Higher Structures

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Abstract

We give an example of a non-trivial linear operad that only admits trivial coalgebras and give sufficient conditions ensuring that the cofree coalgebra functor be faithful.

1. Introduction

In the same way that operads are used to encode algebraic data, they can equally be used for coalgebraic data. For example, the operad governing associative algebras controls at the same time coassociative coalgebras: one operation μ of arity 2 versus one cooperation δ of (co)-arity 2,

$$\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu) \quad \text{versus} \quad (\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta.$$

As we shall see, the theory of coalgebras over an operad is not always as gentle as that of algebras.

The word 'operad' was carefully crafted by Peter May as a portmanteau of 'operations' and 'monad' [3]. Indeed, to every operad P in a cocomplete closed symmetric monoidal category $(\mathfrak{C}, \otimes, \mathbf{1})$ is associated a monad $\widetilde{P}$ with underlying functor

$$X \longmapsto P \triangleleft X := \coprod_{n \in \mathbf{N}} P(n) \otimes_{\mathbf{S}_n} X^{\otimes n}.$$

The category of P-algebras is then the category of $\widetilde{P}$-modules

$$P\text{-alg} = \widetilde{P}\text{-mod}.$$

In many cases the assignment $P \mapsto \widetilde{P}$ is faithful (for example, operads in sets or operads in vector spaces [2]). In particular, P is the trivial operad if and only if $\widetilde{P}$ is the trivial monad. Monads

arising from operads are also well-behaved: when the unit $\mathbf{1} \to P$ is a split monomorphism, the free P-algebra functor is always faithful, that is the unit of the adjunction Free $\dashv$ Forget

$$X \longrightarrow P \triangleleft X$$

is a monomorphism for every $X \in \mathfrak{C}$.

This is no longer the case for coalgebras over P. For one, the forgetful functor $P\text{-}\mathsf{cog} \to \mathfrak{C}$ may not have a right adjoint, that is, there may not exist cofree P-coalgebras. When it does, the category of P-coalgebras is then equivalent to the category of comodules over a comonad

$$P\text{-}\mathsf{cog} = \mathbf{L}^P\text{-}\mathsf{comod}$$

but the functor $P \mapsto \mathbf{L}^P$ may no longer be faithful. In what follows we give an example of a non-trivial linear operad Ins whose associated comonad is the zero comonad

$$\mathsf{Ins} \neq 0 \quad \text{and} \quad \mathbf{L}^{\mathsf{Ins}} = 0.$$

In other terms, all Ins-coalgebras are trivial. In the last section, we give sufficient conditions in the linear setting on operads P so that the cofree coalgebra functor be faithful, i.e. so that the counit of the comonad

$$\mathrm{L}^P X \longrightarrow X$$

be an epimorphism for every X.

2. Coalgebras over an operad

In a closed symmetric monoidal category with tensor product $\otimes$ and internal hom $[-, -]$, a coalgebra over an operad P is the data of an object V together with a morphism of operads $a \colon P \to \mathrm{Coend}_V$ where Coend_V is the coendomorphism operad of V given by

$$\mathrm{Coend}_V(n) := \left[V, V^{\otimes n}\right],$$

with obvious right $\mathbf{S}_n$-action and compositions. Similarly, a P-coalgebra structure on V is the data of maps $P(n) \otimes V \to V^{\otimes n}$ suitably associative, unital and equivariant.

In the cartesian case, the coendomorphism operad simplifies to

$$\mathrm{Coend}_V(n) = [V, V^n] = [V, V]^n = \mathrm{Coend}_V(1)^n.$$

This trivialises the theory of P-coalgebras in cartesian categories as is exemplified by the well-known fact that each set has a unique coassociative counital coalgebra structure, the diagonal. For this reason, we shall focus our attention on the additive case and fix a closed symmetric monoidal cocomplete additive category $(\mathfrak{A}, \otimes, \mathbf{1})$. By a dg-operad, we shall mean an operad in the symmetric monoidal category of chain complexes $(\mathrm{Ch}(\mathfrak{A}), \otimes, \mathbf{1})$.

We shall also assume that cofree coalgebras exist: for every morphism of dg-operads $f \colon P \to P'$, we require that the forgetful functor

$$P'\text{-}\mathsf{cog} \xrightarrow{\ f^* \ } P\text{-}\mathsf{cog}$$

admit a right adjoint f_*. In particular, for every dg-operad P, the unit morphism $\mathbf{1} \to P$ yields the adjunction

$$P\text{-}\mathsf{cog} \underset{\mathrm{Cofree}}{\overset{\mathrm{Forget}}{\rightleftarrows}} \mathrm{Ch}(\mathfrak{A}).$$

This is the case for example when $\mathfrak{A}$ is the category of vector spaces over a field, where cofree coalgebras can be computed for every dg-operad [1]. Thanks to this assumption, the category of P-coalgebras becomes comonadic over the category of chain complexes: the category of P-coalgebras is equivalent to the category of comodules over the comonad $\mathbf{L}^P$ coming from the Forget $\dashv$ Cofree adjunction. Its underlying functor is $\mathbf{L}^P = \text{Forget} \circ \text{Cofree}$. In addition, for every morphism $f \colon P \to P'$, the counit $f^* f_* \to \text{Id}$ induces a morphism of comonads (i.e. a morphism of comonoids in the category of endofunctors)

$$\mathbf{L}^f \colon \mathbf{L}^{P'} \longrightarrow \mathbf{L}^P$$

on $\text{Ch}(\mathfrak{A})$. In other words we assume the existence of a contravariant functor from the category of dg-operads to the category of comonads.

3. An operad without coalgebras

We shall build here an example of an operad without coalgebras in the category of vector spaces over a field $\boldsymbol{K}$ by adding to the unital associative operad uA an infinite number of operations of arity 0, the idea being that a locally finite dimensional coalgebra cannot support an infinite number of independent linear forms.

Let Ins be the operad in $\mathsf{Vect}_{\boldsymbol{K}}$ whose algebras are the uA-algebras $(\Lambda, \mu, \upsilon_0)$, endowed with other arity zero operations $\upsilon_n \colon \boldsymbol{K} \to \Lambda$ for every $n \geqslant 1$ and an arity 1 operation I_λ for every non-zero finitely supported sequence λ of elements of $\boldsymbol{K}$ with relation

$$\mu \circ \left(\sum_{n \in \mathbf{N}} \lambda_n \upsilon_n \otimes \text{id}_\Lambda \right) \circ I_\lambda = \text{id}_\Lambda.$$

The first thing to check is that Ins is not a trivial operad. For this one only needs to exhibit a non-trivial Ins-algebra. Let $\boldsymbol{L}/\boldsymbol{K}$ be a field extension of infinite dimension and let $\upsilon_0 := 1, \upsilon_1, \upsilon_2, \dots$ be a countable family of elements of $\boldsymbol{L}$ linearly independent over $\boldsymbol{K}$. For every λ, the sum $\sum_n \lambda_n \upsilon_n$ is invertible, let I_λ denote the multiplication by its inverse. Then $(\boldsymbol{L}, \times, \upsilon_0, \upsilon_1, \upsilon_2, \dots, I_\lambda)$ is a non-zero Ins-algebra.

The coalgebras over Ins are the uA-coalgebras $(V, \delta, \varepsilon_0)$ endowed with other cooperations ε_n and I_λ such that

$$I_\lambda \circ \left(\sum_{n \in \mathbf{N}} \lambda_n \varepsilon_n \otimes \text{id}_V \right) \circ \delta = \text{id}_V.$$

Any such coalgebra has to be trivial. Indeed, since V is in particular a uA-coalgebra, every element $y \in V$ generates a finite dimensional subcoalgebra $V_y \subset V$. Since V_y is stable by δ, it follows that for every λ, V_y is also stable by

$$\sigma_\lambda := \sum_{n \in \mathbf{N}} (\lambda_n \varepsilon_n \otimes \text{id}_V) \circ \delta.$$

As σ_λ is injective on V and V_y is finite dimensional, we conclude that it restricts to an automorphism of V_y. As a consequence either $V_y = 0$ or the linear forms $(\varepsilon_0, \varepsilon_1, \dots)$ are independent on V_y, which would contradict the finite dimensionality of V_y. As $V_y = 0$ only when $y = 0$, we conclude that $V = 0$.

4. Sane operads

Definition 4.1. We shall say that a dg-operad P is *sane* if the cofree P-coalgebra functor is faithful or equivalently if the counit of the adjunction Forget $\dashv$ Cofree

$$\mathrm{L}^P X \longrightarrow X,$$

is an epimorphism for every chain complex X.

Our goal is to prove that wide classes of dg-operads are sane. The main tool for that is the propagation lemma.

Lemma 4.2 (Propagation). *Let $f \colon P \to P'$ be a morphism of dg-operads. If P' is sane, so is P.*

Proof. Given a chain complex X, the counit map $\mathrm{L}^{P'} X \to X$ factors through the counit map $\mathrm{L}^P X \to X$ as

$$\mathrm{L}^{P'} X \xrightarrow{\ \mathrm{L}^f\ } \mathrm{L}^P X \longrightarrow X.$$

If the composite is epimorphic, so is the second map. $\qquad\square$

Let us say that a dg-operad P is *cofibrant* if for every given morphism $Q \to R$ of dg-operads that is both a degree-wise epimorphism and a quasi-isomorphism, every map $P \to R$ admits a lift

$$
\begin{array}{ccc}
 & & Q \\
 & \nearrow & \downarrow\wr \\
P & \longrightarrow & R.
\end{array}
$$

In particular, a cofibrant dg-operad must lift against $\mathbf{D}^1 \to 0$, where $\mathbf{D}^1$ is the reduced cellular model of the interval

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{1} \xrightarrow{\ \mathrm{id}\ } \mathbf{1} \longrightarrow 0 \longrightarrow \cdots$$

equipped with its canonical dg-algebra structure.

For a chain complex X, let $X^\vee := [X, \mathbf{1}]$. Recall that for every triple A, B, C of object in a symmetric closed monoidal category, one has a canonical morphism $[A, B] \otimes C \to [A, B \otimes C]$ obtained via adjunctions. One says that X is *dualisable* when the canonical map

$$X \otimes X^\vee \longrightarrow [X, X]$$

is an isomorphism. As an example, a chain complex of vector spaces is dualisable if and only if it is (bounded and) finite dimensional. When X is dualisable, there is given a canonical unit map $\mathbf{1} \to X \otimes X^\vee$ and one has the familiar

$$(X \otimes X)^\vee = X^\vee \otimes X^\vee \quad \text{and} \quad \left(X^\vee\right)^\vee = X.$$

Theorem 4.3 (Sanity check). *A dg-operad P is sane when either*
- *P admits an augmentation $P \to \mathbf{1}$;*
- *P is cofibrant;*
- *or $P(0)$ is dualisable and the unit $\mathbf{1} \to P$ is a degree-wise split monomorphism.*

Remark 4.4. We let to the reader the treat of proving that for operads in $\mathsf{Vect}_{\boldsymbol{K}}$ the situation is binary: given a $\boldsymbol{K}$-linear operad P, either all P-coalgebras are zero or for every vector space X, the counit map $\mathrm{L}^P X \to X$ is surjective.

The first two cases follow by propagation since both the unit operad and $\mathbf{D}^1$ are sane. We shall focus on the last case. Let us recall that a dg-operad P is *reduced* when $P(0) = 0$. Every dg-operad P admits a maximal reduced suboperad, that we shall denote by $\overline{P}$.

Lemma 4.5. *If P is a reduced dg-operad and the unit $\mathbf{1} \to P$ is a degree-wise split monomorphism, then P is sane.*

Proof. Since P is reduced, the inclusion $P(1) \subset P$ splits in the category of dg-operads. By propagation one is sane whenever the other is. Since $P(1)$ is concentrated in arity 1, for every chain complex X, the cofree $P(1)$-coalgebra generated by X is given by $[P(1), X]$ with coalgebra structure

$$[P(1), X] \longrightarrow [P(1), [P(1), X]] = [P(1) \otimes P(1), X]$$

induced by the multiplication map $P(1) \otimes P(1) \to P(1)$. The assumption that $\mathbf{1} \to P$ is a degree-wise split monomorphism ensures that for every X, the counit $[P(1), X] \to X$ is a degree-wise split epimorphism, so $P(1)$ is sane. $\hfill\square$

We are thus left to prove that in the case where $\overline{P}$ is sane and $P(0)$ is dualisable, P is sane. For this we shall investigate the adjunction between coalgebras over P and coalgebras over $\overline{P}$.

For every dg-operad, the chain complex $P(0)$ is the initial P-algebra. When $P(0)$ is dualisable, one has a canonical isomorphism of operads

$$\mathrm{End}_{P(0)} = \mathrm{Coend}_{P(0)^\vee}.$$

The P-algebra structure of $P(0)$ composed with the above isomorphism

$$P \longrightarrow \mathrm{End}_{P(0)} = \mathrm{Coend}_{P(0)^\vee}$$

gives a P-coalgebra structure on $P(0)^\vee$. This is the terminal P-coalgebra.

Definition 4.6. A pointed P-coalgebra is the data of a P-coalgebra V together with a morphism of P-coalgebras $P(0)^\vee \to V$, called a pointing. A morphism of pointed P-coalgebras is one that respects the pointing maps. Let $P\text{-}\mathsf{cog}_\bullet$ denote the category of pointed P-coalgebras.

Lemma 4.7. *The category of pointed P-coalgebras is equivalent to the category of $\overline{P}$-coalgebras.*

Proof. For any pointed P-coalgebra V, the cokernel $\overline{V}$ of the pointing map $P(0)^\vee \to V$ admits a canonical $\overline{P}$-coalgebra structure.

Conversely, given a $\overline{P}$-coalgebra $\overline{V}$, the chain complex $\overline{V} \oplus P(0)^\vee$ can be endowed with a canonical P-coalgebra structure. Its restriction to $P(0)^\vee$ is simply the P-coalgebra structure of $P(0)^\vee$. We now describe its restriction to $\overline{V}$. Let $n \geqslant 0$, the data of an $\mathbf{S}_n$-equivariant map $P(n) \otimes \overline{V} \to (\overline{V} \oplus P(0)^\vee)^{\otimes n}$ is equivalent to the data of $\mathbf{S}_p \times \mathbf{S}_q$-equivariant maps $P(n) \otimes \overline{V} \to \overline{V}^{\otimes p} \otimes (P(0)^\vee)^{\otimes q}$ for each decomposition $p + q = n$.

This structure map is zero if $p = 0$, otherwise it is given by the composition

$$\overline{V} \otimes P(n)$$

$$\downarrow \text{ unit of dualisable } P(0)^{\otimes q}$$

$$\overline{V} \otimes P(n) \otimes P(0)^{\otimes q} \otimes (P(0)^\vee)^{\otimes q}$$

$$\downarrow P(n) \otimes \mathbf{1}^{\otimes p} \otimes P(0)^{\otimes q} \to P(p)$$

$$\overline{V} \otimes P(p) \otimes (P(0)^\vee)^{\otimes q}$$

$$\downarrow \text{ coalgebra structure of } \overline{V}$$

$$\overline{V}^{\otimes p} \otimes (P(0)^\vee)^{\otimes q}$$

A straightforward checking shows this actually defines a P-coalgebra structure and that this construction $\overline{V} \mapsto V = \overline{V} \oplus P(0)^\vee$ is inverse to the previous construction $V \mapsto \overline{V}$. $\qquad\square$

The adjunction relating P-coalgebras to $\overline{P}$-coalgebras coming from the inclusion $\overline{P} \subset P$ is actually the following composite adjunction

$$P\text{-cog} \underset{\text{Forget}}{\overset{W \mapsto W \oplus P(0)^\vee}{\rightleftarrows}} P\text{-cog}_\bullet \underset{\overline{V} \mapsto \overline{V} \oplus P(0)^\vee}{\overset{V \mapsto \overline{V}}{\rightleftarrows}} \overline{P}\text{-cog}.$$

As a consequence, for every chain complex X the counit map $\mathrm{L}^P X \to X$ is the composition

$$\mathrm{L}^P X = \mathrm{L}^{\overline{P}} X \oplus P(0)^\vee \longrightarrow \mathrm{L}^{\overline{P}} X \longrightarrow X,$$

which implies that P is sane.

Acknowledgements

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