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ON A ROBIN-DIRICHLET PROBLEM FOR A SYSTEM OF NONLINEAR PSEUDOPARABOLIC EQUATIONS WITH THE VISCOELASTIC TERM

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Abstract. We consider a Robin-Dirichlet problem for a system of nonlinear pseudoparabolic equations with the viscoelastic term. Based on the Faedo-Galerkin method, we first prove existence and uniqueness. Next, we give a sufficient condition for the global existence and decay of weak solutions. Finally, using concavity method, we prove blow-up results for solutions when the initial energy is nonnegative or negative. Furthermore, we establish here the lifespan for the equation via finding the upper bound and the lower bound for the blow-up times.

Keywords: nonlinear pseudoparabolic equation; Faedo-Galerkin method; local existence; blow-up; lifespan; the global existence and decay of weak solutions

MSC 2020: 35K70; 35A01; 35B40, 35B44

1. Introduction

In this paper, we consider the initial-boundary value problem for the system of nonlinear pseudoparabolic equations with Robin-Dirichlet conditions

$$
\begin{aligned}
(1.1)\qquad & u_t-\lambda_1 u_{txx}-\frac{\partial}{\partial x}(\mu_1(x,t)u_x)\\
&\qquad = f_1(x,t,u,v,u_t,v_t,u_x,v_x,u_{xt},v_{xt}),\quad 0<x<1,\quad 0<t<T,\\
& v_t-\lambda_2 v_{txx}-\frac{\partial}{\partial x}(\mu_2(x,t)v_x)+\int_0^t g(t-s)v_{xx}(x,s)\,\mathrm{d}s\\
&\qquad = f_2(x,t,u,v,u_t,v_t,u_x,v_x,u_{xt},v_{xt}),\quad 0<x<1,\quad 0<t<T,\\
(1.2)\qquad & u_x(0,t)-\zeta u(0,t)=u(1,t)=v(0,t)=v(1,t)=0,\\
(1.3)\qquad & (u(x,0),v(x,0))=(\widetilde{u}_0(x),\widetilde{v}_0(x)),
\end{aligned}
$$

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where $\zeta \geqslant 0$, λ_1, $\lambda_2 > 0$ are given constants and g, μ_i, f_i, $(i = 1, 2)$, $\widetilde{u}_0$, $\widetilde{v}_0$ are given functions satisfying conditions specified later.

The pseudoparabolic equation

$$u_t - u_{xxt} = F(x, t, u, u_x, u_{xx}, u_{xt}), \quad 0 < x < 1, \quad t > 0, \tag{1.4}$$

with the initial condition $u(x, 0) = \widetilde{u}_0(x)$ and with the different boundary conditions, has been extensively studied by many authors see for example [1], [4]–[11], [16], [17], [19]–[23], [26], [27], [29], [31], [32], [34]–[36], [38] among others and the references given therein. In these works, many results about existence, asymptotic behavior, blow-up and decay of solutions were obtained.

An important special case of model (1.4) is the Benjamin-Bona-Mahony-Burgers (BBMB) equation

$$u_t + u_x + uu_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0. \tag{1.5}$$

It was studied by Amick et al. in [2] with $\nu > 0$, $\alpha = 1$, $x \in \mathbb{R}$, $t \geqslant 0$, in which the solution of (1.5) with initial data in $L^1 \cap H^2$ decays to zero in L^2 norm as $t \to \infty$. With $\nu > 0$, $\alpha > 0$, $x \in [0, 1]$, $t \geqslant 0$, the model having the form (1.5) was also investigated earlier by Bona and Dougalis [3], where uniqueness, global existence and continuous dependence of solutions on initial and boundary data were established and the solutions were shown to depend continuously on $\nu \geqslant 0$ and on $\alpha > 0$. The results obtained in [2] were developed by many authors such as by Zhang for equations of the form

$$u_t - \nu u_{xx} - u_{xxt} - u_x + u^m u_x = 0,$$

where $m \geqslant 0$, see [17], [37].

The linear version of (1.4) was first studied by Sobolev [28] in 1954. Therefore, the equation of the form (1.4) is also called a Sobolev type equation. Mathematical study of pseudo-parabolic equations goes back to works of Showalter (see [24], [26], [27]) in the seventies. Since then, numerous of interesting results about linear and nonlinear pseudoparabolic equations have been obtained. It is also well known that the work [24] is the first paper on nonlinear pseudoparabolic equation. These equations appear in the study of various problems of hydrodynamics, thermodynamics and filtration theory, see Meyvaci [17] and the references given therein.

Problem (1.1)–(1.3) is a type of viscoelastic pseudoparabolic problems, the Volterra integral in the second equation of (1.1)–(1.3) is a memory term, so called viscoelastic term, responsible for viscoelastic damping. In recent years, a great deal of attention has been paid to the pseudoparabolic equations with memory or viscoelastic term. For instance, Shang and Guo [23] proved the existence, uniqueness, regularities of the global strong solution and gave some conditions of the nonexistence of global solution for the nonlinear pseudoparabolic equation with Volterra integral term

$$\begin{aligned} u_t - f(u)_{xx} - u_{xxt} - \int_0^t \lambda(t-s)(\sigma(u(x,s),u_x(x,s)))_x \,\mathrm{d}s \\ = f(x,t,u,u_x), \quad 0<x<1, \quad t>0. \end{aligned}$$

In [29], Sun et al. considered the Dirichlet problem for the nonlinear pseudoparabolic equation with a power source term and a memory term as

$$\begin{cases} u_t - \Delta u - \Delta u_t + \displaystyle\int_0^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau = |u|^{p-2}u & \text{in } \Omega\times(0,T), \\ u = 0 & \text{on } \partial\Omega\times(0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geqslant 1)$ with smooth boundary $\partial\Omega$, $p>2$, $T\in(0,\infty]$, $u_0\in H^1(\Omega)$ and $g\colon \mathbb{R}_+\to\mathbb{R}_+$ is a positive nonincreasing function. The authors used the Levine's classical concavity method and the improved potential well method to obtain the global existence and the finite time blow-up phenomena of solutions. Recently, in [19], general decay and blow-up results for a nonlinear pseudoparabolic equation containing viscoelastic term

$$u_t - \Big(\mu(t)+\alpha(t)\frac{\partial}{\partial t}\Big)\Big(u_{xx}+\frac{1}{x}u_x\Big) + \int_0^t g(t-s)\Big(u_{xx}(s)+\frac{1}{x}u_x(s)\Big)\,\mathrm{d}s = f(x,t,u),$$

$1<x<R$, $t>0$, with Robin-Dirichlet condition have been also established. For more results on global existence and blow up of solutions to the pseudoparabolic equations with memory or viscoelastic terms, we refer to [16], [36], in which the pseudoparabolic problems with nonlinearities of variable-exponent type were investigated. Moreover, the results given in [16], [36] can be considered a generalization of the above problems with exponential nonlinearity when variable exponent is identified as a constant.

To the best of our knowledge, there are many publications on properties of solutions to single parabolic/pseudoparabolic equations, but it seems that few results of the system of these types are investigated. We refer here to some results as in [8], [12]–[14], [30] and the references therein. Very recently, in [20], Ngoc et al. have just considered the initial-boundary value problem for a system of nonlinear pseudoparabolic equations with Robin-Dirichlet conditions and existence, uniqueness, blow-up and decay estimates were established here. We note more that in [20], by the mixed time and space derivative terms u_{txx}, v_{txx} (strong damping terms), and memory terms in the form of Volterra integrals affect both equations of the system, so the decay or blow up of solutions have been obtained in some cases of the initial energy and datum. Especially, with general decay difference kernels of memory terms, the general decay of the solution has been proved.

Inspired and motivated by the idea of the above mentioned works, because of mathematical context, we study the existence, uniqueness, blow-up and general decay of solutions for Problem (1.1)–(1.3). It consists of five sections as follows.

In Section 2, we present preliminaries.

In Section 3, by using the linear approximating method together with the Galerkin method, we establish the local existence and uniqueness of a weak solution.

In Section 4, we investigate Problem (1.1)–(1.3) with

$$f_i(x,t,u,v,u_t,v_t,u_x,v_x,u_{xt},v_{xt}) = f_i(u,v) + F_i(x,t), \quad i = 1,2.$$

Because of the difficulties, when the memory term in form of a Volterra integral impacts only the second equation of this system, some techniques used in [20] are no longer correct. In order to obtain the decay result as in [20], we need to apply techniques as in [18] with some necessary modifications, that is, adding the new functional $(g_* \diamondsuit u)(t)$ as in (4.5) (see (4.5) below). Then, based on the energy method, a sufficient condition for the global existence and decay of a weak solution is proved in this section.

Finally, in Section 5, we consider the blow-up property for Problem (1.1)–(1.3) in the special case $f_i(x,t,u,v,u_t,v_t,u_x,v_x,u_{xt},v_{xt}) = f_i(u,v)$, $i = 1,2$, with initial data at suitable energy levels. By using concavity method, and applying techniques as in [18], [20] with some necessary modifications, we prove blow-up results for solutions when the initial energy is nonnegative or negative. Furthermore, we establish here the lifespan for the equation via finding the upper bound and the lower bound for the blow-up times.

2. Preliminaries

In this paper, we put $\Omega = (0,1)$, $Q_T = \Omega \times (0,T)$, $T > 0$ and use the usual function spaces $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0,T;X)$,

$1 \leqslant p \leqslant \infty$, the Banach space of real functions $u\colon (0,T) \to X$ measurable such that $\|u\|_{L^p(0,T;X)} < \infty$ with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|_X^p \,\mathrm{d}t\right)^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X & \text{if } p = \infty. \end{cases}$$

Denote

$$u(t) = u(x,t), \quad u'(t) = u_t(t) = \frac{\partial u}{\partial t}(x,t), \quad u''(t) = u_{tt}(t) = \frac{\partial^2 u}{\partial t^2}(x,t),$$
$$u_x(t) = \frac{\partial u}{\partial x}(x,t), \quad u_{xx}(t) = \frac{\partial^2 u}{\partial x^2}(x,t).$$

With $f \in C^k([0,1] \times [0,T^*] \times \mathbb{R}^8)$, $f = f(x,t,y_1,\ldots,y_8)$, we put $D_1 f = \partial f/\partial x$, $D_2 f = \partial f/\partial t$, $D_{i+2} f = \partial f/\partial y_i$, with $i = 1,\ldots,8$ and $D^\alpha f = D_1^{\alpha_1} \ldots D_{10}^{\alpha_{10}} f$, $\alpha = (\alpha_1,\ldots,\alpha_{10}) \in \mathbb{Z}_+^{10}$, $|\alpha| = \alpha_1 + \ldots + \alpha_{10} \leqslant k$, $D^{(0,\ldots,0)} f = D^0 f = f$.

Similarly, with $\mu \in C^k([0,1] \times [0,T^*])$, $\mu = \mu(x,t)$, we put $D_1\mu = \partial\mu/\partial x$, $D_2\mu = \partial\mu/\partial t$, and $D^\beta \mu = D_1^{\beta_1} D_2^{\beta_2} \mu$, $\beta = (\beta_1,\beta_2) \in \mathbb{Z}_+^2$, $|\beta| = \beta_1 + \beta_2 \leqslant k$, $D^{(0,0)}\mu = \mu$.

On H^1, we shall use the norm

$$\|v\|_{H^1} = (\|v\|^2 + \|v_x\|^2)^{1/2},$$

and a closed subspace V of H^1 with

$$V = \{v \in H^1\colon\ v(1) = 0\}. \tag{2.1}$$

Then the following properties are true.

Lemma 2.1 (See [15]). *The imbedding $H^1 \hookrightarrow C^0(\bar\Omega)$ is compact and*

$$\|v\|_{C^0(\bar\Omega)} \leqslant \sqrt{2}\|v\|_{H^1} \quad \text{for all } v \in H^1.$$

Lemma 2.2 (See [15]). *The imbedding $V \hookrightarrow C^0(\bar\Omega)$ is compact. Moreover, we have*

(i) $\|v\|_{C^0(\overline{\Omega})} \leqslant \|v_x\|$ *for all* $v \in V$,
(ii) $\frac{1}{\sqrt{2}}\|v\|_{H^1} \leqslant \|v_x\| \leqslant \|v\|_{H^1}$ *for all* $v \in V$.

Let $\zeta \geqslant 0$ and $\mu_1 \in C^1(\bar{\Omega} \times [0,T^*])$ such that $\mu_1(x,t) \geqslant \mu_{1*} > 0$ for all $(x,t) \in \bar{\Omega}\times[0,T^*]$. On $V\times V$, we consider the symmetric bilinear form $a(\cdot,\cdot)$, and the famillies of symmetric bilinear forms $\{\bar{a}(t;\cdot,\cdot)\}_{t\in[0,T^*]}$ and $\{\bar{a}'(t;\cdot,\cdot)\}_{t\in[0,T^*]}$ defined by

$$
\begin{aligned}
(2.2) \qquad a(u,\varphi) &= \langle u_x, \varphi_x\rangle + \zeta u(0)\varphi(0),\\
\bar{a}(t;u,\varphi) &= \langle \mu_1(t)u_x, \varphi_x\rangle + \zeta\mu_1(0,t)u(0)\varphi(0),\\
\bar{a}'(t;u,\varphi) &= \langle \mu_1'(t)u_x, \varphi_x\rangle + \zeta\mu_1'(0,t)u(0)\varphi(0)\\
&\qquad\qquad \text{for all } (u,\varphi)\in V\times V,\ t\in[0,T^*].
\end{aligned}
$$

Then the following properties are also fulfilled.

Lemma 2.3. *Let $\mu_1 \in C^0(\bar{\Omega}\times[0,T^*])$ such that $\mu_1' \in C^0(\bar{\Omega}\times[0,T^*])$ with $\mu_1(x,t) \geqslant \mu_{1*} > 0$ for all $(x,t)\in\bar{\Omega}\times[0,T^*]$ and $\zeta \geqslant 0$. Then:*

(i) *The symmetric bilinear form $a(\cdot,\cdot)$, and the family of symmetric bilinear forms $\{\bar{a}(t;\cdot,\cdot)\}_{t\in[0,T^*]}$ defined by (2.2) are continuous on $V\times V$ and coercive in V.*

(ii) *The family of symmetric bilinear forms $\{\bar{a}'(t;\cdot,\cdot)\}_{t\in[0,T^*]}$ defined by (2.2) are continuous on $V\times V$.*

Furthermore,

$$
\begin{aligned}
a(u,u) &\geqslant \|u_x\|^2 \quad \text{for all } u\in V,\\
|a(u,\varphi)| &\leqslant (1+\zeta)\|u_x\|\|\varphi_x\| \quad \text{for all } u,\varphi\in V,\\
\bar{a}(t;u,u) &\geqslant \mu_{1*}\|u\|_a^2 \geqslant \mu_{1*}\|u_x\|^2 \quad \text{for all } u\in V,\ t\in[0,T^*],\\
|\bar{a}(t;u,\varphi)| &\leqslant \|\mu_1\|_{C^0(\bar{\Omega}\times[0,T^*])}\|u\|_a\|\varphi\|_a\\
&\leqslant \|\mu_1\|_{C^0(\bar{\Omega}\times[0,T^*])}(1+\zeta)\|u_x\|\|\varphi_x\| \quad \text{for all } u,\varphi\in V,\ t\in[0,T^*],\\
|\bar{a}'(t;u,\varphi)| &\leqslant \|\mu_1'\|_{C^0(\bar{\Omega}\times[0,T^*])}\|u\|_a\|\varphi\|_a\\
&\leqslant \|\mu_1'\|_{C^0(\bar{\Omega}\times[0,T^*])}(1+\zeta)\|u_x\|\|\varphi_x\| \quad \text{for all } u,\varphi\in V,\ t\in[0,T^*],
\end{aligned}
$$

where

$$\|u\|_a = \sqrt{a(u,u)} \quad \text{for all } u\in V.$$

Furthermore, we have the following lemma.

Lemma 2.4. *Let $\zeta \geqslant 0$. Then there exists the Hilbert orthonormal base $\{\widetilde{\phi}_j\}$ of L^2 consisting of the eigenfunctions $\widetilde{\phi}_j$ corresponding to the eigenvalue $\widetilde{\lambda}_j$ such that*

$$
\begin{cases}
0 < \widetilde{\lambda}_1 \leqslant \widetilde{\lambda}_2 \leqslant \ldots \leqslant \widetilde{\lambda}_j \leqslant \ldots,\ \lim\limits_{j\to\infty}\widetilde{\lambda}_j = \infty,\\
a(\widetilde{\phi}_j,\varphi) = \widetilde{\lambda}_j\langle\widetilde{\phi}_j,\varphi\rangle \quad \text{for all } \varphi\in V,\ j=1,2,\ldots
\end{cases}
$$

Furthermore, the sequence $\{\widetilde{\phi}_j/\sqrt{\widetilde{\lambda}_j}\}$ *is also the Hilbert orthonormal base of* V *with respect to the scalar product* $a(\cdot,\cdot)$, *and* $\widetilde{\phi}_j$ *satisfies the following boundary value problem:*

$$\begin{cases} -\Delta\widetilde{\phi}_j = \widetilde{\lambda}_j\widetilde{\phi}_j \quad \textit{in } (0,1), \\ \widetilde{\phi}_{jx}(0) - \zeta\widetilde{\phi}_j(0) = \widetilde{\phi}_j(1) = 0, \\ \widetilde{\phi}_j \in C^\infty(\bar{\Omega}). \end{cases}$$

Proof of Lemma 2.4 can be found in [25], p. 87, Theorem 7.7, with $H = L^2$, V defined by (2.1), and $a(\cdot,\cdot)$ defined by (2.2).

Remark 2.1. The sequence $\{\widetilde{\phi}_j/\sqrt{\widetilde{\lambda}_j + \widetilde{\lambda}_j^2}\}$ is also the Hilbert orthonormal base of $H^2 \cap V$ with respect to the scalar product$(u,\varphi) \mapsto \langle \Delta u, \Delta\varphi\rangle + a(u,\varphi)$.

Remark 2.2.

(i) On V and H_0^1, three norms $v \mapsto \|v_x\|$, $v \mapsto \|v\|_{H^1}$, $v \mapsto \|v\|_a = (\|v_x\|^2 + \zeta v^2(0))^{1/2}$ are equivalent.

(ii) On $H^2 \cap H_0^1$, two norms $v \mapsto \|v\|_{H^2}$, $v \mapsto \|v\|_{H^2\cap H_0^1} = (\|v_x\|^2 + \|v_{xx}\|^2)^{1/2}$ are equivalent.

(iii) On $H^2 \cap V$, two norms $v \mapsto \|v\|_{H^2}$, $v \mapsto \|v\|_{H^2\cap V} = (\|v\|_a^2 + \|v_{xx}\|^2)^{1/2}$ are equivalent.

3. Existence and uniqueness of a weak solution

Consider $T^* > 0$ fixed. For each $M > 0$ given, we put $\bar{\Omega}_M = [0,1] \times [0,T^*] \times [-M,M]^4 \times [-\sqrt{2}M, \sqrt{2}M]^4$. We make the following assumptions:

(H_1) $(\widetilde{u}_0, \widetilde{v}_0) \in (H^2 \cap V) \times (H^2 \cap H_0^1)$, $\widetilde{u}_{0x}(0) - \zeta\widetilde{u}_0(0) = 0$;

(H_2) $\mu_1, \mu_2 \in C^2([0,1] \times [0,T^*])$, and there exist the positive constants μ_{1*}, μ_{2*} such that $\mu_i(x,t) \geqslant \mu_{i*}$ for all $(x,t) \in [0,1] \times [0,T^*]$, $i = 1,2$;

(H_3) $g \in H^1(0,T^*)$;

(H_4) $f_i \in C^1([0,1] \times [0,T^*] \times \mathbb{R}^8)$, $i = 1,2$, such that

  (i) $f_1(1,t,0,y_2,0,y_4,y_5,y_6,y_7,y_8) = f_2(0,t,y_1,0,y_3,0,y_5,y_6,y_7,y_8) = f_2(1,t,y_1,0,y_3,0,y_5,y_6,y_7,y_8) = 0$ for all $t \in [0,T^*]$ for all $y = (y_1,\ldots,y_8) \in \mathbb{R}^8$,

  (ii) there exists a positive constant σ such that

    (j) $0 < \sigma < \bar{\mu}_*/511$, with $\bar{\mu}_* = \min\{1, \mu_{1*}, \mu_{2*}, \lambda_1, \lambda_2\}$,

    (jj) $\max\{\|D^\alpha f_i\|_{C^0(\bar{\Omega}_M)} \colon i = 5,6,9,10\} \leqslant \sigma$ for all $M > 0$.

For each $T \in (0, T^*]$ we denote

$$W_T = \{(u,v) \in L^\infty(0,T; (H^2 \cap V) \times (H^2 \cap H_0^1))\colon \\ (u',v') \in L^\infty(0,T; (H^2 \cap V) \times (H^2 \cap H_0^1))\},$$

a Banach space with respect to the norm

$$\|(u,v)\|_{W_T} = \max\{\|(u,v)\|_{L^\infty(0,T;(H^2\cap V)\times(H^2\cap H_0^1))}, \\ \|(u',v')\|_{L^\infty(0,T;(H^2\cap V)\times(H^2\cap H_0^1))}\}.$$

Definition 3.1. For every $T \in (0, T^*]$, the pair $(u,v) \in W_T$ is a weak solution of Problem (1.1)–(1.3) if (u,v) satisfies the variational problem

$$\text{(3.1)} \qquad \begin{cases} \langle u'(t), \varphi\rangle + \lambda_1 a(u'(t), \varphi) + \bar{a}(t; u(t), \varphi) = \langle f_1[u,v](t), \varphi\rangle, \\ \langle v'(t), \psi\rangle + \lambda_2 \langle v_x'(t), \psi_x\rangle + \langle \mu_2(t) v_x(t), \psi_x\rangle \\ \qquad = \displaystyle\int_0^t g(t-s)\langle v_x(s), \psi_x\rangle \,\mathrm{d}s + \langle f_2[u,v](t), \psi\rangle \end{cases}$$

for all $(\varphi, \psi) \in V \times H_0^1$, and a.e. $t \in (0,T)$, together with the initial condition

$$\text{(3.2)} \qquad (u(0), v(0)) = (\widetilde{u}_0, \widetilde{v}_0),$$

where

$$\text{(3.3)} \qquad f_i[u,v](x,t) = f_i(x,t,u,v,u_t,v_t,u_x,v_x,u_{xt},v_{xt}), \quad i = 1,2.$$

For each $M > 0$ given we define the constants $\overline{K}_M = \overline{K}_M(f_1, f_2)$, $\widetilde{K}_M = \widetilde{K}_M(\mu_1, \mu_2)$ by

$$\begin{cases} \overline{K}_M = \overline{K}_M(f_1,f_2) = \max\limits_{i=1,2} \|f_i\|_{C^1(\bar{\Omega}_M)}, \\ \widetilde{K} = \widetilde{K}(\mu_1,\mu_2) = \max\limits_{i=1,2} \|\mu_i\|_{C^2([0,1]\times[0,T^*])}, \end{cases}$$

with

$$\begin{cases} \|f_i\|_{C^1(\bar{\Omega}_M)} = \max\limits_{|\alpha|\leqslant 1} \|D^\alpha f_i\|_{C^0(\bar{\Omega}_M)} = \overline{K}_M(f_i), \\ \|D^\alpha f_i\|_{C^0(\bar{\Omega}_M)} = \sup\{|D^\alpha f_i(x,t;y_1,\dots,y_8)|\colon (x,t;y_1,\dots,y_8) \in \bar{\Omega}_M\}, \\ \|\mu_i\|_{C^2([0,1]\times[0,T^*])} = \max\limits_{|\beta|\leqslant 2} \|D^\beta \mu_i\|_{C^0([0,1]\times[0,T^*])} = \widetilde{K}(\mu_i) \equiv \widetilde{K}_i, \\ \|D^\beta \mu_i\|_{C^0([0,1]\times[0,T^*])} = \sup\{|D^\beta \mu_i(x,t)|\colon (x,t) \in [0,1]\times[0,T^*]\}. \end{cases}$$

For every $M > 0$ we put

$$B_T(M) = \{(u,v) \in W_T\colon\ \|(u,v)\|_{W_T} \leqslant M\}.$$

Now, we establish the recurrent sequence $\{(u_m, v_m)\}$. First, we choose $(u_0, v_0) \equiv (0,0)$, and we suppose that

$$(u_{m-1}, v_{m-1}) \in B_T(M). \tag{3.4}$$

We will associate (1.1)–(1.3) with the following problem: Find $(u_m, v_m) \in B_T(M)$ $(m \geqslant 1)$ satisfying the linear variational problem

$$\begin{cases} \langle u'_m(t), \varphi\rangle + \lambda_1 a(u'_m(t), \varphi) + \bar{a}(t; u_m(t), \varphi) = \langle F_{1m}(t), \varphi\rangle, \\ \langle v'_m(t), \psi\rangle + \lambda_2 \langle v'_{mx}(t), \psi_x\rangle + \langle \mu_2(t) v_{mx}(t), \psi_x\rangle \\ \qquad = \displaystyle\int_0^t g(t-s)\langle v_{mx}(s), \psi_x\rangle \,\mathrm{d}s + \langle F_{2m}(t), \psi\rangle \\ \qquad\qquad \text{for all } (\varphi, \psi) \in V \times H_0^1, \text{ a.e. } t \in (0,T), \\ (u_m(0), v_m(0)) = (\widetilde{u}_0, \widetilde{v}_0), \end{cases} \tag{3.5}$$

where

$$F_{im}(x,t) = f_i[u_{m-1}, v_{m-1}](x,t), \quad i = 1,2. \tag{3.6}$$

Then we have the following theorem.

Theorem 3.1. *Let* (H_1)–(H_4) *hold. Then there exist constants* M, $T > 0$ *such that for* $(u_0, v_0) \equiv (0,0)$, *there exists a recurrent sequence* $\{(u_m, v_m)\} \subset B_T(M)$ *defined by* (3.4)–(3.6).

Proof. The proof of Theorem 3.1 consists of three steps.

Step 1. The Faedo-Galerkin approximation. Let $\{\bar{\phi}_j\}$ be a basis of H_0^1 formed by eigenfunction ϕ_j of the operator $-\Delta = -\partial^2/\partial x^2$ such as $-\Delta\bar{\phi}_j = \bar{\lambda}_j\bar{\phi}_j$, $\bar{\phi}_j \in H_0^1 \cap C^\infty([0,1])$, $\bar{\phi}_j(x) = \sqrt{2}\sin(j\pi x)$, $\bar{\lambda}_j = (j\pi)^2$, $j = 1,2,\ldots$, and let $\{\widetilde{\phi}_j\}$ be the basis of V as in Lemma 2.4. Put

$$u_m^{(k)}(t) = \sum_{j=1}^k c_{mj}^{(k)}(t)\widetilde{\phi}_j,\ \ v_m^{(k)}(t) = \sum_{j=1}^k d_{mj}^{(k)}(t)\bar{\phi}_j,$$

where the coefficients $c_{mj}^{(k)}$, $d_{mj}^{(k)}$ satisfy the system of linear integrodifferential equations

$$(3.7)\qquad \begin{cases} \langle \dot{u}_m^{(k)}(t), \widetilde{\phi}_j\rangle + \lambda_1 a(\dot{u}_m^{(k)}(t), \widetilde{\phi}_j) + \bar{a}(t; u_m^{(k)}(t), \widetilde{\phi}_j) = \langle F_{1m}(t), \widetilde{\phi}_j\rangle, \\ \langle \dot{v}_m^{(k)}(t), \bar{\phi}_j\rangle + \lambda_2\langle \dot{v}_{mx}^{(k)}(t), \bar{\phi}_{jx}\rangle + \langle \mu_2(t) v_{mx}^{(k)}(t), \bar{\phi}_{jx}\rangle \\ \qquad = \displaystyle\int_0^t g(t-s)\langle v_{mx}^{(k)}(s), \bar{\phi}_{jx}\rangle\,\mathrm{d}s + \langle F_{2m}(t), \bar{\phi}_j\rangle, \quad 1 \leqslant j \leqslant k, \\ (u_m^{(k)}(0), v_m^{(k)}(0)) = (\widetilde{u}_{0k}, \widetilde{v}_{0k}), \end{cases}$$

in which

$$(3.8)\qquad \begin{cases} \widetilde{u}_{0k} = \displaystyle\sum_{j=1}^k \alpha_j^{(k)} \widetilde{\phi}_j \to \widetilde{u}_0 \text{ strongly in } H^2 \cap V, \\ \widetilde{v}_{0k} = \displaystyle\sum_{j=1}^k \beta_j^{(k)} \bar{\phi}_j \to \widetilde{v}_0 \text{ strongly in } H^2 \cap H_0^1. \end{cases}$$

System (3.7) is written as

$$(3.9)\qquad \begin{aligned} c_{mj}^{(k)}(t) &= \alpha_j^{(k)} + \frac{1}{\sigma_{1j}} \int_0^t \langle F_{1m}(s), \widetilde{\phi}_j\rangle\,\mathrm{d}s - \frac{1}{\sigma_{1j}} \sum_{i=1}^k \int_0^t \widetilde{a}_{ij}(s) c_{mi}^{(k)}(s)\,\mathrm{d}s, \\ d_{mj}^{(k)}(t) &= \beta_j^{(k)} + \frac{1}{\sigma_{2j}} \int_0^t \langle F_{2m}(s), \bar{\phi}_j\rangle\,\mathrm{d}s - \frac{1}{\sigma_{2j}} \sum_{i=1}^k \int_0^t \bar{a}_{ij}(s) d_{mi}^{(k)}(s)\,\mathrm{d}s \\ &\quad + \frac{\bar{\lambda}_j}{\sigma_{2j}} \int_0^t \mathrm{d}\tau \int_0^\tau g(\tau - s) d_{mj}^{(k)}(s)\,\mathrm{d}s, \quad 1 \leqslant j \leqslant k, \end{aligned}$$

where

$$\begin{gathered} \sigma_{1j} = 1 + \lambda_1 \widetilde{\lambda}_j, \quad \sigma_{2j} = 1 + \lambda_2 \bar{\lambda}_j, \quad 1 \leqslant j \leqslant k, \\ \widetilde{a}_{ij}(t) = \bar{a}(t; \widetilde{\phi}_i, \widetilde{\phi}_j), \quad \bar{a}_{ij}(t) = \langle \mu_2(t) \bar{\phi}_{ix}, \bar{\phi}_{jx}\rangle, \quad 1 \leqslant i, j \leqslant k. \end{gathered}$$

Omitting the indices m and k, system (3.9) is rewritten in the form of a fixed-point equation as

$$(3.10)\qquad (c, d) = U[(c, d)],$$

where

$$
\begin{aligned}
(c,d)(t) &= (c_1(t),\dots,c_k(t);d_1(t),\dots,d_k(t)),\\
U[(c,d)](t) &= (U_1[(c,d)](t),\dots,U_{2k}[(c,d)](t)) = G(t)+L[(c,d)](t),\\
G(t) &= (G_1(t),\dots,G_{2k}(t)),\\
L[(c,d)](t) &= (L_1[(c,d)](t),\dots,L_{2k}[(c,d)](t)),\\
G_j(t) &= G_j^{(1)}(t) \equiv \alpha_j^{(k)} + \frac{1}{\sigma_{1j}}\int_0^t \langle F_{1m}(s),\widetilde{\phi}_j\rangle\,\mathrm{d}s, \quad 1\leqslant j\leqslant k,\\
G_{k+j}(t) &= G_j^{(2)}(t) \equiv \beta_j^{(k)} + \frac{1}{\sigma_{2j}}\int_0^t \langle F_{2m}(s),\bar{\phi}_j\rangle\,\mathrm{d}s, \quad 1\leqslant j\leqslant k,\\
L_j[(c,d)](t) &= L_j^{(1)}[c](t) \equiv -\frac{1}{\sigma_{1j}}\sum_{i=1}^k\int_0^t \widetilde{a}_{ij}(s)c_i(s)\,\mathrm{d}s, \quad 1\leqslant j\leqslant k,\\
L_{k+j}[(c,d)](t) &= L_j^{(2)}[d](t) \equiv -\frac{1}{\sigma_{2j}}\sum_{i=1}^k\int_0^t \bar{a}_{ij}(s)d_i(s)\,\mathrm{d}s\\
&\quad + \frac{\bar{\lambda}_j}{\sigma_{2j}}\int_0^t \mathrm{d}\tau\int_0^\tau g(\tau-s)d_j(s)\,\mathrm{d}s, \quad 1\leqslant j\leqslant k.
\end{aligned}
$$

Applying the contraction principle, system (3.10) has a unique solution (c,d) in $[0,T]$. The proof is given below.

Let $\gamma>0$ such that

$$
\theta_\gamma = \frac{1}{\gamma}\left(\frac{\widetilde{A}_{\max}}{\sigma_{11}} + \frac{\bar{A}_{\max}}{\sigma_{21}}\right) + \frac{1}{\sqrt{2\gamma}}\frac{\bar{\lambda}_k}{\sigma_{21}}T^*\|g\|_{L^2(0,T^*)} < 1,
$$

where we denote

$$
\widetilde{A}_{\max} \equiv \sup_{0\leqslant t\leqslant T}\left(\max_{1\leqslant i\leqslant k}\sum_{j=1}^k|\widetilde{a}_{ij}(t)|\right), \quad \bar{A}_{\max} \equiv \sup_{0\leqslant t\leqslant T}\left(\max_{1\leqslant i\leqslant k}\sum_{j=1}^k|\bar{a}_{ij}(t)|\right).
$$

It is well known that $X = C^0([0,T];\mathbb{R}^{2k})$ is a Banach space with respect to the norm

$$
\begin{aligned}
\|(c,d)\|_{\gamma,X} &= \sup_{0\leqslant t\leqslant T} \mathrm{e}^{-\gamma t}|(c,d)(t)|_1,\\
|(c,d)(t)|_1 &= \sum_{j=1}^k(|c_j(t)|+|d_j(t)|), \quad (c,d)\in X.
\end{aligned}
$$

Clearly, $U\colon X\to X$. We will prove that $U\colon X\to X$ is contractive as follows.

First we note that for all (c,d), $(\bar c,\bar d)\in X$, $(z,\overline{z})=(c,d)-(\bar c,\bar d)=(c-\bar c,d-\bar d)$, we have

$$\begin{aligned}U[(c,d)]-U[(\bar c,\bar d)]&=L[(c,d)]-L[(\bar c,\bar d)]\\&=L[(c,d)-(\bar c,\bar d)]=L[(z,\overline{z})]=(L^{(1)}[z];L^{(2)}[\overline{z}]),\end{aligned}$$

so

$$\begin{aligned}&|U[(c,d)](t)-U[(\bar c,\bar d)](t)|_1\\&\quad=\sum_{j=1}^{2k}|U_j[(c,d)](t)-U_j[(\bar c,\bar d)](t)|=\sum_{j=1}^{k}(|L_j^{(1)}[z](t)|+|L_j^{(2)}[\overline{z}](t)|)\\&\quad\leqslant\sum_{j=1}^{k}\left(\left|\frac{1}{\sigma_{1j}}\sum_{i=1}^{k}\int_0^t\widetilde{a}_{ij}(s)z_i(s)\,\mathrm{d}s\right|\right)\\&\qquad+\sum_{j=1}^{k}\left(\left|-\frac{1}{\sigma_{2j}}\sum_{i=1}^{k}\int_0^t\bar a_{ij}(s)\overline{z}_i(s)\,\mathrm{d}s+\frac{\bar\lambda_k}{\sigma_{2j}}\int_0^t\mathrm{d}\tau\int_0^\tau g(\tau-s)\overline{z}_j(s)\,\mathrm{d}s\right|\right)\\&\quad\leqslant\frac{1}{\sigma_{11}}\int_0^t\max_{1\leqslant i\leqslant k}\sum_{j=1}^{k}|\widetilde{a}_{ij}(s)|\sum_{i=1}^{k}|z_i(s)|\,\mathrm{d}s+\frac{1}{\sigma_{21}}\int_0^t\max_{1\leqslant i\leqslant k}\sum_{j=1}^{k}|\bar a_{ij}(s)|\sum_{i=1}^{k}|\overline{z}_i(s)|\,\mathrm{d}s\\&\qquad+\frac{\bar\lambda_k}{\sigma_{21}}\int_0^t\mathrm{d}\tau\int_0^\tau|g(\tau-s)|\left|\sum_{j=1}^{k}|\overline{z}_j(s)|\right|\mathrm{d}s\\&\quad\leqslant\left(\frac{\widetilde{A}_{\max}}{\sigma_{11}}+\frac{\bar A_{\max}}{\sigma_{21}}\right)\int_0^t|(z,\overline{z})(s)|_1\,\mathrm{d}s+\frac{\bar\lambda_k}{\sigma_{21}}\int_0^t\mathrm{d}\tau\int_0^\tau|g(\tau-s)||(z,\overline{z})(s)|_1\,\mathrm{d}s\\&\quad\leqslant\left(\frac{\widetilde{A}_{\max}}{\sigma_{11}}+\frac{\bar A_{\max}}{\sigma_{21}}\right)\int_0^t\mathrm{e}^{\gamma s}\|(z,\overline{z})\|_{\gamma,X}\,\mathrm{d}s\\&\qquad+\frac{\bar\lambda_k}{\sigma_{21}}\int_0^t\mathrm{d}\tau\int_0^\tau|g(\tau-s)|\mathrm{e}^{\gamma s}\|(z,\overline{z})\|_{\gamma,X}\,\mathrm{d}s\\&\quad\leqslant\left(\frac{\widetilde{A}_{\max}}{\sigma_{11}}+\frac{\bar A_{\max}}{\sigma_{21}}\right)\frac{\mathrm{e}^{\gamma t}}{\gamma}\|(z,\overline{z})\|_{\gamma,X}+\frac{\bar\lambda_k}{\sigma_{21}}T^*\|g\|_{L^2(0,T^*)}\frac{\mathrm{e}^{\gamma t}}{\sqrt{2\gamma}}\|(z,\overline{z})\|_{\gamma,X}\\&\quad=\theta_\gamma\mathrm{e}^{\gamma t}\|(z,\overline{z})\|_{\gamma,X}=\theta_\gamma\mathrm{e}^{\gamma t}\|(c,d)-(\bar c,\bar d)\|_{\gamma,X},\end{aligned}$$

where

$$\theta_\gamma=\frac{1}{\gamma}\left(\frac{\widetilde{A}_{\max}}{\sigma_{11}}+\frac{\bar A_{\max}}{\sigma_{21}}\right)+\frac{1}{\sqrt{2\gamma}}\frac{\bar\lambda_k}{\sigma_{21}}T^*\|g\|_{L^2(0,T^*)}.$$

It follows that

$$\mathrm{e}^{-\gamma t}|U[(c,d)](t)-U[(\bar c,\bar d)](t)|_1\leqslant\theta_\gamma\|(c,d)-(\bar c,\bar d)\|_{\gamma,X},$$

which leads to

$$\|U[(c,d)] - U[(\bar{c},\bar{d})]\|_{\gamma,X} \leqslant \theta_\gamma \|(c,d) - (\bar{c},\bar{d})\|_{\gamma,X} \quad \text{for all } (c,d), \quad (\bar{c},\bar{d}) \in X.$$

By $0 < \theta_\gamma < 1$, $U\colon X \to X$ is contractive. Then (3.10) has a unique solution $(c,d) \in X$. Thus, system (3.7) has a unique solution $(u_m^{(k)}(t), v_m^{(k)}(t))$ on interval $[0,T]$.

Step 2. A priori estimate. We put

$$\begin{aligned}(3.11)\qquad S_m^{(k)}(t) &= \|\dot{u}_m^{(k)}(t)\|_a^2 + \|\dot{v}_{mx}^{(k)}(t)\|^2 + \lambda_1 \|\Delta \dot{u}_m^{(k)}(t)\|^2 + \lambda_2 \|\Delta \dot{v}_m^{(k)}(t)\|^2 \\ &+ \bar{a}(t; u_m^{(k)}(t), u_m^{(k)}(t)) + \|\sqrt{\mu_1(t)}\Delta u_m^{(k)}(t)\|^2 \\ &+ \|\sqrt{\mu_2(t)} v_{mx}^{(k)}(t)\|^2 + \|\sqrt{\mu_2(t)}\Delta v_m^{(k)}(t)\|^2 \\ &+ 2\int_0^t [\|\dot{u}_m^{(k)}(s)\|^2 + (1+\lambda_1)\|\dot{u}_m^{(k)}(s)\|_a^2 + \lambda_1 \|\Delta \dot{u}_m^{(k)}(s)\|^2]\,\mathrm{d}s \\ &+ 2\int_0^1 [\|\dot{v}_m^{(k)}(s)\|^2 + (1+\lambda_2)\|\dot{v}_{mx}^{(k)}(s)\|^2 + \lambda_2 \|\Delta \dot{v}_m^{(k)}(s)\|^2]\,\mathrm{d}s.\end{aligned}$$

Then it follows from (3.7), (3.11) that

$$\begin{aligned}(3.12)\ \bar{\mu}_* \overline{S}_m^{(k)}(t) &\leqslant S_m^{(k)}(t) = \overline{S}_{0mk} \\ &+ \int_0^t \left[\bar{a}'(s; u_m^{(k)}(s), u_m^{(k)}(s)) + \int_0^1 \mu_1'(x,s)|\Delta u_m^{(k)}(x,s)|^2\,\mathrm{d}x\right]\mathrm{d}s \\ &+ \int_0^t \mathrm{d}s \int_0^1 \mu_2'(x,s)(|v_{mx}^{(k)}(x,s)|^2 + |\Delta v_m^{(k)}(x,s)|^2)\,\mathrm{d}x \\ &- 2g(0)\int_0^t [\|v_{mx}^{(k)}(s)\|^2 + \|\Delta v_m^{(k)}(s)\|^2]\,\mathrm{d}s \\ &+ \int_0^t g(t-s)[2\langle v_{mx}^{(k)}(s), v_{mx}^{(k)}(t)\rangle + \langle \Delta v_m^{(k)}(s), 2\Delta v_m^{(k)}(t) + \Delta \dot{v}_m^{(k)}(t)\rangle]\,\mathrm{d}s \\ &- 2\int_0^t \mathrm{d}\tau \int_0^\tau g'(\tau - s)[\langle v_{mx}^{(k)}(s), v_{mx}^{(k)}(\tau)\rangle + \langle \Delta v_m^{(k)}(s), \Delta v_m^{(k)}(\tau)\rangle]\,\mathrm{d}s \\ &- 2\langle \mu_{1x}(t) u_{mx}^{(k)}(t), \Delta u_m^{(k)}(t)\rangle + 2\int_0^t \left\langle \frac{\partial}{\partial s}(\mu_{1x}(s) u_{mx}^{(k)}(s)), \Delta u_m^{(k)}(s)\right\rangle \mathrm{d}s \\ &- 2\langle \mu_{2x}(t) v_{mx}^{(k)}(t), \Delta v_m^{(k)}(t)\rangle + 2\int_0^t \left\langle \frac{\partial}{\partial s}(\mu_{2x}(s) v_{mx}^{(k)}(s)), \Delta v_m^{(k)}(s)\right\rangle \mathrm{d}s \\ &- \left\langle \frac{\partial}{\partial x}(\mu_1(t) u_{mx}^{(k)}(t)), \Delta \dot{u}_m^{(k)}(t)\right\rangle - \left\langle \frac{\partial}{\partial x}(\mu_2(t) v_{mx}^{(k)}(t)), \Delta \dot{v}_m^{(k)}(t)\right\rangle\end{aligned}$$

$$+ 2\int_0^t [\langle F_{1m}(s), \dot u_m^{(k)}(s) - \Delta\dot u_m^{(k)}(s)\rangle + \langle F_{2m}(s), \dot v_m^{(k)}(s) - \Delta\dot v_m^{(k)}(s)\rangle]\,\mathrm{d}s$$
$$+ \langle F_{1m}(t), -\Delta\dot u_m^{(k)}(t)\rangle + \langle F_{2m}(t), -\Delta\dot v_m^{(k)}(t)\rangle$$
$$= \overline{S}_{0mk} + \sum_{j=1}^{14} I_j,$$

where $\bar\mu_* = \min\{1, \mu_{1*}, \mu_{2*}, \lambda_1, \lambda_2\}$,

$$\overline{S}_m^{(k)}(t) = \|u_m^{(k)}(t)\|^2_{H^2\cap V} + \|v_m^{(k)}(t)\|^2_{H^2\cap H^1_0} + \|\dot u_m^{(k)}(t)\|^2_{H^2\cap V} + \|\dot v_m^{(k)}(t)\|^2_{H^2\cap H^1_0} + 2\int_0^t (\|\dot u_m^{(k)}(s)\|^2_{H^2\cap V} + \|\dot v_m^{(k)}(s)\|^2_{H^2\cap H^1_0})\,\mathrm{d}s, \tag{3.13}$$
$$\overline{S}_{0mk} = S_{0mk} + 2\langle \mu_{1x}(0)\widetilde u_{0kx}, \Delta\widetilde u_{0k}\rangle + 2\langle \mu_{2x}(0)\widetilde v_{0kx}, \Delta\widetilde v_{0k}\rangle,$$
$$S_{0mk} = \bar a(0;\widetilde u_{0k}, \widetilde u_{0k}) + \|\sqrt{\mu_2(0)}\widetilde v_{0kx}\|^2 + \|\sqrt{\mu_1(0)}\Delta\widetilde u_{0k}\|^2 + \|\sqrt{\mu_2(0)}\Delta\widetilde v_{0k}\|^2.$$

Using the inequality $2ab \leqslant \beta a^2 + \beta^{-1}b^2$ for all $a, b \in \mathbb{R}$ with $\beta = \bar\mu_*/12$ and the inequalities

$$|\mu_i'(x,s)| \leqslant \widetilde K_i \leqslant \widetilde K, \quad i = 1,2,$$
$$\bar a'(t; u_m^{(k)}(t), u_m^{(k)}(t)) \leqslant \widetilde K_1 \|u_m^{(k)}(t)\|_a^2,$$
$$\begin{aligned}
&2\langle v_{mx}^{(k)}(s), v_{mx}^{(k)}(t)\rangle + \langle \Delta v_m^{(k)}(s), 2\Delta v_m^{(k)}(t) + \Delta\dot v_m^{(k)}(t)\rangle\\
&\quad\leqslant 2[\|v_{mx}^{(k)}(s)\|\|v_{mx}^{(k)}(t)\| + \|\Delta v_m^{(k)}(s)\|(\|\Delta v_m^{(k)}(t)\| + \|\Delta\dot v_m^{(k)}(t)\|)]\\
&\quad\leqslant 2(\|v_{mx}^{(k)}(s)\|^2 + \|\Delta v_m^{(k)}(s)\|^2 + \|\Delta v_m^{(k)}(s)\|^2)^{1/2}\\
&\qquad\times (\|v_{mx}^{(k)}(t)\|^2 + \|\Delta v_m^{(k)}(t)\|^2 + \|\Delta\dot v_m^{(k)}(t)\|^2)^{1/2}\\
&\quad\leqslant 2(\|v_m^{(k)}(s)\|^2_{H^2\cap H^1_0} + \|v_m^{(k)}(s)\|^2_{H^2\cap H^1_0})^{1/2}\\
&\qquad\times (\|v_m^{(k)}(s)\|^2_{H^2\cap H^1_0} + \|\dot v_m^{(k)}(t)\|^2_{H^2\cap H^1_0})^{1/2}\\
&\quad\leqslant 2\sqrt2\sqrt{\overline{S}_m^{(k)}(s)}\sqrt{\overline{S}_m^{(k)}(t)},
\end{aligned}$$
$$|\langle v_{mx}^{(k)}(s), v_{mx}^{(k)}(\tau)\rangle + \langle \Delta v_m^{(k)}(s), \Delta v_m^{(k)}(\tau)\rangle| \leqslant \sqrt{\overline{S}_m^{(k)}(s)}\sqrt{\overline{S}_m^{(k)}(\tau)},$$

we shall estimate respectively the terms I_1–I_5 on the right-hand side of (3.12) as follows:

$$\begin{aligned}
I_1 + I_2 &= \int_0^t \Big[\bar a'(s; u_m^{(k)}(s), u_m^{(k)}(s)) + \int_0^1 \mu_1'(x,s)|\Delta u_m^{(k)}(x,s)|^2\,\mathrm{d}x\Big]\,\mathrm{d}s\\
&\quad + \int_0^t \mathrm{d}s\int_0^1 \mu_2'(x,s)(|v_{mx}^{(k)}(x,s)|^2 + |\Delta v_m^{(k)}(x,s)|^2)\,\mathrm{d}x\\
&\leqslant \widetilde K\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s,
\end{aligned}\tag{3.14}$$

$$
\begin{aligned}
I_3 = & -2g(0)\int_0^t [\|v_{mx}^{(k)}(s)\|^2 + \|\Delta v_m^{(k)}(s)\|^2]\,\mathrm{d}s \leqslant 2|g(0)|\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s,\\
I_4 = & \int_0^t g(t-s)[2\langle v_{mx}^{(k)}(s), v_{mx}^{(k)}(t)\rangle + \langle \Delta v_m^{(k)}(s), 2\Delta v_m^{(k)}(t) + \Delta \dot{v}_m^{(k)}(t)\rangle]\,\mathrm{d}s\\
& \leqslant 2\sqrt{2}\int_0^t |g(t-s)|\sqrt{\overline{S}_m^{(k)}(s)}\sqrt{\overline{S}_m^{(k)}(t)}\,\mathrm{d}s\\
& \leqslant \varepsilon \overline{S}_m^{(k)}(t) + \frac{2}{\varepsilon}\left(\int_0^t |g(t-s)|\sqrt{\overline{S}_m^{(k)}(s)}\,\mathrm{d}s\right)^2\\
& \leqslant \varepsilon \overline{S}_m^{(k)}(t) + \frac{2}{\varepsilon}\|g\|_{L^2(0,T^*)}^2 \int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s,\\
I_5 = & -2\int_0^t \mathrm{d}\tau \int_0^\tau g'(\tau-s)[\langle v_{mx}^{(k)}(s), v_{mx}^{(k)}(\tau)\rangle + \langle \Delta v_m^{(k)}(s), \Delta v_m^{(k)}(\tau)\rangle]\,\mathrm{d}s\\
& \leqslant 2\int_0^t \mathrm{d}\tau \int_0^\tau |g'(\tau-s)|\sqrt{\overline{S}_m^{(k)}(s)}\sqrt{\overline{S}_m^{(k)}(\tau)}\,\mathrm{d}s\\
& \leqslant 2\sqrt{T^*}\|g'\|_{L^2(0,T^*)}\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s.
\end{aligned}
$$

In order to estimate the terms I_6–I_{14}, we need the following properties. Let us omit the details of proof.

Lemma 3.1. *The following estimates are fulfilled:*

(i) $\left\|\dfrac{\partial}{\partial t}(\mu_{1x}(t)u_{mx}^{(k)}(t))\right\| \leqslant \sqrt{2}\widetilde{K}\sqrt{\overline{S}_m^{(k)}(t)}$,

(ii) $\left\|\dfrac{\partial}{\partial t}(\mu_{2x}(t)v_{mx}^{(k)}(t))\right\| \leqslant \sqrt{2}\widetilde{K}\sqrt{\overline{S}_m^{(k)}(t)}$,

(iii) $\|\mu_{1x}(t)u_{mx}^{(k)}(t)\| \leqslant \|\mu_{1x}(0)\widetilde{u}_{0kx}\| + \sqrt{2}\widetilde{K}\displaystyle\int_0^t \sqrt{\overline{S}_m^{(k)}(s)}\,\mathrm{d}s$,

(iv) $\|\mu_{2x}(t)v_{mx}^{(k)}(t)\| \leqslant \|\mu_{2x}(0)\widetilde{v}_{0kx}\| + \sqrt{2}\widetilde{K}\displaystyle\int_0^t \sqrt{\overline{S}_m^{(k)}(s)}\,\mathrm{d}s$,

(v) $\left\|\dfrac{\partial^2}{\partial x\partial t}(\mu_1(t)u_{mx}^{(k)}(t))\right\| \leqslant 2\widetilde{K}\sqrt{\overline{S}_m^{(k)}(t)}$,

(vi) $\left\|\dfrac{\partial^2}{\partial x\partial t}(\mu_2(t)v_{mx}^{(k)}(t))\right\| \leqslant 2\widetilde{K}\sqrt{\overline{S}_m^{(k)}(t)}$,

(vii) $\left\|\dfrac{\partial}{\partial x}(\mu_1(t)u_{mx}^{(k)}(t))\right\| \leqslant \left\|\dfrac{\partial}{\partial x}(\mu_1(0)\widetilde{u}_{0kx})\right\| + 2\widetilde{K}\displaystyle\int_0^t \sqrt{\overline{S}_m^{(k)}(s)}\,\mathrm{d}s$,

(viii) $\left\|\dfrac{\partial}{\partial x}(\mu_2(t)v_{mx}^{(k)}(t))\right\| \leqslant \left\|\dfrac{\partial}{\partial x}(\mu_2(0)\widetilde{v}_{0kx})\right\| + 2\widetilde{K}\displaystyle\int_0^t \sqrt{\overline{S}_m^{(k)}(s)}\,\mathrm{d}s$,

(ix) $\|F_{im}(t)\| \leqslant 4\sigma M + \|f_i(\cdot,0,\widetilde{u}_0,\widetilde{v}_0,0,0,\widetilde{u}_{0x},\widetilde{v}_{0x},0,0)\| + T(1+4M)\overline{K}_M,\ i=1,2.$

Using Lemma 3.1, we have

$$
\begin{aligned}
(3.15)\quad I_6+I_7 = & -2\langle \mu_{1x}(t)u_{mx}^{(k)}(t), \Delta u_m^{(k)}(t)\rangle \\
& + 2\int_0^t \left\langle \frac{\partial}{\partial s}(\mu_{1x}(s)u_{mx}^{(k)}(s)), \Delta u_m^{(k)}(s)\right\rangle \mathrm{d}s \\
\leqslant & \, 2\|\mu_{1x}(t)u_{mx}^{(k)}(t)\|\sqrt{\overline{S}_m^{(k)}(t)} + 2\int_0^t \left\|\frac{\partial}{\partial s}(\mu_{1x}(s)u_{mx}^{(k)}(s))\right\|\sqrt{\overline{S}_m^{(k)}(s)}\,\mathrm{d}s \\
\leqslant & \, \varepsilon\overline{S}_m^{(k)}(t) + \frac{1}{\varepsilon}\|\mu_{1x}(t)u_{mx}^{(k)}(t)\|^2 + 2\sqrt{2}\widetilde{K}\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s \\
\leqslant & \, \varepsilon\overline{S}_m^{(k)}(t) + \frac{2}{\varepsilon}\left(\|\mu_{1x}(0)\widetilde{u}_{0kx}\|^2 + 2\widetilde{K}^2T^*\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s\right) \\
& + 2\sqrt{2}\widetilde{K}\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s \\
= & \, \varepsilon\overline{S}_m^{(k)}(t) + \frac{2}{\varepsilon}\|\mu_{1x}(0)\widetilde{u}_{0kx}\|^2 + 2\sqrt{2}\left(1+\frac{1}{\varepsilon}\sqrt{2}\widetilde{K}T^*\right)\widetilde{K}\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s, \\
I_8+I_9 = & -2\langle \mu_{2x}(t)v_{mx}^{(k)}(t), \Delta v_m^{(k)}(t)\rangle \\
& + 2\int_0^t \left\langle \frac{\partial}{\partial s}(\mu_{2x}(s)v_{mx}^{(k)}(s)), \Delta v_m^{(k)}(s)\right\rangle \mathrm{d}s \\
\leqslant & \, \varepsilon\overline{S}_m^{(k)}(t) + \frac{2}{\varepsilon}\|\mu_{1x}(0)\widetilde{u}_{0kx}\|^2 + 2\sqrt{2}\left(1+\frac{1}{\varepsilon}\sqrt{2}\widetilde{K}T^*\right)\widetilde{K}\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s, \\
I_{10}+I_{11} = & -\left\langle \frac{\partial}{\partial x}(\mu_1(t)u_{mx}^{(k)}(t)), \Delta \dot{u}_m^{(k)}(t)\right\rangle - \left\langle \frac{\partial}{\partial x}(\mu_2(t)v_{mx}^{(k)}(t)), \Delta \dot{v}_m^{(k)}(t)\right\rangle \\
\leqslant & \left\|\frac{\partial}{\partial x}(\mu_1(t)u_{mx}^{(k)}(t))\right\|\sqrt{\overline{S}_m^{(k)}(t)} + \left\|\frac{\partial}{\partial x}(\mu_2(t)v_{mx}^{(k)}(t))\right\|\sqrt{\overline{S}_m^{(k)}(t)} \\
\leqslant & \, \varepsilon\overline{S}_m^{(k)}(t) + \frac{2}{\varepsilon}\left(\left\|\frac{\partial}{\partial x}(\mu_1(t)u_{mx}^{(k)}(t))\right\|^2 + \left\|\frac{\partial}{\partial x}(\mu_2(t)v_{mx}^{(k)}(t))\right\|^2\right) \\
\leqslant & \, \varepsilon\overline{S}_m^{(k)}(t) + \frac{2}{\varepsilon}\left(\left\|\frac{\partial}{\partial x}(\mu_1(0)\widetilde{u}_{0kx})\right\| + 2\widetilde{K}\int_0^t \sqrt{\overline{S}_m^{(k)}(s)}\,\mathrm{d}s\right)^2 \\
& + \frac{2}{\varepsilon}\left(\left\|\frac{\partial}{\partial x}(\mu_2(0)\widetilde{v}_{0kx})\right\| + 2\widetilde{K}\int_0^t \sqrt{\overline{S}_m^{(k)}(s)}\,\mathrm{d}s\right)^2 \\
\leqslant & \, \varepsilon\overline{S}_m^{(k)}(t) + \frac{4}{\varepsilon}\left(\left\|\frac{\partial}{\partial x}(\mu_1(0)\widetilde{u}_{0kx})\right\|^2 + 4\widetilde{K}^2T^*\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s\right) \\
& + \frac{4}{\varepsilon}\left(\left\|\frac{\partial}{\partial x}(\mu_2(0)\widetilde{v}_{0kx})\right\|^2 + 4\widetilde{K}^2T^*\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s\right) \\
= & \, \varepsilon\overline{S}_m^{(k)}(t) + \frac{4}{\varepsilon}\left(\left\|\frac{\partial}{\partial x}(\mu_1(0)\widetilde{u}_{0kx})\right\|^2 + \left\|\frac{\partial}{\partial x}(\mu_2(0)\widetilde{v}_{0kx})\right\|^2\right) \\
& + \frac{32}{\varepsilon}\widetilde{K}^2T^*\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
I_{12} &= 2\int_0^t [\langle F_{1m}(s), \dot{u}_m^{(k)}(s) - \Delta\dot{u}_m^{(k)}(s)\rangle + \langle F_{2m}(s), \dot{v}_m^{(k)}(s) - \Delta\dot{v}_m^{(k)}(s)\rangle]\,\mathrm{d}s \\
&\leqslant 2\int_0^t [\|F_{1m}(s)\|(\|\dot{u}_m^{(k)}(s)\| + \|\Delta\dot{u}_m^{(k)}(s)\|) \\
&\quad + \|F_{2m}(s)\|(\|\dot{v}_m^{(k)}(s)\| + \|\Delta\dot{v}_m^{(k)}(s)\|)]\,\mathrm{d}s \\
&\leqslant 4\overline{K}_M \int_0^t \sqrt{\overline{S}_m^{(k)}(s)}\,\mathrm{d}s \leqslant 4T\overline{K}_M^2 + \int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s, \\
I_{13} &= \langle F_{1m}(t), -\Delta\dot{u}_m^{(k)}(t)\rangle \leqslant \|F_{1m}(t)\|\|\Delta\dot{u}_m^{(k)}(t)\| \\
&\leqslant \varepsilon\overline{S}_m^{(k)}(t) + \frac{1}{4\varepsilon}\|F_{1m}(t)\|^2 \\
&\leqslant \varepsilon\overline{S}_m^{(k)}(t) + \frac{1}{4\varepsilon}(4\sigma M + \|f_1(\cdot,0,\widetilde{u}_0,\widetilde{v}_0,0,0,\widetilde{u}_{0x},\widetilde{v}_{0x},0,0)\| \\
&\quad + T(1+4M)\overline{K}_M)^2 \\
&\leqslant \varepsilon\overline{S}_m^{(k)}(t) + \frac{12}{\varepsilon}\sigma^2 M^2 + \frac{3}{4\varepsilon}\|f_1(\cdot,0,\widetilde{u}_0,\widetilde{v}_0,0,0,\widetilde{u}_{0x},\widetilde{v}_{0x},0,0)\|^2 \\
&\quad + \frac{3}{4\varepsilon}T^2(1+4M)^2\overline{K}_M^2, \\
I_{14} &= \langle F_{2m}(t), -\Delta\dot{v}_m^{(k)}(t)\rangle \\
&\leqslant \varepsilon\overline{S}_m^{(k)}(t) + \frac{12}{\varepsilon}\sigma^2 M^2 + \frac{3}{4\varepsilon}\|f_2(\cdot,0,\widetilde{u}_0,\widetilde{v}_0,0,0,\widetilde{u}_{0x},\widetilde{v}_{0x},0,0)\|^2 \\
&\quad + \frac{3}{4\varepsilon}T^2(1+4M)^2\overline{K}_M^2.
\end{aligned}
$$

Combining (3.12), (3.14), and (3.15) with $\varepsilon = \bar{\mu}_*/12$ leads to

$$
\overline{S}_m^{(k)}(t) \leqslant \widehat{S}_{0mk} + \overline{\sigma}_* M^2 + TD_1(M) + D_2(M)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s, \tag{3.16}
$$

with $\overline{\sigma}_* = 576\sigma^2/\bar{\mu}_*^2$ and

$$
\begin{aligned}
\widehat{S}_{0mk} &= \frac{2}{\bar{\mu}_*}\overline{S}_{0mk} + \frac{18}{\bar{\mu}_*^2}\sum_{i=1}^2 \|f_i(\cdot,0,\widetilde{u}_0,\widetilde{v}_0,0,0,\widetilde{u}_{0x},\widetilde{v}_{0x},0,0)\|^2 \\
&\quad + \frac{48}{\bar{\mu}_*^2}(\|\mu_{1x}(0)\widetilde{u}_{0kx}\|^2 + \|\mu_{1x}(0)\widetilde{u}_{0kx}\|^2) \\
&\quad + \frac{96}{\bar{\mu}_*^2}\Big(\Big\|\frac{\partial}{\partial x}(\mu_1(0)\widetilde{u}_{0kx})\|^2 + \Big\|\frac{\partial}{\partial x}(\mu_2(0)\widetilde{v}_{0kx})\Big\|^2\Big),
\end{aligned} \tag{3.17}
$$

$$
\begin{aligned}
D_1(M) &= \frac{2}{\bar{\mu}_*}\Big(4 + \frac{18}{\bar{\mu}_*}T^*(1+4M)^2\Big)\overline{K}_M^2, \\
D_2(M) &= \frac{2}{\bar{\mu}_*}\Big[1 + \widetilde{K} + 4\sqrt{2}\Big(1 + \frac{12}{\bar{\mu}_*}\sqrt{2}\widetilde{K}T^*\Big)\widetilde{K} + 2|g(0)| + 2\sqrt{T^*}\|g'\|_{L^2(0,T^*)}\Big] \\
&\quad + \frac{48}{\bar{\mu}_*^2}[\|g\|_{L^2(0,T^*)}^2 + 16\widetilde{K}^2 T^*].
\end{aligned}
$$

Note that $0 < \sigma < \bar{\mu}_*/511$, which yields

$$\overline{\sigma}_* = \frac{576\sigma^2}{\bar{\mu}_*^2} \leqslant \frac{576}{\bar{\mu}_*^2}\left(\frac{\bar{\mu}_*}{511}\right)^2 = \frac{576}{(511)^2} < 1. \tag{3.18}$$

The convergence given in (3.8) implies that there exists a constant $M > 0$ independent of k and m such that

$$\widehat{S}_{0mk} \leqslant \frac{1-\overline{\sigma}_*}{2} M^2 \quad \text{for all } m,\ k \in \mathbb{N}. \tag{3.19}$$

In the case of $0 < \sigma < \bar{\mu}_*/511$, we have the following lemma.

Lemma 3.2. *For every $T \in (0, T^*]$ we put*

$$k_T = 4\sqrt{\Delta^{(1)}(M,T,\sigma)}\exp(T\Delta^{(2)}(M,\sigma)), \tag{3.20}$$

where

$$\Delta^{(1)}(M,T,\sigma) = \frac{2}{\bar{\mu}_* - \sigma}\left(\frac{32}{\bar{\mu}_* - \sigma}T\overline{K}_M^2 + 16\sigma\right),$$
$$\Delta^{(2)}(M,\sigma) = \frac{1}{\bar{\mu}_* - \sigma}\left(\widetilde{K} + 2|g(0)| + 2\sqrt{T^*}\|g'\|_{L^2(0,T^*)} + \frac{4\|g\|^2_{L^2(0,T^*)}}{\bar{\mu}_* - \sigma}\right).$$

Then we can choose $T \in (0, T^*]$ such that

$$\begin{aligned} &\text{(i)} \quad \left(\frac{1+\overline{\sigma}_*}{2}M^2 + TD_1(M)\right)\exp(TD_2(M)) \leqslant M^2, \\ &\text{(ii)} \quad k_T < 1. \end{aligned} \tag{3.21}$$

Proof. Note that $0 < \sigma < \bar{\mu}_*/511$ is equivalent to $\sqrt{512\sigma/(\bar{\mu}_* - \sigma)} < 1$. We have

$$\lim_{T\to 0_+}\left(\frac{1+\overline{\sigma}_*}{2}M^2 + TD_1(M)\right)\exp(TD_2(M)) = \frac{1+\overline{\sigma}_*}{2}M^2 < M^2,$$

and

$$\lim_{T\to 0_+} k_T = 4\lim_{T\to 0_+}\sqrt{\Delta^{(1)}(M,T,\sigma)}\exp(T\Delta^{(2)}(M,\sigma)) = \sqrt{\frac{512\sigma}{\bar{\mu}_* - \sigma}} < 1.$$

Therefore, Lemma 3.2 is proved. □

By (3.16), (3.19) and $(3.21)_1$, we obtain

$$\overline{S}_m^{(k)}(t) \leqslant M^2 \mathrm{e}^{-TD_2(M)} + D_2(M) \int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s. \tag{3.22}$$

By using Gronwall's Lemma, we deduce from (3.22) that

$$\overline{S}_m^{(k)}(t) \leqslant M^2 \mathrm{e}^{-TD_2(M)} \mathrm{e}^{tD_2(M)} \leqslant M^2 \tag{3.23}$$

for all $t \in [0,T]$, for all m, $k \in \mathbb{N}$. Therefore, we have

$$(u_m^{(k)}, v_m^{(k)}) \in B_T(M) \quad \text{for all } m \quad \text{and} \quad k \in \mathbb{N}. \tag{3.24}$$

Step 3. Limiting process. From (3.24) we deduce the existence of a subsequence of $\{(u_m^{(k)}, v_m^{(k)})\}$, which we still denote by $\{(u_m^{(k)}, v_m^{(k)})\}$, such that

$$\begin{cases} (u_m^{(k)}, v_m^{(k)}) \to (u_m, v_m) & \text{in } L^\infty(0,T;(H^2\cap V)\times(H^2\cap V)) \text{ weak*}, \\ (\dot{u}_m^{(k)}, \dot{v}_m^{(k)}) \to (u_m', v_m') & \text{in } L^\infty(0,T;(H^2\cap V)\times(H^2\cap H_0^1)) \text{ weak*}, \\ (u_m, v_m) \in B_T(M). \end{cases} \tag{3.25}$$

Passing to the limit in (3.7), we have (u_m, v_m) satisfying (3.5), (3.6) in $L^2(0,T)$. This will be checked as follows:

Multiplying both sides in $(3.7)_1$ and $(3.7)_2$ by $\varphi \in C_c^\infty(0,T)$, then integrating with respect to time, we obtain

$$\begin{aligned} &\int_0^T \langle \dot{u}_m^{(k)}(t), \widetilde{\phi}_j\rangle \varphi(t)\,\mathrm{d}t + \lambda_1 \int_0^T a(\dot{u}_m^{(k)}(t), \widetilde{\phi}_j)\varphi(t)\,\mathrm{d}t \\ &\qquad + \int_0^T \bar{a}(t; u_m^{(k)}(t), \widetilde{\phi}_j)\varphi(t)\,\mathrm{d}t \\ &\qquad = \int_0^T \langle F_{1m}(t), \widetilde{\phi}_j\rangle \varphi(t)\,\mathrm{d}t, \\ &\int_0^T \langle \dot{v}_m^{(k)}(t), \bar{\phi}_j\rangle \varphi(t)\,\mathrm{d}t + \lambda_2 \int_0^T \langle \dot{v}_{mx}^{(k)}(t), \bar{\phi}_{jx}\rangle \varphi(t)\,\mathrm{d}t \\ &\qquad + \int_0^T \langle \mu_2(t) v_{mx}^{(k)}(t), \bar{\phi}_{jx}\rangle \varphi(t)\,\mathrm{d}t \\ &\qquad = \int_0^T \Big(\int_0^t g(t-s)\langle v_{mx}^{(k)}(s), \bar{\phi}_{jx}\rangle\,\mathrm{d}s\Big)\varphi(t)\,\mathrm{d}t \\ &\qquad + \int_0^T \langle F_{2m}(t), \bar{\phi}_j\rangle \varphi(t)\,\mathrm{d}t, \quad 1 \leqslant j \leqslant k. \end{aligned} \tag{3.26}$$

By $(3.25)_{1,2}$, we obtain

$$
\begin{aligned}
(3.27)\qquad &\int_0^T \bar{a}(t;u_m^{(k)}(t),\widetilde{\phi}_j)\varphi(t)\,\mathrm{d}t \to \int_0^T \bar{a}(t;u_m(t),\widetilde{\phi}_j)\varphi(t)\,\mathrm{d}t,\\
&\int_0^T \langle \mu_2(t)v_{mx}^{(k)}(t),\bar{\phi}_{jx}\rangle\varphi(t)\,\mathrm{d}t \to \int_0^T \langle \mu_2(t)v_{mx}(t),\bar{\phi}_{jx}\rangle\varphi(t)\,\mathrm{d}t,\\
&\int_0^T \langle \dot{u}_m^{(k)}(t),\widetilde{\phi}_j\rangle\varphi(t)\,\mathrm{d}t \to \int_0^T \langle u_m'(t),\widetilde{\phi}_j\rangle\varphi(t)\,\mathrm{d}t,\\
&\int_0^T \langle \dot{v}_m^{(k)}(t),\bar{\phi}_j\rangle\varphi(t)\,\mathrm{d}t \to \int_0^T \langle v_m'(t),\bar{\phi}_j\rangle\varphi(t)\,\mathrm{d}t,\\
&\lambda_1\int_0^T a(\dot{u}_m^{(k)}(t),\widetilde{\phi}_j)\varphi(t)\,\mathrm{d}t \to \lambda_1\int_0^T a(u_m'(t),\widetilde{\phi}_j)\varphi(t)\,\mathrm{d}t,\\
&\lambda_2\int_0^T \langle \dot{v}_{mx}^{(k)}(t),\bar{\phi}_{jx}\rangle\varphi(t)\,\mathrm{d}t \to \lambda_2\int_0^T \langle v_{mx}'(t),\bar{\phi}_{jx}\rangle\varphi(t)\,\mathrm{d}t.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\int_0^T\Big(\int_0^t g(t-s)\langle v_{mx}^{(k)}(s),\bar{\phi}_{jx}\rangle\,\mathrm{d}s\Big)\varphi(t)\,\mathrm{d}t\\
&\qquad=\int_0^T \langle v_{mx}^{(k)}(s),\bar{\phi}_{jx}\int_s^T g(t-s)\varphi(t)\,\mathrm{d}t\rangle\,\mathrm{d}s\\
&\qquad=\int_0^T \langle v_{mx}^{(k)}(s),\Phi(s)\rangle\,\mathrm{d}s,
\end{aligned}
$$

where $\Phi(s)=\bar{\phi}_{jx}\int_s^T g(t-s)\varphi(t)\,\mathrm{d}t$ and $\Phi\in L^1(0,T;(H^2\cap V)')$. Therefore,

$$
\begin{aligned}
(3.28)\qquad &\int_0^T\Big(\int_0^t g(t-s)\langle v_{mx}^{(k)}(s),\bar{\phi}_{jx}\rangle\,\mathrm{d}s\Big)\varphi(t)\,\mathrm{d}t\\
&\qquad=\int_0^T \langle v_{mx}^{(k)}(s),\Phi(s)\rangle\,\mathrm{d}s \to \int_0^T \langle v_{mx}(s),\Phi(s)\rangle\,\mathrm{d}s\\
&\qquad=\int_0^T\Big(\int_0^t g(t-s)\langle v_{mx}(s),\bar{\phi}_{jx}\rangle\,\mathrm{d}s\Big)\varphi(t)\,\mathrm{d}t.
\end{aligned}
$$

It follows from (3.26), (3.27) and (3.28) that (3.5), (3.6) holds.

Theorem 3.1 is proved. □

Based on Theorem 3.1 and the compact imbedding theorems, in the following we prove the existence and uniqueness of a weak local solution of Problem (1.1)–(1.3). We first note that the space

$$W_1(T)=\{(u,v)\in C^0([0,T];V\times H_0^1)\colon\ (u',v')\in L^2(0,T;V\times H_0^1)\}$$

is a Banach space with respect to the norm (see Lions [15])

$$\|(u,v)\|_{W_1(T)} = \|(u,v)\|_{C^0([0,T];V\times H_0^1)} + \|(u',v')\|_{L^2(0,T;V\times H_0^1)}.$$

Theorem 3.2. *Let* (H_1)–(H_4) *hold. Then the recurrent sequence* $\{(u_m,v_m)\}$ *defined by* (3.4)–(3.6) *converges strongly to a function* (u,v) *in* $W_1(T)$ *and* $(u,v)\in B_T(M)$ *is the unique weak solution of Problem* (1.1)–(1.3). *Moreover, we have the estimate*

$$\|(u_m,v_m)-(u,v)\|_{W_1(T)} \leqslant C_T k_T^m \quad \text{for all } m\in\mathbb{N},$$

where $k_T\in[0,1)$ *is defined by* (3.20) *and* C_T *is a constant depending only on* T, f_1, f_2, g, μ_1, μ_2, $\widetilde{u}_0$, $\widetilde{v}_0$ *and* k_T.

P r o o f. First, we prove the local existence of Problem (1.1)–(1.3). We need to prove that $\{(u_m,v_m)\}$ is a Cauchy sequence in $W_1(T)$. Let $\bar{u}_m = u_{m+1}-u_m$, $\overline{v}_m = v_{m+1}-v_m$. Then $(\bar{u}_m,\overline{v}_m)$ satisfies the variational problem

$$(3.29)\qquad \begin{cases} \langle \bar{u}_m'(t),\varphi\rangle + \lambda_1 a(\bar{u}_m'(t),\varphi) + \bar{a}(t;\bar{u}_m(t),\varphi) = \langle \overline{F}_{1m}(t),\varphi\rangle,\\ \langle \overline{v}_m'(t),\psi\rangle + \lambda_2\langle \overline{v}_{mx}'(t),\psi_x\rangle + \langle \mu_2(t)\overline{v}_{mx}(t),\psi_x\rangle \\ \qquad = \displaystyle\int_0^t g(t-s)\langle \overline{v}_{mx}(s),\psi_x\rangle\,\mathrm{d}s \\ \qquad\quad + \langle \overline{F}_{2m}(t),\psi\rangle \text{ for all } (\varphi,\psi)\in V\times H_0^1, \text{ a.e. } t\in(0,T),\\ (\bar{u}_m(0),\overline{v}_m(0)) = (0,0), \end{cases}$$

with

$$\overline{F}_{im}(t) = F_{im+1}(t) - F_{im}(t), \quad i=1,2.$$

Taking $(\varphi,\psi) = (\bar{u}_m'(t),\overline{v}_m'(t))$ in $(3.29)_{1,2}$ and then integrating in t, we have

(3.30)

$$\begin{aligned} \bar{\mu}_*\overline{S}_m(t) \leqslant{}& \int_0^t \Big[\bar{a}'(s;\bar{u}_m(s),\bar{u}_m(s)) + \int_0^1 \mu_2'(x,s)\overline{v}_{mx}^2(x,s)\,\mathrm{d}x - 2g(0)\|\overline{v}_{mx}(s)\|^2\Big]\,\mathrm{d}s \\ &+ 2\int_0^t g(t-s)\langle \overline{v}_{mx}(s),\overline{v}_{mx}(t)\rangle\,\mathrm{d}s - 2\int_0^t \mathrm{d}\tau \\ &\times \int_0^\tau g'(\tau-s)\langle \overline{v}_{mx}(s),\overline{v}_{mx}(\tau)\rangle\,\mathrm{d}s \\ &+ 2\int_0^t [\langle \overline{F}_{1m}(s),\bar{u}_m'(s)\rangle + \langle \overline{F}_{2m}(s),\overline{v}_m'(s)\rangle]\,\mathrm{d}s \\ ={}& \sum_{j=1}^4 \overline{I}_j, \end{aligned}$$

where $\bar{\mu}_* = \min\{1, \mu_{1*}, \mu_{2*}, \lambda_1, \lambda_2\}$ and

$$\overline{S}_m(t) = \|\bar{u}_m(t)\|_a^2 + \|\overline{v}_{mx}(t)\|^2 + \int_0^t (\|\bar{u}_m'(s)\|^2 + \|\overline{v}_m'(s)\|^2 + \|\bar{u}_m'(s)\|_a^2 + \|\overline{v}_{mx}'(s)\|^2)\,\mathrm{d}s. \tag{3.31}$$

Next, with $\gamma = (\bar{\mu}_* - \sigma)/4$, we will estimate the integrals on the right-hand side of (3.31) as follows:

$$\begin{aligned}
\overline{I}_1 &= \int_0^t \Big[\bar{a}'(s; \bar{u}_m(s), \bar{u}_m(s)) + \int_0^1 \mu_2'(x,s)\overline{v}_{mx}^2(x,s)\,\mathrm{d}x - 2g(0)\|\overline{v}_{mx}(s)\|^2\Big]\,\mathrm{d}s \\
&\leqslant (\widetilde{K} + 2|g(0)|)\int_0^t \overline{S}_m(s)\,\mathrm{d}s, \\
\overline{I}_2 &= 2\int_0^t g(t-s)\langle \overline{v}_{mx}(s), \overline{v}_{mx}(t)\rangle\,\mathrm{d}s \\
&\leqslant 2\int_0^t |g(t-s)|\sqrt{\overline{S}_m(s)}\sqrt{\overline{S}_m(t)}\,\mathrm{d}s \\
&\leqslant \gamma\overline{S}_m(t) + \frac{1}{\gamma}\Big(\int_0^t |g(t-s)|\sqrt{\overline{S}_m(s)}\,\mathrm{d}s\Big)^2 \\
&\leqslant \gamma\overline{S}_m(t) + \frac{1}{\gamma}\|g\|_{L^2(0,T^*)}^2\int_0^t \overline{S}_m(s)\,\mathrm{d}s, \\
\overline{I}_3 &= -2\int_0^t \mathrm{d}\tau\int_0^\tau g'(\tau-s)\langle \overline{v}_{mx}(s), \overline{v}_{mx}(\tau)\rangle\,\mathrm{d}s \\
&\leqslant 2\int_0^t \mathrm{d}\tau\int_0^\tau |g'(\tau-s)|\sqrt{\overline{S}_m(s)}\sqrt{\overline{S}_m(\tau)}\,\mathrm{d}s \\
&\leqslant 2\sqrt{T^*}\|g'\|_{L^2(0,T^*)}\int_0^t \overline{S}_m(s)\,\mathrm{d}s.
\end{aligned}$$

With the integral $\overline{I}_4$, applying the mean value theorem to the function f_1, we get

$$\begin{aligned}
\overline{F}_{1m}(t) &= F_{1m+1}(t) - F_{1m}(t) \\
&= D_3f[\overline{z}_m^*](t)\bar{u}_{m-1}(t) + D_4f[\overline{z}_m^*](t)\overline{v}_{m-1}(t) \\
&\quad + D_5f[\overline{z}_m^*](t)\bar{u}_{m-1}'(t) + D_6f[\overline{z}_m^*](t)\overline{v}_{m-1}'(t) \\
&\quad + D_7f[\overline{z}_m^*](t)\nabla\bar{u}_{m-1}(t) + D_8f[\overline{z}_m^*](t)\nabla\overline{v}_{m-1}(t) \\
&\quad + D_9f[\overline{z}_m^*](t)\nabla\bar{u}_{m-1}'(t) + D_{10}f[\overline{z}_m^*](t)\nabla\overline{v}_{m-1}'(t),
\end{aligned}$$

where

$$\begin{aligned}
\overline{z}_m^* = x, t, u_{m-1} + \theta\bar{u}_{m-1}, v_{m-1} + \theta\overline{v}_{m-1}, u_{m-1}' + \theta\bar{u}_{m-1}', v_{m-1}' + \theta\overline{v}_{m-1}', \\
\nabla u_{m-1} + \theta\nabla\bar{u}_{m-1}, \nabla v_{m-1} + \theta\nabla\overline{v}_{m-1}, \nabla u_{m-1}' + \theta\nabla\bar{u}_{m-1}', \\
\nabla v_{m-1}' + \theta\nabla\overline{v}_{m-1}', \quad 0 < \theta < 1.
\end{aligned}$$

Therefore

$$\begin{aligned}\|\overline{F}_{1m}(t)\| &\leqslant \overline{K}_M(f_1)(\|\bar{u}_{m-1}(t)\| + \|\overline{v}_{m-1}(t)\|) + \sigma(\|\bar{u}'_{m-1}(t)\| + \|\overline{v}'_{m-1}(t)\|)\\ &\quad + \overline{K}_M(f_1)(\|\nabla\bar{u}_{m-1}(t)\| + \|\nabla\overline{v}_{m-1}(t)\|)\\ &\quad + \sigma(\|\nabla\bar{u}'_{m-1}(t)\| + \|\nabla\overline{v}'_{m-1}(t)\|)\\ &\leqslant 2\overline{K}_M\|(\bar{u}_{m-1},\overline{v}_{m-1})\|_{W_1(T)} + 2\sigma\sqrt{2}(\|\nabla\bar{u}'_{m-1}(t)\|^2\\ &\quad + \|\nabla\overline{v}'_{m-1}(t)\|^2)^{1/2}.\end{aligned}$$

Similarly

$$\begin{aligned}\|\overline{F}_{2m}(t)\| &\leqslant 2\overline{K}_M\|(\bar{u}_{m-1},\overline{v}_{m-1})\|_{W_1(T)}\\ &\qquad + 2\sigma\sqrt{2}(\|\nabla\bar{u}'_{m-1}(t)\|^2 + \|\nabla\overline{v}'_{m-1}(t)\|^2)^{1/2}.\end{aligned}$$

Hence

$$\begin{aligned}\overline{I}_4 &= 2\int_0^t [\langle\overline{F}_{1m}(s),\bar{u}'_m(s)\rangle + \langle\overline{F}_{2m}(s),\overline{v}'_m(s)\rangle]\,\mathrm{d}s\\ &\leqslant 2\int_0^t [\|\overline{F}_{1m}(s)\|\|\bar{u}'_m(s)\| + \|\overline{F}_{2m}(s)\|\|\overline{v}'_m(s)\|]\,\mathrm{d}s\\ &\leqslant \left(\frac{8}{\gamma}T\overline{K}_M^2 + 16\sigma\right)\|(\bar{u}_{m-1},\overline{v}_{m-1})\|^2_{W_1(T)} + (\gamma+\sigma)\overline{S}_m(t).\end{aligned}$$

From the estimates for $\overline{I}_1$, $\overline{I}_2$, $\overline{I}_3$ and $\overline{I}_4$, we deduce that

$$\overline{S}_m(t) \leqslant \Delta^{(1)}(M,T,\sigma)\|(\bar{u}_{m-1},\overline{v}_{m-1})\|^2_{W_1(T)} + \Delta^{(2)}(M,\sigma)\int_0^t \overline{S}_m(s)\,\mathrm{d}s, \tag{3.32}$$

with

$$\begin{aligned}\Delta^{(1)}(M,T,\sigma) &= \frac{2}{\bar{\mu}_* - \sigma}\left(\frac{32T\overline{K}_M^2}{\bar{\mu}_* - \sigma} + 16\sigma\right),\\ \Delta^{(2)}(M,\sigma) &= \frac{1}{\bar{\mu}_* - \sigma}\left(\widetilde{K} + 2|g(0)| + 2\sqrt{T^*}\|g'\|_{L^2(0,T^*)} + \frac{4\|g\|^2_{L^2(0,T^*)}}{\bar{\mu}_* - \sigma}\right).\end{aligned}$$

Using Gronwall's Lemma, it implies from (3.32) that

$$\overline{S}_m(t) \leqslant \Delta^{(1)}(M,T,\sigma)\|(\bar{u}_{m-1},\overline{v}_{m-1})\|^2_{W_1(T)}\exp(2T\Delta^{(2)}(M,\sigma)).$$

Thus

$$\|(\bar{u}_m,\overline{v}_m)\|_{W_1(T)} \leqslant k_T\|(\bar{u}_{m-1},\overline{v}_{m-1})\|_{W_1(T)} \quad \text{for all } m \in \mathbb{N},$$

where the constant $k_T = 4\sqrt{\Delta^{(1)}(M,T,\sigma)}\exp(T\Delta^{(2)}(M,\sigma)) \in [0,1)$ is defined as in Lemma 3.2, which gives

$$\|(u_{m+p}, v_{m+p}) - (u_m, v_m)\|_{W_1(T)} \leqslant \frac{M}{1-k_T}k_T^m \quad \text{for all } m,p \in \mathbb{N}.$$

It follows that $\{(u_m, v_m)\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $(u,v) \in W_1(T)$ such that

$$(u_m, v_m) \to (u,v) \text{ strongly in } W_1(T). \tag{3.33}$$

Note that $(u_m, v_m) \in B_T(M)$, then there exists a subsequence $\{(u_{m_j}, v_{m_j})\}$ of $\{(u_m, v_m)\}$ such that

$$\begin{cases} (u_{m_j}, v_{m_j}) \to (u,v) & \text{in } L^\infty(0,T;(H^2\cap V)\times(H^2\cap H_0^1)) \text{ weakly*},\\ (u'_{m_j}, v'_{m_j}) \to (u',v') & \text{in } L^\infty(0,T;(H^2\cap V)\times(H^2\cap H_0^1)) \text{ weakly*},\\ (u,v) \in B_T(M). \end{cases} \tag{3.34}$$

We note more that

$$\begin{aligned}\|F_{im} - f_i[u,v]\|^2_{L^2(Q_T)} \leqslant 8(T\overline{K}^2_M + 2\sigma^2)&\\ \times \|(u_{m-1}-u, v_{m-1}-v)\|^2_{W_1(T)},& \quad i=1,2.\end{aligned} \tag{3.35}$$

Therefore it follows from (3.33), (3.35) that

$$F_{im} \to f_i[u,v] \text{ strongly in } L^2(Q_T), \quad i=1,2. \tag{3.36}$$

Letting $m = m_j \to \infty$ in (3.5), (3.6) and using (3.33), $(3.34)_{1,2}$ and (3.36), it is clear that there exists $(u,v) \in B_T(M)$ satisfying (3.1)–(3.3). The existence is proved.

It remains to prove the uniqueness. Let (u_1, v_1), $(u_2, v_2) \in B_T(M)$ be two weak solutions of Problem (1.1)–(1.3). Then $(u,v) = (u_1,v_1)-(u_2,v_2) = (u_1-u_2, v_1-v_2)$ satisfies the variational problem

$$\begin{cases} \langle u'(t),\varphi\rangle + \lambda_1 a(u'(t),\varphi) + \bar{a}(t;u(t),\varphi) = \langle \overline{F}_1(t),\varphi\rangle,\\ \langle v'(t),\psi\rangle + \lambda_2\langle v'_x(t),\psi_x\rangle + \langle \mu_2(t)v_x(t),\psi_x\rangle\\ \quad = \displaystyle\int_0^t g(t-s)\langle v_x(s),\psi_x\rangle\,\mathrm{d}s + \langle \overline{F}_2(t),\psi\rangle\\ \qquad\qquad \text{for all } (\varphi,\psi)\in V\times H_0^1, \text{ a.e. } t\in(0,T),\\ (u(0),v(0)) = (0,0), \end{cases} \tag{3.37}$$

where

$$\overline{F}_i(t) = f_i[u_1,v_1](t) - f_i[u_2,v_2](t), \quad i=1,2.$$

Taking $(\varphi, \psi) = (u'(t), v'(t))$ in $(3.37)_{1,2}$ and integrating in time from 0 to t, we obtain

$$\begin{aligned}
\bar{\mu}_* \overline{Z}(t) \leqslant & \int_0^t \bar{a}'(s; u(s), u(s))\,\mathrm{d}s \\
& + \int_0^t \left(\int_0^1 \mu_2'(x,s) v_x^2(x,s)\,\mathrm{d}x - 2g(0)\|v_x(s)\|^2 \right) \mathrm{d}s \\
& + 2\int_0^t g(t-s)\langle v_x(s), v_x(t)\rangle\,\mathrm{d}s \\
& - 2\int_0^t \mathrm{d}\tau \int_0^\tau g'(\tau - s)\langle v_x(s), v_x(\tau)\rangle\,\mathrm{d}s \\
& + 2\int_0^t [\langle \overline{F}_1(s), u'(s)\rangle + \langle \overline{F}_2(s), v'(s)\rangle]\,\mathrm{d}s,
\end{aligned} \tag{3.38}$$

where $\bar{\mu}_* = \min\{1, \mu_{1*}, \mu_{2*}, \lambda_1, \lambda_2\}$ and

$$\overline{Z}(t) = \|u(t)\|_a^2 + \|v_x(t)\|^2 + \int_0^t (\|u'(s)\|_a^2 + \|v_x'(s)\|^2)\,\mathrm{d}s. \tag{3.39}$$

Put

$$K_* = \frac{2}{\bar{\mu}_* - 4\sigma}\left(\widetilde{K} + 2|g(0)| + 2\sqrt{T^*}\|g'\|_{L^2(0,T^*)} + \frac{2(\|g\|_{L^2(0,T^*)}^2 + 4\overline{K}_M^2)}{\bar{\mu}_* - 4\sigma}\right).$$

From (3.38) and (3.39) we get

$$\overline{Z}(t) \leqslant K_* \int_0^t \overline{Z}(s)\,\mathrm{d}s.$$

By Gronwall lemma, it follows that $\overline{Z}(t) \equiv 0$, i.e., $(u, v) = (u_1, v_1) - (u_2, v_2) = 0$. The uniqueness is proved. Theorem 3.2 is proved completely. □

4. General decay of the solution

In this section, Problem (1.1)–(1.3) is considered with the form as follows:

$$\begin{cases}
u_t - \lambda_1 u_{txx} - \dfrac{\partial}{\partial x}(\mu_1(x,t)u_x) = f_1(u,v) + F_1(x,t),\ 0 < x < 1,\ t > 0, \\
v_t - \lambda_2 v_{txx} - \dfrac{\partial}{\partial x}(\mu_2(x,t)v_x) + \displaystyle\int_0^t g(t-s)v_{xx}(x,s)\,\mathrm{d}s \\
\qquad = f_2(u,v) + F_2(x,t),\ 0 < x < 1,\ t > 0, \\
u_x(0,t) - \zeta u(0,t) = u(1,t) = v(0,t) = v(1,t) = 0, \\
(u(x,0), v(x,0)) = (\widetilde{u}_0(x), \widetilde{v}_0(x)),
\end{cases} \tag{4.1}$$

where $\zeta \geqslant 0$; λ_1, $\lambda_2 > 0$ are given constants and g, μ_i, f_i, F_i, $(i = 1, 2)$, $\widetilde{u}_0$, $\widetilde{v}_0$ are given functions satisfying conditions specified later.

a. Local existence and uniqueness.

Based on the results obtained in Section 3, we can propose the following assumptions to obtain the local existence and uniqueness of a weak solution for Problem (4.1).

($\widetilde{\mathrm{H}}_1$) $(\widetilde{u}_0, \widetilde{v}_0) \in V \times H_0^1$;

($\widetilde{\mathrm{H}}_2$) μ_1, $\mu_2 \in C^1([0,1] \times \mathbb{R}_+)$, and there exist the positive constants μ_{1*}, μ_{2*} such that $\mu_i(x,t) \geqslant \mu_{i*}$ for all $(x,t) \in [0,1] \times \mathbb{R}_+$, $i = 1, 2$;

($\widetilde{\mathrm{H}}_3$) $g \in C^1(\mathbb{R}_+; \mathbb{R}_+)$;

($\widetilde{\mathrm{H}}_4$) there exist the function $\mathcal{F} \in C^2(\mathbb{R}^2; \mathbb{R})$ and the positive constants

(i) $\partial\mathcal{F}/\partial u = f_1(u,v)$, $\partial\mathcal{F}/\partial v = f_2(u,v)$ for all $(u,v) \in \mathbb{R}^2$,

(ii) $\mathcal{F}(u,v) \leqslant \bar{d}_2(1 + |u|^\alpha + |v|^\beta)$ for all $(u,v) \in \mathbb{R}^2$;

($\widetilde{\mathrm{H}}_5$) $F_i \in L^2(\mathbb{R}_+; L^2)$, $i = 1, 2$.

Using the standard arguments of density, based on Theorem 3.2, we obtain the following theorem.

Theorem 4.1. *Let* ($\widetilde{\mathrm{H}}_1$)–($\widetilde{\mathrm{H}}_5$) *hold. Then there exists* $T > 0$ *such that Problem* (4.1) *has a unique weak solution* $(u,v) \in C^0([0,T]; V \times H_0^1)$, *and* $(u', v') \in L^2(0,T; V \times H_0^1)$.

b. Global existence and General decay of the solution.

We strengthen the above assumptions as follows.

($\widetilde{\mathrm{H}}_1$) $(\widetilde{u}_0, \widetilde{v}_0) \in V \times H_0^1$;

(H_2^d) μ_1, $\mu_2 \in C^1([0,1] \times \mathbb{R}_+)$, and there exist the positive constants μ_{1*}, μ_{2*} such that $\mu_i(x,t) \geqslant \mu_{i*}$, $\mu_i'(x,t) \leqslant 0$ for all $(x,t) \in [0,1] \times \mathbb{R}_+$, $i = 1, 2$;

(H_3^d) $g \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$, and there exists the function $\zeta \in C^1(\mathbb{R}_+)$ such that

(i) $\zeta'(t) \leqslant 0 < \zeta(t)$ for all $t \geqslant 0$, $\int_0^\infty \zeta(t)\,\mathrm{d}t = \infty$,

(ii) $g'(t) \leqslant -\zeta(t)g(t)$, $0 < g(t) \leqslant g(0)$ for all $t \geqslant 0$,

(iii) $L_* \equiv \mu_{2*} - \int_0^\infty g(s)\,\mathrm{d}s > 0$;

(H_4^d) there exist the function $\mathcal{F} \in C^2(\mathbb{R}^2; \mathbb{R})$ and the positive constants

(i) $\partial\mathcal{F}/\partial u = f_1(u,v)$, $\partial\mathcal{F}/\partial v = f_2(u,v)$ for all $(u,v) \in \mathbb{R}^2$,

(ii) $uf_1(u,v) + vf_2(u,v) \leqslant d_2\mathcal{F}(u,v)$ for all $(u,v) \in \mathbb{R}^2$,

(iii) $\mathcal{F}(u,v) \leqslant \bar{d}_2 \sum\limits_{i=1}^N (|u|^{q_i} + |v|^{\bar{q}_i})$ for all $(u,v) \in \mathbb{R}^2$;

(H_5^d) $F_i \in L^2(\mathbb{R}_+; L^2)$ such that there exist two constants $\overline{C}_0 > 0$, $\overline{\gamma}_0 > 0$, satisfying $\|F_1(t)\|^2 + \|F_2(t)\|^2 \leqslant \overline{C}_0 \mathrm{e}^{-\overline{\gamma}_0 t}$ for all $t \geqslant 0$.

(H_6^d) $p > \max\{2, d_2\}$.

R e m a r k 4.1. The following specific functions $g(t)$, $f_1(u,v)$, $f_2(u,v)$ satisfy (H_3^d), (H_4^d):

$$\begin{aligned} g(t) &= \sigma \exp\Big(-\int_0^t \zeta(s)\,\mathrm{d}s\Big), \\ f_1(u,v) &= \alpha k_1|u|^{\alpha-2}u + \alpha_1 k_3|u|^{\alpha_1-2}u|v|^{\beta_1}, \\ f_2(u,v) &= \beta k_2|v|^{\beta-2}v + \beta_1 k_3|u|^{\alpha_1}|v|^{\beta_1-2}v, \end{aligned}$$

where σ, k_1, k_2, $k_3 > 0$, α, β, α_1, $\beta_1 > 2$ are constants, $\zeta \in C^1(\mathbb{R}_+)$ such that $\zeta'(t) \leqslant 0 < \zeta(t)$ for all $t \geqslant 0$, $\int_0^\infty \zeta(t)\,\mathrm{d}t = \infty$. Indeed, by $g'(t) = -\sigma\zeta(t)\exp(-\int_0^t \zeta(s)\,\mathrm{d}s) = -\zeta(t)g(t)$, (H_3^d) holds. On the other hand, the function $\mathcal{F} \in C^2(\mathbb{R}^2;\mathbb{R})$ with $\mathcal{F}(u,v) = k_1|u|^\alpha + k_2|v|^\beta + k_3|u|^{\alpha_1}|v|^{\beta_1}$ has the following properties:

$$\begin{gathered} D_1\mathcal{F}(u,v) = f_1(u,v), \quad D_2\mathcal{F}(u,v) = f_2(u,v) \quad \text{for all } (u,v) \in \mathbb{R}^2, \\ uf_1(u,v) + vf_2(u,v) \leqslant d_2\mathcal{F}(u,v) \quad \text{for all } (u,v) \in \mathbb{R}^2, \end{gathered}$$

where $d_2 = \max\{\alpha, \beta, \alpha_1+\beta_1\}$. Moreover, the inequality $|u|^{\alpha_1}|v|^{\beta_1} \leqslant (|u|^{2\alpha_1}+|v|^{2\beta_1})/2$ implies that

$$\mathcal{F}(u,v) \leqslant \bar{d}_2(|u|^\alpha + |u|^{2\alpha_1} + |v|^\beta + |v|^{2\beta_1}) = \bar{d}_2\sum_{i=1}^2(|u|^{q_i} + |v|^{\bar{q}_i}) \quad \text{for all } (u,v) \in \mathbb{R}^2,$$

where $\bar{d}_2 = \max\{k_1, k_2, k_3/2\}$, $q_1 = \alpha$, $q_2 = 2\alpha_1$, $\bar{q}_1 = \beta$, $\bar{q}_2 = 2\beta_1$. Hence, (H_4^d) holds.

We now consider the Lyapunov functional as

$$\mathcal{L}(t) = E(t) + \delta\Psi(t), \quad t > 0, \tag{4.2}$$

where δ is a positive real number, which will be chosen later, and

$$E(t) = \frac{1}{2}\widetilde{E}(t) - \overline{\mathcal{F}}(t) = \frac{1}{2}\widetilde{E}(t) - \overline{\mathcal{F}}(t) = \Big(\frac{1}{2} - \frac{1}{p}\Big)\widetilde{E}(t) + \frac{1}{p}I(t), \tag{4.3}$$

$$\Psi(t) = \frac{1}{2}\|u(t)\|^2 + \frac{1}{2}\|v(t)\|^2 + \frac{\lambda_1}{2}\|u(t)\|_a^2 + \frac{\lambda_2}{2}\|v_x(t)\|^2, \tag{4.4}$$

where

$$\begin{aligned} \widetilde{E}(t) &= (g_*\Diamond u)(t) + (g*v)(t) + \bar{a}(t;u(t),u(t)) + \|\sqrt{\mu_2(t)}v_x(t)\|^2 - \overline{g}(t)\|v_x(t)\|^2, \qquad (4.5) \\ \overline{\mathcal{F}}(t) &= \int_0^1 \mathcal{F}(u(x,t),v(x,t))\,\mathrm{d}x, \quad I(t) = I(u(t)) = \widetilde{E}(t) - p\overline{\mathcal{F}}(t), \\ (g*v)(t) &= \int_0^t g(t-s)\|v_x(t) - v_x(s)\|^2\,\mathrm{d}s, \quad \overline{g}(t) = \int_0^t g(s)\,\mathrm{d}s, \\ (g_*\Diamond u)(t) &= \int_0^t g_*(t-s)\|u'(s)\|^2\,\mathrm{d}s, \end{aligned}$$

with $g_*(t) = 2\bar{\lambda}_*\mathrm{e}^{-2\overline{k}_*t}$, $\overline{k}_*$, $\bar{\lambda}_*$ are constants with $\overline{k}_* > 0$, $0 < \bar{\lambda}_* < 1$.

In the following, we prove that if

$$I(0)=\bar{a}(0;\widetilde{u}_0,\widetilde{u}_0)+\|\sqrt{\mu_2(0)}\widetilde{v}_{0x}\|^2-p\int_0^1\mathcal{F}(\widetilde{u}_0(x),\widetilde{u}_0(x))\,\mathrm{d}x>0,$$

and if the initial energy is small enough, then global existence is obtained and the energy of the solution decays as $t\to\infty$.

We first estimate $E'(t)$.

Lemma 4.1. *Suppose that the hypotheses* $(\widetilde{\mathrm{H}}_1)$, (H_2^d)–(H_6^d) *hold. Then*

$$\begin{aligned}(4.6)\qquad E'(t)\leqslant&-\Big(1-\bar{\lambda}_*-\frac{\varepsilon_1}{2}\Big)(\|u'(t)\|^2+\|v'(t)\|^2)\\&-\overline{k}_*(g_*\diamondsuit u)(t)-\frac{1}{2}\zeta(t)(g*v)(t)+\frac{1}{2\varepsilon_1}\varrho_1(t)\end{aligned}$$

for all $\varepsilon_1>0$, *for all* $t>0$, *where* $\varrho_1(t)=\|F_1(t)\|^2+\|F_2(t)\|^2$.

Proof. Multiplying equation (4.1) by $(u'(x,t),v'(x,t))$ and integrating on $(0,1)$, we obtain

$$\begin{aligned}(4.7)\qquad E'(t)=&\frac{1}{2}\Big[\bar{a}'(t;u(t),u(t))+\int_0^1\mu_2'(x,t)v_x^2(x,t)\,\mathrm{d}x\Big]-(1-\bar{\lambda}_*)\|u'(t)\|^2\\&-\|v'(t)\|^2-\lambda_1\|u'(t)\|_a^2-\lambda_2\|v_x'(t)\|^2-\overline{k}_*(g_*\diamondsuit u)(t)\\&+\frac{1}{2}(g'*v)(t)-\frac{1}{2}\overline{g}(t)\|v_x(t)\|^2+\langle F_1(t),u'(t)\rangle+\langle F_2(t),v'(t)\rangle.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}(4.8)\qquad &\bar{a}'(t;u(t),u(t))+\int_0^1\mu_2'(x,t)v_x^2(x,t)\,\mathrm{d}x\leqslant 0,\\&\langle F_1(t),u'(t)\rangle+\langle F_2(t),v'(t)\rangle\leqslant\frac{\varepsilon_1}{2}(\|u'(t)\|^2+\|v'(t)\|^2)+\frac{1}{2\varepsilon_1}\varrho_1(t),\\&\frac{1}{2}(g'*v)(t)\leqslant-\frac{1}{2}\zeta(t)(g*v)(t)\quad\text{for all }\varepsilon_1>0\quad\text{for all }t>0.\end{aligned}$$

Then (4.7) and (4.8) leads to (4.6). Lemma 4.1 is proved. □

Next, using Lemma 4.1, we prove the following lemma to obtain the global existence.

Lemma 4.2. *Assume that* (H_2^d)–(H_6^d) *hold. Let* $(\widetilde{u}_0,\widetilde{v}_0)\in V\times H_0^1$ *such that* $I(0)>0$ *and the initial energy* $E(0)$ *satisfies*

$$\begin{aligned}(4.9)\qquad &\eta^*\equiv L_*-p\bar{d}_2\max\Big\{\sum_{i=1}^N R_*^{q_i-2},\sum_{i=1}^N R_*^{\bar{q}_i-2}\Big\}>0,\\&0<\overline{g}(\infty)=\int_0^\infty g(s)\,\mathrm{d}s<\mu_{2*},\end{aligned}$$

where

$$(4.10)\qquad L_* = \min\{\mu_{1*}, \mu_{2*} - \overline{g}(\infty)\} > 0, \quad R_* = \sqrt{\frac{2pE_*}{(p-2)L_*}},$$

$$E_* = E(0) + \frac{1}{4(1-\bar{\lambda}_*)}\int_0^\infty (\|F_1(t)\|^2 + \|F_2(t)\|^2)\,\mathrm{d}t.$$

Then $I(t) > 0$ *for all* $t \geqslant 0$.

Proof. By the continuity of $I(t)$ and $I(0) > 0$, there exists $T_1 > 0$ such that

$$(4.11)\qquad I(t) = I(u(t), v(t)) > 0 \quad \text{for all } t \in [0, T_1].$$

From (4.3), (4.11), we get

$$(4.12)\qquad \begin{aligned} E(t) &\geqslant \Big(\frac{1}{2} - \frac{1}{p}\Big)\widetilde{E}(t) \\ &\geqslant \frac{(p-2)}{2p}[(g_*\diamondsuit u)(t) + (g * v)(t) + L_*(\|u(t)\|_a^2 + \|v_x(t)\|^2)] \\ &\qquad\qquad \text{for all } t \in [0, T_1]. \end{aligned}$$

Combining (4.6) in Lemma 4.1 and (4.12), we get

$$(4.13)\qquad \|u(t)\|_a^2 + \|v_x(t)\|^2 \leqslant \frac{2pE(t)}{(p-2)L_*} \leqslant \frac{2pE_*}{(p-2)L_*} \equiv R_*^2 \quad \text{for all } t \in [0, T_1].$$

By $(\mathrm{H}_4^d)_{(\mathrm{iii})}$, (4.13) implies that

$$(4.14)\qquad \begin{aligned} p\overline{\mathcal{F}}(t) &= p\int_0^1 \mathcal{F}(u(x,t), v(x,t))\,\mathrm{d}x \leqslant p\bar{d}_2 \sum_{i=1}^N (\|u(t)\|_{L^{q_i}}^{q_i} + \|v(t)\|_{L^{\bar{q}_i}}^{\bar{q}_i}) \\ &\leqslant p\bar{d}_2 \max\Big\{\sum_{i=1}^N R_*^{q_i-2}, \sum_{i=1}^N R_*^{\bar{q}_i-2}\Big\}(\|u(t)\|_a^2 + \|v_x(t)\|^2). \end{aligned}$$

Thus

$$(4.15)\quad I(t) \geqslant (g_*\diamondsuit u)(t) + (g * v)(t) + \eta^*(\|u(t)\|_a^2 + \|v_x(t)\|^2) \geqslant 0 \quad \text{for all } t \in [0, T_1].$$

Put $T_\infty = \sup\{T_1 > 0\colon\ I(t) > 0 \text{ for all } t \in [0, T_1]\}$. Suppose that $T_\infty < \infty$. By the continuity of $I(t)$, we have $I(T_\infty) \geqslant 0$.

If $I(T_\infty) = 0$, (4.15) gives $(g_*\diamondsuit u)(T_\infty) = (g * v)(T_\infty) = \|u(T_\infty)\|_a^2 = \|v_x(T_\infty)\|^2 = 0$, which leads to $\int_0^{T_\infty} g_*(T_\infty - s)\|u'(s)\|^2\,\mathrm{d}s = \int_0^{T_\infty} g(T_\infty - s)\|v_x(s)\|^2\,\mathrm{d}s = 0$. Because of the continuity of function $s \mapsto g_*(T_\infty - s)\|u'(s)\|^2$ on $[0, T_\infty]$ and $g_*(T_\infty - s) > 0$ for all $s \in [0, T_\infty]$, it follows that $u'(s) = 0$ for all $s \in [0, T_\infty]$. Then $u(0) = u(T_\infty) = 0$. Similarly $v(0) = 0$, so $I(0) = 0$. We have a contradiction since $I(0) > 0$.

Therefore, $I(T_\infty) > 0$. By the same arguments as above, we can deduce that there exists $\widetilde{T}_\infty > T_\infty$ such that $I(t) > 0$ for all $t \in [0, \widetilde{T}_\infty]$. This is a contradiction to the definition of T_∞.

Thus $T_\infty = \infty$, i.e., $I(t) > 0$ for all $t \geqslant 0$. Lemma 4.2 is proved. □

Now, we investigate the decay for Problem (4.1). Put

$$E_1(t) = (g_* \diamondsuit u)(t) + (g * v)(t) + \|u(t)\|_a^2 + \|v_x(t)\|^2 + I(t). \tag{4.16}$$

We will prove Lemmas 4.3, 4.4 as below.

Lemma 4.3. *There exist the positive constants* β_1, β_2, $\overline{\beta}_1$, $\overline{\beta}_2$ *such that*

(i) $\beta_1 E_1(t) \leqslant \mathcal{L}(t) \leqslant \beta_2 E_1(t)$ *for all* $t \geqslant 0$,

(ii) $\overline{\beta}_1 E_1(t) \leqslant E(t) \leqslant \overline{\beta}_2 E_1(t)$ *for all* $t \geqslant 0$.

Proof. (i) Obviously, with

$$\beta_1 = \min\left\{\frac{(p-2)L_*}{2p}, \frac{p-2}{2p}, \frac{1}{p}\right\}$$

we have

$$\begin{aligned}
\mathcal{L}(t) \geqslant E(t) &= \left(\frac{1}{2} - \frac{1}{p}\right)\widetilde{E}(t) + \frac{1}{p} I(t) \\
&\geqslant \left(\frac{1}{2} - \frac{1}{p}\right)[L_*(\|u(t)\|_a^2 + \|v_x(t)\|^2) + (g_* \diamondsuit u)(t) + (g * v)(t)] + \frac{1}{p} I(t) \\
&\geqslant \beta_1 E_1(t).
\end{aligned}$$

Similarly, by $0 < \mu_i(x,t) \leqslant \mu_i(x,0) \leqslant \max\limits_{0\leqslant x\leqslant 1} \mu_i(x,0) = \mu_i^* \leqslant \mu^* \equiv \max\limits_{i=1,2} \mu_i^* \equiv \mu^*$, we get

$$\begin{aligned}
\mathcal{L}(t) &= \left(\frac{1}{2} - \frac{1}{p}\right)\widetilde{E}(t) + \frac{1}{p} I(t) \\
&\quad + \frac{\delta}{2}(\|u(t)\|^2 + \|v(t)\|^2 + \lambda_1 \|u(t)\|_a^2 + \lambda_2 \|v_x(t)\|^2) \\
&\leqslant \left[\left(\frac{1}{2} - \frac{1}{p}\right)\mu^* + \frac{\delta}{2}(1 + \lambda^*)\right](\|u(t)\|_a^2 + \|v_x(t)\|^2) \\
&\quad + \left(\frac{1}{2} - \frac{1}{p}\right)[(g_* \diamondsuit u)(t) + (g * v)(t)] + \frac{1}{p} I(t) \\
&\leqslant \beta_2 E_1(t),
\end{aligned}$$

where

$$\beta_2 = \max\left\{\left(\frac{1}{2} - \frac{1}{p}\right), \left(\frac{1}{2} - \frac{1}{p}\right)\mu^* + \frac{\delta}{2}\left(1 + \lambda^*\right), \frac{1}{p}\right\}, \quad \lambda^* = \max\{\lambda_1, \lambda_2\}.$$

(ii) It is similar to (i). Lemma 4.3 is proved. □

Lemma 4.4. *The functional* $\Psi(t)$ *satisfies the estimation*

$$
\begin{aligned}
(4.17)\qquad \Psi'(t) \leqslant{} & \Big(\frac{1}{2\varepsilon_3}+\frac{d_2}{p}\Big)(g*v)(t)+\frac{d_2}{p}(g_*\Diamond u)(t)-\frac{\delta_1 d_2}{p}I(t)\\
&-\Big[(1-\delta_1)\frac{d_2}{p}\eta^*+\Big(1-\frac{d_2}{p}\Big)\mu_{1*}-\frac{\varepsilon_2}{2}\Big]\|u(t)\|_a^2\\
&-\Big[(1-\delta_1)\frac{d_2}{p}\eta^*+\Big(1-\frac{d_2}{p}\Big)(\mu_{2*}-\overline{g}(\infty))-\frac{1}{2}(\varepsilon_2+\varepsilon_3\overline{g}(\infty))\Big]\\
&\times\|v_x(t)\|^2+\frac{1}{2\varepsilon_2}\varrho_1(t)
\end{aligned}
$$

for all $\delta_1\in(0,1)$ *and* $\varepsilon_2,\varepsilon_3>0$, *where* $\varrho_1(t)=\|F_1(t)\|^2+\|F_2(t)\|^2$.

Proof. By multiplying (4.1) by $(u(x,t),v(x,t))$ and integrating over $(0,1)$, we obtain

$$
\begin{aligned}
(4.18)\qquad \Psi'(t) ={} & -\bar{a}(t;u(t),u(t))-\|\sqrt{\mu_2(t)}v_x(t)\|^2+\int_0^t g(t-s)\langle v_x(s),v_x(t)\rangle\,\mathrm{d}s\\
&+\langle f_1(u(t),v(t)),u(t)\rangle+\langle f_2(u(t),v(t)),v(t)\rangle\\
&+\langle F_1(t),u(t)\rangle+\langle F_2(t),v(t)\rangle.
\end{aligned}
$$

We also have

$$
\begin{aligned}
(4.19)\qquad & \langle F_1(t),u(t)\rangle+\langle F_2(t),v(t)\rangle\leqslant\frac{\varepsilon_2}{2}(\|u(t)\|_a^2+\|v_x(t)\|^2)+\frac{1}{2\varepsilon_2}\varrho_1(t),\\
& \int_0^t g(t-s)\langle v_x(s),v_x(t)\rangle\,\mathrm{d}s\leqslant\frac{1}{2\varepsilon_3}(g*v)(t)+\Big(\frac{\varepsilon_3}{2}+1\Big)\overline{g}(t)\|v_x(t)\|^2,
\end{aligned}
$$

$$
\begin{aligned}
(4.20)\qquad & \langle f_1(u(t),v(t)),u(t)\rangle+\langle f_2(u(t),v(t)),v(t)\rangle\\
&\leqslant d_2\overline{\mathcal{F}}(t)=\frac{d_2}{p}[\widetilde{E}(t)-I(t)]=\frac{d_2}{p}\widetilde{E}(t)-(1-\delta_1)\frac{d_2}{p}I(t)-\frac{\delta_1 d_2}{p}I(t)\\
&\leqslant\frac{d_2}{p}\widetilde{E}(t)-(1-\delta_1)\frac{d_2}{p}\eta^*(\|u(t)\|_a^2+\|v_x(t)\|^2)-\frac{\delta_1 d_2}{p}I(t)\\
&\leqslant\frac{d_2}{p}[(g_*\Diamond u)(t)+(g*v)(t)+\bar{a}(t;u(t),u(t))\\
&\quad+\|\sqrt{\mu_2(t)}v_x(t)\|^2-\overline{g}(t)\|v_x(t)\|^2]\\
&\quad-(1-\delta_1)\frac{d_2}{p}\eta^*(\|u(t)\|_a^2+\|v_x(t)\|^2)-\frac{\delta_1 d_2}{p}I(t).
\end{aligned}
$$

Therefore, by (4.19), (4.20), it leads to (4.17). Lemma 4.4 is proved. □

The main result in this section is stated as follows.

Theorem 4.2. *Assume that* (H_2^d)–(H_6^d) *hold. Let* $(\widetilde{u}_0, \widetilde{v}_0) \in V \times H_0^1$ *such that* $I(0) > 0$ *and let the initial energy* $E(0)$ *satisfy* (4.9). *Then there exist positive constants* $\overline{C}$, $\overline{\gamma}$ *such that*

$$\|u(t)\|_a^2 + \|v_x(t)\|^2 \leqslant \overline{C} \exp\Big(-\overline{\gamma} \int_0^t \zeta(s)\,\mathrm{d}s\Big) \quad \text{for all } t \geqslant 0. \tag{4.21}$$

P r o o f. First, by definition of $\mathcal{L}(t)$ and inequalities (4.6), (4.17), we obtain

$$\begin{aligned} \mathcal{L}'(t) \leqslant & -(1 - \bar{\lambda}_* - \frac{\varepsilon_1}{2})(\|u'(t)\|^2 + \|v'(t)\|^2) - \widetilde{\theta}_3(g_* \Diamond u)(t) \\ & + \delta d_3 (g * v)(t) - \frac{\delta \delta_1 d_2}{p} I(t) - \delta \widetilde{\theta}_1 \|u(t)\|_a^2 - \delta \widetilde{\theta}_2 \|v_x(t)\|^2 \\ & + \frac{1}{2}\Big(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\Big)\varrho_1(t), \end{aligned} \tag{4.22}$$

with

$$\begin{aligned} d_3 &= \frac{1}{2\varepsilon_3} + \frac{d_2}{p}, \\ \widetilde{\theta}_1 &= \widetilde{\theta}_1(\delta_1, \varepsilon_2) = (1 - \delta_1)\frac{d_2}{p}\eta^* + \Big(1 - \frac{d_2}{p}\Big)\mu_{1*} - \frac{\varepsilon_2}{2}, \\ \widetilde{\theta}_2 &= \widetilde{\theta}_2(\delta_1, \varepsilon_2, \varepsilon_3) = (1 - \delta_1)\frac{d_2}{p}\eta^* + \Big(1 - \frac{d_2}{p}\Big)(\mu_{2*} - \overline{g}(\infty)) \\ & \quad - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 \overline{g}(\infty)), \\ \widetilde{\theta}_3 &= \overline{k}_* - \frac{\delta d_2}{p}. \end{aligned} \tag{4.23}$$

By $p > d_2$ and $0 < \overline{g}(\infty) < \mu_{2*}$, we also get

$$\begin{aligned} \lim_{\delta_1 \to 0_+, \varepsilon_2 \to 0_+} \widetilde{\theta}_1(\delta_1, \varepsilon_2) &= \frac{d_2}{p}\eta^* + \Big(1 - \frac{d_2}{p}\Big)\mu_{1*} > 0, \\ \lim_{\delta_1 \to 0_+, \varepsilon_2 \to 0_+, \varepsilon_3 \to 0_+} \widetilde{\theta}_2(\delta_1, \varepsilon_2, \varepsilon_3) &= \Big[\frac{d_2}{p}\eta^* + (1 - \frac{d_2}{p})\mu_{2*} - \Big(1 - \frac{d_2}{p}\Big)\overline{g}(\infty)\Big] > 0. \end{aligned} \tag{4.24}$$

Thus, we can choose $\delta_1 \in (0,1)$ and ε_2, $\varepsilon_3 > 0$ small enough such that

$$\widetilde{\theta}_1 = \widetilde{\theta}_1(\delta_1, \varepsilon_2) > 0, \quad \widetilde{\theta}_2 = \widetilde{\theta}_2(\delta_1, \varepsilon_2, \varepsilon_3) > 0. \tag{4.25}$$

By $1 - \bar{\lambda}_* > 0$, we also can choose $\delta > 0$ and $\varepsilon_1 > 0$ small enough such that

$$\widetilde{\theta}_3 = \overline{k}_* - \frac{\delta d_2}{p} > 0, \quad 1 - \bar{\lambda}_* - \frac{\varepsilon_1}{2} > 0. \tag{4.26}$$

Put

$$\theta_* = \min\left\{\delta\widetilde{\theta}_1, \delta\widetilde{\theta}_2, \widetilde{\theta}_3, \frac{\delta\delta_1 d_2}{p}\right\}, \tag{4.27}$$

it follows from (4.22), (4.25), (4.26), (4.27) that

$$\begin{aligned} \mathcal{L}'(t) \leqslant\ & -\theta_*[\|u(t)\|_a^2 + \|v_x(t)\|^2 + (g_*\Diamond u)(t) + I(t)] \\ & + \delta d_3(g*v)(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\varrho_1(t) \\ \leqslant\ & -\theta_* E_1(t) + (\theta_* + \delta d_3)(g*v)(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\varrho_1(t). \end{aligned} \tag{4.28}$$

Combining (4.6) and (4.28), we have

$$\begin{aligned} \zeta(t)\mathcal{L}'(t) \leqslant\ & -\theta_*\zeta(t)E_1(t) + (\theta_* + \delta d_3)\zeta(t)(g*v)(t) \\ & + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\zeta(0)\varrho_1(t) \\ \leqslant\ & -\theta_*\zeta(t)E_1(t) + 2(\theta_* + \delta d_3)\left[-E'(t) + \frac{1}{2\varepsilon_1}\varrho_1(t)\right] \\ & + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\zeta(0)\varrho_1(t) \\ =\ & -\theta_*\zeta(t)E_1(t) - 2(\theta_* + \delta d_3)E'(t) \\ & + \left[\frac{\theta_* + \delta d_3}{\varepsilon_1} + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\zeta(0)\right]\varrho_1(t) \\ \leqslant\ & -\theta_*\zeta(t)E_1(t) - 2(\theta_* + \delta d_3)E'(t) + \overline{C}_1 \mathrm{e}^{-\overline{\gamma}_0 t}, \end{aligned}$$

where

$$\overline{C}_1 = \left[\frac{\theta_* + \delta d_3}{\varepsilon_1} + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\zeta(0)\right]\overline{C}_0.$$

Considering the functional

$$L(t) = \zeta(t)\mathcal{L}(t) + 2(\theta_* + \delta d_3)E(t),$$

then

$$L(t) \leqslant [\zeta(0)\beta_2 + 2(\theta_* + \delta d_3)\overline{\beta}_2]E_1(t) \equiv \overline{\beta}_3 E_1(t),$$

and

$$\begin{aligned} L'(t) &= \zeta'(t)\mathcal{L}(t) + \zeta(t)\mathcal{L}'(t) + 2(\theta_* + \delta d_3)E'(t) \\ &\leqslant -\theta_*\zeta(t)E_1(t) + \overline{C}_1\mathrm{e}^{-\overline{\gamma}_0 t} \leqslant -\frac{\theta_*}{\overline{\beta}_3}\zeta(t)L(t) + \overline{C}_1\mathrm{e}^{-\overline{\gamma}_0 t}. \end{aligned} \tag{4.29}$$

Choosing $\overline{\gamma}$, $0 < \overline{\gamma} < \min\{\theta_*/\overline{\beta}_3, \overline{\gamma}_0/\zeta(0)\}$ from (4.29), we obtain

$$L'(t) + \overline{\gamma}\zeta(t)L(t) \leqslant \overline{C}_1 \mathrm{e}^{-\overline{\gamma}_0 t}. \tag{4.30}$$

Integrating (4.30) it leads to

$$L(t) \leqslant \Big(L(0) + \frac{\overline{C}_1}{\overline{\gamma}_0 - \overline{\gamma}\zeta(0)}\Big) \exp\Big(-\overline{\gamma}\int_0^t \zeta(s)\,\mathrm{d}s\Big). \tag{4.31}$$

Moreover,

$$\begin{aligned} L(t) &\geqslant 2(\theta_* + \delta d_3)E(t) \geqslant 2(\theta_* + \delta d_3)\overline{\beta}_1 E_1(t) \\ &\geqslant 2(\theta_* + \delta d_3)\overline{\beta}_1(\|u(t)\|_a^2 + \|v_x(t)\|^2). \end{aligned} \tag{4.32}$$

Therefore, (4.31) and (4.32) imply that (4.21). Theorem 4.2 is proved. □

5. Blow-up and lifespan of the solution

First, we make the following assumptions:

($\widetilde{\mathrm{H}}_1$) $(\widetilde{u}_0, \widetilde{v}_0) \in V \times H_0^1$;

(H_2^B) μ_1, $\mu_2 \in C^1([0,1] \times \mathbb{R}_+)$, and there exist positive constants μ_{1*}, μ_{2*} such that $\mu_i(x,t) \geqslant \mu_{i*}$, $\mu_i'(x,t) \leqslant 0$ for all $(x,t) \in [0,1] \times \mathbb{R}_+$, $i = 1,2$;

(H_3^B) $g \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ such that
   (i) $g(t) \geqslant 0$, $g'(t) \leqslant 0$ for all $t \geqslant 0$,
   (ii) $\int_0^\infty g(s)\,\mathrm{d}s \leqslant p(p-2)\mu_{2*}/(p-1)^2$, where $p > 2$;

(H_4^B) there exists the function $\mathcal{F} \in C^2(\mathbb{R}^2; \mathbb{R})$ and a positive constant $d_1 > p$ such that
   (i) $\partial\mathcal{F}/\partial u = f_1(u,v)$, $\partial\mathcal{F}/\partial v = f_2(u,v)$ for all $(u,v) \in \mathbb{R}^2$, $i = 1,2$,
   (ii) $uf_1(u,v) + vf_2(u,v) \geqslant d_1\mathcal{F}(u,v) \geqslant 0$ for all $(u,v) \in \mathbb{R}^2$.

Remark 5.1. The functions $f_1(u,v)$, $f_2(u,v)$ given in the example of Remark 4.1 also satisfy (H_4^B) (ii), because of

$$\begin{aligned} uf_1(u,v) + vf_2(u,v) &= \alpha k_1|u|^\alpha + \beta k_2|v|^\beta + (\alpha_1 + \beta_1)k_3|u|^{\alpha_1}|v|^{\beta_1} \\ &\geqslant \min\{\alpha, \beta, \alpha_1 + \beta_1\}(k_1|u|^\alpha + k_2|v|^\beta + k_3|u|^{\alpha_1}|v|^{\beta_1}) \\ &= d_1\mathcal{F}(u,v) \geqslant 0 \quad \text{for all } (u,v) \in \mathbb{R}^2, \end{aligned}$$

where $d_1 = \min\{\alpha, \beta, \alpha_1 + \beta_1\}$.

We now consider the symmetric bilinear forms $\widehat{a}_i(\cdot,\cdot)$ $(i=1,2)$ defined by

$$\widehat{a}_1(u,\varphi)=\langle u,\varphi\rangle+\lambda_1 a(u,\varphi),\quad (u,\varphi)\in V\times V,$$
$$\widehat{a}_2(v,\psi)=\langle v,\psi\rangle+\lambda_2\langle v_x,\psi_x\rangle,\quad (v,\psi)\in H_0^1\times H_0^1,$$

with $\Psi(t)$ defined as in (4.4). Denoting

$$\|u\|_{\widehat{a}_1}=\sqrt{\widehat{a}_1(u,u)},\quad u\in V,$$
$$\|v\|_{\widehat{a}_2}=\sqrt{\widehat{a}_2(v,v)},\quad v\in H_0^1,$$

we can rewrite $\Psi(t)$ as

$$\Psi(t)=\frac{1}{2}(\|u(t)\|^2_{\widehat{a}_1}+\|v(t)\|^2_{\widehat{a}_2}).$$

We also consider the functional $\overline{E}(t)$ defined as

$$\overline{E}(t)=\frac{1}{2}(g*v)(t)+\frac{1}{2}\bar{a}(t;u(t),u(t))+\frac{1}{2}(\|\sqrt{\mu_2(t)}v_x(t)\|^2-\overline{g}(t)\|v_x(t)\|^2)-\overline{\mathcal{F}}(t) \tag{5.1}$$

with

$$\overline{\mathcal{F}}(t)=\int_0^1\mathcal{F}(u(x,t),v(x,t))\,\mathrm{d}x.$$

We note more that $\overline{E}(t)$ is the functional $E(t)$ as in (4.3) corresponding to $g_*\equiv 0$. Furthermore $\overline{E}(0)=E(0)$.

Lemma 5.1. *Assume that* $(\widetilde{H}_1)$, (H_2^B), (H_3^B) *and* (H_4^B) *hold. Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\overline{E}(t)+\int_0^t(\|u'(s)\|^2_{\widehat{a}_1}+\|v'(s)\|^2_{\widehat{a}_2})\,\mathrm{d}s\Big]\leqslant 0.$$

Moreover, the following energy inequality holds:

$$\overline{E}(t)+\int_0^t(\|u'(s)\|^2_{\widehat{a}_1}+\|v'(s)\|^2_{\widehat{a}_2})\,\mathrm{d}s\leqslant\overline{E}(0). \tag{5.2}$$

P r o o f. Multiplying equation (4.1) by $(u'(x,t),v'(x,t))$ and integrating on $(0,1)$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\overline{E}(t)&+\int_0^t(\|u'(s)\|^2_{\widehat{a}_1}+\|v'(s)\|^2_{\widehat{a}_2})\,\mathrm{d}s\Big]\\&=\frac{1}{2}\Big[\bar{a}'(t;u(t),u(t))+\int_0^1\mu_2'(x,t)v_x^2(x,t)\,\mathrm{d}x\Big]\\&\quad+\frac{1}{2}(g'*v)(t)-\frac{1}{2}\overline{g}(t)\|v_x(t)\|^2\leqslant 0\end{aligned} \tag{5.3}$$

for any regular solution (u,v). We can extend (5.3) to weak solutions by using density arguments. Combining $(\widetilde{H}_1)$, (H_2^B), (H_3^B) and (H_4^B), Lemma 5.1 is proved. □

a. Blow-up solutions with negative initial energy.

Theorem 5.1. *Assume that* (H_2^B), (H_3^B) *and* (H_4^B) *hold. Then for any initial conditions* $(\widetilde{u}_0, \widetilde{v}_0) \in (V \times H_0^1)$ *such that* $E(0) < 0$, *the weak solution* (u, v) *of Problem* (4.1) *blows up at finite time and the lifespan* T_∞ *of this solution satisfies*

$$T_\infty \leqslant \frac{-8(p-1)\Psi(0)}{p(p-2)^2E(0)} \equiv T_{\infty\,\max}. \tag{5.4}$$

Furthermore, if in addition:

(H_{4*}^B) *there exist the constants* $d_2 > d_1$, $\bar{d}_2 > 0$, $q_i > 2$, $\bar{q}_i > 2$ $(i = 1, \ldots, N)$, *such that*

(i) $uf_1(u,v) + vf_2(u,v) \leqslant d_2\mathcal{F}(u,v)$ *for all* $(u,v) \in \mathbb{R}^2$, *for all* $(u,v) \in \mathbb{R}^2$,

(ii) $\mathcal{F}(u,v) \leqslant \bar{d}_2 \sum\limits_{i=1}^N (|u|^{q_i} + |v|^{\bar{q}_i})$ *for all* $(u,v) \in \mathbb{R}^2$; *for all* $(u,v) \in \mathbb{R}^2$,

(H_{5*}^B) $\int_{\Psi(0)}^\infty \mathrm{d}z/G_1(z) \leqslant -8(p-1)(\Psi(0)/p(p-2)^2E(0))$,

where

$$G_1(z) = \frac{4\overline{g}(\infty)}{\lambda_*}z + (1+d_2)d_3\sum_{i=1}^N(1 + z^{q_i/2} + z^{\bar{q}_i/2}), \tag{5.5}$$

$$\lambda_* = \min\{\lambda_1, \lambda_2\}, \quad d_3 = \bar{d}_2 \max\left\{\left(\frac{2}{\lambda_*}\right)^{q_i/2}, \left(\frac{2}{\lambda_*}\right)^{\bar{q}_i/2}, \ i = 1, \ldots, N\right\},$$

then T_∞ *satisfies*

$$T_\infty \geqslant \int_{\Psi(0)}^\infty \frac{\mathrm{d}z}{G_1(z)} \equiv T_{\infty\,\min}. \tag{5.6}$$

P r o o f. Before proving Theorem 5.1, we will explain the validity of the hypothesis (H_{5*}^B).

We note that

$$\Psi(0) = \frac{1}{2}(\|\widetilde{u}_0\|^2 + \|\widetilde{v}_0\|^2 + \lambda_1\|\widetilde{u}_0\|_a^2 + \lambda_2\|\widetilde{v}_{0x}\|^2) \geqslant \frac{1}{2}(\|\widetilde{u}_0\|^2 + \|\widetilde{v}_0\|^2),$$

$$E(0) = \frac{1}{2}(\bar{a}(0; \widetilde{u}_0, \widetilde{u}_0) + \|\sqrt{\mu_2(0)}\widetilde{v}_{0x}\|^2) - \int_0^1 \mathcal{F}(\widetilde{u}_0(x), \widetilde{v}_0(x))\,\mathrm{d}x < 0,$$

therefore

$$\frac{-8(p-1)\Psi(0)}{p(p-2)^2E(0)} \geqslant \frac{4(p-1)(\|\widetilde{u}_0\|^2 + \|\widetilde{v}_0\|^2)}{p(p-2)^2(-E(0))}, \tag{5.7}$$

while the right-hand side of (5.7) is independent of $\lambda_* = \min\{\lambda_1, \lambda_2\}$.

On the other hand, from (5.5) we can deduce that

$$d_3 = d_3(\lambda_*) = \bar{d}_2 \max\left\{\left(\frac{2}{\lambda_*}\right)^{q_i/2}, \left(\frac{2}{\lambda_*}\right)^{\bar{q}_i/2},\ i = 1,\dots,N\right\} \to \infty,$$

as $\lambda_* \to 0_+$,

$$G_1(z) = \frac{4\overline{g}(\infty)}{\lambda_*} z + (1+d_2)d_3 \sum_{i=1}^{N} (1 + z^{q_i/2} + z^{\bar{q}_i/2}) \geqslant d_3(\lambda_*)(1 + z^{q_1/2}).$$

Hence

$$\int_{\Psi(0)}^{\infty} \frac{\mathrm{d}z}{G_1(z)} \leqslant \frac{1}{d_3(\lambda_*)} \int_{\Psi(0)}^{\infty} \frac{\mathrm{d}z}{1 + z^{q_1/2}} \to 0, \quad \text{as } \lambda_* \to 0_+. \tag{5.8}$$

This leads to the existence of $\lambda_* > 0$ sufficiently small such that

$$\frac{1}{d_3(\lambda_*)} \int_{\Psi(0)}^{\infty} \frac{\mathrm{d}z}{1 + z^{q_1/2}} < \frac{4(p-1)(\|\widetilde{u}_0\|^2 + \|\widetilde{v}_0\|^2)}{p(p-2)^2(-E(0))}. \tag{5.9}$$

From (5.7)–(5.9), we conclude that (H_{5*}^{B}) holds true.

Now, we first prove that the solution (u, v) obtained here is not a global solution in $\mathbb{R}_+$. Indeed, by contradiction, we will assume that the weak solution exists in the whole interval $\mathbb{R}_+$.

For $\overline{E}(0) < 0$,

$$0 < \beta \leqslant \frac{-p\overline{E}(0)}{p-1}, \quad \tau > \frac{2\Psi(0)}{\beta(p-2)},$$

and

$$T_0 \geqslant \frac{\beta\tau^2}{(p-2)\beta\tau - 2\Psi(0)},$$

we define the auxiliary functional $\Gamma\colon\ [0, T_0] \to \mathbb{R}$ as

$$\Gamma(t) = 2\int_0^t \Psi(s)\,\mathrm{d}s + 2(T_0 - t)\Psi(0) + \beta(t+\tau)^2, \quad 0 \leqslant t \leqslant T_0. \tag{5.10}$$

By direct computation, we have

$$\begin{aligned} \Gamma'(t) &= 2\Psi(t) - 2\Psi(0) + 2\beta(t+\tau) = 2\int_0^t \Psi'(s)\,\mathrm{d}s + 2\beta(t+\tau) \\ &= 2\int_0^t \widehat{a}_1(u'(s), u(s))\,\mathrm{d}s + 2\int_0^t \widehat{a}_2(v'(s), v(s))\,\mathrm{d}s + 2\beta(t+\tau), \end{aligned} \tag{5.11}$$

and

$$\Gamma''(t) = 2\Psi'(t) + 2\beta. \tag{5.12}$$

Since (5.10) and (5.11), we see that $\Gamma(t) > 0$ for all $t \in [0, T_0]$ and $\Gamma'(0) = 2\beta\tau > 0$. By the Cauchy-Schwarz inequality, we get from (5.11) that

$$\begin{aligned}
\frac{1}{2}|\Gamma'(t)| &\leqslant \left[\int_0^t |\widehat{a}_1(u'(s), u(s))|\,\mathrm{d}s + \int_0^t |\widehat{a}_2(v'(s), v(s))|\,\mathrm{d}s + \beta(t+\tau)\right] \\
&\leqslant \left[\int_0^t \|u'(s)\|_{\widehat{a}_1}\|u(s)\|_{\widehat{a}_1}\,\mathrm{d}s + \int_0^t \|v'(s)\|_{\widehat{a}_2}\|v(s)\|_{\widehat{a}_2}\,\mathrm{d}s + \beta(t+\tau)\right] \\
&\leqslant \left(\int_0^t \|u'(s)\|^2_{\widehat{a}_1}\,\mathrm{d}s\right)^{1/2}\left(\int_0^t \|u(s)\|^2_{\widehat{a}_1}\,\mathrm{d}s\right)^{1/2} \\
&\quad + \left(\int_0^t \|v'(s)\|^2_{\widehat{a}_2}\,\mathrm{d}s\right)^{1/2}\left(\int_0^t \|v(s)\|^2_{\widehat{a}_2}\,\mathrm{d}s\right)^{1/2} + \beta(t+\tau) \\
&\leqslant \left[\int_0^t \|u'(s)\|^2_{\widehat{a}_1}\,\mathrm{d}s + \int_0^t \|v'(s)\|^2_{\widehat{a}_2}\,\mathrm{d}s + \beta\right]^{1/2} \\
&\quad \times \left[\int_0^t \|u(s)\|^2_{\widehat{a}_1}\,\mathrm{d}s + \left(\int_0^t \|v(s)\|^2_{\widehat{a}_2}\,\mathrm{d}s\right) + \beta(t+\tau)^2\right]^{1/2} \\
&= \sqrt{\sigma_1(t)}\sqrt{\sigma_2(t)} = \sqrt{\sigma(t)},
\end{aligned} \tag{5.13}$$

where

$$\begin{aligned}
\sigma(t) &= \sigma_1(t)\sigma_2(t), \\
\sigma_1(t) &= \int_0^t \|u'(s)\|^2_{\widehat{a}_1}\,\mathrm{d}s + \int_0^t \|v'(s)\|^2_{\widehat{a}_2}\,\mathrm{d}s + \beta, \\
\sigma_2(t) &= \int_0^t \|u(s)\|^2_{\widehat{a}_1}\,\mathrm{d}s + \int_0^t \|v(s)\|^2_{\widehat{a}_2}\,\mathrm{d}s + \beta(t+\tau)^2 \\
&= 2\int_0^t \Psi(s)\,\mathrm{d}s + \beta(t+\tau)^2,
\end{aligned} \tag{5.14}$$

then it follows from (5.14) that

$$\sigma(t) \geqslant \frac{1}{4}|\Gamma'(t)|^2 \quad \text{for all } t \in [0, T_0]. \tag{5.15}$$

Therefore, since $\Gamma(t) = \sigma_2(t) + 2(T_0 - t)\Psi(0) \geqslant \sigma_2(t)$, we get

$$2p\Gamma(t)\sigma_1(t) \geqslant 2p\sigma_2(t)\sigma_1(t) = 2p\sigma(t) \geqslant \frac{p}{2}|\Gamma'(t)|^2. \tag{5.16}$$

It follows from (5.16) that

$$\Gamma''(t)\Gamma(t) - \frac{p}{2}|\Gamma'(t)|^2 \geqslant 2\Gamma(t)\left[\frac{1}{2}\Gamma''(t) - p\sigma_1(t)\right] = 2\Gamma(t)D(t) \tag{5.17}$$

with

$$D(t) = \frac{1}{2}\Gamma''(t) - p\sigma_1(t). \tag{5.18}$$

Furthermore, by multiplying equations in (4.1) by $(u(x,t),\ v(x,t))$, and then integrating over $(0,1)$, we deduce from (5.12) that

$$\begin{aligned} D(t) = \frac{1}{2}\Gamma''(t) - p\sigma_1(t) &= \beta + \Psi'(t) - p\sigma_1(t) \\ &= \beta - \bar{a}(t; u(t), u(t)) - \|\sqrt{\mu_2(t)}v_x(t)\|^2 \\ &\quad + \int_0^t g(t-s)\langle v_x(s), v_x(t)\rangle \,\mathrm{d}s \\ &\quad + \langle f_1(u(t), v(t)), u(t)\rangle + \langle f_2(u(t), v(t)), v(t)\rangle \\ &\quad - p\left[\int_0^t \|u'(s)\|^2_{\hat{a}_1}\,\mathrm{d}s + \int_0^t \|v'(s)\|^2_{\hat{a}_2}\,\mathrm{d}s + \beta\right]. \end{aligned} \tag{5.19}$$

The terms on the right-hand side of (5.19) can be estimated as

$$\int_0^t g(t-s)\langle v_x(s), v_x(t)\rangle \,\mathrm{d}s \geqslant -\frac{p}{2}(g*v)(t) + \left(1 - \frac{1}{2p}\right)(\overline{g}(t)\|v_x(t)\|^2), \tag{5.20}$$

and

$$\langle f_1(u(t), v(t)), u(t)\rangle + \langle f_2(u(t), v(t)), v(t)\rangle \geqslant d_1\overline{\mathcal{F}}(t). \tag{5.21}$$

It yields from (5.1), (5.2), (5.20) and (5.21) that

$$\begin{aligned} D(t) &\geqslant \beta - \bar{a}(t; u(t), u(t)) - \|\sqrt{\mu_2(t)}v_x(t)\|^2 \\ &\quad - \frac{p}{2}(g*v)(t) + \left(1 - \frac{1}{2p}\right)(\overline{g}(t)\|v_x(t)\|^2) \\ &\quad + d_1\overline{\mathcal{F}}(t) - p\left[\overline{E}(t) + \int_0^t \|u'(s)\|^2_{\hat{a}_1}\,\mathrm{d}s + \int_0^t \|v'(s)\|^2_{\hat{a}_2}\,\mathrm{d}s + \beta\right] \\ &\quad + \frac{p}{2}[(g*v)(t) + \bar{a}(t; u(t), u(t)) \\ &\quad + \|\sqrt{\mu_2(t)}v_x(t)\|^2 - \overline{g}(t)\|v_x(t)\|^2 - 2\overline{\mathcal{F}}(t)] \\ &\geqslant \beta - \bar{a}(t; u(t), u(t)) - \|\sqrt{\mu_2(t)}v_x(t)\|^2 + \left(1 - \frac{1}{2p}\right)(\overline{g}(t)\|v_x(t)\|^2) \\ &\quad + d_1\overline{\mathcal{F}}(t) - p\overline{E}(0) - p\beta \\ &\quad + \frac{p}{2}(\bar{a}(t; u(t), u(t)) + \|\sqrt{\mu_2(t)}v_x(t)\|^2) - \frac{p}{2}\overline{g}(t)\|v_x(t)\|^2 - p\overline{\mathcal{F}}(t) \\ &= -p\overline{E}(0) - (p-1)\beta + (d_1 - p)\overline{\mathcal{F}}(t) \end{aligned} \tag{5.22}$$

$$
\begin{aligned}
&+\frac{p-2}{2}(\bar{a}(t;u(t),u(t))+\|\sqrt{\mu_2(t)}v_x(t)\|^2)-\frac{(p-1)^2}{2p}\overline{g}(t)\|v_x(t)\|^2\\
\geqslant &-p\overline{E}(0)-(p-1)\beta+(d_1-p)\overline{\mathcal{F}}(t)\\
&+\frac{p-2}{2}(\mu_{1*}\|u(t)\|_a^2+\mu_{2*}\|v_x(t)\|^2)-\frac{(p-1)^2}{2p}\overline{g}(\infty)\|v_x(t)\|^2\\
= &-p\overline{E}(0)-(p-1)\beta+(d_1-p)\overline{\mathcal{F}}(t)+\frac{(p-2)\mu_{1*}}{2}\|u(t)\|_a^2\\
&+\frac{(p-1)^2}{2p}\left[\frac{p(p-2)\mu_{2*}}{(p-1)^2}-\overline{g}(\infty)\right]\|v_x(t)\|^2\geqslant 0,
\end{aligned}
$$

because of $d_1 > p > 2$, $0 < \overline{g}(\infty) \leqslant p(p-2)\mu_{2*}/(p-1)^2$ and $0 < \beta \leqslant -p\overline{E}(0)/(p-1)$.

From (5.17) and (5.22) we obtain

$$
\Gamma^{p/2-1}(t) \geqslant \frac{2\Gamma^{p/2}(0)}{(p-2)\Gamma'(0)}\frac{1}{T_*-t} \quad \text{for all } t\in[0,t_{\min}), \tag{5.23}
$$

where $t_{\min} = \min\{T_*, T_0,\}$, with $T_* = 2\Gamma(0)/((p-2)\Gamma'(0))$.

By

$$
0<\beta\leqslant\frac{-p\overline{E}(0)}{p-1}, \quad \tau>\frac{2\Psi(0)}{\beta(p-2)}
$$

and

$$
T_0 \geqslant \frac{\beta\tau^2}{(p-2)\beta\tau-2\Psi(0)},
$$

we have

$$
T_* = \frac{2\Gamma(0)}{(p-2)\Gamma'(0)} = \frac{2T_0\Psi(0)+\beta\tau^2}{(p-2)\beta\tau} \in (0,T_0]. \tag{5.24}
$$

From (5.23) we get $\lim_{t\to T_*^-}\Gamma(t)=\infty$. This is a contradiction. Therefore, the solution (u,v) blows up at finite time.

Now, we find an upper bound for T_∞. It is clear to see that

$$
T_\infty \leqslant \frac{2T_\infty\Psi(0)+\beta\tau^2}{(p-2)\beta\tau}
$$

is equivalent to

$$
T_\infty \leqslant \frac{\beta\tau^2}{(p-2)\beta\tau-2\Psi(0)} \quad \text{for all } (\beta,\tau)\in\widetilde{D}(\widetilde{u}_0,\widetilde{v}_0), \tag{5.25}
$$

where

$$
\widetilde{D}(\widetilde{u}_0,\widetilde{v}_0)=\left\{(\beta,\tau)\in\mathbb{R}_+^2\colon\ 0<\beta\leqslant\frac{-p\overline{E}(0)}{p-1}, \quad \tau>\frac{2\Psi(0)}{\beta(p-2)}\right\}.
$$

Considering the function

$$h(\tau,\beta)=\frac{\beta\tau^2}{(p-2)\beta\tau-2\Psi(0)}=\frac{\tau^2}{(p-2)(\tau-\tau_*)},\quad (\beta,\tau)\in\widetilde{D}(\widetilde{u}_0,\widetilde{v}_0),$$

with

$$\tau_*=\frac{2\Psi(0)}{\beta(p-2)}.$$

Let β, $0<\beta\leqslant -p\overline{E}(0)/(p-1)$ be fixed. We have

$$\frac{\partial h}{\partial\tau}(\tau,\beta)=\frac{\tau(\tau-2\tau_*)}{(p-2)(\tau-\tau_*)^2}\quad\text{for all } \tau>\tau_*,$$

which implies that the function $\tau\mapsto h(\tau,\beta)$ is decreasing in $(\tau_*,2\tau_*)$, and increasing in $(2\tau_*,\infty)$, so

$$\begin{aligned}h(\tau,\beta)\geqslant h(2\tau_*,\beta)&=\frac{4\tau_*}{p-2}=\frac{8\Psi(0)}{\beta(p-2)^2}\\&\geqslant\frac{8\Psi(0)}{-p\overline{E}(0)(p-2)^2/(p-1)}\\&=\frac{-8(p-1)\Psi(0)}{p(p-2)^2\overline{E}(0)}=T_{\infty\max}\quad\text{for all } (\beta,\tau)\in\widetilde{D}(\widetilde{u}_0,\widetilde{v}_0).\end{aligned}\tag{5.26}$$

From (5.25) and (5.26), we get $T_\infty\leqslant -8(p-1)\Psi(0)/(p(p-2)^2\overline{E}(0))=T_{\infty\max}$. Thus, (5.4) is proved.

Next, we seek a lower bound for the blow-up time T_∞. We have

$$\begin{aligned}\Psi'(t)=&-\bar{a}(t;u(t),u(t))-\|\sqrt{\mu_2(t)}v_x(t)\|^2+\int_0^t g(t-s)\langle v_x(s),v_x(t)\rangle\,\mathrm{d}s\\&+\langle f_1(u(t),v(t)),u(t)\rangle+\langle f_2(u(t),v(t)),v(t)\rangle.\end{aligned}\tag{5.27}$$

The terms on the right-hand side of (5.27) are also estimated as

$$\int_0^t g(t-s)\langle v_x(s),v_x(t)\rangle\,\mathrm{d}s\leqslant\frac{1}{2}(g*v)(t)+\frac{3}{2}\overline{g}(t)\|v_x(t)\|^2,\tag{5.28}$$

$$\langle f_1(u(t),v(t)),u(t)\rangle+\langle f_2(u(t),v(t)),v(t)\rangle\leqslant d_2\overline{\mathcal{F}}(t),\tag{5.29}$$

$$\|u(t)\|_a^2+\|v_x(t)\|^2\leqslant\frac{2}{\lambda_*}\Psi(t),\ \text{with } \lambda_*=\min\{\lambda_1,\lambda_2\},\tag{5.30}$$

$$(5.31)\qquad \overline{\mathcal{F}}(t) \leqslant \bar{d}_2 \sum_{i=1}^{N}(\|u(t)\|_{L^{q_i}}^{q_i} + \|v(t)\|_{L^{\bar{q}_i}}^{\bar{q}_i}) \leqslant \bar{d}_2 \sum_{i=1}^{N}(\|u(t)\|_{a_1}^{q_i} + \|v(t)\|_{a_2}^{\bar{q}_i})$$
$$\leqslant \bar{d}_2 \sum_{i=1}^{N}\left[\left(\frac{2}{\lambda_*}\Psi(t)\right)^{q_i/2} + \left(\frac{2}{\lambda_*}\Psi(t)\right)^{\bar{q}_i/2}\right]$$
$$\leqslant d_3 \sum_{i=1}^{N}[(\Psi(t))^{q_i/2} + (\Psi(t))^{\bar{q}_i/2}]$$
$$\leqslant d_3 \sum_{i=1}^{N}[1 + (\Psi(t))^{q_i/2} + (\Psi(t))^{\bar{q}_i/2}],$$

where

$$d_3 = \bar{d}_2 \max\left\{\left(\frac{2}{\lambda_*}\right)^{q_i/2}, \left(\frac{2}{\lambda_*}\right)^{\bar{q}_i/2},\ i = 1, \dots, N\right\}.$$

On the other hand,

$$(5.32)\quad \overline{E}(t)+\overline{\mathcal{F}}(t)-\frac{1}{2}\bar{a}(t; u(t), u(t))-\frac{1}{2}(\|\sqrt{\mu_2(t)}v_x(t)\|^2-\overline{g}(t)\|v_x(t)\|^2) = \frac{1}{2}(g{*}v)(t).$$

Combining (5.2), (5.28)–(5.32), it leads to

(5.33)

$$\Psi'(t) \leqslant -\bar{a}(t; u(t), u(t)) - \|\sqrt{\mu_2(t)}v_x(t)\|^2 + \frac{1}{2}(g * v)(t)$$
$$+ \frac{3}{2}\overline{g}(\infty)\|v_x(t)\|^2 + d_2\overline{\mathcal{F}}(t)$$
$$= -\bar{a}(t; u(t), u(t)) - \|\sqrt{\mu_2(t)}v_x(t)\|^2$$
$$+ \left[\overline{E}(t) + \overline{\mathcal{F}}(t) - \frac{1}{2}\bar{a}(t; u(t), u(t)) - \frac{1}{2}(\|\sqrt{\mu_2(t)}v_x(t)\|^2 - \overline{g}(t)\|v_x(t)\|^2)\right]$$
$$+ \frac{3}{2}\overline{g}(\infty)\|v_x(t)\|^2 + d_2\overline{\mathcal{F}}(t)$$
$$\leqslant \overline{E}(t) - \frac{3}{2}\bar{a}(t; u(t), u(t)) - \frac{3}{2}\|\sqrt{\mu_2(t)}v_x(t)\|^2 + 2\overline{g}(\infty)\|v_x(t)\|^2 + (1 + d_2)\overline{\mathcal{F}}(t)$$
$$\leqslant 2\overline{g}(\infty)\|v_x(t)\|^2 + (1 + d_2)\overline{\mathcal{F}}(t) \leqslant 2\overline{g}(\infty)\|v_x(t)\|^2 + (1 + d_2)\overline{\mathcal{F}}(t)$$
$$\leqslant \frac{4\overline{g}(\infty)}{\lambda_*}\Psi(t) + (1 + d_2)d_3 \sum_{i=1}^{N}[1 + (\Psi(t))^{q_i/2} + (\Psi(t))^{\bar{q}_i/2}]$$
$$= G_1(\Psi(t)),$$

where

$$G_1(z) = \frac{4\overline{g}(\infty)}{\lambda_*}z + (1 + d_2)d_3 \sum_{i=1}^{N}(1 + z^{q_i/2} + z^{\bar{q}_i/2})$$

is defined as in (5.5).

From (5.33), it gives that

$$t \geqslant \int_0^t \frac{\Psi'(s)\,\mathrm{d}s}{G_1(\Psi(s))} = \int_{\Psi(0)}^{\Psi(t)} \frac{\mathrm{d}z}{G_1(z)}.$$

Therefore, we derive the lower bound for T_∞ as

$$T_\infty \geqslant \int_{\Psi(0)}^{\infty} \frac{\mathrm{d}z}{G_1(z)} = T_{\infty\,\min}.$$

Theorem 5.1 is proved. □

b. Blow-up solutions with nonnegative initial energy.

First, we put

$$y(t) \equiv \sqrt{2\Psi(t)} = \sqrt{\|u(t)\|_{\widehat{a}_1}^2 + \|v(t)\|_{\widehat{a}_2}^2}, \tag{5.34}$$

then

$$\sqrt{\lambda_*}\sqrt{\|u(t)\|_a^2 + \|v_x(t)\|^2} \leqslant y(t) \leqslant \sqrt{1+\lambda^*}\sqrt{\|u(t)\|_a^2 + \|v_x(t)\|^2} \tag{5.35}$$

with

$$\lambda_* = \min\{\lambda_1, \lambda_2\}, \quad \lambda^* = \max\{\lambda_1, \lambda_2\}.$$

By $(\widetilde{\mathrm{H}}_1)$, (H_2^B)–(H_4^B), (H_{4*}^B) and (5.35), we have

$$\begin{aligned}
\overline{E}(t) &= \frac{1}{2}(g*v)(t) + \frac{1}{2}\bar{a}(t; u(t), u(t)) \\
&\quad + \frac{1}{2}(\|\sqrt{\mu_2(t)}v_x(t)\|^2 - \overline{g}(t)\|v_x(t)\|^2) - \overline{\mathcal{F}}(t) \\
&\geqslant \frac{1}{2}L_*(\|u(t)\|_a^2 + \|v_x(t)\|^2) - \overline{d}_2 \sum_{i=1}^N (\|u(t)\|_{L^{q_i}}^{q_i} + \|v(t)\|_{L^{\bar{q}_i}}^{\bar{q}_i}) \\
&\geqslant \frac{1}{2}L_* \frac{y^2(t)}{1+\lambda^*} - \overline{d}_2 \operatorname*{Max}_{1\leqslant i\leqslant N}\left\{\frac{1}{(\sqrt{\lambda^*})^{q_i}} + \frac{1}{(\sqrt{\lambda^*})^{\bar{q}_i}}\right\} \sum_{i=1}^N [(y(t))^{q_i} + (y(t))^{\bar{q}_i}] \\
&= \frac{L_*}{1+\lambda^*}\left[\frac{1}{2}y^2(t) - \bar{d}_* \sum_{i=1}^N [(y(t))^{q_i} + (y(t))^{\bar{q}_i}]\right] = G(y(t)),
\end{aligned}$$

where

$$\begin{aligned}
G(\lambda) &= \frac{L_*}{1+\lambda^*}\left[\frac{1}{2}\lambda^2 - \bar{d}_* \sum_{i=1}^N (\lambda^{q_i} + \lambda^{\bar{q}_i})\right] \quad \text{for all } \lambda \geqslant 0, \\
\bar{d}_* &= \frac{(1+\lambda^*)\overline{d}_2}{L_*} \operatorname*{Max}_{1\leqslant i\leqslant N}\left\{\frac{1}{(\sqrt{\lambda^*})^{q_i}} + \frac{1}{(\sqrt{\lambda^*})^{\bar{q}_i}}\right\}.
\end{aligned}$$

Then we can prove the following lemma, let us omit its proof.

Lemma 5.2.

(i) *The equation* $G'(\lambda)=0$ *has a unique positive solution* λ_0 *satisfying*

$$1-\bar{d}_*\sum_{i=1}^{N}(q_i\lambda_0^{q_i-2}+\bar{q}_i\lambda_0^{\bar{q}_i-2})=0;$$

(ii) $G(0)=0,\ \lim\limits_{\lambda\to\infty}G(\lambda)=-\infty$;

(iii) $G'(\lambda)>0$ *if* $\lambda\in(0,\lambda_0)$ *and* $G'(\lambda)<0$ *if* $\lambda>\lambda_0$.

We need more the following lemma, which is similar to the lemma used firstly by Vitillaro in [33].

Lemma 5.3. *Assume that* $\overline{E}(0)<G(\lambda_0)$. *Then:*

(i) *if* $y(0)=\sqrt{\|\widetilde{u}_0\|^2_{\widehat{a}_1}+\|\widetilde{v}_0\|^2_{\widehat{a}_2}}>\lambda_0$, *then there exists* $\widehat{\lambda}_2>\lambda_0$ *such that*

$$y(t)\geqslant\widehat{\lambda}_2\quad\textit{for all } t\in[0,T_\infty);$$

(ii) *if* $y(0)=\sqrt{\|\widetilde{u}_0\|^2_{\widehat{a}_1}+\|\widetilde{v}_0\|^2_{\widehat{a}_2}}<\lambda_0$ *and* $\overline{E}(0)\geqslant0$, *then there exists* $\widehat{\lambda}_1\in[0,\lambda_0)$ *such that*

$$y(t)\leqslant\widehat{\lambda}_1\quad\textit{for all } t\in[0,T_\infty).$$

Proof. The argument used is similar to that of [19], [20], [33]. □

Theorem 5.2. *Let* (H_2^B)–(H_4^B) *and* (H_{4*}^B) *hold. For any initial conditions* $(\widetilde{u}_0,\widetilde{v}_0)\in V\times H_0^1$ *such that* $\sqrt{\|\widetilde{u}_0\|^2_{\widehat{a}_1}+\|\widetilde{v}_0\|^2_{\widehat{a}_2}}>\lambda_0$, *in addition assume that*

$$\begin{aligned}
&0\leqslant\overline{E}(0)<\min\Big\{G(\lambda_0),\frac{1}{p}\varrho\widehat{\lambda}_2^2\Big\},\\
&\varrho=\min\Big\{\frac{(p-2)\mu_{1*}}{2},\frac{(p-1)^2}{2p}\Big[\frac{p(p-2)\mu_{2*}}{(p-1)^2}-\overline{g}(\infty)\Big]\Big\},\\
&\int_{\Psi(0)}^{\infty}\frac{\mathrm{d}z}{G_2(z)}\leqslant\frac{8(p-1)\Psi(0)}{(p-2)^2(\varrho\widehat{\lambda}_2^2-p\overline{E}(0))}
\end{aligned}$$

with

$$G_2(z)=\overline{E}(0)+\frac{4\overline{g}(\infty)}{\lambda_*}z+(1+d_2)d_3\sum_{i=1}^{N}(z^{q_i/2}+z^{\bar{q}_i/2}), \tag{5.36}$$

where the constants λ_*, d_3 are defined as in (5.5). *Then the weak solution* (u,v) *of Problem* (4.1) *blows up at finite time and the lifespan* T_∞ *is estimated by*

$$\int_{\Psi(0)}^{\infty} \frac{\mathrm{d}z}{G_2(z)} \equiv T_{\infty\,\min} \leqslant T_\infty \leqslant \frac{8(p-1)\Psi(0)}{(p-2)^2(\varrho\widehat{\lambda}_2^2 - p\overline{E}(0))} \equiv T_{\infty\,\max}.$$

Proof. We also prove that the solution (u,v) obtained here is not a global solution in $\mathbb{R}_+$. By contradiction, we also assume that the weak solution exists in the whole interval $\mathbb{R}_+$. By the same method as in proofs of Theorem 5.1, with

$$\begin{aligned}
&0 \leqslant \overline{E}(0) \leqslant \frac{1}{p}\varrho\widehat{\lambda}_2^2,\\
&0<\beta \leqslant \frac{\varrho\widehat{\lambda}_2^2 - p\overline{E}(0)}{p-1},\\
&\varrho = \min\left\{\frac{(p-2)\mu_{1*}}{2}, \frac{(p-1)^2}{2p}\left[\frac{p(p-2)\mu_{2*}}{(p-1)^2} - \overline{g}(\infty)\right]\right\},\\
&\tau > \frac{2\Psi(0)}{\beta(p-2)} \text{ and } T_0 \geqslant \frac{\beta\tau^2}{(p-2)\beta\tau - 2\Psi(0)},
\end{aligned}$$

we can define the auxiliary functional $\Gamma\colon\ [0,T_0] \to \mathbb{R}$ as

$$\Gamma(t) = 2\int_0^t \Psi(s)\,\mathrm{d}s + 2(T_0 - t)\Psi(0) + \beta(t+\tau)^2, \quad 0 \leqslant t \leqslant T_0.$$

From (5.22) and Lemma 5.3 (i), it follows that

$$\begin{aligned}
(5.37)\qquad D(t) &\geqslant -p\overline{E}(0) - (p-1)\beta + \frac{(p-2)\mu_{1*}}{2}\|u(t)\|_a^2\\
&\quad + \frac{(p-1)^2}{2p}\left[\left(1 - \frac{1}{(p-1)^2}\right)\mu_{2*} - \overline{g}(\infty)\right]\|v_x(t)\|^2\\
&\geqslant -p\overline{E}(0) - (p-1)\beta + \varrho(\|u(t)\|_a^2 + \|v_x(t)\|^2)\\
&\geqslant -p\overline{E}(0) - (p-1)\beta + \varrho\widehat{\lambda}_2^2 \geqslant 0,
\end{aligned}$$

since $0 < (p-1)\beta \leqslant \varrho\widehat{\lambda}_2^2 - p\overline{E}(0)$.

Because of $0 < \beta \leqslant (\varrho\widehat{\lambda}_2^2 - p\overline{E}(0))/(p-1)$,

$$\varrho = \min\left\{\frac{(p-2)\mu_{1*}}{2}, \frac{(p-1)^2}{2p}\left[\frac{p(p-2)\mu_{2*}}{(p-1)^2} - \overline{g}(\infty)\right]\right\},$$

which implies from (5.17) and (5.37) that

$$(5.38)\qquad \Gamma^{p/2-1}(t) \geqslant \frac{2\Gamma^{p/2}(0)}{(p-2)\Gamma'(0)}\frac{1}{T_* - t} \quad \text{for all } t \in [0, t_{\min}),$$

where $t_{\min} = \min\{T_*, T_0,\}$, with $T_* = 2\Gamma(0)/((p-2)\Gamma'(0))$.

By

$$0 < \beta \leqslant \frac{\varrho\widehat{\lambda}_2^2 - p\overline{E}(0)}{p-1}, \quad \tau > \frac{2\Psi(0)}{\beta(p-2)}$$

and

$$T_0 \geqslant \frac{\beta\tau^2}{(p-2)\beta\tau - 2\Psi(0)}$$

we get

$$T_* = \frac{2\Gamma(0)}{(p-2)\Gamma'(0)} = \frac{2T_0\Psi(0) + \beta\tau^2}{(p-2)\beta\tau} \in (0, T_0].$$

From (5.38), it leads to $\lim_{t\to T_*^-} \Gamma(t) = \infty$. This is also a contradiction, so the solution blows up at finite time.

As in proofs of Theorem 5.1, we also have

$$T_\infty \leqslant \frac{8(p-1)\Psi(0)}{(p-2)^2(\varrho\widehat{\lambda}_2^2 - p\overline{E}(0))} = T_{\infty\,\max}.$$

Finally, we will find a lower bound for the blow-up time T_∞ for the solution (u, v). We have

$$\begin{aligned} \Psi'(t) &\leqslant \overline{E}(t) - \frac{3}{2}\bar{a}(t; u(t), u(t)) - \frac{3}{2}\|\sqrt{\mu_2(t)}v_x(t)\|^2 \\ &\quad + 2\overline{g}(\infty)\|v_x(t)\|^2 + (1+d_2)\overline{\mathcal{F}}(t) \\ &\leqslant \overline{E}(0) + 2\overline{g}(\infty)\|v_x(t)\|^2 + (1+d_2)\overline{\mathcal{F}}(t) \\ &\leqslant \overline{E}(0) + \frac{4\overline{g}(\infty)}{\lambda_*}\Psi(t) + (1+d_2)d_3\sum_{i=1}^N[(\Psi(t))^{q_i/2} + (\Psi(t))^{\bar{q}_i/2}] \\ &= G_2(\Psi(t)), \end{aligned} \tag{5.39}$$

where $G_2(z)$ is defined as in (5.36). From (5.39) we get

$$T_\infty \geqslant \int_{\Psi(0)}^\infty \frac{\mathrm{d}z}{G_2(z)} = T_{\infty\,\min}.$$

Theorem 5.2 is proved. □

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