_0 \in (0, 1)$ with $g(s_0) = \mu$. Hence, setting $c = (1 - s_0)a_* + s_0b_*$ and $t = (1 - s_0)t_1 + s_0t_2 > 0$, we have $F(c, t) = \mu$, whence the solution of our question for $\xi_1 = c, \xi_2 = c + t, \dots, \xi_n = c + (n - 1)t$. \square

Similarly, we get the following result that extends [12].

Proposition 7. *Let $f: [a, b] \rightarrow \mathbb{R}$ be an integrable function on $[a, b]$, continuous on (a, b) and such that $\int_a^b f \neq 0$. Then for any $n \in \mathbb{N}$ and positive numbers $\lambda_1, \dots, \lambda_n$ whose sum equals 1, there exist numbers $c_1, c_2, \dots, c_n \in (a, b)$ that form a nontrivial arithmetic progression and satisfy the equality*

$$\frac{1}{b - a} \int_a^b f = \frac{1}{\lambda_1/f(c_1) + \lambda_2/f(c_2) + \dots + \lambda_n/f(c_n)}.$$

Proof. Let $c \in (a, b)$ with $(\int_a^b f)/(b - a) = f(c)$. Without loss in generality we may take $f(c) > 0$. Hence, since f is continuous on (a, b) , there exists $\varepsilon > 0$ with $\varepsilon < \min\{c - a, b - c\}$ such that $f > 0$ on the interval $[c - \varepsilon, c + \varepsilon]$. Then the proof concludes along the lines of proof of Proposition 6 for the function $1/f$ on the interval $[c - \varepsilon, c + \varepsilon]$. \square

In the same vein, we have the next result for which we omit the details of the proof.

Proposition 8. *Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function. Set $K = f([a, b])$, and let $\Phi: K^n \rightarrow \mathbb{R}$ be a continuous function. Then for any point c of $[a, b]$ such that $\Phi(f(c), \dots, f(c))$ is not an extremum of $\Phi(K^n)$ there are points c_1, \dots, c_n of (a, b) which form a nontrivial arithmetic progression and such that*

$$\Phi(f(c), \dots, f(c)) = \Phi(f(c_1), \dots, f(c_n)).$$

The following result is due to Pompeiu [7] for the case of two functions. His proof differs from the one we give subsequently.

Proposition 9. *Let $f_j: [a, b] \rightarrow \mathbb{R}$ be continuous functions that are differentiable on (a, b) , $j \in \{1, \dots, n\}$. If each function f_j is ascending and convex, then there exists a point ξ that satisfies*

$$(2) \quad \prod_{j=1}^n (f_j(b) - f_j(a)) = (b - a)^n \prod_{j=1}^n f'_j(\xi).$$

Proof. First, let us look at the following example due to Pompeiu [7]. Take two functions $f_1, f_2: [-1, 1] \rightarrow \mathbb{R}$ defined by setting

$$f_1(x) = \begin{cases} 0 & \text{for } -1 \leq x \leq 0, \\ x^2 & \text{for } 0 < x \leq 1, \end{cases}$$

and $f_2(x) = x^2 - f_1(x)$ for $x \in [-1, 1]$. Both functions are differentiable and convex, the left-hand side term in (2) is -1 and the right-hand term is 0 since the product $f'_1 f'_2 = 0$. Thus, in general, (2) fails to hold true without extra conditions upon the involved functions.

Now, coming to the proof of (2), recall that a differentiable, convex function on an open interval of \mathbb{R} is of class \mathcal{C}^1 . Then (2) follows immediately by the following claim.

For finitely many ascending, continuous functions $g_j: (a, b) \rightarrow [0, \infty)$ and points $\xi_j \in (a, b)$ there exists a point $\xi \in (a, b)$ such that

$$\prod g_j(\xi_j) = \prod g_j(\xi).$$

We prove this by induction over the number n of functions g_j . Since the initial step ($n = 1$) is trivial, consider $n + 1$ functions g_1, \dots, g_n, g and points $\xi_1, \dots, \xi_n, \zeta \in (a, b)$ as above. By the inductive hypothesis, there exists $\eta \in (a, b)$ such that

$$\prod_{j=1}^n g_j(\xi_j) = \prod_{j=1}^n g_j(\eta).$$

Let h denote the product of the functions g_1, \dots, g_n . Obviously, h is non-negative, continuous and ascending. Therefore, to conclude we need to prove that there exists a point $\xi \in (a, b)$ such that

$$h(\eta)g(\zeta) = h(\xi)g(\xi).$$

In other words, we reduced our induction step to the case of two functions that is contained in [7] with a different proof. Below we provide a simpler proof. The essential case is when $\eta \neq \zeta$, say $\eta < \zeta$. To conclude the claim in this case, consider the continuous function $\mathfrak{d}: [\eta, \zeta] \rightarrow \mathbb{R}$ given by

$$\mathfrak{d}(x) = \mathfrak{h}(x)\mathfrak{g}(x) - \mathfrak{h}(\eta)\mathfrak{g}(\zeta), \quad x \in [\eta, \zeta].$$

Then the hypotheses on \mathfrak{h} and \mathfrak{g} imply easily that

$$\mathfrak{d}(\eta)\mathfrak{d}(\zeta) \leq 0,$$

hence thanks to Bolzano there exists a point $\xi \in [\eta, \zeta]$ with $\mathfrak{d}(\xi) = 0$, which ends our induction. \square

5. OPEN PROBLEMS

Related to the questions discussed previously we mention the following open questions.

Let $\mathfrak{f}: [a, b] \rightarrow \mathbb{R}$ be a continuous function which is differentiable on (a, b) , and let $n \in \mathbb{N}$, $n \geq 3$.

- ▷ If $\mathfrak{f}(a) \neq \mathfrak{f}(b)$, then there exist points $\xi_1, \xi_2, \dots, \xi_n$ of (a, b) that form a nontrivial arithmetic progression and satisfy

$$\frac{\mathfrak{f}(b) - \mathfrak{f}(a)}{b - a} = \frac{n}{1/\mathfrak{f}'(\xi_1) + 1/\mathfrak{f}'(\xi_2) + \dots + 1/\mathfrak{f}'(\xi_n)}.$$

- ▷ There exists a nontrivial arithmetic progression c_1, c_2, \dots, c_n of points in (a, b) such that

$$\left(\frac{\mathfrak{f}(b) - \mathfrak{f}(a)}{b - a} \right)^n = \prod_{j=1}^n \mathfrak{f}'(c_j).$$

The answer is “Yes” for both questions if \mathfrak{f} is of class \mathcal{C}^1 on (a, b) . In fact, *mutatis mutandis*, the arguments in the proof of Proposition 5 can be easily adapted. Furthermore, by Ampère’s chord theorem, the above questions are also answered in the affirmative provided that \mathfrak{f}' has constant sign on a neighborhood of a Lagrange point c obtained via Pompeiu’s approach, or that \mathfrak{f} is strictly monotone.

A c k n o w l e d g e m e n t s . We would like to thank the referees for carefully reading our manuscript, constructive and pertinent comments, and for drawing our attention to the problem proposed in [5].

References

- [1] *A. M. Ampère*: Recherche sur quelques points de la théorie des fonctions dérivées qui conduisent à une nouvelle démonstration de la série de Taylor, et à l'expression finie des termes qu'on néglige lorsqu'on arrête cette série à un terme quelconque. *J. Éc. Polytech.* 6 (1806), 148–181. (In French.)
- [2] *Z. Chen*: A higher mean value theorem: 10935. *Am. Math. Mon.* 110 (2003), 544–545. [doi](#)
- [3] *T. M. Flett*: Ampère and the horizontal chord theorem. *Bull. Inst. Math. Appl.* 11 (1975), 34. [MR](#)
- [4] *D. Ş. Marinescu, M. Monea*: Some mean value theorems as consequences of the Darboux property. *Math. Bohem.* 142 (2017), 211–224. [zbl](#) [MR](#) [doi](#)
- [5] *R. Mortini*: Problem 12487. *Am. Math. Mon.* 131 (2024), 723. [doi](#)
- [6] *P. Pangsruiam*: Problem 11753. *Am. Math. Mon.* 121 (2014), 84. [doi](#)
- [7] *D. Pompeiu*: Sur le théorème des accroissements finis. *Ann. Sci. Univ. Jassy* 3 (1906), 244–250. (In French.)
- [8] *D. Pompeiu*: Remarque sur une démonstration du théorème des accroissements finis. *Ann. Sci. Univ. Jassy* 9 (1915), 46–53. (In French.)
- [9] *D. Pompeiu*: Sur une forme du théorème des accroissements finis. 9 (1925), 61–64. (In French.) [zbl](#)
- [10] *D. Pompeiu*: Sur une proposition analogue au théorème des accroissements finis. *Mathematica, Timișoara* 22 (1946), 143–146. (In French.) [zbl](#) [MR](#)
- [11] *D. Pompeiu*: Oeuvre mathématique. Editura Academiei Republicii Populare Romîne, București, 1959. (In Romanian.) [zbl](#) [MR](#)
- [12] *T. Precupanu*: Problem 5.119. Olimpiadele Naționale de Matematică pentru liceu 1954–2003. Editura Enciclopedică, București, 2004, pp. 146. (In Romanian.)
- [13] *V. Vîjîitu*: Hyers-Ulam stability of mean value points. *Monatsh. Math.* 204 (2024), 625–640. [zbl](#) [MR](#) [doi](#)

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