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COEFFICIENT ESTIMATES AND FEKETE-SZEGÖ FUNCTIONAL
FOR SUBCLASSES OF BI-UNIVALENT FUNCTIONS
WITH RESPECT TO SYMMETRIC POINTS ASSOCIATED
WITH GEGENBAUER POLYNOMIALS

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Abstract. In the present article, the authors introduce two new subclasses of holomorphic and bi-univalent functions with respect to the symmetric points defined in the domain of open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ by making use of subordination between two analytic functions and also using the Gegenbauer polynomials. We investigate bounds of some of the initial Taylor-Maclaurin coefficients belonging to this newly constructed holomorphic and bi-univalent function class. Moreover, we derive the well-known Fekete-Szegö functional for the above said classes. Some of the corollaries of the main results are pointed out.

Keywords: analytic function; bi-univalent function; subordination; Fekete-Szegö functional; Gegenbauer polynomial

MSC 2020: 30C45, 30C50, 30C80

1. INTRODUCTION AND MOTIVATION

Let \mathcal{A} represent the family of all holomorphic functions $h(z)$ normalized by $h(0) = h'(0) - 1 = 0$ defined in the domain of the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. In view of the above normalization, the function $h(z)$ admits a Taylor-Maclaurin series expansion of the form

$$(1.1) \quad h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \Delta.$$

Let \mathcal{S} be the subclass of \mathcal{A} consisting of all functions $h \in \mathcal{A}$ that are univalent in Δ .

One of the important concept in the theory of univalent function is the principle of subordination between two analytic functions. Let the functions $h(z)$ and $g(z)$

be analytic in Δ . Then we say the function h is subordinate to g if there exists a Schwarz function ω with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \Delta$ such that $h(z) = g(\omega(z))$. We denote this subordination by $h(z) \prec g(z)$, $z \in \Delta$. Further, if the function g is an univalent function in Δ , then the following equivalence holds (see [11]):

$$h(z) \prec g(z) \iff h(0) = g(0) \quad \text{and} \quad h(\Delta) \subset g(\Delta).$$

Sakaguchi [16] introduced and studied the class of functions starlike with respect to symmetric points denoted by \mathcal{S}_s^* , which consists of functions $h \in \mathcal{S}$ satisfying the condition

$$(1.2) \quad \Re \left\{ \frac{zh'(z)}{[h(z) - h(-z)]} \right\} > 0, \quad z \in \Delta.$$

Further, Wang et al. [21] introduced another class \mathcal{C}_s , namely convex functions with respect to symmetric points, which consists of functions $h \in \mathcal{S}$ satisfying the condition

$$(1.3) \quad \Re \left\{ \frac{[zh'(z)]'}{[h(z) - h(-z)]'} \right\} > 0, \quad z \in \Delta.$$

In terms of subordination, Goel and Mehrok [8] introduced the subclass of \mathcal{S} denoted by $\mathcal{S}_s^*(A, B)$ defined to be the class of functions of the form (1.1) that satisfies the condition

$$(1.4) \quad \frac{2zh'(z)}{[h(z) - h(-z)]} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in \Delta.$$

In the similar manner, we can define the class $\mathcal{C}_s(A, B)$ to be the class of functions of the form (1.1) satisfying the following subordination condition:

$$(1.5) \quad \frac{2[zh'(z)]'}{[h(z) - h(-z)]'} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in \Delta.$$

Following them, many authors introduced the analogue definitions by extension (see [9]).

The Koebe one-quarter theorem [5] ensures that the image of Δ under every univalent function $h \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. This means that every function $h \in \mathcal{S}$ retains an inverse $h^{-1}: h(\Delta) \rightarrow \Delta$ such that

$$h^{-1}(h(z)) = z, \quad z \in \Delta$$

and

$$h(h^{-1}(\omega)) = \omega, \quad |\omega| < r_0(h), \quad r_0(h) \geq \frac{1}{4},$$

where

$$(1.6) \quad h^{-1}(\omega) = g(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \dots$$

A function $h \in \mathcal{A}$ is said to be bi-univalent in Δ if both the functions h and h^{-1} are univalent in Δ . The collection of bi-univalent functions within the domain Δ is denoted by \mathfrak{B} . The class \mathfrak{B} is nonempty as it contains at least the function like

$$h_1(z) = \frac{z}{1-z}, \quad h_2(z) = -\log(1-z), \quad h_3(z) = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right)$$

with their corresponding inverse functions

$$h_1^{-1}(\omega) = \frac{\omega}{1+\omega}, \quad h_2^{-1}(\omega) = \frac{e^\omega - 1}{e^\omega}, \quad h_3^{-1}(\omega) = \frac{e^{2\omega} - 1}{e^{2\omega} + 1},$$

respectively. It may be noted that the familiar Koebe function $K(z) = z/(1-z)^2 \notin \mathfrak{B}$ as it maps the open unit disk $\Delta \subset \mathbb{C}$ to $K(\Delta) = \mathbb{C} \setminus (-\infty, -\frac{1}{4}]$, which does not include Δ .

A good amount of literature are available for the bounds of the coefficients estimate for the class of bi-univalent functions. The concept of bi-univalent analytic functions was introduced by Lewin [10] and he showed that $|a_2| < 1.51$. Later on, Brannan and Clunie [3] conjectured that $|a_2| < \sqrt{2}$ for $h \in \mathfrak{B}$. Netanyahu [13] showed that $\max_{h \in \mathfrak{B}} |a_2| = \frac{4}{3}$. The coefficient estimate problem for the class \mathfrak{B} is about half a century old. Any good outcome on this topic is recognized world wide as a significant contribution. The work of Srivastava et al. [18] on bi-univalent function instead of closing a chapter, opened new direction to the researchers to work on it. Inspired by their works, many researchers investigated the coefficient bounds for various subclasses of bi-univalent function classes [1], [4], [12], [14], [15], [19], [20]. However, the bounds of $|a_n|$, $n \in \mathbb{N} \setminus \{2, 3\}$ known as coefficient estimation problem has become an open problem. Under certain conditions, in some works the bounds of the higher order coefficients are determined by applying Faber polynomial method [2], [6]. Further, for finding an upper bound for the functional $|a_3 - \mu a_2^2|$ of $f \in \mathcal{S}$ constitute the Fekete-Szegő type inequality [7]. It initiates from the disproof of the Little-Paley conjecture that the coefficients of odd univalent function are bounded by one.

Orthogonal polynomials play a significant role in all areas of mathematics and engineering such as approximation theory, theory of differential and integral equations, mathematical statistics, quantum mechanics, automatic control, signal analysis and others. Gegenbauer polynomials or ultraspherical polynomial named after Leopold Gegenbauer are orthogonal polynomials on the interval $[-1, 1]$ with respect to the

weight function $(1-x^2)^{\alpha-1/2}$. They generalize Legendre polynomial and Chebyshev polynomial and are special cases of Jacobi polynomials.

For nonzero real constant α , the generating function of Gegenbauer polynomial is defined to be

$$(1.7) \quad H_\alpha(x, z) = \frac{1}{(1-2xz+z^2)^\alpha}, \quad x \in [-1, 1], \quad z \in \Delta.$$

For fixed x , the function H_α is analytic in Δ . So it can be expanded in a Taylor series as

$$(1.8) \quad H_\alpha(x, z) = \sum_{n=0}^{\infty} c_n^\alpha(x) z^n,$$

where $c_n^\alpha(x)$ is a Gegenbauer polynomial of degree n . Clearly, H_α generates nothing when $\alpha = 0$. Therefore, the generating function of the Gegenbauer polynomial is set to be

$$H_0(x, z) = 1 - \log(1-2xz+z^2) = \sum_{n=0}^{\infty} c_n^0(x) z^n.$$

It is worthy to mention here that a normalization of α to be greater than $-\frac{1}{2}$ is desirable. These polynomials can also be defined in terms of recurrence relation as

$$(1.9) \quad c_n^\alpha(x) = \frac{1}{n} [2x(n+\alpha-1)c_{n-1}^\alpha(x) - (n+2\alpha-2)c_{n-2}^\alpha(x)], \quad n \geq 2$$

with the initial values

$$(1.10) \quad c_0^\alpha(x) = 1, \quad c_1^\alpha(x) = 2\alpha x \quad \text{and} \quad c_2^\alpha(x) = 2\alpha(1+\alpha)x^2 - \alpha.$$

These polynomials are the particular solution of Gegenbauer differential equation given by

$$(1.11) \quad (1-x^2)y'' - (2\alpha+1)xy' + n(n+2\alpha)y = 0.$$

Remark 1.1.

- (i) When $\alpha = \frac{1}{2}$, equation (1.11) reduces to Legendre differential equation and the Gegenbauer polynomial reduces to Legendre polynomial.
- (ii) When $\alpha = 1$, equation (1.11) reduces to the Chebyshev differential equation and Gegenbauer polynomial reduces to Chebyshev polynomial of second kind.

Recently, researchers have introduced several bi-univalent function classes associated with orthogonal polynomials like Legendre polynomial, Lagurre polynomial, Hordam polynomial, Chebyshev polynomial, Lucas polynomials, balancing polynomial and so on. Few research articles are devoted to Gegenbauer polynomial.

Motivated by aforementioned works, we introduce two new subclasses of bi-univalent functions class \mathfrak{B} by using generating function of Gegenbauer polynomial.

Definition 1.2. Let α be a nonzero real constant. A function $h \in \mathfrak{B}$ of the form (1.1) is said to be in the function class $R_s(\alpha)$ if it satisfies the following condition:

$$(1.12) \quad \frac{2zh'(z)}{h(z) - h(-z)} \prec H_\alpha(x, z), \quad z \in \Delta,$$

and

$$(1.13) \quad \frac{2wg'(\omega)}{g(\omega) - g(-\omega)} \prec H_\alpha(x, \omega), \quad \omega \in \Delta,$$

where $x \in (\frac{1}{2}, 1]$, the function $g = h^{-1}(\omega)$ is defined in (1.6) and H_α is the generating function of Gegenbauer polynomial given by (1.8).

Remark 1.3. Taking $\alpha = 1$ and $\alpha = \frac{1}{2}$ in Definition 1.2, the above function class reduces to $R_s(1)$ and $R_s(\frac{1}{2})$ related to Chebyshev polynomial of second kind and Legendre polynomial, respectively.

Definition 1.4. A function $h \in \mathfrak{B}$ of the form (1.1) is a member of the class $R_c(\alpha)$, $\alpha \neq 0 \in \mathbb{R}$ if it satisfies the following subordination condition:

$$(1.14) \quad \frac{2[zh'(z)]'}{[h(z) - h(-z)]'} \prec H_\alpha(x, z), \quad z \in \Delta,$$

and

$$(1.15) \quad \frac{2[wg'(\omega)]'}{[g(\omega) - g(-\omega)]'} \prec H_\alpha(x, \omega), \quad \omega \in \Delta,$$

where $x \in (\frac{1}{2}, 1]$, the function $g = h^{-1}(\omega)$ is defined in (1.6) and H_α is the generating function of Gegenbauer polynomial given by (1.8).

Remark 1.5. Letting $\alpha = 1$ and $\alpha = \frac{1}{2}$ in Definition 1.4, we get the class $R_c(1)$ and $R_c(\frac{1}{2})$ related to Chebyshev polynomial of second kind and Legendre polynomial, respectively.

In this paper by virtue of generating function of Gegenbauer polynomial, we derive estimates for the initial coefficients of the subclasses of bi-univalent functions. Further, we investigate the Fekete-Szegö problem for the functions in this classes.

Unless otherwise stated, we assume throughout our sequel that $x \in (\frac{1}{2}, 1]$ and α is a nonzero real constant.

2. COEFFICIENT BOUNDS FOR THE CLASS $R_s(\alpha)$

In this section, we determine the initial coefficient bounds for the class $R_s(\alpha)$ of bi-univalent functions.

Theorem 2.1. *Let the function $h \in \mathfrak{B}$ given by (1.1) belong to the function class $R_s(\alpha)$. Then*

$$(2.1) \quad |a_2| \leq \frac{\sqrt{2}|\alpha|x^{3/2}}{\sqrt{|1-2x^2|}}$$

and

$$(2.2) \quad |a_3| \leq \alpha^2 x^2 + |\alpha|x.$$

Proof. Suppose that the function $h \in \mathfrak{B}$ given by (1.1) belongs to the class $R_s(\alpha)$. Then by Definition 1.2 there exist two holomorphic functions $u, v: \Delta \rightarrow \Delta$ given by

$$(2.3) \quad u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \dots, \quad z \in \Delta$$

and

$$(2.4) \quad v(\omega) = v_1 \omega + v_2 \omega^2 + v_3 \omega^3 + \dots, \quad \omega \in \Delta$$

with $u(0) = v(0)$ and $|u(z)| < 1$, $z \in \Delta$ and $|v(\omega)| < 1$, $\omega \in \Delta$ such that

$$(2.5) \quad \frac{2zh'(z)}{h(z) - h(-z)} = H_\alpha(x, u(z))$$

and

$$(2.6) \quad \frac{2wg'(\omega)}{g(\omega) - g(-\omega)} = H_\alpha(x, v(\omega)).$$

A simple computation shows that

$$(2.7) \quad \frac{2zh'(z)}{h(z) - h(-z)} = 1 + 2a_2z + 2a_3z^2 + \dots$$

and

$$(2.8) \quad \frac{2wg'(\omega)}{g(\omega) - g(-\omega)} = 1 - 2a_2\omega + (4a_2^2 - 2a_3)\omega^2 + \dots$$

Making use of generating function of Gegenbauer polynomial H_α defined in (1.8), equations (2.5) and (2.6) become

$$(2.9) \quad 1 + 2a_2z + 2a_3z^2 + \dots = 1 + c_1^\alpha(x)u_1z + [c_1^\alpha(x)u_2 + c_2^\alpha(x)u_1^2]z^2 + \dots$$

and

$$(2.10) \quad 1 - 2a_2\omega + [4a_2^2 - 2a_3]\omega^2 + \dots = 1 + c_1^\alpha(x)v_1\omega + [c_1^\alpha(x)v_2 + c_2^\alpha(x)v_1^2]\omega^2 + \dots$$

Comparing the corresponding coefficients in (2.9) and (2.10), we find that

$$(2.11) \quad 2a_2 = c_1^\alpha(x)u_1,$$

$$(2.12) \quad 2a_3 = c_1^\alpha(x)u_2 + c_2^\alpha(x)u_1^2$$

and

$$(2.13) \quad -2a_2 = c_1^\alpha(x)v_1,$$

$$(2.14) \quad 4a_2^2 - 2a_3 = c_1^\alpha(x)v_2 + c_2^\alpha(x)v_1^2.$$

From relations (2.11) and (2.13) we get

$$(2.15) \quad u_1 = -v_1$$

and

$$(2.16) \quad 8a_2^2 = [c_1^\alpha(x)]^2[u_1^2 + v_1^2] \implies a_2^2 = \frac{[c_1^\alpha(x)]^2[u_1^2 + v_1^2]}{8}.$$

Further, adding (2.12) and (2.14) and putting the value of $(u_1^2 + v_1^2)$ from (2.16) in the resulting expression, after simplification, one may get

$$(2.17) \quad a_2^2 = \frac{[c_1^\alpha(x)]^3(u_2 + v_2)}{4[(c_1^\alpha(x))^2 - 2c_2^\alpha(x)]}.$$

It is known that if $|u(z)| < 1$ and $|v(\omega)| < 1$ (see [5]), then

$$(2.18) \quad |u_j| \leq 1 \quad \text{and} \quad |v_j| \leq 1 \quad \text{for all } j \in \mathbb{N}.$$

By virtue of (1.10) and (2.18), equation (2.17) reduces to the desired estimates as mentioned in (2.1).

Next, subtracting (2.14) from (2.12) we have

$$(2.19) \quad 4(a_3 - a_2^2) = c_1^\alpha(x)(u_2 - v_2) + c_2^\alpha(x)(u_1^2 - v_1^2).$$

In view of (2.15), it follows from (2.19) that

$$(2.20) \quad a_3 = a_2^2 + \frac{1}{4}c_1^\alpha(x)(u_2 - v_2).$$

Taking modulus on both sides of (2.20) and making use of relations (2.16) and (2.18) in (2.20) we obtain the required bound for a_3 . The proof of Theorem 2.1 is thus completed. \square

Letting $\alpha = 1$ in Theorem 2.1 we get the following result in the form of the corollary:

Corollary 2.2. *Let the function $h \in \mathfrak{B}$ given by (1.1) be in the function class $R_s(1)$. Then*

$$|a_2| \leq \frac{\sqrt{2}x^{3/2}}{\sqrt{|1 - 2x^2|}}$$

and

$$|a_3| \leq x^2 + x.$$

Putting $\alpha = \frac{1}{2}$ in Theorem 2.1 we get the following result:

Corollary 2.3. *Let the function $h \in \mathfrak{B}$ given by (1.1) be in the function class $R_s(\frac{1}{2})$. Then*

$$|a_2| \leq \frac{x^{3/2}}{\sqrt{2|1 - 2x^2|}}$$

and

$$|a_3| \leq \frac{x(x + 2)}{4}.$$

3. FEKETE-SZEGÖ FUNCTIONAL FOR THE CLASS $R_s(\alpha)$

One of the famous coefficient problem of univalent analytic function is the Fekete-Szegö inequality. Fekete-Szegö (see [7]) states that if $h \in \mathcal{S}$, then

$$|a_3 - \nu a_2^2| \leq 1 + 2e^{-2\nu/(1-\nu)}, \quad 0 \leq \nu < 1.$$

The elementary inequality $|a_3 - a_2^2| \leq 1$ is obtained as $\nu \rightarrow 1$. The coefficient functional $F_\nu(h) = a_3 - \nu a_2^2$ on the normalized analytic functions h in the open unit disk Δ has a significant effect in geometric function theory. The Fekete-Szegö problem is known as the maximization problem for the functional $|F_\nu(h)|$.

Now we obtain Fekete-Szegö inequality $|a_3 - \nu a_2^2|$ for the class $R_s(\alpha)$ of bi-univalent function, where ν is a real parameter.

Theorem 3.1. *If the function $h(z)$ given by (1.1) is in the function class $R_s(\alpha)$, then for any real parameter ν we have*

$$(3.1) \quad |a_3 - \nu a_2^2| \leq \begin{cases} |\alpha|x, & |1 - \nu| \leq \left| \frac{1 - 2x^2}{2\alpha x^2} \right|, \\ \frac{2\alpha^2 x^3 |1 - \nu|}{|1 - 2x^2|}, & |1 - \nu| > \left| \frac{1 - 2x^2}{2\alpha x^2} \right|. \end{cases}$$

Proof. Let $h \in R_s(\alpha)$. From (2.17) and (2.20) and after simplification we get

$$(3.2) \quad \begin{aligned} a_3 - \nu a_2^2 &= (1 - \nu) \frac{[c_1^\alpha(x)]^3(u_2 + v_2)}{4[(c_1^\alpha(x))^2 - 2c_2^\alpha(x)]} + \frac{1}{4}c_1^\alpha(x)(u_2 - v_2) \\ &= \left[\frac{[c_1^\alpha(x)]^3(1 - \nu)}{4[(c_1^\alpha(x))^2 - 2c_2^\alpha(x)]} + \frac{c_1^\alpha(x)}{4} \right] u_2 \\ &\quad + \left[\frac{[c_1^\alpha(x)]^3(1 - \nu)}{4[(c_1^\alpha(x))^2 - 2c_2^\alpha(x)]} - \frac{c_1^\alpha(x)}{4} \right] v_2 \\ &= c_1^\alpha(x) \left[\left(K(\nu) + \frac{1}{4} \right) u_2 + \left(K(\nu) - \frac{1}{4} \right) v_2 \right], \end{aligned}$$

where

$$(3.3) \quad K(\nu) = \frac{[c_1^\alpha(x)]^2(1 - \nu)}{4[(c_1^\alpha(x))^2 - 2c_2^\alpha(x)]}.$$

By virtue of (1.10) and (2.18) we conclude that

$$(3.4) \quad |a_3 - \nu a_2^2| \leq \begin{cases} |\alpha|x, & |K(\nu)| \leq \frac{1}{4}, \\ 4|\alpha|x|K(\nu)|, & |K(\nu)| \geq \frac{1}{4}. \end{cases}$$

Estimate (3.1) follows from (3.4). This completes the proof of Theorem 3.1. \square

Putting $\alpha = 1$ in Theorem 3.1, we obtain Fekete-Szegő inequality for the class $R_s(1)$ related to Chebyshev polynomial of second kind.

Corollary 3.2. *Let the function $h \in \mathfrak{B}$ given by (1.1) be the member of the class $R_s(1)$. Then*

$$|a_3 - \nu a_2^2| \leq \begin{cases} x, & |1 - \nu| \leq \left| \frac{1 - 2x^2}{2x^2} \right|, \\ \frac{2x^3|1 - \nu|}{|1 - 2x^2|}, & |1 - \nu| > \left| \frac{1 - 2x^2}{2x^2} \right|. \end{cases}$$

Taking $\alpha = \frac{1}{2}$ in Theorem 3.1 we obtain the result of Fekete-Szegő problem for the class $R_s(\frac{1}{2})$ associated with Legendre polynomial as below.

Corollary 3.3. *Let the function $h \in \mathfrak{B}$ given by (1.1) be the member of the class $R_s(\frac{1}{2})$. Then*

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{x}{2}, & |1 - \nu| \leq \left| \frac{1 - 2x^2}{x^2} \right|, \\ \frac{x^3|1 - \nu|}{2|1 - 2x^2|}, & |1 - \nu| > \left| \frac{1 - 2x^2}{x^2} \right|. \end{cases}$$

4. COEFFICIENT BOUNDS FOR THE CLASS $R_c(\alpha)$

This section is devoted to finding the initial coefficient bounds of the function class $R_c(\alpha)$ of bi-univalent function.

Theorem 4.1. *Let the function $h \in \mathfrak{B}$ given by (1.1) be the member of the class $R_c(\alpha)$. Then*

$$(4.1) \quad |a_2| \leq \frac{|\alpha|x^{3/2}}{\sqrt{|2 - (4 + \alpha)x^2|}}$$

and

$$(4.2) \quad |a_3| \leq \frac{|\alpha|x}{3} + \frac{\alpha^2 x^2}{4}.$$

Proof. Let $h \in R_c(\alpha)$. Then by Definition 1.4, there exist two holomorphic functions u, v as stated in Theorem 2.1 such that

$$(4.3) \quad \frac{2[zh'(z)]'}{[h(z) - h(-z)]'} = H_\alpha(x, u(z)), \quad z \in \Delta$$

and

$$(4.4) \quad \frac{2[\omega g'(\omega)]'}{[g(\omega) - g(-\omega)]'} = H_\alpha(x, v(\omega)), \quad \omega \in \Delta.$$

A simple calculation shows that

$$(4.5) \quad \frac{2[zh'(z)]'}{[h(z) - h(-z)]'} = 1 + 4a_2z + 6a_3z^2 + \dots$$

and

$$(4.6) \quad \frac{2[\omega g'(\omega)]'}{[g(\omega) - g(-\omega)]'} = 1 - 4a_2\omega + (12a_2^2 - 6a_3)\omega^2 + \dots$$

Using (1.8), (4.5) in (4.3) and (1.8), (4.6) in (4.4) and comparing the corresponding coefficients we get

$$(4.7) \quad 4a_2 = c_1^\alpha(x)u_1,$$

$$(4.8) \quad 6a_3 = c_1^\alpha(x)u_2 + c_2^\alpha(x)u_1^2$$

and

$$(4.9) \quad -4a_2 = c_1^\alpha(x)v_1,$$

$$(4.10) \quad 12a_2^2 - 6a_3 = c_1^\alpha(x)v_2 + c_2^\alpha(x)v_1^2.$$

From relations (4.7) and (4.9) we get

$$(4.11) \quad u_1 = -v_1$$

and

$$(4.12) \quad 32a_2^2 = [c_1^\alpha(x)]^2(u_1^2 + v_1^2).$$

Adding relations (4.8) and (4.10), one can get

$$(4.13) \quad 12a_2^2 = c_1^\alpha(x)(u_2 + v_2) + c_2^\alpha(x)(u_1^2 + v_1^2).$$

Putting the value of $(u_1^2 + v_1^2)$ from (4.12) in (4.13), we obtain

$$(4.14) \quad a_2^2 = \frac{[c_1^\alpha(x)]^3(u_2 + v_2)}{4[3(c_1^\alpha(x))^2 - 8c_2^\alpha(x)]}.$$

Taking modulus on both sides of (4.14) and making use of (2.18) in the resulting equation we get

$$(4.15) \quad |a_2|^2 \leq \frac{|c_1^\alpha(x)|^3}{2|3(c_1^\alpha(x))^2 - 8c_2^\alpha(x)|}.$$

Further, subtracting (4.10) from (4.8) and using (4.11) we get

$$(4.16) \quad a_3 = a_2^2 + \frac{c_1^\alpha(x)(u_2 - v_2)}{12}.$$

Applying modulus on both sides of relation (4.16) and then using (4.12) and (2.18) in (4.16) we get the desired estimate as stated in (4.2). This proves the result of Theorem 4.1. \square

5. FEKETE-SZEGÖ FUNCTIONAL FOR THE CLASS $R_c(\alpha)$

In this section, we determine the upper bound of Fekete-Szegö functional $(a_3 - \nu a_2^2)$ for the function class $R_c(\alpha)$ of bi-univalent functions.

Theorem 5.1. *Suppose the function $h \in \mathfrak{B}$ given by (1.1) is a member of the class $R_c(\alpha)$. Then*

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{|\alpha|x}{3}, & |1 - \nu| \leq \frac{|2 - (4 + \alpha)x^2|}{3|\alpha|x^2}, \\ \frac{|1 - \nu|\alpha^2 x^3}{|2 - (4 + \alpha)x^2|}, & |1 - \nu| \geq \frac{|2 - (4 + \alpha)x^2|}{3|\alpha|x^2}. \end{cases}$$

Proof. Let $h \in R_c(\alpha)$. Then by using (4.14) and (4.16) for some $\nu \in \mathbb{R}$, we obtain

$$(5.1) \quad \begin{aligned} a_3 - \nu a_2^2 &= a_2^2 + \frac{c_1^\alpha(x)(u_2 - v_2)}{12} - \nu a_2^2 \\ &= (1 - \nu)a_2^2 + \frac{c_1^\alpha(x)(u_2 - v_2)}{12} \\ &= c_1^\alpha(x) \left[\left(\frac{(1 - \nu)(c_1^\alpha(x))^2}{4[3(c_1^\alpha(x))^2 - 8c_2^\alpha(x)]} + \frac{1}{12} \right) u_2 \right. \\ &\quad \left. + \left(\frac{(1 - \nu)(c_1^\alpha(x))^2}{4[3(c_1^\alpha(x))^2 - 8c_2^\alpha(x)]} - \frac{1}{12} \right) v_2 \right] \\ &= c_1^\alpha(x) \left[\left(\varphi(\nu) + \frac{1}{12} \right) u_2 + \left(\varphi(\nu) - \frac{1}{12} \right) v_2 \right], \end{aligned}$$

where

$$(5.2) \quad \varphi(\nu) = \frac{(1-\nu)(c_1^\alpha(x))^2}{4[3(c_1^\alpha(x))^2 - 8c_2^\alpha(x)]}.$$

Now we deduce that

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{|\alpha|x}{3}, & |\varphi(\nu)| \leq \frac{1}{12}, \\ 4|\alpha|x|\varphi(\nu)|, & |\varphi(\nu)| \geq \frac{1}{12}. \end{cases}$$

This proves the result of Theorem 5.1. \square

Putting $\alpha = 1$ in Theorem 5.1, we obtain Fekete-Szegő inequality for the class $R_c(1)$ related to Chebyshev polynomial of second kind.

Corollary 5.2. *Let the function $h \in \mathfrak{B}$ given by (1.1) be the member of the class $R_c(1)$. Then*

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{x}{3}, & |1-\nu| \leq \left| \frac{2-5x^2}{3x^2} \right|, \\ \frac{x^3|1-\nu|}{|2-5x^2|}, & |1-\nu| > \left| \frac{2-5x^2}{3x^2} \right|. \end{cases}$$

Taking $\alpha = \frac{1}{2}$ in Theorem 5.1 we obtain the result of Fekete-Szegő problem for the class $R_c(\frac{1}{2})$ associated with Legendre polynomial as below.

Corollary 5.3. *Let the function $h \in \mathfrak{B}$ given by (1.1) be the member of the class $R_c(\frac{1}{2})$. Then*

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{x}{6}, & |1-\nu| \leq \left| \frac{4-9x^2}{3x^2} \right|, \\ \frac{x^3|1-\nu|}{2|4-9x^2|}, & |1-\nu| > \left| \frac{4-9x^2}{3x^2} \right|. \end{cases}$$

Concluding Remarks. A good amount of literature motivates us to considered two new subclasses of holomorphic and bi-univalent functions defined in the open unit disk Δ which satisfy a certain subordination condition and are linked with the generating function of Gegenbauer or ultra spherical polynomial $c_n^\alpha(x)$ of order α and degree n in x . For functions belonging to this class, we have determined the estimate for initial Taylor-Macaurin coefficients $|a_2|$ and $|a_3|$ and also the Fekete-Szegő functional $|a_3 - \nu a_2^2|$ for real parameter ν . Here we may recall a recently

published survey-cum-expository review article in which Srivastava [17] explored the applications of quantum calculus or q -calculus, the fractional q -calculus and fractional q -derivative operator in geometric function theory of complex analysis on the study of coefficient problems. Researchers can generalize the results of the paper by making use of (p, q) -analogue of Gegenbauer polynomial.

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