Higher Structures

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Abstract

We introduce and develop a categorification of the theory of Real representations of finite groups. In particular, we generalize the categorical character theory of Ganter–Kapranov and Bartlett to the Real setting. Given a Real representation of a finite group G, or more generally a categorical group, on a linear category, we associate a number, the modified secondary trace, to each graded commuting pair $(g, \omega) \in \mathsf{G} \times \hat{\mathsf{G}}$, where $\hat{\mathsf{G}}$ is the background Real structure on G. This collection of numbers defines the Real 2-character of the Real representation. We also define various forms of induction for Real representations of categorical groups and compute their effect on Real 2-characters. We formulate our results geometrically using gerbes, vector bundles and functions on iterated unoriented loop spaces. This perspective leads to connections with the representation theory of unoriented versions of the twisted Drinfeld double of G and with discrete torsion in M-theory with orientifolds. We speculate on the interpretation of our results as a Hopkins–Kuhn–Ravenel-type character theory in Real equivariant homotopy theory.

1. Introduction

Let G be a finite group. The complex representation theory of G is a classical and well-understood subject. In this paper we are interested in two variations of this theory. The first variation, also classical, is the real representation theory of G. More generally, following Atiyah–Segal [2] and Karoubi [27], after fixing a short exact sequence of finite groups

$$1 \to \mathsf{G} \to \hat{\mathsf{G}} \xrightarrow{\pi} \mathbb{Z}_2 \to 1$$

we can consider Real representations of G, that is, a complex vector space together with an action of $\hat{\mathsf{G}}$ in which elements of G (resp. $\hat{\mathsf{G}} \backslash \mathsf{G}$) act $\mathbb{C}$-linearly (resp. $\mathbb{C}$-antilinearly). The second, more

recent variation is the 2-representation theory of G, in which G acts by autoequivalences of an object of a bicategory. More generally, the group G itself can be categorified to a categorical group. In this paper we introduce and develop the Real representation theory of finite categorical groups.

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Apart from its intrinsic importance, the representation theory of higher groups has been studied because of its connections to other areas of mathematics and physics. For example, the works of Bartlett [4] and Ganter–Kapranov [17] were motivated by oriented topological field theory and equivariant homotopy theory, respectively. In both examples, the connections are strengthened by considering categorified character theory. Related appearances of categorical traces in geometry and representation theory can be found in the works of Toën–Vezzosi [41], Ben-Zvi–Nadler [5] and Hoyois–Scherotzke–Sibilla [22]. We expect Real 2-representations, and the resulting categorified Real character theory which we develop in this paper, to be related to $\mathbb{Z}_2$-equivariant refinements of the connections appearing in 2-representation theory. In the above examples, the applications are to unoriented topological field theory and, conjecturally, Real equivariant homotopy theory.

In the remainder of this introduction we explain our main results. We restrict attention to Real 2-representations of finite groups on categories; in the body of the paper, we work more generally with linear Real representations of $\mathbb{Z}_2$-graded categorical groups determined by a twisted 3-cocycle $\hat{\alpha} \in Z^3(B\hat{\mathsf{G}}, \mathbb{C}_\pi^\times)$. A Real 2-representation of G on a $\mathbb{C}$-linear category $\mathcal{C}$ is the data of autoequivalences

$$\rho(g) : \mathcal{C} \to \mathcal{C}, \qquad g \in \mathsf{G}$$

and antiautoequivalences

$$\rho(\omega) : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}, \qquad \omega \in \hat{\mathsf{G}} \backslash \mathsf{G}$$

together with coherently associative natural isomorphisms encoding their compositions. Here $\mathcal{C}^{\mathrm{op}}$ is the category opposite to $\mathcal{C}$. Retaining only the information attached to G recovers the notion of a 2-representation of G, as studied by Elgueta [12], Ganter–Kapranov [17], Bartlett [4] and others. The above definition, which categorifies the Grothendieck–Witt approach to Real representation theory, admits a variation in which elements of G (resp. $\hat{\mathsf{G}} \backslash \mathsf{G}$) act by $\mathbb{C}$-linear (resp. $\mathbb{C}$-antilinear) autoequivalences. This variation categorifies the standard approach to Real representation theory. It is a matter of preference which categorification one uses, as all results of the paper hold in either approach.

Associated to an ordinary 2-representation ρ is a collection of vector spaces of natural trans-

formations,

$$\mathbb{Tr}_\rho(g) = 2\mathrm{Hom}_{\mathsf{Cat}}(1_\mathcal{C}, \rho(g)), \qquad g \in \mathsf{G}.$$

Ganter–Kapranov [17] and Bartlett [4] categorified the conjugation invariance of the character of a representation by constructing a compatible system of bijections

$$\beta_{g,h} : \mathbb{Tr}_\rho(g) \xrightarrow{\sim} \mathbb{Tr}_\rho(hgh^{-1}), \qquad g, h \in \mathsf{G}.$$

The data $(\mathbb{Tr}_\rho, \beta)$ is called the categorical character of ρ. When ρ is a Real 2-representation of G, we construct an enhancement of $\mathbb{Tr}_\rho$ to the Real categorical character, by which we mean a system of bijections

$$\beta_{g,\omega} : \mathbb{Tr}_\rho(g) \xrightarrow{\sim} \mathbb{Tr}_\rho(\omega g^{\pi(\omega)} \omega^{-1}), \qquad g \in \mathsf{G}, \, \omega \in \hat{\mathsf{G}}.$$

In particular, the Real categorical character contains strictly more information than the categorical character of the underlying 2-representation. This is in contrast to the fact that, one category level down, the character of a Real representation is an ordinary character subject to additional constraints. Our first main result is a geometric characterization of the compatibility of the maps $\{\beta_{g,\omega}\}_{(g,\omega)\in\mathsf{G}\times\hat{\mathsf{G}}}$.

Theorem 1.1 (Theorem 6.9)**.** *The Real categorical character $\mathbb{Tr}_\rho$ of a Real 2-representation defines a flat vector bundle over the unoriented loop groupoid of $B\mathsf{G}$.*

Here we regard $B\mathsf{G}$ as the total space of the double cover $B\mathsf{G} \to B\hat{\mathsf{G}}$ classified by $B\pi : B\hat{\mathsf{G}} \to B\mathbb{Z}_2$. The unoriented loop groupoid of $B\mathsf{G}$ is then the quotient of the loop groupoid of $B\mathsf{G}$ by the simultaneous action of deck transformations of $B\mathsf{G}$ and loop reflection. It is worth emphasizing that $\mathbb{Tr}_\rho$ is an ordinary, as opposed to Real, vector bundle; the Real information is encoded entirely in the base of the vector bundle. This allows us to apply techniques from the ordinary representation theory of groupoids to study Real 2-representations.

Call a pair $(g, \omega) \in \mathsf{G} \times \hat{\mathsf{G}}$ graded commuting if the equality $\omega g^{\pi(\omega)} = g\omega$ holds. We associate to each such pair a number

$$\chi_\rho(g, \omega) = \mathrm{tr}_{\mathbb{Tr}_\rho(g)}(\beta_{g,\omega}) \in \mathbb{C}.$$

The Real 2-character χ_ρ is defined to be this collection of joint traces. Geometrically, χ_ρ is the holonomy of $\mathbb{Tr}_\rho$. Using this perspective, we prove the following result.

Theorem 1.2 (Theorem 6.11)**.** *Real 2-characters are Real $\hat{\mathsf{G}}$-conjugation invariant functions on the set of grading commuting pairs in $\mathsf{G} \times \hat{\mathsf{G}}$:*

$$\chi(g, \omega) = \chi(\sigma g^{\pi(\sigma)} \sigma^{-1}, \sigma \omega \sigma^{-1}), \qquad \sigma \in \hat{\mathsf{G}}.$$

We call such $\hat{\mathsf{G}}$-invariant functions Real 2-class functions on G. In particular, Real 2-representation theory naturally produces functions on various moduli spaces of G-bundles on the the 2-torus and Klein bottle, corresponding to graded commuting pairs with $\pi(\omega) = 1$ and $\pi(\omega) = -1$, respectively. See [44] for a precise description of these moduli spaces. The Klein bottle sector is not accessible from the point of view of ordinary 2-representation theory. Real 2-class functions first appeared in [32], where they were realized as characters of unoriented versions of (twisted) Drinfeld doubles of G. Corollary 6.10 explains the precise relationship between Real 2-representations and unoriented Drinfeld doubles. In Section 6.3 we study in detail the case in which $\mathcal{C}$ is a 2-vector space, giving in Theorem 6.12 a cohomological classification of

equivalence classes of Real 2-representations on $2\mathsf{Vect}_{\mathbb{C}}$ and computing in Proposition 6.13 all Real 2-characters.

We also study induction of Real 2-representations, of which there are two forms. The first categorifies the hyperbolic functor from complex representation theory to Real representation theory; the second categorifies induction internal to Real representation theory. Our final main results computes the result of 2-induction at the level of categorical character theory.

Theorem 1.3 (Theorems 8.7 and 8.10). *Let $\hat{\mathsf{H}} \leq \hat{\mathsf{G}}$ be a subgroup compatible with the structure maps to $\mathbb{Z}_2$ and ρ a Real 2-representation of H. Then the Real 2-character of the induced Real 2-representation of G is*

$$\chi_{\mathrm{RInd}_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}}(\rho)}(g,\omega) = \frac{1}{2|\mathsf{H}|} \sum_{\substack{\sigma \in \hat{\mathsf{G}} \\ \sigma(g,\omega)\sigma^{-1} \in \hat{\mathsf{H}}^2}} \chi_\rho(\sigma g^{\pi(\sigma)} \sigma^{-1}, \sigma\omega\sigma^{-1}).$$

A similar result holds for hyperbolically induced Real 2-characters.

Theorem 1.3 is proved by first proving the analogous result at the level of Real categorical characters, which is of independent interest; see Theorems 8.3 and 8.8.

Our geometric set-up allows for a systematic generalization of Theorems 1.1, 1.2 and 1.3 in which the twist $\hat{\alpha}$ is non-trivial. The relevant twists of $\mathbb{Tr}_\rho$ and χ_ρ are described in terms of the unoriented loop transgression map of [32]. Twisted loop transgression realizes a sort of dimensional reduction from Real 2-representation theory to the twisted representation theory of unoriented loop groupoids.

As mentioned above, one motivation for Ganter and Kapranov [17] to develop 2-character theory was to relate 2-representation theory to higher chromatic phenomena in equivariant homotopy theory. Denote by $\mathbf{B}\mathsf{G}$ the classifying space of G. Hopkins, Kuhn and Ravenel [20] showed that the equivariant Morava E-theory $E_n^\bullet(\mathbf{B}\mathsf{G})$, $n \geq 1$, at a prime p admits a generalized character theoretic description. Generalized characters are conjugation invariant functions on commuting n-tuples in G; the values of these functions and the pth order condition on the commuting elements will be ignored in this introduction. When $n = 1$ this recovers a p-completed version of the character theoretic description of $K^\bullet(\mathbf{B}\mathsf{G})$ given by Atiyah and Segal [2]. When $n = 2$ this gives a generalized character theoretic description of the equivariant elliptic cohomology $E_2^\bullet(\mathbf{B}\mathsf{G})$. Ganter and Kapranov showed that the 2-character theory of G also produces such generalized characters, although with values in the ground field and without the pth order condition. Moreover, Ganter and Kapranov showed that 2-induction of 2-representations is given at the level of 2-characters by the same formula as transfer for Hopkins–Kuhn–Ravenel characters. This analogy persists in the twisted case. Indeed, twisted elliptic characters appear in both Devoto's approach to the twisted elliptic cohomology of $\mathbf{B}\mathsf{G}$ [8] and, by combining the works of Ganter–Usher [19] and Willerton [43], in the twisted 2-character theory of finite groups. Moreover, we expect the transfer maps in twisted equivariant elliptic cohomology to behave like induction of twisted elliptic characters, as described in Corollary 8.2. The analogy between $E_3^\bullet(\mathbf{B}\mathsf{G})$ and 3-representations of G was established by Wang [42].

In view of the above analogy, we expect Real 2-representation theory to shed light on Real versions of Morava E-theory at $p = 2$. More precisely, we conjecture the existence of a real oriented generalized cohomology theory $R_n^\bullet$ such that $R_n^\bullet(\mathbf{B}\mathsf{G})$ can be described in terms of Real n-class functions of G which satisfy a pth order condition. At height one, $R_1^\bullet(\mathbf{B}\mathsf{G})$ should reduce

to Atiyah's Real K-theory $KR^\bullet(\mathbf{B}\mathsf{G})$ localized at $p = 2$ (see [1]), which, by the work of Atiyah–Segal [2], is known to admit a character theoretic description. At height two, the group $R_2^\bullet(\mathbf{B}\mathsf{G})$, which we envisage as a Real equivariant elliptic cohomology theory, should be described as a space of Real 2-class functions on G. Moreover, our results on hyperbolic and Real 2-induction should agree with the corresponding transfer maps in equivariant $R_2^\bullet$-theory. We expect that $R_n^\bullet$ is closely related to the theory $E\mathbb{R}_n^\bullet$ constructed by Hu–Kriz [23], but we do not establish a direct link in this paper.

Finally, we describe some applications of Real 2-representation theory to mathematical physics. Let $\mathcal{G}$ be the categorical group determined by a finite group G and a cocycle $\alpha \in Z^3(B\mathsf{G}, \mathbb{C}^\times)$. A Real structure $\hat{\mathsf{G}}$ on G and a lift of α to $\hat{\alpha} \in Z^3(B\hat{\mathsf{G}}, \mathbb{C}_\pi^\times)$ determine a Real structure on $\mathcal{G}$. On the other hand, the pair $(\hat{\mathsf{G}}, \hat{\alpha})$ determines an unoriented lift $\mathcal{Z}_{\hat{\mathsf{G}}}^{\hat{\alpha}}$ of three dimensional α-twisted oriented Dijkgraaf–Witten theory [44]. These two constructions are related by the fact that the bicategory of Real 2-representations of $\mathcal{G}$ on $2\mathsf{Vect}_\mathbb{C}$ is the homotopy fixed points for the action of $\pi_0(\mathsf{O}(3)) \simeq \mathbb{Z}_2$ on $\mathcal{Z}_{\hat{\mathsf{G}}}^{\hat{\alpha}}(\mathrm{pt})$. The theory $\mathcal{Z}_{\hat{\mathsf{G}}}^{\hat{\alpha}}$, and its generalization to categorical groups, appears in the theory of symmetry protected topological phases with time reversal symmetry. See the work of Kapustin–Thorngren [26] and Sharpe [38] related discussions in the oriented setting. Not unrelated, the precise form of the twisted $\hat{\mathsf{G}}$-equivariance of Real 2-characters recovers Sharpe's expressions [37] for discrete torsion phase factors in M-theory with orientifolds. The relevance of twisted Real elliptic cohomology, and $E\mathbb{R}_n^\bullet$-theory more generally, to string and M-theory with orientifolds has been conjectured by H. Sati. Finally, the prominence of unoriented loop transgression in this paper is particularly natural from the point of view of field theory, where it is an instance of the 'quantization via cohomological push-pull' procedure, which has been used extensively in the oriented setting. See, for example, the work of Freed [13] and Freed–Hopkins–Teleman [14].

A brief overview of the paper is as follows. Section 2 collects background material. Section 3 contains relevant results from the twisted Real representation theory of finite groups in a form which is convenient for categorification. Section 4 introduces Real representations of a finite categorical group and Section 5 develops their character theory. Section 6 then considers the general case of (linear) Real representations of finite categorical groups. Section 7, which serves as preparation for the following section, contains basic, but not widely available, material about induction of twisted Real representations of finite groups. In Section 8 we introduce 2-induction of Real 2-representations and compute its effect at the level of Real categorical and 2-characters. In Section 9 we describe our conjectural applications to Real equivariant homotopy theory.

2. Background material

2.1 Bicategories We establish our notation for bicategories. For a detailed introduction to bicategories the reader is referred to [6].

A bicategory $\mathcal{V}$ consists of the following data:

(i) A class $\mathrm{Obj}(\mathcal{V})$ of objects.
(ii) For each pair $x, y \in \mathrm{Obj}(\mathcal{V})$, a small category $1\mathrm{Hom}_\mathcal{V}(x, y)$, objects and morphisms of which are called 1-morphisms and 2-morphisms, respectively.
(iii) For each triple $x, y, z \in \mathrm{Obj}(\mathcal{V})$, a composition bifunctor

$$- \circ_0 - : 1\mathrm{Hom}_\mathcal{V}(y, z) \times 1\mathrm{Hom}_\mathcal{V}(x, y) \to 1\mathrm{Hom}_\mathcal{V}(x, z).$$

(iv) For each $x \in \mathrm{Obj}(\mathcal{V})$, an identity 1-morphism $1_x : x \to x$.

(v) For each triple of composable 1-morphisms f, g, h, an associator 2-isomorphism

$$\alpha_{f,g,h} : (f \circ_0 g) \circ_0 h \Longrightarrow f \circ_0 (g \circ_0 h).$$

(vi) For each 1-morphism $f : x \to y$, a pair of unitor 2-isomorphisms

$$\lambda_f : 1_y \circ_0 f \Longrightarrow f, \qquad \rho_f : f \circ_0 1_x \Longrightarrow f.$$

The data (i)-(vi) is subject to a number of coherence conditions which we do not recall here.

Composition of 2-morphisms within the same 1-morphism category will be denoted by $- \circ_1 -$. When it will not lead to confusion we will write $- \circ -$ in place of $- \circ_0 -$ or $- \circ_1 -$. The set of 2-morphisms $\mathrm{Hom}_{1\mathrm{Hom}_{\mathcal{V}}(x,y)}(f,g)$ will be denoted by $2\mathrm{Hom}_{\mathcal{V}}(f,g)$. Given 1-morphisms $f_1, f_2 : x \to y$ and $g : y \to z$ and a 2-morphism $u : f_1 \Rightarrow f_2$, the left whiskering of u by g, namely $1_g \circ_0 u : g \circ_0 f_1 \Rightarrow g \circ_0 f_2$, will be denoted by $g \circ_0 u$. We adopt analogous notation for right whiskering.

A (strict) 2-category is a bicategory in which all associator 2-isomorphisms $\alpha_{f,g,h}$ and all unitor 2-isomorphisms λ_f, ρ_f are identity maps. Coherence for bicategories asserts that any bicategory is biequivalent to a 2-category.

Example 2.1. Small categories, functors and natural transformations form a 2-category Cat. For each field k, there is a sub-2-category Cat_k of Cat consisting of k-linear categories, k-linear functors and natural transformations.

Example 2.2. Let k be a field. Kapranov and Voevodsky defined the bicategory $2\mathsf{Vect}_k$ of finite dimensional 2-vector spaces over k [25]. This is a 2-categorical analogue of the category Vect_k of finite dimensional vector spaces over k. There are a number of standard variants of this bicategory. One definition takes $2\mathsf{Vect}_k$ to be the bicategory of k-linear additive finitely semisimple categories, k-linear functors and natural transformations. We will use the following slightly different model. Objects of $2\mathsf{Vect}_k$ are non-negative integers $[n]$, $n \in \mathbb{Z}_{\geq 0}$. A 1-morphism $[n] \to [m]$ is an $m \times n$ matrix $A = (A_{ij})$ whose entries are finite dimensional vector spaces over k. The composition of the 1-morphisms $A : [m] \to [n]$ and $B : [n] \to [p]$ is defined by

$$(B \circ_0 A)_{ik} = \bigoplus_{j=1}^{n} B_{ij} \otimes_k A_{jk}.$$

A 2-morphism $u : A \Rightarrow B$ is a collection of k-linear maps $(u_{ij} : A_{ij} \to B_{ij})$. The compositions of 2-morphisms are defined by

$$(v \circ_0 u)_{ik} = \bigoplus_{j=1}^{n} v_{ij} \otimes u_{jk}, \qquad (u' \circ_1 u)_{ij} = u'_{ij} \circ u_{ij}.$$

The composition $- \circ_0 -$ is not strictly associative.

2.2 Duality involutions on bicategories

We introduce the categorical background required for our formulation of Real 2-representation theory. Our main approach uses contravariant involutions on categories and bicategories. An alternative approach, described in Sections 3.1 and 6.4, uses antilinear covariant involutions. These approaches are parallel and one can easily translate between the two.

We begin by recalling some basic categorical notions of duality. For further details the reader is referred to [36, §3]. Given a category $\mathcal{C}$, we denote by $\mathcal{C}^{\mathrm{op}}$ its opposite category. A category with duality is then a category $\mathcal{C}$ together with a functor $(-)^* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ and a natural isomorphism $\Theta : 1_{\mathcal{C}} \Rightarrow (-)^* \circ ((-)^*)^{\mathrm{op}}$ which satisfy

$$\Theta_x^* \circ \Theta_{x^*} = 1_{x^*} \tag{1}$$

for each $x \in \mathrm{Obj}(\mathcal{C})$. A morphism $(\mathcal{C}, (-)^*, \Theta) \to (\mathcal{D}, (-)^*, \Xi)$ of categories with duality, sometimes called a form functor, consists of a functor $F : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\varphi : F \circ (-)^* \Rightarrow (-)^* \circ F$ which satisfy

$$\varphi_x^* \circ \Xi_{F(x)} = \varphi_{x^*} \circ F(\Theta_x)$$

for each $x \in \mathrm{Obj}(\mathcal{C})$. A symmetric form in $(\mathcal{C}, (-)^*, \Theta)$ is an object $x \in \mathrm{Obj}(\mathcal{C})$ together with an isomorphism $\psi : x \to x^*$ which satisfies

$$\psi^* \circ \Theta_x = \psi.$$

Symmetric forms and their partial isometries, that is, morphisms $\phi : x \to y$ which satisfy $\phi^* \circ \psi_y \circ \phi = \psi_x$, define the homotopy fixed point[1] category $\mathcal{C}^{h\mathbb{Z}_2}$. Form functors induce functors between homotopy fixed point categories.

We now turn to the categorification of the above notions. We will use the 2-cell dual $\mathcal{V}^{\mathrm{co}}$ of a bicategory $\mathcal{V}$, that is, the bicategory obtained from $\mathcal{V}$ by reversing its 2-cells. Hence, if

is a 2-morphism in $\mathcal{V}$, then

is a 2-morphism in $\mathcal{V}^{\mathrm{co}}$.

Definition 2.3 ([39, Definition 2.1]). *A bicategory with weak duality involution is a bicategory $\mathcal{V}$ together with*

  (i) *a pseudofunctor $(-)^\circ : \mathcal{V}^{\mathrm{co}} \to \mathcal{V}$,*

 (ii) *a pseudonatural adjoint equivalence $\eta : 1_{\mathcal{V}} \Rightarrow (-)^\circ \circ_0 ((-)^\circ)^{\mathrm{co}}$, and*

(iii) *an invertible modification $\zeta : \eta \circ_0 (-)^\circ \implies (-)^\circ \circ_0 \eta^{\mathrm{co}}$*

such that, for each $x \in \mathrm{Obj}(\mathcal{V})$, the equality

$$\zeta_{x^\circ} \circ_0 \eta_x = (\zeta_x^\circ \circ_0 \eta_x) \circ_1 \eta(x) \tag{2}$$

of 2-morphisms holds. Here $\eta(x) : \eta_{x^{\circ\circ}} \circ \eta_x \Rightarrow \eta_x^{\circ\circ} \circ \eta_x$ is a pseudonaturality constraint for η.

If $\mathcal{V}$ is a 2-category, $(-)^\circ$ is a strict 2-functor and η and ζ are the identities, then the above

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[1]The term 'homotopy fixed points' is usually reserved for covariant group actions, but we will use it more generally.

data is said to define a strict duality involution on $\mathcal{V}$.

Definition 2.4 ([39, Definition 2.2])**.** *A duality pseudofunctor* $(\mathcal{V}, (-)^\circ, \eta) \to (\mathcal{W}, (-)^\circ, \lambda)$ *between bicategories with weak duality involution is a pseudofunctor* $F : \mathcal{V} \to \mathcal{W}$ *together with*

(i) *a pseudonatural adjoint equivalence* $\mathfrak{i} : (-)^\circ \circ_0 F^{\mathrm{co}} \Rightarrow F \circ_0 (-)^\circ$, *and*

(ii) *an invertible modification*

such that, for each $x \in \mathrm{Obj}(\mathcal{V})$, *a coherence constraint, which we omit, is satisfied.*

Coherence for bicategories generalizes to the present setting. Indeed, any bicategory with weak duality involution is biequivalent via a duality pseudofunctor to a 2-category with strict duality involution [39, Theorem 2.3].

Example 2.5. The strict 2-functor $(-)^{\mathrm{op}} : \mathsf{Cat}^{\mathrm{co}} \to \mathsf{Cat}$ which sends categories, functors and natural transformations to their opposites is a strict duality involution. The restriction of $(-)^{\mathrm{op}}$ to Cat_k is a k-linear strict duality involution.

Example 2.6. The bicategory $2\mathsf{Vect}_k$ has a weak duality involution $(-)^\vee$ which is a 2-categorical analogue of the k-linear duality functor on Vect_k. On objects let $[n]^\vee = [n]$ and on 1- and 2-morphisms let $(-)^\vee$ be given by k-linear duality. Explicitly, we have $(A_{ij})^\vee = (A_{ij}^\vee)$ and $(u_{ij})^\vee = (u_{ij}^\vee)$. The adjoint equivalence η is induced by ev, the canonical evaluation isomorphism from a finite dimensional vector space to its double dual. The modification ζ is the identity.

Next, we define homotopy fixed point objects.

Definition 2.7. *A symmetric form in a bicategory with weak duality involution* $(\mathcal{V}, (-)^\circ, \eta, \zeta)$ *is an object* $x \in \mathrm{Obj}(\mathcal{V})$ *together with*

(i) *an equivalence* $\psi : x^\circ \to x$, *and*

(ii) *a 2-isomorphism* $\mu : 1_x \Rightarrow \psi \circ_0 \psi^\circ \circ_0 \eta_x$

such that the 2-morphism given by the diagram

is equal to 1_ψ.

Symmetric forms in $(\mathcal{V}, (-)^\circ, \eta, \zeta)$ are the objects of a homotopy fixed point bicategory $\mathcal{V}^{h\mathbb{Z}_2}$. As we will not have the occasion to use 1- and 2-morphisms of $\mathcal{V}^{h\mathbb{Z}_2}$, we omit their explicit definitions.

Example 2.8. The homotopy fixed point bicategory $\mathsf{Cat}^{h\mathbb{Z}_2}$, with respect to the duality involution $(-)^{\mathrm{op}}$, is the bicategory of categories with duality, form functors and natural transformations of form functors.

Given an object x of a bicategory with weak duality involution, put

$$
{}^\epsilon x = \begin{cases} x & \text{if } \epsilon = 1, \\ x^\circ & \text{if } \epsilon = -1. \end{cases} \tag{3}
$$

Similar notation will be used for the action of $(-)^\circ$ on 1- and 2-morphisms.

Closely related to bicategories with duality involutions are bicategories with contravariance [39, §4]. Roughly speaking, these are bicategories which have both covariant and contravariant 1-morphisms. More precisely, a bicategory with contravariance consists of the following data:

(i) A class $\mathrm{Obj}(\mathcal{V})$ of objects.
(ii) For each pair $x, y \in \mathrm{Obj}(\mathcal{V})$ and each $\epsilon \in \{\pm 1\}$, a small category $1\mathrm{Hom}_\mathcal{V}^\epsilon(x, y)$.
(iii) For each triple $x, y, z \in \mathrm{Obj}(\mathcal{V})$ and each pair $\epsilon_1, \epsilon_2 \in \{\pm 1\}$, a composition bifunctor

$$
- \circ_0 - \; : 1\mathrm{Hom}_\mathcal{V}^{\epsilon_2}(y, z) \times {}^{\epsilon_2} 1\mathrm{Hom}_\mathcal{V}^{\epsilon_1}(x, y) \to 1\mathrm{Hom}_\mathcal{V}^{\epsilon_2 \epsilon_1}(x, z).
$$

Here we apply equation (3) to $(\mathsf{Cat}, (-)^{\mathrm{op}})$, so that the left superscript ϵ_2 indicates whether or not we consider opposite categories.
(iv) For each $x \in \mathrm{Obj}(\mathcal{V})$, an identity 1-morphism $1_x \in \mathrm{Hom}_\mathcal{V}^1(x, x)$.
(v) Associator and unitor 2-isomorphisms.

This data is required to satisfy coherence constraints similar to those of a bicategory.

By keeping only the data associated to $1 \in \{\pm 1\}$, each bicategory $\mathcal{V}$ with contravariance defines a bicategory $\mathcal{V}_1$.

A pseudofunctor preserving contravariance between bicategories with contravariance is defined as in the case of a pseudofunctor, but with the additional requirement that the sign $\epsilon \in \{\pm 1\}$ of 1-morphisms be preserved. Similarly, one defines pseudonatural transformations respecting contravariance, the components of which are required to be 1-morphisms of degree $+1$.

There is an obvious strictification of the above definition which leads to the notion of a 2-category with contravariance. Any bicategory with contravariance is biequivalent via a pseudofunctor preserving contravariance to a 2-category with contravariance [39, Theorems 8.1, 8.2].

Example 2.9.   (i) A bicategory $\mathcal{V}$ with weak duality involution defines a bicategory $\underline{\mathcal{V}}$ with contravariance by setting

$$
\mathrm{Obj}(\underline{\mathcal{V}}) = \mathrm{Obj}(\mathcal{V}), \qquad 1\mathrm{Hom}_{\underline{\mathcal{V}}}^\epsilon(x, y) = 1\mathrm{Hom}_\mathcal{V}({}^\epsilon x, y).
$$

Composition of 1- and 2-morphisms in $\underline{\mathcal{V}}$ is induced by the corresponding compositions in $\mathcal{V}$. See [39, Theorem 7.2] for details.
(ii) Applying the construction from part (i) to $(\mathsf{Cat}, (-)^{\mathrm{op}})$ yields a 2-category with contravariance whose objects are small categories and whose 1-morphisms are covariant ($\epsilon = 1$) and contravariant ($\epsilon = -1$) functors.

2.3 String diagrams For a detailed introduction to string diagrams the reader is referred to [3, §4].

String diagrams, which will be used to perform calculations in 2-categories, are Poincaré dual to globular diagrams for 2-categories. Two dimensional regions of a string diagram are therefore labelled by objects of the 2-category while strings and nodes are labelled by 1- and 2-morphisms, respectively. Our conventions are such that string diagrams are read from right to left and bottom to top. Below is a globular diagram (left) together with its corresponding string diagram (right):
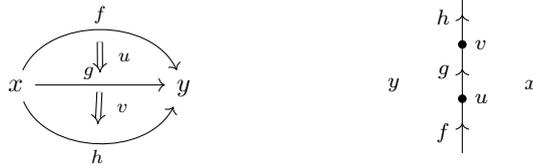
Compositions of 1- and 2-morphisms are represented by the appropriate concatenations of string diagrams. Although arrows drawn on strings are redundant, we will often include them if they clarify diagrams. We sometimes omit labels of two dimensional regions and we do not draw identity 1-morphisms. For example, a 2-morphism $u : 1_x \Rightarrow f$ may be depicted as

String diagrams can also be used for calculations in bicategories. If the bicategory is skeletal, as will be the case in all relevant examples below, then the only additional complication is that we must keep track of associators.

2.4 $\mathbb{Z}_2$-graded groups Denote by $\mathbb{Z}_2$ the multiplicative group $\{\pm 1\}$. A group homomorphism $\pi : \hat{\mathsf{G}} \to \mathbb{Z}_2$ is called a $\mathbb{Z}_2$-graded group. Morphisms of $\mathbb{Z}_2$-graded groups are group homomorphisms which respect the structure maps to $\mathbb{Z}_2$. We will always assume that a given $\mathbb{Z}_2$-graded group is non-trivially graded in the sense that the structure map is surjective. A non-trivially graded $\mathbb{Z}_2$-graded group $\hat{\mathsf{G}}$ is necessarily an extension

$$1 \to \mathsf{G} \to \hat{\mathsf{G}} \xrightarrow{\pi} \mathbb{Z}_2 \to 1. \tag{4}$$

The subgroup $\mathsf{G} = \ker(\pi)$ is called the ungraded subgroup of $\hat{\mathsf{G}}$.

Alternatively, if we are given a group G, then an extension of the form (4) is called a Real structure on G.

Example 2.10. An involutive group homomorphism $\varsigma : \mathsf{G} \to \mathsf{G}$ defines a split Real structure $\hat{\mathsf{G}} = \mathsf{G} \rtimes_\varsigma \mathbb{Z}_2$ on G. Atiyah and Segal restrict attention to such Real structures in their study of equivariant KR-theory [2].

Remark 2.11. All constructions of this paper apply to general, as opposed to just split, Real structures. We give here two motivations for this level of generality.

   (i) Even if one is ultimately concerned only with split Real structures, inductive arguments in KR-theory often involve general Real structures.

  (ii) The assumption that the Real structure is split is often too restrictive for applications to mathematical physics. For example, non-split Real structure are central to the orientifolding of string theory on orbifold backgrounds. See, for example, [10].

Let $\hat{\mathsf{G}}$ be a $\mathbb{Z}_2$-graded group. Denote by $\mathrm{Aut}^{\mathrm{gen}}_{\mathsf{Grp}}(\mathsf{G})$ the $\mathbb{Z}_2$-graded group of automorphisms and antiautomorphisms of G. Define a map of $\mathbb{Z}_2$-graded groups by

$$\varphi : \hat{\mathsf{G}} \to \mathrm{Aut}^{\mathrm{gen}}_{\mathsf{Grp}}(\mathsf{G}), \qquad \omega \mapsto (g \mapsto \omega g^{\pi(\omega)} \omega^{-1}).$$

The induced $\hat{\mathsf{G}}$-action on G is called Real conjugation.

Example 2.12. Let $(\mathcal{C}, (-)^*, \Theta)$ be a category with duality. Given $x \in \mathrm{Obj}(\mathcal{C})$, let $\mathrm{Aut}^{\mathrm{gen}}_{\mathcal{C}}(x)$ be the set of all automorphisms and antiautomorphisms of x, the latter being by definition an isomorphism $x^* \to x$. For $f \in \mathrm{Aut}^{\mathrm{gen}}_{\mathcal{C}}(x)$, define $\pi(f) \in \mathbb{Z}_2$ so that $f : {}^{\pi(f)}x \to x$, the notation as in equation (3). Then $\mathrm{Aut}^{\mathrm{gen}}_{\mathcal{C}}(x)$ becomes a $\mathbb{Z}_2$-graded group with ungraded subgroup $\mathrm{Aut}_{\mathcal{C}}(x)$ when given the product and inverse

$$f_2 f_1 = f_2 \circ {}^{\pi(f_2)} f_1^{\pi(f_2)} \circ \Theta_x^{\delta_{\pi(f_2), \pi(f_1), -1}}$$

and

$$I(f) = \begin{cases} f^{-1} & \text{if } \pi(f) = 1, \\ \Theta_x^{-1} \circ f^* & \text{if } \pi(f) = -1, \end{cases}$$

respectively, where we have introduced the notation

$$\delta_{\epsilon_2, \epsilon_1, -1} = \begin{cases} 1 & \text{if } \epsilon_1 = \epsilon_2 = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Explicitly, the group law of $\mathrm{Aut}^{\mathrm{gen}}_{\mathcal{C}}(x)$ reads

$$f_2 f_1 = \begin{cases} f_2 \circ f_1 & \text{if } \pi(f_2) = +1, \\ f_2 \circ (f_1^{-1})^* & \text{if } \pi(f_2) = -1 \text{ and } \pi(f_1) = +1, \\ f_2 \circ (f_1^{-1})^* \circ \Theta_x & \text{if } \pi(f_2) = \pi(f_1) = -1. \end{cases}$$

For example, the associativity of the composition of three antiautomorphisms follows from equation (1).

2.5 Loop groupoids Recall that a groupoid is a category in which all morphisms are isomorphisms. A groupoid is called finite if it has only finitely many objects and morphisms.

If a group G acts on a set X, then we denote by $X /\!\!/ \mathsf{G}$ the groupoid with

$$\mathrm{Obj}(X /\!\!/ \mathsf{G}) = X, \qquad \mathrm{Hom}_{X /\!\!/ \mathsf{G}}(x, y) = \{g \in \mathsf{G} \mid gx = y\}.$$

We will write $B\mathsf{G}$ in place of $\mathrm{pt} /\!\!/ \mathsf{G}$.

Definition 2.13 ([43, §1.3])**.** *The loop groupoid of a finite groupoid $\mathfrak{G}$ is the functor category* $\Lambda\mathfrak{G} = 1\mathrm{Hom}_{\mathsf{Cat}}(B\mathbb{Z}, \mathfrak{G})$.

Concretely, an object (x, γ) of $\Lambda\mathfrak{G}$ is a loop $\gamma : x \to x$ in $\mathfrak{G}$ while a morphism $(x_1, \gamma_1) \to (x_2, \gamma_2)$ is a morphism $g : x_1 \to x_2$ which satisfies $\gamma_2 = g\gamma_1 g^{-1}$.

A finite groupoid over $B\mathbb{Z}_2$ is a morphism $\pi : \hat{\mathfrak{G}} \to B\mathbb{Z}_2$ of finite groupoids. The functor π classifies an equivalence class of double covers $\pi : \mathfrak{G} \to \hat{\mathfrak{G}}$; we fix a choice of such a double cover

in what follows. The relevant analogue of the loop groupoid in the setting of finite groupoids over $B\mathbb{Z}_2$ is the following.

Definition 2.14 ([32, §1.4]). *The unoriented loop groupoid $\Lambda_\pi^{\mathrm{ref}}\hat{\mathfrak{G}}$ of a finite groupoid $\hat{\mathfrak{G}}$ over $B\mathbb{Z}_2$ has objects the degree one loops in $\hat{\mathfrak{G}}$ and morphisms $(x_1, \gamma_1) \to (x_2, \gamma_2)$ the morphisms $\omega : x_1 \to x_2$ which satisfy $\gamma_2 = \omega\gamma_1^{\pi(\omega)}\omega^{-1}$.*

The superscript 'ref' stands for reflection, since $\Lambda_\pi^{\mathrm{ref}}\hat{\mathfrak{G}}$ is equivalent to the quotient of $\Lambda\mathfrak{G}$ by the diagonal $\mathbb{Z}_2$-action coming from deck transformations of $\mathfrak{G}$ and reflection of the circle $B\mathbb{Z}$.

Example 2.15.   (i) Let G be a finite group. The loop groupoid $\Lambda B\mathsf{G}$ is equivalent to the conjugation action groupoid $\mathsf{G}/\!\!/\mathsf{G}$.

  (ii) Let $\hat{\mathsf{G}}$ be a finite $\mathbb{Z}_2$-graded group. The functor $B\pi : B\hat{\mathsf{G}} \to B\mathbb{Z}_2$ classifies the double cover $B\mathsf{G} \to B\hat{\mathsf{G}}$. The unoriented loop groupoid $\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}}$ is equivalent to the Real conjugation action groupoid $\mathsf{G}/\!\!/_\varphi\hat{\mathsf{G}}$.

2.6   Twisted loop transgression Loop transgression for finite groupoids was studied by Willerton [43]. We recall a version of loop transgression for finite groupoids over $B\mathbb{Z}_2$ [32].

Let $\hat{\mathfrak{G}}$ be a finite groupoid over $B\mathbb{Z}_2$. The double cover $\pi : \mathfrak{G} \to \hat{\mathfrak{G}}$ can be used to twist local systems on (the simplicial complex associated to) $\hat{\mathfrak{G}}$. Given an abelian group A, viewed as a $\mathbb{Z}_2$-module via inversion, denote by $C^\bullet(\hat{\mathfrak{G}}, \mathsf{A}_\pi)$ the complex of π-twisted A-valued simplicial cochains on $\hat{\mathfrak{G}}$. Write $[\omega_n|\cdots|\omega_1]$ for the n-simplex of $\hat{\mathfrak{G}}$ determined by the diagram $x_1 \xrightarrow{\omega_1} \cdots \xrightarrow{\omega_n} x_{n+1}$ in $\hat{\mathfrak{G}}$. In this notation, the differential of $\hat{\beta} \in C^n(\hat{\mathfrak{G}}, \mathsf{A}_\pi)$ is defined by

$$d\hat{\beta}([\omega_{n+1}|\cdots|\omega_1]) = \hat{\beta}([\omega_n|\cdots|\omega_1])^{\pi(\omega_{n+1})}\hat{\beta}([\omega_{n+1}|\cdots|\omega_2])^{(-1)^{n+1}} \times$$
$$\prod_{i=1}^n \hat{\beta}([\omega_{n+1}|\cdots|\omega_{i+2}|\omega_{i+1}\omega_i|\omega_{i-1}|\cdots|\omega_1])^{(-1)^i}.$$

The notation $Z^\bullet \subset C^\bullet$ indicates the subgroup of cocycles. Without loss of generality, we will assume that all cochains are normalized in the sense that they evaluate to the identity on chains in which one of the morphisms ω_i is an identity map.

We also write $[\omega_n|\cdots|\omega_1]\gamma$ for the n-simplex of $\Lambda_\pi^{\mathrm{ref}}\hat{\mathfrak{G}}$ determined by the diagram

$$\gamma \xrightarrow{\omega_1} \omega_1\gamma^{\pi(\omega_1)}\omega_1^{-1} \xrightarrow{\omega_2} \cdots \xrightarrow{\omega_n} (\omega_n\cdots\omega_1)\gamma^{\pi(\omega_n\cdots\omega_1)}(\omega_n\cdots\omega_1)^{-1}.$$

Let k be a field. Reflection twisted loop transgression is a cochain map

$$\tau_\pi^{\mathrm{ref}} : C^\bullet(\hat{\mathfrak{G}}, k_\pi^\times) \to C^{\bullet-1}(\Lambda_\pi^{\mathrm{ref}}\hat{\mathfrak{G}}, k^\times).$$

The map τ_π^{ref} is defined by a push-pull procedure, the main point being that the pushforward is along an unoriented map. This leads to the change in coefficient systems. We do not require a full description of τ_π^{ref}. Instead, we record that for a 2-cochain $\hat{\theta} \in C^2(\hat{\mathfrak{G}}, k_\pi^\times)$ we have

$$\tau_\pi^{\mathrm{ref}}(\hat{\theta})([\omega]\gamma) = \hat{\theta}([\gamma^{-1}|\gamma])^{\frac{\pi(\omega)-1}{2}} \frac{\hat{\theta}([\omega\gamma^{\pi(\omega)}\omega^{-1}|\omega])}{\hat{\theta}([\omega|\gamma^{\pi(\omega)}])}$$

and for a 3-cochain $\hat{\alpha} \in C^3(\hat{\mathfrak{G}}, k_\pi^\times)$ we have

$$\tau_\pi^{\mathrm{ref}}(\hat{\alpha})([\omega_2|\omega_1]\gamma) = \hat{\alpha}([\gamma|\gamma^{-1}|\gamma])^{\delta_{\pi(\omega_2),\pi(\omega_1),-1}} \times$$

$$\left(\frac{\hat{\alpha}([\omega_1\gamma^{-\pi(\omega_1)}\omega_1^{-1}|\omega_1\gamma^{\pi(\omega_1)}\omega_1^{-1}|\omega_1])\hat{\alpha}([\omega_1|\gamma^{-\pi(\omega_1)}|\gamma^{\pi(\omega_1)}])}{\hat{\alpha}([\omega_1\gamma^{-\pi(\omega_1)}\omega_1^{-1}|\omega_1|\gamma^{\pi(\omega_1)}])} \right)^{\frac{\pi(\omega_2)-1}{2}} \times$$

$$\frac{\hat{\alpha}([\omega_2|\omega_1|\gamma^{\pi(\omega_2\omega_1)}])\hat{\alpha}([\omega_2\omega_1\gamma^{\pi(\omega_2\omega_1)}(\omega_2\omega_1)^{-1}|\omega_2|\omega_1])}{\hat{\alpha}([\omega_2|\omega_1\gamma^{\pi(\omega_2\omega_1)}\omega_1^{-1}|\omega_1])}.$$

If $\hat{\theta}$ is in fact a 2-cocycle, then $\tau_\pi^{\mathrm{ref}}(\hat{\theta})$ is a 1-cocycle, meaning the equality

$$\tau_\pi^{\mathrm{ref}}(\hat{\theta})([\omega_2]\omega_1\gamma^{\pi(\omega_1)}\omega_1^{-1})\tau_\pi^{\mathrm{ref}}(\hat{\theta})([\omega_1]\gamma) = \tau_\pi^{\mathrm{ref}}(\hat{\theta})([\omega_2\omega_1]\gamma) \qquad (5)$$

holds for each 2-chain $[\omega_2|\omega_1]\gamma$. This follows from [32, §2.3], but can also be verified directly. This and the corresponding statement for 3-cocycles are the only facts about τ_π^{ref} that we will assume. In particular, the expressions for $\tau_\pi^{\mathrm{ref}}(\hat{\theta})$ and $\tau_\pi^{\mathrm{ref}}(\hat{\alpha})$ will be derived from the point of view of Real (2-)representation theory.

Finally, we note that the twisted transgression map τ_π^{ref} restricts to Willerton's transgression map $\tau : C^\bullet(\mathfrak{G}, k^\times) \to C^{\bullet-1}(\Lambda\mathfrak{G}, k^\times)$.

3. Twisted Real representation theory of finite groups

As motivation for the remainder of the paper, we recall the basics of the theory of twisted, or projective, Real representations of finite groups, in both its linear and antilinear formulation. For twisted representations of finite group(oid)s, see [28], [43].

3.1   The antilinear theory   In the case of untwisted real representations, the material in this section is standard [16]. Aspects of the untwisted Real case are treated in [2], [27]. A general reference is [32].

Let k be a field which is a quadratic extension of a field k_0. We regard k_0 as the fixed subfield of a k_0-linear Galois involution $k \to k$. A standard example is $k_0 = \mathbb{R} \subset k = \mathbb{C}$. A map $V \to W$ of vector spaces over k is called $+1$-linear (resp. -1-linear) if it is k-linear (resp. k-antilinear).

Let G be a finite group with Real structure $\hat{\mathsf{G}}$. Let $\hat{\theta} \in Z^2(B\hat{\mathsf{G}}, k_\pi^\times)$, where G acts trivially on $k^\times$ and $\hat{\mathsf{G}}\backslash\mathsf{G}$ acts by the Galois involution. Write $\theta \in Z^2(B\mathsf{G}, k^\times)$ for the restriction of $\hat{\theta}$ to $B\mathsf{G}$.

Definition 3.1. *A $\hat{\theta}$-twisted Real representation of G is a finite dimensional vector space N over k together with $\pi(\omega)$-linear maps $\rho(\omega) : N \to N$, $\omega \in \hat{\mathsf{G}}$, which satisfy $\rho(e) = 1_N$ and*

$$\rho(\omega_2) \circ \rho(\omega_1) = \hat{\theta}([\omega_2|\omega_1])\rho(\omega_2\omega_1).$$

Twisted Real representations of G and their $\hat{\mathsf{G}}$-equivariant k-linear maps form a k_0-linear additive category $\mathsf{RRep}_k^{\hat{\theta}}(\mathsf{G})$. Despite the notation, $\mathsf{RRep}_k^{\hat{\theta}}(\mathsf{G})$ depends on the Real structure $\hat{\mathsf{G}}$. Let $KR^{0+\hat{\theta}}(B\mathsf{G})$ be the Grothendieck group of $\mathsf{RRep}_k^{\hat{\theta}}(\mathsf{G})$.

The Real character of a $\hat{\theta}$-twisted Real representation ρ is the function

$$\chi_\rho : \mathsf{G} \to k, \qquad g \mapsto \mathrm{tr}_N(\rho(g)).$$

In other words, χ_ρ is the character of the underlying θ-twisted representation of G. The new feature of Real characters is their Real conjugation equivariance,

$$\chi_\rho(\omega g^{\pi(\omega)}\omega^{-1}) = \tau_\pi^{\mathrm{ref}}(\hat{\theta})([\omega]g) \cdot \chi_\rho(g), \qquad \omega \in \hat{\mathsf{G}} \tag{6}$$

which refines the conjugation equivariance of characters of θ-twisted representations. The Real character map extends to a k-linear map

$$\chi : KR^{0+\hat{\theta}}(B\mathsf{G}) \otimes_{\mathbb{Z}} k \to \Gamma_{\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}}}(\tau_\pi^{\mathrm{ref}}(\hat{\theta})_k). \tag{7}$$

Adopting the notation of [43, §2.2], the right hand side denotes the space of flat sections of the transgressed line bundle $\tau_\pi^{\mathrm{ref}}(\hat{\theta})_k \to \Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}}$. Explicitly, $\Gamma_{\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}}}(\tau_\pi^{\mathrm{ref}}(\hat{\theta})_k)$ is the space of all functions $\chi : \mathsf{G} \to k$ which satisfy equation (6). When $k = \mathbb{C}$ with complex conjugation as the involution, the map (7) is an isomorphism [32, Theorem 3.10].

Example 3.2. The real setting is $k_0 = \mathbb{R} \subset k = \mathbb{C}$ with $\pi : \hat{\mathsf{G}} = \mathsf{G} \times \mathbb{Z}_2 \to \mathbb{Z}_2$ the projection and $\hat{\theta} = 1$. Then $\mathsf{RRep}_{\mathbb{C}}(\mathsf{G}) \simeq \mathsf{Rep}_{\mathbb{R}}(\mathsf{G})$ and $KR^0(B\mathsf{G}) \simeq RO(\mathsf{G})$. Equation (6) becomes the statement that characters of real representations are real valued class functions and the isomorphism (7) identifies $RO(\mathsf{G}) \otimes_{\mathbb{Z}} \mathbb{C}$ with the space of functions on G which are constant on conjugacy classes and their inverses.

3.2 The linear theory We describe a linear approach to the twisted Real representation theory of a finite group. Aspects of the untwisted real case are discussed in [45]. This section will be the basis for our categorification in later sections.

We keep the notation from Section 3.1, although k is now an arbitrary field and $\hat{\mathsf{G}}\backslash\mathsf{G}$ acts on $k_\pi^\times$ by inversion. We give two linear versions of the notion of a Real representation of G. The first is less natural, requiring the choice of an element $\varsigma \in \hat{\mathsf{G}}\backslash\mathsf{G}$, but has the benefit that it fits into the framework of Grothendieck–Witt theory.

Lemma 3.3. *Let (V, ρ) be a θ-twisted representation of G. For each $\varsigma \in \hat{\mathsf{G}}\backslash\mathsf{G}$, the pair $(V^\vee, \rho^\varsigma)$, where $V^\vee$ is the k-linear dual of V and*

$$\rho^\varsigma(g) = \tau_\pi^{\mathrm{ref}}(\hat{\theta})([\varsigma^{-1}]g)^{-1}\rho(\varsigma^{-1}g^{-1}\varsigma)^\vee, \qquad g \in \mathsf{G}$$

is a θ-twisted representation of G.

Proof. The key point is the following identity, valid for $g_1, g_2 \in \mathsf{G}$ and $\omega \in \hat{\mathsf{G}}$:

$$\frac{\hat{\theta}([\omega g_2 \omega^{-1}|\omega g_1 \omega^{-1}])}{\hat{\theta}([g_2|g_1])^{\pi(\omega)}} = \frac{\hat{\theta}([\omega|\omega^{-1}])\hat{\theta}([\omega g_2 g_1|\omega^{-1}])\hat{\theta}([\omega|g_2 g_1])}{\hat{\theta}([\omega|g_2])\hat{\theta}([\omega g_2|\omega^{-1}])\hat{\theta}([\omega|g_1])\hat{\theta}([\omega g_1|\omega^{-1}])}. \tag{8}$$

$\square$

Each element $\varsigma \in \hat{\mathsf{G}}\backslash\mathsf{G}$ determines a k-linear exact duality structure $(P^\varsigma, \Theta^\varsigma)$ on $\mathsf{Rep}_k^\theta(\mathsf{G})$. The functor $P^\varsigma : \mathsf{Rep}_k^\theta(\mathsf{G})^{\mathrm{op}} \to \mathsf{Rep}_k^\theta(\mathsf{G})$ is given on objects by $P^\varsigma(\rho) = \rho^\varsigma$. The natural isomorphism $\Theta^\varsigma : 1_{\mathsf{Rep}_k^\theta(\mathsf{G})} \Rightarrow P^\varsigma \circ (P^\varsigma)^{\mathrm{op}}$ has components

$$\Theta_\rho^\varsigma = \hat{\theta}([\varsigma^{-1}|\varsigma^{-1}])\mathrm{ev}_\rho \circ \rho(\varsigma^{-2}).$$

Given two elements $\varsigma_1, \varsigma_2 \in \hat{\mathsf{G}} \backslash \mathsf{G}$, the natural transformation $\nu^{\varsigma_1, \varsigma_2} : P^{\varsigma_1} \Rightarrow P^{\varsigma_2}$ with components $\nu_\rho^{\varsigma_1, \varsigma_2} = \rho(\varsigma_1^{-1} \varsigma_2)^\vee$ lifts to a non-singular form functor

$$(\mathsf{Rep}_k^\theta(\mathsf{G}), P^{\varsigma_1}, \Theta^{\varsigma_1}) \xrightarrow{\sim} (\mathsf{Rep}_k^\theta(\mathsf{G}), P^{\varsigma_2}, \Theta^{\varsigma_2}).$$

In this way, the pair $(\hat{\mathsf{G}}, \hat{\theta})$ determines a G-torsor of duality structures on $\mathsf{Rep}_k^\theta(\mathsf{G})$.

Definition 3.4. *A $\hat{\theta}$-twisted symmetric representation of G is a symmetric form in $(\mathsf{Rep}_k^\theta(\mathsf{G}), P^\varsigma, \Theta^\varsigma)$.*

Twisted symmetric representations are objects of a homotopy fixed point category, which we denote by $\mathsf{Rep}_k^{\hat{\theta}, \varsigma}(\mathsf{G})$.

Example 3.5. Let $\hat{\mathsf{G}} = \mathsf{G} \times \mathbb{Z}_2$ with ς the generator of $\mathbb{Z}_2$. An untwisted symmetric representation is a representation together with a G-invariant nondegenerate symmetric bilinear form. If instead the twisting is $\hat{\theta}([\omega_2|\omega_1]) = (-1)^{\delta_{\pi(\omega_2), \pi(\omega_1), -1}}$, then the bilinear form is skew-symmetric

We now give a more invariant definition.

Definition 3.6. *A $\hat{\theta}$-twisted generalized symmetric representation of G is a vector space N together with linear maps $\rho(\omega) : {}^{\pi(\omega)}N \to N$, $\omega \in \hat{\mathsf{G}}$, which satisfy $\rho(e) = 1_N$ and*

$$\rho(\omega_2) \circ {}^{\pi(\omega_2)}\rho(\omega_1)^{\pi(\omega_2)} \circ \mathrm{ev}_N^{\delta_{\pi(\omega_1), \pi(\omega_2), -1}} = \hat{\theta}([\omega_2|\omega_1])\rho(\omega_2\omega_1).$$

Remark 3.7. More generally, a Real representation of G on an object x of a category with duality $(\mathcal{C}, (-)^*, \Theta)$ is defined to be a $\mathbb{Z}_2$-graded group homomorphism $\rho : \hat{\mathsf{G}} \to \mathrm{Aut}_\mathcal{C}^{\mathrm{gen}}(x)$. To relate this to the previous definition, let ${}^\theta\hat{\mathsf{G}}$ be the $\mathbb{Z}_2$-graded group which is equal to $k^\times \times \hat{\mathsf{G}}$ as a set and has product

$$(z_2, \omega_2) \cdot (z_1, \omega_1) = (\hat{\theta}([\omega_2|\omega_1])z_2 z_1^{\pi(\omega_2)}, \omega_2\omega_1).$$

The ungraded subgroup of ${}^\theta\hat{\mathsf{G}}$ is ${}^\theta\mathsf{G}$, the central extension of G by $k^\times$ determined by θ. A $\hat{\theta}$-twisted generalized symmetric representation of G is then a Real representation of ${}^\theta\mathsf{G}$ on an object N of $(\mathsf{Vect}_k, (-)^\vee, \mathrm{ev})$, that is, a $\mathbb{Z}_2$-graded group homomorphism $\rho : {}^\theta\hat{\mathsf{G}} \to \mathrm{Aut}_{\mathsf{Vect}}^{\mathrm{gen}}(N)$, which has the additional property that $k^\times \leq {}^\theta\hat{\mathsf{G}}$ acts on N by scalar multiplication.

Finite dimensional twisted generalized symmetric representations form a category $\mathsf{SRep}_k^{\hat{\theta}}(\mathsf{G})$, morphisms $\phi : N \to M$ being morphisms of twisted representations which satisfy $\phi \circ \rho_N(\omega) \circ \phi^\vee = \rho_M(\omega)$ for each $\omega \in \hat{\mathsf{G}} \backslash \mathsf{G}$.

Proposition 3.8. *The categories $\mathsf{Rep}_k^{\hat{\theta}, \varsigma}(\mathsf{G})$ and $\mathsf{SRep}_k^{\hat{\theta}}(\mathsf{G})$ are equivalent.*

Proof. An equivalence $F^\varsigma : \mathsf{Rep}_k^{\hat{\theta}, \varsigma}(\mathsf{G}) \to \mathsf{SRep}_k^{\hat{\theta}}(\mathsf{G})$ is defined on objects by assigning to a twisted symmetric representation (N, ψ_N) the twisted generalized symmetric representation which is equal to N as a twisted representation and has

$$\rho(\omega) = \hat{\theta}([\omega|\varsigma^{-1}])\rho(\omega\varsigma^{-1}) \circ \psi_N^{-1}, \qquad \omega \in \hat{\mathsf{G}} \backslash \mathsf{G}.$$

On morphisms F^ς acts as the identity. $\square$

Let $GW_0^{\hat{\theta}}(\mathsf{G})$ be the Grothendieck–Witt group of $(\mathsf{Rep}_k^\theta(\mathsf{G}), P^\varsigma, \Theta^\varsigma)$. Since non-singular form functors induce isomorphisms of Grothendieck–Witt groups, up to isomorphism, $GW_0^{\hat{\theta}}(\mathsf{G})$ is independent of the choice of $\varsigma \in \hat{\mathsf{G}} \backslash \mathsf{G}$.

Characters of twisted (generalized) symmetric representations of G are defined in the same way as Section 3.1. Real conjugation equivariance (6) continues to hold. When $k = \mathbb{C}$, the isomorphism (7) is replaced by the isomorphism

$$\chi : GW_0^{\hat{\theta}}(\mathsf{G}) \otimes_{\mathbb{Z}} \mathbb{C} \to \Gamma_{\Lambda_{\pi}^{\mathrm{ref}} B\hat{\mathsf{G}}}(\tau_{\pi}^{\mathrm{ref}}(\hat{\theta})_{\mathbb{C}}).$$

In fact, by picking a $\hat{\mathsf{G}}$-invariant Hermitian metric on each twisted symmetric representation, we obtain an isomorphism of abelian groups

$$GW_0^{\hat{\theta}}(\mathsf{G}) \to KR^{0+\hat{\theta}}(B\mathsf{G}).$$

So while the linear and antilinear Real representation categories are not equivalent,[2] the relevant Grothendieck(–Witt) groups are isomorphic.

4. Real representations of finite categorical groups

4.1 Categorical groups The concept of a group can be categorified in a number of ways. A detailed discussion of these categorifications, and the relations between them, can be found in [3].

A categorical group, called a weak 2-group in [3], is a monoidal groupoid $(\mathcal{G}, \otimes, \mathbf{1})$ in which every object admits a weak inverse. Explicitly, this means that for each object x of $\mathcal{G}$ there exists a second object y such that both $x \otimes y$ and $y \otimes x$ are equivalent to the monoidal unit $\mathbf{1}$. A morphism of categorical groups is a monoidal functor. By considering also monoidal natural transformations between monoidal functors, categorical groups assemble to a 2-category.

The monoidal structure $\otimes$ gives the set of connected components $\pi_0(\mathcal{G})$ the structure of a group. The group $\pi_1(\mathcal{G})$ of autoequivalences of the monoidal unit $\mathbf{1}$ is, by an Eckmann–Hilton argument, abelian. As described in Section 4.2 below, the groups $\pi_0(\mathcal{G})$ and $\pi_1(\mathcal{G})$, together with some additional data, determine $\mathcal{G}$ up to equivalence.

Example 4.1. Any group G, considered as a discrete category with object set G and monoidal structure determined (on objects) by its group law, defines a categorical group. By a slight abuse of notation, we will denote this categorical group by G.

Example 4.2. Let A be an abelian group. The action groupoid $B\mathsf{A}$ is a categorical group, the monoidal structure determined (on morphisms) by the group law of A.

Example 4.3. Let x be an object of a bicategory $\mathcal{V}$. Then $1\mathrm{Aut}_{\mathcal{V}}(x)$, the groupoid of autoequivalences of x and the 2-isomorphisms between them, is a categorical group, called the weak automorphism 2-group of x [3, §8.1]. If $\mathcal{V}$ is k-linear and we restrict attention to k-linear autoequivalences and their k-linear 2-isomorphisms, then we obtain a categorical group $\mathsf{GL}_k(x)$ [15, §3.3.2].

Definition 4.4. *A categorical group $\mathcal{G}$ is called finite if $\pi_0(\mathcal{G})$ is finite.*

4.2 Sinh's theorem The following classification indicates that categorical groups can be viewed as twisted extended versions of groups.

Theorem 4.5 ([40]; see also [3, §8.3]). *Categorical groups are classified up to equivalence by the following data:*

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[2]For example, the latter is an $\mathbb{R}$-linear category, while the former is not a linear category.

   (i) *A group* G.
   (ii) *An abelian group* A.
   (iii) *A group homomorphism* $\Pi : \mathsf{G} \to \mathrm{Aut}_{\mathsf{Grp}}(\mathsf{A})$.
   (iv) *A cohomology class* $[\alpha] \in H^3(B\mathsf{G}, \mathsf{A}_\Pi)$.

   In a similar way, equivalence classes of finite categorical groups are classified by the data (i)-(iv), with the additional condition that G be finite.

   An explicit model for the categorical group determined by Theorem 4.5 is as follows. Fix a normalized representative $\alpha \in Z^3(B\mathsf{G}, \mathsf{A}_\Pi)$ of $[\alpha]$. Let $\mathcal{G}(\mathsf{G}, \mathsf{A}, \Pi, \alpha)$ be the skeletal groupoid with set of objects G, a morphism $g \xrightarrow{a} g$ for each pair $(g, a) \in \mathsf{G} \times \mathsf{A}$ and composition law

$$(g \xrightarrow{a_2} g) \circ (g \xrightarrow{a_1} g) = (g \xrightarrow{a_1 \cdot a_2} g).$$

The monoidal structure $\otimes$ is determined on objects by the group law of G and on morphisms by

$$(g \xrightarrow{a} g) \otimes (g' \xrightarrow{a'} g') = (gg' \xrightarrow{a \cdot \Pi(g)(a')} gg').$$

The associator is given by the maps $g_3 g_2 g_1 \xrightarrow{\alpha([g_3|g_2|g_1])} g_3 g_2 g_1$. Since α is normalized, the unitors can be taken to be identity maps.

Example 4.6. If A is trivial, then $\mathcal{G}(\mathsf{G}, \mathsf{A}, \Pi, \alpha)$ is simply the group G, viewed as a categorical group. If A is non-trivial but α is trivial, then $\mathcal{G}(\mathsf{G}, \mathsf{A}, \Pi, \alpha)$ is the categorical group extension of G by $B\mathsf{A}$ determined by Π.

   The next example describes a particularly important and well-studied class of finite categorical groups.

Example 4.7. Let k be a field. Let G be a group and let $\Pi : \mathsf{G} \to \mathrm{Aut}_{\mathsf{Grp}}(k^\times)$ be the trivial map. The associated categorical group, denoted simply by $\mathcal{G}(\mathsf{G}, \alpha)$, is a twisted categorical group extension of G by $Bk^\times$.

4.3 $\mathbb{Z}_2$-graded categorical groups Before introducing Real representations of categorical groups, we categorify the notion of a Real structure on a group. We recall from Section 4.1 that any group defines a categorical group.

Definition 4.8. *A morphism of categorical groups* $\pi : \hat{\mathcal{G}} \to \mathbb{Z}_2$ *is called a* $\mathbb{Z}_2$-*graded categorical group.*

   A morphism of $\mathbb{Z}_2$-graded categorical groups is a morphism of categorical groups which commutes with the structure maps to $\mathbb{Z}_2$.

   The ungraded categorical group of (a non-trivially graded) $\hat{\mathcal{G}}$ is the full subcategory $\mathcal{G} \subset \hat{\mathcal{G}}$ on objects which map via π to $1 \in \mathbb{Z}_2$. There are morphisms of categorical groups

$$1 \to \mathcal{G} \xrightarrow{i} \hat{\mathcal{G}} \xrightarrow{\pi} \mathbb{Z}_2 \tag{9}$$

with i an isomorphism onto its image and π surjective on objects and full. Alternatively, given a categorical group $\mathcal{G}$, a diagram of the form (9) having the above properties is called a Real structure on $\mathcal{G}$.

   Since the groupoid $\mathbb{Z}_2$ is discrete, a $\mathbb{Z}_2$-grading of $\hat{\mathcal{G}}$ is simply a $\mathbb{Z}_2$-grading of $\mathrm{Obj}(\hat{\mathcal{G}})$ which is compatible with the monoidal structure. In particular, a $\mathbb{Z}_2$-grading of $\mathcal{G}(\mathsf{G}, \mathsf{A}, \Pi, \alpha)$ deter-

mines, and is determined by, a $\mathbb{Z}_2$-grading of G. The ungraded categorical group is obtained by restriction to the ungraded subgroup of G.

The following example plays an important role in the remainder of the paper.

Example 4.9. Let k be a field and let $\pi : \hat{\mathsf{G}} \to \mathbb{Z}_2$ be a $\mathbb{Z}_2$-graded group. Let $\Pi : \hat{\mathsf{G}} \to \mathrm{Aut}_{\mathsf{Grp}}(k^\times)$ be the map $\Pi(\omega)(a) = a^{\pi(\omega)}$, so that $k_\Pi^\times$ is $k_\pi^\times$ in the notation of Section 3.2. Let $\hat{\alpha} \in Z^3(B\hat{\mathsf{G}}, k_\pi^\times)$. The categorical group $\mathcal{G}(\hat{\mathsf{G}}, k^\times, \Pi, \hat{\alpha})$ defined by Theorem 4.5, henceforth denoted by $\mathcal{G}(\hat{\mathsf{G}}, \hat{\alpha})$, is $\mathbb{Z}_2$-graded with ungraded categorical group $\mathcal{G}(\mathsf{G}, \alpha)$, where $\alpha \in Z^3(B\mathsf{G}, k^\times)$ is the restriction of $\hat{\alpha}$ to $B\mathsf{G}$.

The following example categorifies the $\mathbb{Z}_2$-graded group $\mathrm{Aut}_{\mathcal{C}}^{\mathrm{gen}}(x)$ of Section 2.4.

Example 4.10. Let x be an object of a bicategory $\mathcal{V}$ with weak duality involution. Then $1\mathrm{Aut}_{\mathcal{V}}^{\mathrm{gen}}(x)$, the collection of all equivalences $x \to x$ and $x^\circ \to x$, together with the 2-isomorphisms between them, is a $\mathbb{Z}_2$-graded categorical group. The monoidal structure $\otimes$ is defined on objects by

$$f_2 \otimes f_1 = f_2 \circ_0 (^{\pi(f_2)}f_1 \circ_0 \eta_x^{\delta_{\pi(f_2),\pi(f_1),-1}}),$$

where $\pi(f) \in \mathbb{Z}_2$ is such that $f : {}^{\pi(f)}x \to x$. The definition of $\otimes$ on morphisms is similar. The associator for three antiautoequivalences is

$$(f_3 \otimes f_2) \otimes f_1 = (f_3 \circ (f_2^\circ \circ \eta_x)) \circ f_1 \xrightarrow{\alpha} f_3 \circ (f_2^\circ \circ (\eta_x \circ f_1)) \xrightarrow{\eta}$$
$$f_3 \circ (f_2^\circ \circ (f_1^{\circ\circ} \circ \eta_{x^\circ})) \xrightarrow{\zeta_x} f_3 \circ (f_2^\circ \circ (f_1^{\circ\circ} \circ \eta_x^\circ)) = f_3 \otimes (f_2 \otimes f_1),$$

where α is a composition of associators for $\mathcal{V}$ and the arrow labelled by η is a pseudo-naturality constraint for η. The remaining associators are constructed in a similar way, but do not use the modification ζ. The pentagon identity is verified using the constraint (2). If x has at least one antiautoequivalence, then the morphism $\pi : 1\mathrm{Aut}_{\mathcal{V}}^{\mathrm{gen}}(x) \to \mathbb{Z}_2$ fits into an exact sequence of categorical groups:

$$1 \to 1\mathrm{Aut}_{\mathcal{V}}(x) \to 1\mathrm{Aut}_{\mathcal{V}}^{\mathrm{gen}}(x) \xrightarrow{\pi} \mathbb{Z}_2 \to 1.$$

If $\mathcal{V}$ is k-linear and we restrict attention to k-linear (anti)autoequivalences and 2-isomorphisms, then we obtain a $\mathbb{Z}_2$-graded categorical group $\mathsf{GL}_k^{\mathrm{gen}}(x)$ whose ungraded categorical group is $\mathsf{GL}_k(x)$.

Example 4.11. The previous example has a variation in which the bicategory with duality involution is replaced by a bicategory $\mathcal{V}$ with contravariance. In this way, for each $x \in \mathrm{Obj}(\mathcal{V})$ we obtain a $\mathbb{Z}_2$-graded categorical group $1\mathrm{Aut}_{\mathcal{V}}^{\mathrm{gen}}(x)$ whose ungraded categorical group is $1\mathrm{Aut}_{\mathcal{V}_1}(x)$.

A $\mathbb{Z}_2$-graded categorical group $\pi : \hat{\mathcal{G}} \to \mathbb{Z}_2$ defines a bicategory $\underline{\hat{\mathcal{G}}}$ with contravariance as follows. Let $\mathrm{Obj}(\underline{\hat{\mathcal{G}}}) = \{\mathrm{pt}\}$. For each $\epsilon \in \mathbb{Z}_2$, let $1\mathrm{Hom}_{\underline{\hat{\mathcal{G}}}}^\epsilon(\mathrm{pt}, \mathrm{pt})$ be the full subcategory of $\hat{\mathcal{G}}$ on objects which map via π to ϵ. The horizontal compositions and associators in $\underline{\hat{\mathcal{G}}}$ are induced by the monoidal structure of $\hat{\mathcal{G}}$.

Remark 4.12. It may be interesting to consider gradings of categorical groups by non-trivial categorifications of $\mathbb{Z}_2$. In the k-linear setting, one possibility is to use the symmetric monoidal category $\mathsf{Pic}^{\mathbb{Z}_2}(k^\times)$ of Ganter–Kapranov [18, Example 3.1.2(d)].

4.4 Real representations of finite categorical groups We introduce Real representations of finite categorical groups, categorifying the linear approach of Section 3.2. An antilinear approach can be found in Section 6.4.

Let $\mathcal{G}$ be a finite categorical group with Real structure $\hat{\mathcal{G}}$.

Definition 4.13. *A Real representation of $\mathcal{G}$ on a bicategory $\mathcal{V}$ with contravariance is a contravariance preserving pseudofunctor $\rho : \underline{\hat{\mathcal{G}}} \to \mathcal{V}$, where $\underline{\hat{\mathcal{G}}}$ is the bicategory with contravariance defined is as in Section 4.3.*

Real representations of $\mathcal{G}$ on $\mathcal{V}$ assemble to a bicategory $\mathsf{RRep}_{\mathcal{V}}(\mathcal{G})$ whose 1- and 2-morphisms are pseudonatural transformations and modifications which respect contravariance, respectively. More compactly, we can define

$$\mathsf{RRep}_{\mathcal{V}}(\mathcal{G}) = 1\mathrm{Hom}_{\mathsf{Bicat}_{\mathsf{con}}}(\underline{\hat{\mathcal{G}}}, \mathcal{V}),$$

where $\mathsf{Bicat}_{\mathsf{con}}$ is the tricategory of bicategories with contravariance described in [39]. If $\mathcal{V}$ is in fact a 2-category, then so too is $\mathsf{RRep}_{\mathcal{V}}(\mathcal{G})$.

Let $\mathsf{Bicat}_{\mathsf{con}}^{\leq 1}$ be the category of bicategories with contravariance and their pseudofunctors preserving contravariance. Taking 1-morphism bicategories defines a functor

$$1\mathrm{Hom}_{\mathsf{Bicat}_{\mathsf{con}}}(-, -) : (\mathsf{Bicat}_{\mathsf{con}}^{\leq 1})^{\mathrm{op}} \times \mathsf{Bicat}_{\mathsf{con}}^{\leq 1} \to \mathsf{Bicat}^{\leq 1}.$$

Using this functor, it can be verified that if $\mathcal{V}$ and $\mathcal{V}'$ are biequivalent bicategories with contravariance and $\hat{\mathcal{G}}$ and $\hat{\mathcal{G}}'$ are equivalent $\mathbb{Z}_2$-graded categorical groups, then $\mathsf{RRep}_{\mathcal{V}}(\mathcal{G})$ and $\mathsf{RRep}_{\mathcal{V}'}(\mathcal{G}')$ are biequivalent. Compare with [12, §3.5]. In view of coherence theorem for bicategories with contravariance, it follows that there is no loss of generality in restricting attention to Real representations on 2-categories with contravariance.

We will use the following interpretation of Real representations.

Lemma 4.14. *A Real representation of $\mathcal{G}$ on a bicategory $\mathcal{V}$ with contravariance is the data of an object $V \in \mathrm{Obj}(\mathcal{V})$ together with a morphism of $\mathbb{Z}_2$-graded categorical groups $\rho : \hat{\mathcal{G}} \to 1\mathrm{Aut}_{\mathcal{V}}^{\mathrm{gen}}(V)$.*

Proof. This is straightforward. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Motivated by Lemma 4.14, define a Real representation of $\mathcal{G}$ on a bicategory $\mathcal{V}$ with weak duality involution to be an object $V \in \mathrm{Obj}(\mathcal{V})$ together with a morphism $\rho : \hat{\mathcal{G}} \to 1\mathrm{Aut}_{\mathcal{V}}^{\mathrm{gen}}(V)$ of $\mathbb{Z}_2$-graded categorical groups.

Finally, we state a k-linear version of the above definitions. We restrict to categorical groups of the form $\mathcal{G}(\mathsf{G}, \alpha)$ with Real structure $\mathcal{G}(\hat{\mathsf{G}}, \hat{\alpha})$.

Definition 4.15. *A linear Real representation of $\mathcal{G}(\mathsf{G}, \alpha)$ on a k-linear bicategory $\mathcal{V}$ with contravariance is a contravariance preserving pseudofunctor $\rho : \underline{\mathcal{G}(\hat{\mathsf{G}}, \hat{\alpha})} \to \mathcal{V}$ with the additional property that $\mathrm{Aut}_{\hat{\mathcal{G}}}(\mathbf{1}) \simeq k^{\times}$ acts by scalar multiplication.*

Linear Real representations of $\mathcal{G}(\mathsf{G}, \alpha)$ form a bicategory $\mathsf{RRep}_{\mathcal{V},k}(\mathcal{G})$. The obvious analogue of Lemma 4.14, with $1\mathrm{Aut}_{\mathcal{V}}^{\mathrm{gen}}(V)$ replaced by $\mathsf{GL}_k^{\mathrm{gen}}(V)$, holds.

To close this section, we describe an interpretation of Real representations of finite categorical groups which categorifies the homotopy fixed point perspective of Section 3.2. Fix an element $\varsigma \in \mathrm{Obj}(\hat{\mathcal{G}})$ such that $\pi(\varsigma) = -1$ together with a weak inverse $\overline{\varsigma}$. Define a biequivalence $F^{\varsigma} : \mathcal{G}^{\mathrm{co}} \to \mathcal{G}$ by assigning to $x : \mathrm{pt} \to \mathrm{pt}$ and $f : x \Rightarrow y$ in $\mathcal{G}^{\mathrm{co}}$ the 1- and 2-morphisms $(\varsigma \otimes x) \otimes \overline{\varsigma}$ and $(\varsigma \otimes f^{-1}) \otimes \overline{\varsigma}$ in $\mathcal{G}$, respectively. Noting that $F^{\varsigma} \circ_0 (F^{\varsigma})^{\mathrm{co}}$ is the adjoint action $\mathrm{Ad}_{\varsigma^2} = (\varsigma^2 \otimes -) \otimes \overline{\varsigma}^2$, the element ς^2 and the associator for $\mathcal{G}$ induce a pseudonatural isomorphism $\varsigma^2 : 1_{\mathcal{G}} \Rightarrow F^{\varsigma} \circ_0 (F^{\varsigma})^{\mathrm{co}}$. The biequivalence F^{ς} can be used to define a weak duality involution on $\mathsf{Rep}_{\mathcal{V}}(\mathcal{G})$ as follows, giving a

ς-twisted version of [39, Example 2.6]. The duality involution takes a pseudofunctor $\rho : \mathcal{G} \to \mathcal{V}$ to the composition

$$\mathcal{G} \xrightarrow{(F^\varsigma)^{\mathrm{co}}} \mathcal{G}^{\mathrm{co}} \xrightarrow{\rho^{\mathrm{co}}} \mathcal{V}^{\mathrm{co}} \xrightarrow{(-)^\circ} \mathcal{V}.$$

The required adjoint equivalence $\tilde{\eta}$ and modification $\tilde{\zeta}$ are induced by whiskering with ς^2 and the duality involution data of $\mathcal{V}$. For example, the component of $\tilde{\eta}$ at ρ_V assigns to pt the 1-morphism $\eta_V \circ_0 \rho_V(\varsigma^2) : V \to V^{\circ\circ}$.

Proposition 4.16. *(i) Up to duality biequivalence, the weak duality involution on $\mathsf{Rep}_\mathcal{V}(\mathcal{G})$ is independent of the choice of $\varsigma \in \mathrm{Obj}(\hat{\mathcal{G}})$.*

(ii) For any ς as above, there is a biequivalence $\mathsf{Rep}_\mathcal{V}(\mathcal{G})^{h\mathbb{Z}_2} \simeq \mathsf{RRep}_\mathcal{V}(\mathcal{G})$.

Proof. Let $\varsigma_1, \varsigma_2 \in \mathrm{Obj}(\hat{\mathcal{G}})$ be as above with associated biequivalences $F^{\varsigma_1}, F^{\varsigma_2} : \mathcal{G}^{\mathrm{co}} \to \mathcal{G}$. After fixing an equivalence $\overline{\varsigma}_1 \otimes \varsigma_1 \simeq \mathbf{1}$, the 1-morphism $\varsigma_2 \otimes \overline{\varsigma}_1$ defines a pseudonatural isomorphism $\mathrm{Ad}_{\varsigma_2 \otimes \overline{\varsigma}_1} : F^{\varsigma_1} \Rightarrow F^{\varsigma_2}$. The remaining components of the duality biequivalence are induced by whiskering.

The second statement is proved in the same way as Proposition 3.8. We will describe a biequivalence at the level of objects, leaving the description on 1- and 2-morphisms to the reader. Given a symmetric form (ρ, ψ, μ) in $\mathsf{Rep}_\mathcal{V}(\mathcal{G})$, with $\rho(\mathrm{pt}) = V$, the map $\psi(\mathrm{pt})$ is an equivalence $V^\circ \to V$. For $\omega \in \mathrm{Obj}(\hat{\mathcal{G}})$ with $\pi(\omega) = -1$, define $\rho(\omega)$ to be the composition $\rho(\omega \otimes \overline{\varsigma}) \circ_0 \psi(\mathrm{pt})$. The monoidal coherence 2-isomorphisms $\psi_{\bullet,\bullet}$ are induced by μ and $\tilde{\eta}$. It is straightforward to verify that this indeed defines a Real representation of $\mathcal{G}$. $\qquad\square$

Remark 4.17. (i) When restricted to trivially $\mathbb{Z}_2$-graded categorical groups, the above definitions recover the representation theory of finite categorical groups, as studied in [12], [17], [4].

(ii) While the above definitions make sense for categorical groups which are not finite, in the continuous case they should be supplemented with topological coherence conditions.

5. Real 2-representation theory of finite groups

We study the definitions of Section 4.4 when the categorical group is a finite group. The more technical case of categorical groups is the focus of Section 6.

5.1 Real 2-representations Lemma 4.14 leads to an explicit description of a Real representation of the categorical group determined by a finite group, which we state as a new definition.

Definition 5.1. *A Real 2-representation of a finite group G on a 2-category $\mathcal{V}$ with strict duality involution consists of the following data:*

(i) *An object V of $\mathcal{V}$.*

(ii) *For each $\omega \in \hat{\mathsf{G}}$, an equivalence $\rho(\omega) : {}^{\pi(\omega)}V \to V$.*

(iii) *For each pair $\omega_1, \omega_2 \in \hat{\mathsf{G}}$, a 2-isomorphism*

$$\psi_{\omega_2,\omega_1} : \rho(\omega_2) \circ {}^{\pi(\omega_2)}\rho(\omega_1) \Longrightarrow \rho(\omega_2 \omega_1).$$

(iv) *A 2-isomorphism $\psi_e : \rho(e) \Rightarrow 1_V$.*

This data is required to satisfy the following conditions:

(a) *For each triple $\omega_1, \omega_2, \omega_3 \in \hat{\mathsf{G}}$, the equality*

$$\psi_{\omega_3\omega_2,\omega_1} \circ_1 \left(\psi_{\omega_3,\omega_2} \circ {}^{\pi(\omega_3\omega_2)}\rho(\omega_1) \right) = \psi_{\omega_3,\omega_2\omega_1} \circ_1 \left(\rho(\omega_3) \circ {}^{\pi(\omega_3)}\psi_{\omega_2,\omega_1}^{\pi(\omega_3)} \right) \tag{10}$$

of 2-isomorphisms $\rho(\omega_3) \circ {}^{\pi(\omega_3)}\rho(\omega_2) \circ {}^{\pi(\omega_3\omega_2)}\rho(\omega_1) \Longrightarrow \rho(\omega_3\omega_2\omega_1)$ holds.

(b) *For each $\omega \in \hat{\mathsf{G}}$, the equalities*

$$\psi_{e,\omega} = \psi_e \circ \rho(\omega), \qquad \psi_{\omega,e} = \rho(\omega) \circ {}^{\pi(\omega)}\psi_e \tag{11}$$

of 2-isomorphisms $\rho(\omega) \Rightarrow \rho(\omega)$ hold.

Denote by $\psi_{\omega_3,\omega_2,\omega_1}$ the 2-isomorphism defined by either side of equation (10).

5.2   Real conjugation invariance of categorical traces We study categorical traces, as introduced by Ganter–Kapranov [17] and Bartlett [4], in the presence of duality involutions.

Let x be an object of a 2-category $\mathcal{V}$. As in [17, §3.1] and [4, §4.1], the categorical trace $\mathbb{Tr}(f)$ of a 1-morphism $f : x \to x$ is the set of all 2-morphisms from 1_x to f:

$$\mathbb{Tr}(f) = 2\mathrm{Hom}_{\mathcal{V}}(1_x, f).$$

Given a 2-morphism $u : f_1 \Rightarrow f_2$, define $\mathbb{Tr}(u) : \mathbb{Tr}(f_1) \to \mathbb{Tr}(f_2)$ to be $u \circ_1 (-)$. These definitions extend the categorical trace to a functor

$$\mathbb{Tr} : 1\mathrm{End}_{\mathcal{V}}(x) \to \mathsf{Set}.$$

If $\mathcal{V}$ is enriched in a category $\mathcal{A}$, then $\mathbb{Tr}$ takes values in $\mathcal{A}$. For example, when $\mathcal{V}$ is k-linear the functor $\mathbb{Tr}$ is Vect_k-valued.

In [17, §4.3] and [4, §4.3] a kind of conjugation invariance of categorical traces is established. We generalize this result in what follows by showing that categorical traces in 2-categories with duality involutions (or contravariance) are invariant under Real conjugation.

Suppose then that $\mathcal{V}$ is a 2-category with strict duality involution. Fix a sign $\epsilon \in \mathbb{Z}_2$. Let $f : x \to x$ be an equivalence. When $\epsilon = -1$ we also fix a quasi-inverse $\tilde{f}$ of f and a 2-isomorphism $\mu : \tilde{f} \circ f \Rightarrow 1_x$. Write

$$f^\nu = \begin{cases} f & \text{if } \nu = 1, \\ \tilde{f} & \text{if } \nu = -1. \end{cases}$$

Let $h : {}^\epsilon x \to y$ be an equivalence with quasi-inverse $k : y \to {}^\epsilon x$ and 2-isomorphisms $u : 1_y \Rightarrow h \circ k$ and $v : 1_{\epsilon x} \Rightarrow k \circ h$. This data can be used to define a map

$$\Psi(h, k, u, v; \mu) : \mathbb{Tr}(f) \to \mathbb{Tr}(h \circ {}^\epsilon f^\epsilon \circ k),$$

henceforth denoted by $\Psi(h)$. The map μ is required only when $\epsilon = -1$. Suppose that we are given a 2-morphism $\phi \in \mathbb{Tr}(f)$. Interpret u as a 2-morphism $1_y \Longrightarrow h \circ 1_{\epsilon x} \circ k$. When $\epsilon = 1$ the map $\Psi(h)$ is defined by post-composing u with ϕ:

$$\Psi(h)(\phi) = (h \circ_0 \phi \circ_0 k) \circ_1 u.$$

This is the definition of [17], [4]. If instead $\epsilon = -1$, then we can form the composition

$$1_{x^\circ} \xrightarrow{\mu^\circ} \tilde{f}^\circ \circ f^\circ \xrightarrow{\tilde{f}^\circ \circ_0 \phi^\circ} \tilde{f}^\circ.$$

The map $\Psi(h)$ is defined by further pre-composing with u:

$$\Psi(h)(\phi) = \left(h \circ_0 \left((\tilde{f}^\circ \circ_0 \phi^\circ) \circ_1 \mu^\circ \right) \circ_0 k \right) \circ_1 u.$$

The following result generalizes [17, Proposition 4.10] and [4, Proposition 4.3]. A further generalization (with a different proof) will be given in Theorem 6.9 below.

Proposition 5.2. *For each pair of equivalences $f : x \to x$ and $h : {}^\epsilon x \to y$ with quasi-inverse data as above, the map*

$$\Psi(h) : \mathbb{Tr}(f) \to \mathbb{Tr}(h \circ {}^\epsilon f^\epsilon \circ k)$$

is a bijection. Moreover, $\Psi(1_x) = 1_{\mathbb{Tr}(f)}$ and, given equivalences ${}^{\epsilon_1} x \xrightarrow{h_1} y_1$ and ${}^{\epsilon_2} y_1 \xrightarrow{h_2} y_2$ with quasi-inverse data, the equality $\Psi(h_2) \circ \Psi(h_1) = \Psi(h_2 \circ {}^{\epsilon_2} h_1)$ holds.

Proof. That $\Psi(h)$ is a bijection follows from the assumption that h is an equivalence. The equality $\Psi(1_x) = 1_{\mathbb{Tr}(f)}$ is clear from the construction.

To explain the precise meaning of the final statement, we need to describe the quasi-inverse data implicit in the definition of $\Psi(h_2 \circ {}^{\epsilon_2} h_1)$. Since h_1 and h_2 are equivalences, so too is $h_2 \circ {}^{\epsilon_2} h_1$. We take ${}^{\epsilon_2} k_1 \circ k_2$ as the quasi-inverse of $h_2 \circ {}^{\epsilon_2} h_1$ with 2-isomorphisms

$$u : 1_{y_2} \xrightarrow{u_2} h_2 \circ k_2 \xrightarrow{h_2 \circ_0 {}^{\epsilon_2} u_1^{\epsilon_2} \circ_0 k_2} h_2 \circ {}^{\epsilon_2} h_1 \circ {}^{\epsilon_2} k_1 \circ k_2$$

and

$$v : 1_{{}^{\epsilon_2 \epsilon_1} x} \xrightarrow{{}^{\epsilon_2} v_1^{\epsilon_2}} {}^{\epsilon_2} k_1 \circ {}^{\epsilon_2} h_1 \xrightarrow{{}^{\epsilon_2} k_1 \circ_0 v_2 \circ_0 {}^{\epsilon_2} h_1} {}^{\epsilon_2} k_1 \circ k_2 \circ h_2 \circ {}^{\epsilon_2} h_1.$$

When $\epsilon_1 \epsilon_2 = 1$ no additional data is needed to define $\Psi(h_2 \circ {}^{\epsilon_2} h_1)$. If $\epsilon_1 = -1$ and $\epsilon_2 = 1$, then we take for $\mu : \tilde{f} \circ f \Rightarrow 1_x$ the data used to define $\Psi(h_1)$. If instead $\epsilon_1 = 1$ and $\epsilon_2 = -1$, then part of the data used to define $\Psi(h_2)$ is a quasi-inverse $\tilde{f}'$ of $f' = h_1 \circ f \circ k_1$ and a 2-isomorphism $\mu' : \tilde{f}' \circ f' \implies 1_{y_1}$. Set $\tilde{f} = k_1 \circ \tilde{f}' \circ h_1$ with 2-isomorphism $\mu : \tilde{f} \circ f \Rightarrow 1_x$ given by the composition

$$\tilde{f} \circ f \xrightarrow{\tilde{f} \circ f \circ v_1^{-1}} \tilde{f} \circ f \circ k_1 \circ h_1 = k_1 \circ \tilde{f}' \circ h_1 \circ f \circ k_1 \circ h_1 \xrightarrow{k_1 \circ_0 \mu' \circ_0 h_1} k_1 \circ h_1 \xrightarrow{v_1} 1_x.$$

Then $\Psi(h_2 \circ {}^{\epsilon_2} h_1)$ is defined to be $\Psi(h_2 \circ {}^{\epsilon_2} h_1, {}^{\epsilon_2} k_1 \circ k_2, u, v; \mu)$. With the above definitions in place, it is now straightforward to verify the claimed equality $\Psi(h_2) \circ \Psi(h_1) = \Psi(h_2 \circ {}^{\epsilon_2} h_1)$. $\square$

Remark 5.3. While the categorical trace of an arbitrary 1-morphism $f : x \to x$ is defined, Proposition 5.2 requires that f be an equivalence.

Keeping the above notation, let us further assume that $x = y$ and that the 1-morphisms f and h graded commute in the sense that we are given a 2-isomorphism $\kappa : h \circ {}^\epsilon f^\epsilon \Rightarrow f \circ h$. We can then define $(h, \kappa)_* : \mathbb{Tr}(f) \to \mathbb{Tr}(f)$ to be the composition

$$\mathbb{Tr}(f) \xrightarrow{\Psi(h)} \mathbb{Tr}(h \circ {}^\epsilon f^\epsilon \circ k) \xrightarrow{\mathbb{Tr}(\kappa \circ_0 k)} \mathbb{Tr}(f \circ h \circ k) \xrightarrow{\mathbb{Tr}(f \circ_0 u^{-1})} \mathbb{Tr}(f).$$

When $\epsilon = 1$ this reduces to a construction of Ganter–Kapranov. Note that if $\mathcal{V}$ is enriched in $\mathcal{A}$, then $(h, \kappa)_*$ is a morphism in $\mathcal{A}$. In particular, when $\mathcal{A} = \mathsf{Vect}_k$ we can make the following definition, generalizing that of [17, §3.6].

Definition 5.4. *Let $\mathcal{V}$ be a k-linear 2-category with strict duality involution and let $f : x \to x$ and $h : {}^\epsilon x \to x$ be graded commuting equivalences. Assuming that the vector space $\mathrm{Tr}(f)$ is finite dimensional, the joint trace of (f, h) is*

$$\mathrm{tr}(f, h) = \mathrm{tr}\left((h, \kappa)_* : \mathrm{Tr}(f) \to \mathrm{Tr}(f)\right).$$

5.3   Real categorical characters Let ρ be a Real 2-representation of a finite group G on a 2-category $\mathcal{V}$ with strict duality involution. For $g \in \mathsf{G}$, write $\mathrm{Tr}_\rho(g)$ for the set $\mathrm{Tr}(\rho(g))$. Fix $g \in \mathsf{G}$ and $\omega \in \hat{\mathsf{G}}$. By applying Proposition 5.2 to the equivalences

$$f = \rho(g), \qquad \tilde{f} = \rho(g^{-1}), \qquad h = \rho(\omega), \qquad k = {}^{\pi(\omega)}\rho(\omega^{-1}),$$

and the 2-isomorphisms

$$u = \psi_{\omega,\omega^{-1}}^{-1} \circ_0 \psi_e^{-1}, \qquad \mu = \psi_e \circ_0 \psi_{g^{-1},g}$$

we obtain a map

$$\mathrm{Tr}(\rho(g)) \to \mathrm{Tr}\left(\rho(\omega) \circ {}^{\pi(\omega)}\rho(g^{\pi(\omega)}) \circ {}^{\pi(\omega)}\rho(\omega^{-1})\right).$$

Post composing with $\mathrm{Tr}(\psi_{\omega,g^{\pi(\omega)},\omega^{-1}})$ then gives a map $\beta_{g,\omega} : \mathrm{Tr}_\rho(g) \to \mathrm{Tr}_\rho(\omega g^{\pi(\omega)}\omega^{-1})$.

Definition 5.5. *The Real categorical character of ρ is the assignment*

$$g \mapsto \mathrm{Tr}_\rho(g), \qquad g \in \mathsf{G}$$

together with the bijections

$$\beta_{g,\omega} : \mathrm{Tr}_\rho(g) \to \mathrm{Tr}_\rho(\omega g^{\pi(\omega)}\omega^{-1}), \qquad (g, \omega) \in \mathsf{G} \times \hat{\mathsf{G}}.$$

The sets $\{\mathrm{Tr}_\rho(g)\}_{g \in \mathsf{G}}$, together with the bijections $\{\beta_{g_1,g_2}\}_{(g_1,g_2) \in \mathsf{G}^2}$, define the categorical character of the underlying 2-representation of G [17], [4]. In particular, unlike the case of Real characters, Real categorical characters contain strictly more information than the categorical character of the underlying 2-representation.

Proposition 5.6. *The Real categorical character of a Real 2-representation ρ of G on $\mathcal{V}$ defines a functor*

$$\mathrm{Tr}(\rho) : \Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}} \to \mathsf{Set}.$$

Moreover, if $\mathcal{V}$ is enriched in $\mathcal{A}$, then the functor $\mathrm{Tr}(\rho)$ takes values in $\mathcal{A}$.

Proof. Recall that objects of $\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}$ are labelled by elements $g \in \mathsf{G}$. Set $\mathrm{Tr}(\rho)(g) = \mathrm{Tr}_\rho(g)$. Given a morphism $\omega : g \to \omega g^{\pi(\omega)}\omega^{-1}$ in $\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}$, set $\mathrm{Tr}(\rho)(\omega) = \beta_{g,\omega}$. That these assignments define a functor $\mathrm{Tr}(\rho) : \Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}} \to \mathsf{Set}$ follows from Proposition 5.2. The final statement is clear. $\qquad\square$

For example, when $\mathcal{V}$ is k-linear, Proposition 5.6 states that the Real categorical character of a linear Real 2-representation of G is a vector bundle over $\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}$.

Example 5.7. Let $\mathsf{Vect}_{\mathbb{F}_1}(X)$ be the category of $\mathbb{F}_1$-vector bundles over a finite set X, as in [24]. An action of a finite $\mathbb{Z}_2$-graded group $\hat{\mathsf{G}}$ on X defines a Real 2-representation ρ of G on $\mathsf{Vect}_{\mathbb{F}_1}(X) \in \mathrm{Obj}(\mathsf{Cat})$ by setting

$$\rho(g) = (g^{-1})^*, \qquad \rho(\omega) = (-)^\vee \circ (\omega^{-1})^*$$

for $g \in \mathsf{G}$ and $\omega \in \hat{\mathsf{G}} \backslash \mathsf{G}$, where $(-)^\vee = \mathrm{Hom}_{\mathbb{F}_1}(-, \mathbb{F}_1)$. As $(-)^\vee$ squares to the identify functor, the 2-isomorphisms $\psi_{\bullet,\bullet}$ are canonical. The Real categorical character $\mathbb{Tr}(\rho)$ is the unoriented loop groupoid of the double cover $X /\!\!/ \mathsf{G} \to X /\!\!/ \hat{\mathsf{G}}$, viewed as a groupoid over $\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}$. Interpreting the trace of an endomorphism of a finite set as the cardinality of its fixed point set, the joint traces $\mathrm{tr}(\rho(g), \rho(\omega)) = \chi_\rho(g, \omega)$ of Section 5.2 compute the cardinality of joint fixed point sets:

$$\chi_\rho(g, \omega) = \# X^{g,\omega}.$$

In physical terminology, this example describes an orientifold of the $\mathbb{F}_1$-analogue of the G-equivariant B-model on X. From this point of view, the Real part of χ_ρ computes the charge of the orientifold plane.

6. Twisted Real 2-representation theory of finite groups

In this section we study linear Real representations of finite categorical groups. This recovers the k-linear version of the results of Section 5 when the categorical group is a trivial extension of a finite group by $Bk^\times$.

6.1 Basic definitions Fix a field k. The following is a Real variant of definitions of Frenkel–Zhu [15, Definition 2.8] and Ganter–Usher [19, Definition 4.1].

Definition 6.1. *A twisted Real 2-representation of a finite group* G *on a k-linear 2-category* $\mathcal{V}$ *with strict duality involution consists of data* $V \in \mathrm{Obj}(\mathcal{V})$, $\rho(\omega)$, ψ_{ω_2,ω_1} *and* ψ_e *as in Section 5.1, with the constraint* (11) *unchanged but with the constraint* (10) *replaced by the condition that*

$$\hat{\alpha}([\omega_3|\omega_2|\omega_1]) \cdot \psi_{\omega_3\omega_2,\omega_1} \left(\psi_{\omega_3,\omega_2} \circ {}^{\pi(\omega_3\omega_2)}\rho(\omega_1) \right) = \psi_{\omega_3,\omega_2\omega_1} \left(\rho(\omega_3) \circ {}^{\pi(\omega_3)}\psi_{\omega_2,\omega_1}^{\pi(\omega_3)} \right) \quad (12)$$

for some function $\hat{\alpha} : \hat{\mathsf{G}} \times \hat{\mathsf{G}} \times \hat{\mathsf{G}} \to k^\times$.

We call $\hat{\alpha}$, which we regard as a 3-cochain on $B\hat{\mathsf{G}}$, the Real 2-Schur multiplier of the twisted Real 2-representation ρ.

In terms of string diagrams, the 2-isomorphisms ψ_{ω_2,ω_1} and ψ_e are

respectively. Equation (12) will be written as

(13)

the arrow indicating that the 2-morphism on the right is $\hat{\alpha}([\omega_3|\omega_2|\omega_1])$ times that on the left. When computing with string diagrams, labels of 1-morphisms will often be omitted when they can be reconstructed from the labelled data in a string diagram. For example, the 2-morphism $\psi_e \circ \psi_{\omega,\omega^{-1}}$ will often be drawn as


Lemma 6.2. *The Real 2-Schur multiplier defines a 3-cocycle $\hat{\alpha} \in Z^3(B\hat{\mathsf{G}}, k_\pi^\times)$.*

Proof. We need to verify that the equality

$$\hat{\alpha}([\omega_4\omega_3|\omega_2|\omega_1])\hat{\alpha}([\omega_4|\omega_3|\omega_2\omega_1]) = \hat{\alpha}([\omega_3|\omega_2|\omega_1])^{\pi(\omega_4)}\hat{\alpha}([\omega_4|\omega_3\omega_2|\omega_1])\hat{\alpha}([\omega_4|\omega_3|\omega_2])$$

holds for all $\omega_1, \omega_2, \omega_3, \omega_4 \in \hat{\mathsf{G}}$. This can be proved using string diagrams, similarly to [19, Proposition 4.3], the corresponding statement for twisted 2-representations. Repeated application of equation (13) gives the following commutative diagram of string diagrams:
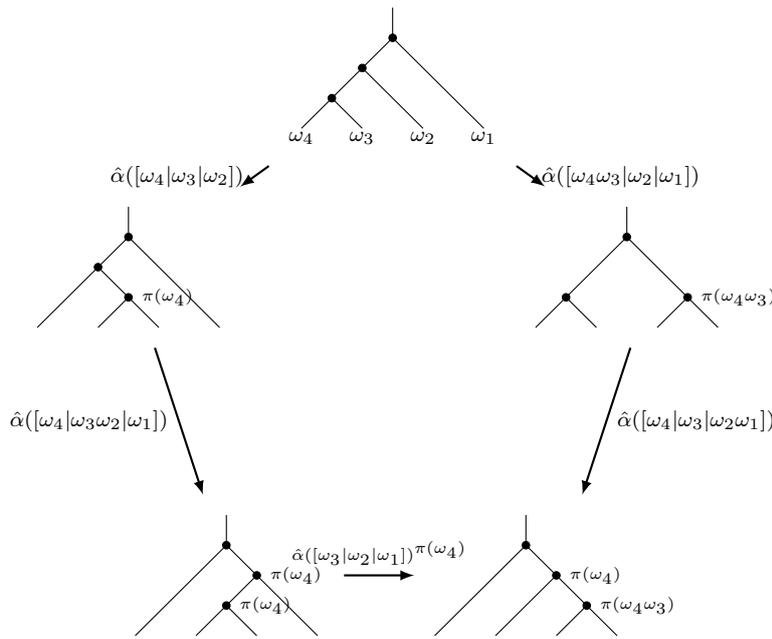
A node labelled by -1 indicates that it is $\psi_{\bullet,\bullet}^{-\mathrm{op}}$, instead of $\psi_{\bullet,\bullet}$, which is applied. For example, in the bottom right string diagram of the above diagram the node labelled by $\pi(\omega_4)$ corresponds to the 2-isomorphism ${}^{\pi(\omega_4)}\psi_{\omega_3,\omega_2\omega_1}^{\pi(\omega_4)}$. The bottom arrow is multiplication by $\hat{\alpha}([\omega_3|\omega_2|\omega_1])^{\pi(\omega_4)}$, since it is the $\pi(\omega_4)^{\mathrm{th}}$ power of equation (13) which is being applied. Commutativity of the above diagram implies the desired cocycle condition. $\square$

By combining Theorem 4.5 and Lemma 6.2, we find that an $\hat{\alpha}$-twisted Real 2-representation of G determines a $\mathbb{Z}_2$-graded categorical group $\mathcal{G}(\hat{\mathsf{G}}, \hat{\alpha})$. The following proposition shows that the corresponding Real (2-)representation theories are equivalent.

Proposition 6.3. *There is a canonical biequivalence between* $\mathsf{RRep}_{\mathcal{V},k}(\mathcal{G}(\mathsf{G}, \alpha))$ *and the bicategory of* $\hat{\alpha}$-*twisted Real* 2-*representations of* G *on* $\mathcal{V}$.

Proof. By construction, $\hat{\alpha} \in Z^3(B\hat{\mathsf{G}}, k_\pi^\times)$ determines the associator of the monoidal groupoid $\mathcal{G}(\hat{\mathsf{G}}, \hat{\alpha})$. After observing that equation (13) encodes the hexagon diagram for a monoidal functor $\mathcal{G}(\hat{\mathsf{G}}, \hat{\alpha}) \to \mathsf{GL}_k^{\mathrm{gen}}(V)$ which is compatible with the structure maps to $\mathbb{Z}_2$, the remainder of the proof is straightforward. $\qquad\square$

To end this section, we record some basic string diagram identities.

Lemma 6.4. *For all* $g \in \mathsf{G}$ *and* $\omega_1, \omega_2 \in \hat{\mathsf{G}}$, *the following identities hold:*

(i)

$$(14)$$

(ii)

$$(15)$$

(iii)

$$(16)$$

(iv)

$$(17)$$

(v)

$$(18)$$

Proof. The first two identities are obvious. For the remaining identities, see [19, Lemma 4.8], [19, Corollary 4.9] and [19, Corollary 4.10]. See also [4, §§3.2.1, 3.4] $\qquad\square$

6.2   Real 2-characters and 2-class functions We extend the theory of Real categorical characters to twisted Real 2-representations. Instead of the direct approach of Section 5.3, we use string diagrams. See [19, §4] for the ungraded case.

We begin with some terminology.

Definition 6.5. *A pair $(g, \omega) \in \mathsf{G} \times \hat{\mathsf{G}}$ is said to graded commute if $\omega g^{\pi(\omega)} = g\omega$.*

The group $\hat{\mathsf{G}}$ acts on $\mathsf{G} \times \hat{\mathsf{G}}$ by $\sigma \cdot (g, \omega) = (\sigma g^{\pi(\sigma)} \sigma^{-1}, \sigma \omega \sigma^{-1})$. This action preserves the subset $\hat{\mathsf{G}}^{(2)}$ of graded commuting pairs.

Definition 6.6. *The Real categorical character of a twisted Real 2-representation ρ of G is the assignment*

$$g \mapsto \mathbb{T}\mathrm{r}_\rho(g) \in \mathrm{Obj}(\mathsf{Vect}_k), \qquad g \in \mathsf{G}$$

together with the collection of k-linear isomorphisms

$$\beta_{g,\omega} : \mathbb{T}\mathrm{r}_\rho(g) \to \mathbb{T}\mathrm{r}_\rho(\omega g^{\pi(\omega)} \omega^{-1}), \qquad (g, \omega) \in \mathsf{G} \times \hat{\mathsf{G}}$$

defined by the string diagrams

$$\phi \longmapsto \qquad\qquad\qquad \omega \in \mathsf{G} \qquad\qquad (19)$$

and

$$\phi \longmapsto \qquad\qquad\qquad \omega \in \hat{\mathsf{G}} \backslash \mathsf{G}. \qquad\qquad (20)$$

This definition recovers that of Section 5.3 in the case of trivial Real 2-Schur multiplier. The next definition formulates the joint trace of Section 5.2 in the context of Real 2-representation theory.

Definition 6.7. *Assume that each vector space $\mathbb{T}\mathrm{r}_\rho(g)$, $g \in \mathsf{G}$, is finite dimensional. Then the Real 2-character of ρ is the collection of joint traces*

$$\chi_\rho(g, \omega) = \mathrm{tr}_{\mathbb{T}\mathrm{r}_\rho(g)}(\beta_{g,\omega}), \qquad (g, \omega) \in \hat{\mathsf{G}}^{(2)}.$$

Before proceeding, we note that the theory of 2-characters is weaker than its non-categorical counterpart, in that inequivalent 2-representations may have the same 2-character. For an explicit example, see [33, §5]. Analogous statements apply to Real 2-characters as can be seen, for example, by using Proposition 6.13 below.

Example 6.8. In this example we assume basic familiarity with homological matrix factorizations. For all necessary background, see [34], [11], [7].

Let k be a field of characteristic zero. Denote by LG_k the bicategory of Landau–Ginzburg models over k, as in [7, §2.2]. Objects of LG_k are pairs (R, W) consisting of a ring R of the form $k[\![x_1, \ldots, x_n]\!]$ for some $n \geq 0$ and a potential $W \in R$. The 1-morphism category $1\mathrm{Hom}_{\mathsf{LG}_k}((R_1, W_1), (R_2, W_2))$ is the 2-periodic triangulated category of finite rank matrix factorizations of $W_2 - W_1$:

$$1\mathrm{Hom}_{\mathsf{LG}_k}((R_1, W_1), (R_2, W_2)) = \mathsf{HMF}(R_1 \hat{\otimes}_k R_2, W_2 - W_1).$$

The composition $- \circ_0 -$ is tensor product of matrix factorizations. For example, the identity 1-morphism $1_{(R,W)} : (R, W) \to (R, W)$ is represented by the stabilized diagonal

$$\Delta_W = \bigwedge\nolimits_{R^e}^{\bullet} \left(\bigoplus_{i=1}^{n} R^e \cdot \theta_i \right), \qquad d_{\Delta_W} = \sum_{i=1}^{n} (x_i - x_i') \cdot \theta_i^{\vee} + \sum_{i=1}^{n} \partial_{[i]}^{x, x'} W \cdot \theta_i \wedge -.$$

Here θ_i are Grassmann variables, $R^e = R \hat{\otimes}_k R \simeq k[\![x_1, \ldots, x_n, x_1', \ldots, x_n']\!]$ and

$$\partial_{[i]}^{x, x'} W = \frac{W(x_1', \ldots, x_{i-1}', x_i, \ldots, x_n, x') - W(x_1', \ldots, x_i', x_{i+1}, \ldots, x_n, x')}{x_i - x_i'}.$$

Define a weak duality involution $(-)^{\vee}$ on LG_k as follows. On objects set $(R, W)^{\vee} = (R, -W)$. On 1-morphism categories, $(-)^{\vee}$ acts as the linear (with respect to the ground ring) dual of matrix factorizations. Because of the Koszul sign rule, the dual of a matrix factorization of W is canonically a matrix factorization of $-W$. The adjoint equivalence η is induced by the canonical evaluation isomorphism from a finite rank free module to its double dual.

Suppose now that a finite $\mathbb{Z}_2$-graded group $\hat{\mathsf{G}}$ acts on $R = k[\![x_1, \ldots, x_n]\!]$ by unital algebra automorphisms. Assume that $W \in R$ is a potential which satisfies

$$\omega(W) = \pi(\omega)W, \qquad \omega \in \hat{\mathsf{G}}.$$

That is, G and $\hat{\mathsf{G}} \backslash \mathsf{G}$ act by symmetries and antisymmetries of W, respectively. A potential together with a finite group of symmetries defines a Landau–Ginzburg orbifold model. Mathematically, such models can be studied within the framework of equivariant matrix factorizations. On the other hand, a potential with an action of $\hat{\mathsf{G}}$ as above defines a Landau–Ginzburg orientifold model [21]. Such models have not been studied in the mathematical literature. As an explicit example, take $a, b \geq 1$ and let $W = x^{2a+1} + xy^{2b} \in k[\![x, y]\!]$. Consider the exact sequence of multiplicative groups

$$1 \to \mathbb{Z}_b \to \mathbb{Z}_{2b} \xrightarrow{\pi} \mathbb{Z}_2 \to 1.$$

Let ξ be a primitive $2b^{\text{th}}$ root of unity, which we assume to lie in k. Then a W-compatible action of $\mathbb{Z}_{2b}$ on $k[\![x, y]\!]$ is given by $\xi \cdot (x, y) = (-x, \xi y)$.

Define a Real 2-representation ρ of G on $(R, W) \in \mathrm{Obj}(\mathsf{LG}_k)$ by letting $\omega \in \hat{\mathsf{G}}$ act by the 1-morphism $_{\omega}\Delta_W \in \mathsf{HMF}(R \hat{\otimes}_k R, W - \pi(\omega)W)$ which is the pullback of Δ_W by $\omega \otimes 1$. Explicitly, $_{\omega}\Delta_W$ is equal to Δ_W as an R^e-module but has the twisted differential

$$d_{_{\omega}\Delta_W} = \sum_{i=1}^{n} (\omega(x_i) - x_i')\theta_i^{\vee} + \sum_{i=1}^{n} \partial_{[i]}^{\omega(x), x'} W \cdot \theta_i \wedge -.$$

The coherence 2-isomorphisms $\psi_{\bullet, \bullet}$ are induced by the associators in LG_k. More generally, the

maps $\psi_{\bullet,\bullet}$ can be twisted by a 2-cocycle $\hat{\theta} \in Z^2(B\hat{\mathsf{G}}, k_\pi^\times)$, thereby incorporating discrete torsion. We will not do this here; see, however, Section 6.3.

Using [34, Lemma 2.5.3], the Real categorical character is the Hochschild homology,

$$\mathbb{Tr}_\rho(g) \simeq HH_\bullet(\mathsf{MF}(R^g, W^g)), \qquad g \in \mathsf{G}$$

and the maps

$$\beta_{g,\omega} : HH_\bullet(\mathsf{MF}(R^g, W^g)) \to HH_\bullet(\mathsf{MF}(R^{\omega g^{\pi(\omega)}\omega^{-1}}, W^{\omega g^{\pi(\omega)}\omega^{-1}}))$$

are the canonical k-linear isomorphisms. The pair (R^g, W^g) is defined as follows. Choose coordinates $x_1, \dots, x_n$ of R in which g acts linearly and such that $\mathrm{Span}_k\{x_1, \dots, x_n\}^g = \mathrm{Span}_k\{x_{t+1}, \dots, x_n\}$. Then $R^g = R/(x_1, \dots, x_t)$ and W^g is the image of W in R^g. By [11, §6.3], the Hochschild homology $HH_\bullet(\mathsf{MF}(R^g, W^g))$ is isomorphic to the Milnor algebra of W^g, supported in degree $n-t$ mod 2.

As for Real 2-characters, we do not know a geometric interpretation of each joint trace $\chi_\rho(g, \omega)$. However, it follows from [34, Theorem 2.5.4] that we have

$$\frac{1}{|\mathsf{G}|} \sum_{(g,h) \in \mathsf{G}^{(2)}} \chi_\rho(g, h) = \dim_k HH_\bullet^\mathsf{G}(\mathsf{MF}(R, W)),$$

where $\mathsf{G}^{(2)} \subset \mathsf{G}^2$ is the subset of commuting pairs. Similarly, the Real 2-character computes the dimension of the G-equivariant involutive Hochschild homology:

$$\frac{1}{2|\mathsf{G}|} \sum_{(g,\omega) \in \hat{\mathsf{G}}^{(2)}} \chi_\rho(g, \omega) = \dim_k HH_\bullet^{\mathsf{G},+}(\mathsf{MF}(R, W)).$$

The cohomology $HH_\bullet^{\mathsf{G},+}(\mathsf{MF}(R, W))$ controls the deformation theory of the equivariant matrix factorization category $\mathsf{MF}_\mathsf{G}(R, W)$, considered as a category with duality determined by $\hat{\mathsf{G}}$, and so controls the deformation theory of the corresponding Landau–Ginzburg orientifold model.

Our next goal is to give geometric interpretations of $\mathbb{Tr}(\rho)$ and χ_ρ. We require some preliminary material. Let $\mathfrak{G}$ be a finite groupoid. Following Willerton [43, §2.3.1], a 2-cocycle $\theta \in Z^2(\mathfrak{G}, k^\times)$ defines a $k^\times$-gerbe $^\theta\mathfrak{G}$ over $\mathfrak{G}$. Explicitly, $^\theta\mathfrak{G}$ is the category with

$$\mathrm{Obj}(^\theta\mathfrak{G}) = \mathrm{Obj}(\mathfrak{G}), \qquad \mathrm{Hom}_{\theta\mathfrak{G}}(x_1, x_2) = k^\times \times \mathrm{Hom}_\mathfrak{G}(x_1, x_2)$$

and composition law

$$(z_2, g_2) \circ (z_1, g_1) = (\theta([g_2|g_1])z_2 z_1, g_2 g_1), \qquad z_1, z_2 \in k^\times, \ g_1, g_2 \in \mathrm{Mor}(\mathfrak{G}).$$

A vector bundle over $^\theta\mathfrak{G}$, also called a θ-twisted vector bundle over $\mathfrak{G}$, is a functor $^\theta\mathfrak{G} \to \mathsf{Vect}_k$ with the additional property that each subgroupoid $(Bk^\times)_{|x} \subset {}^\theta\mathfrak{G}$, $x \in \mathrm{Obj}(\mathfrak{G})$, acts by scalar multiplication.

The following result generalizes Proposition 5.6 to finite categorical groups. The analogous result in the ungraded setting is [19, Theorem 4.17].

Theorem 6.9. *The Real categorical character of an $\hat{\alpha}$-twisted Real 2-representation ρ of G defines*

a $\tau_\pi^{\mathrm{ref}}(\hat{\alpha})$-*twisted vector bundle over* $\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}$,

$$\mathbb{Tr}(\rho) : {}^{\tau_\pi^{\mathrm{ref}}(\hat{\alpha})}\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}} \to \mathsf{Vect}_k.$$

Proof. The theorem is equivalent to twisted commutativity of the diagram

for all $g \in \mathsf{G}$ and $\omega_1, \omega_2 \in \hat{\mathsf{G}}$. The vertical double arrow indicates that the top arrow is $\tau_\pi^{\mathrm{ref}}(\hat{\alpha})([\omega_2|\omega_1]g)$ times the bottom composition. To prove twisted commutativity, suppose first that $\pi(\omega_2) = 1$. In this case the expression for $\tau_\pi^{\mathrm{ref}}(\hat{\alpha})([\omega_2|\omega_1]g)$ differs from that of $\tau(\alpha)([\omega_2|\omega_1]g)$ only through the replacement of g with $g^{\pi(\omega_1)}$ (see Section 2.6). The desired equality can therefore be verified by a straightforward modification of the arguments used to prove [19, Theorem 4.17].

Suppose then that $\pi(\omega_2) = -1$. Consider first the case $\pi(\omega_1) = 1$. Let $\phi \in \mathbb{Tr}_\rho(g)$. Then $\beta_{\omega_1 g^{\pi(\omega_1)}\omega_1^{-1}, \omega_2}(\beta_{g,\omega_1}(\phi))$ is computed by the string diagram
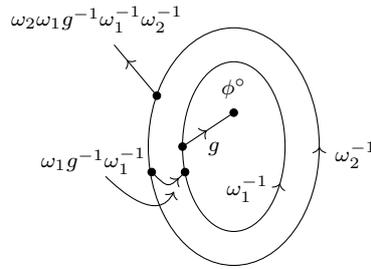
In this diagram, and those which follow, the exterior region is labelled by the category V while the interior regions are labelled by V^{op}. Using equations (15) and (18), the previous string diagram is seen to equal
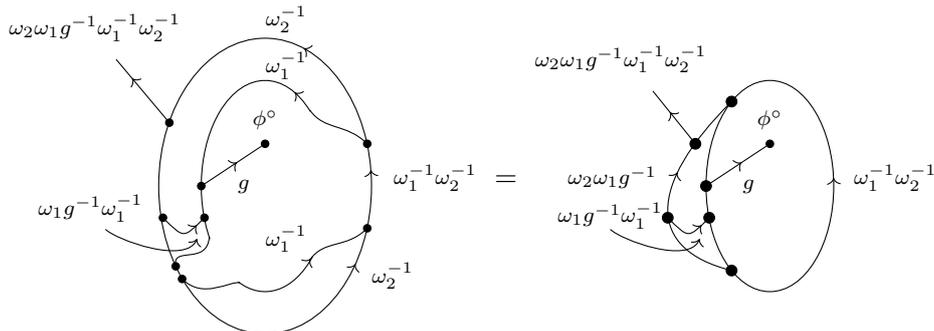
which, by equation (15), is equal to

$$\omega_2\omega_1 g^{-1}\omega_1^{-1}\omega_2^{-1}$$

Repeatedly applying equation (13) gives

$$\xrightarrow{\hat{\alpha}([\omega_2\omega_1 g^{-1}\omega_1^{-1}\omega_2^{-1}|\omega_2|\omega_1])^{-1}}$$
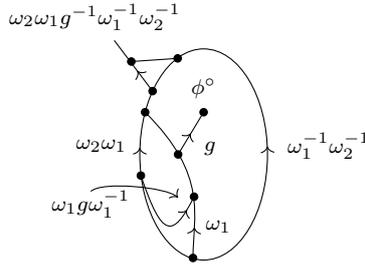
$$\xrightarrow{\hat{\alpha}([\omega_2|\omega_1 g^{-1}\omega_1^{-1}|\omega_1])} \qquad \xrightarrow{\hat{\alpha}([\omega_1 g^{-1}\omega_1^{-1}|\omega_1 g\omega_1^{-1}|\omega_1])}$$
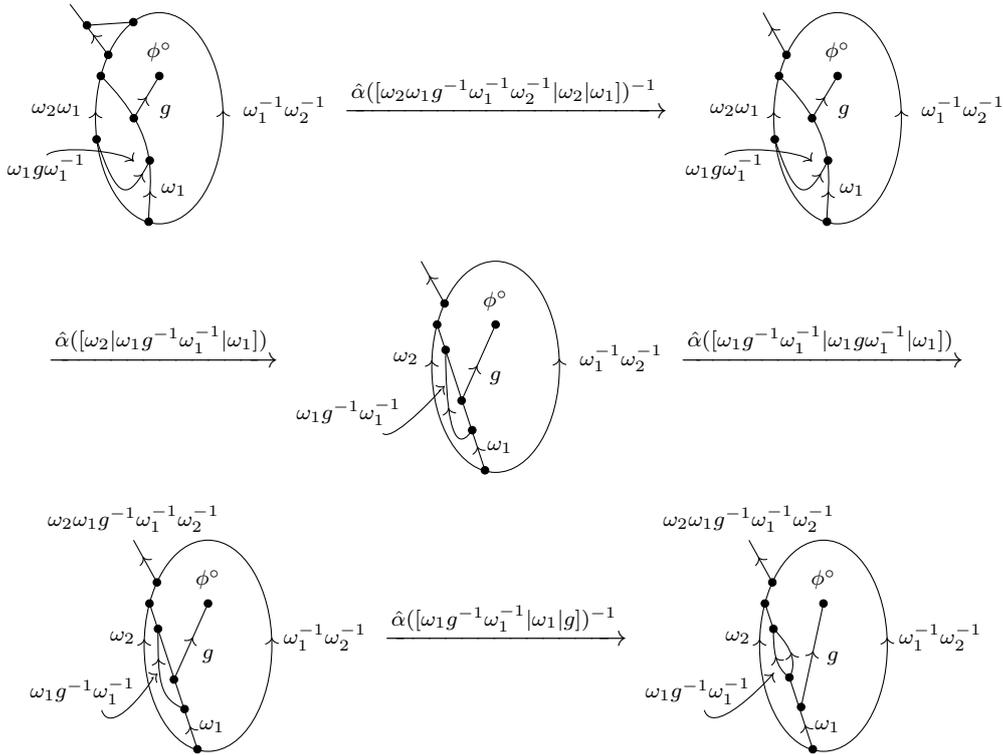
$$\xrightarrow{\hat{\alpha}([\omega_1 g^{-1}\omega_1^{-1}|\omega_1|g])^{-1}}$$

More precisely, the first two arrows arise from the inverted form of equation (13) while the last two arrows used equation (13) but applied in the category V^{op}. This explains why we multiply by $\hat{\alpha}([\omega_1 g^{-1}\omega_1^{-1}|\omega_1 g\omega_1^{-1}|\omega_1])$ and $\hat{\alpha}([\omega_1 g^{-1}\omega_1^{-1}|\omega_1|g])^{-1}$, rather than their inverses. Continuing, by first removing the loop and then adding a different loop (see equation (14)), the previous diagram becomes

$$=$$
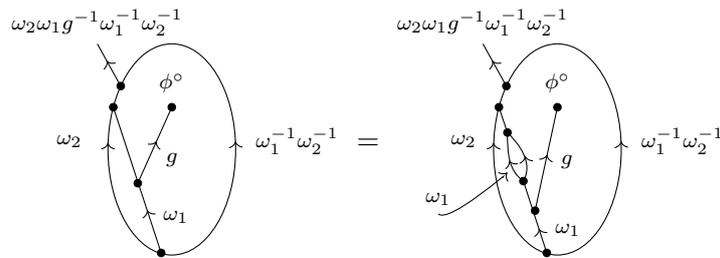
Repeatedly applying equation (13) then gives
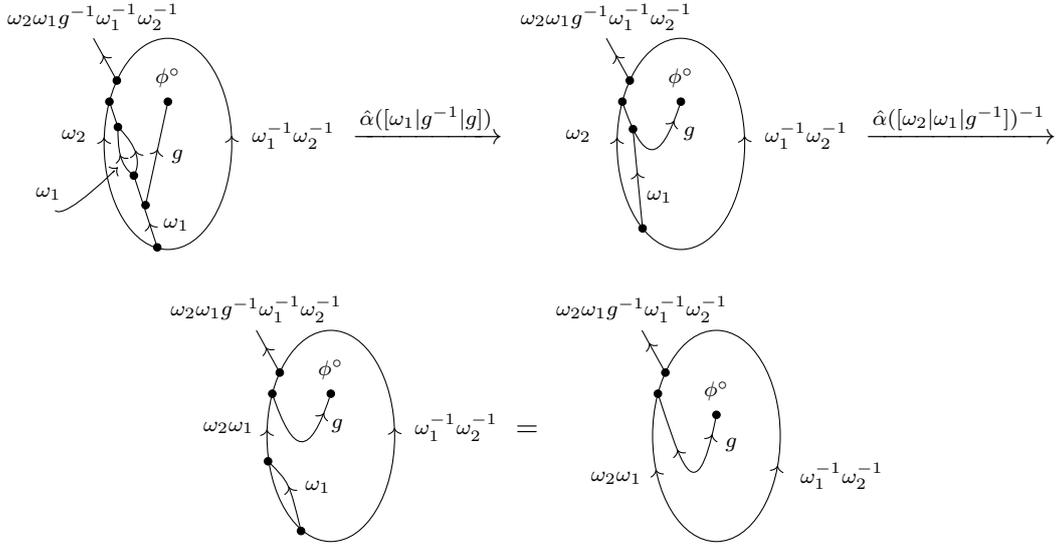
The first arrow used equation (13) in V^{op}. By definition, the final diagram computes $\beta_{g,\omega_2\omega_1}(\phi)$. The scalar introduced in the entire computation is thus

$$\frac{\hat{\alpha}([\omega_1|g^{-1}|g])\hat{\alpha}([\omega_1 g^{-1}\omega_1^{-1}|\omega_1 g\omega_1^{-1}|\omega_1])}{\hat{\alpha}([\omega_1 g^{-1}\omega_1^{-1}|\omega_1|g])} \times \frac{\hat{\alpha}([\omega_2|\omega_1 g^{-1}\omega_1^{-1}|\omega_1])}{\hat{\alpha}([\omega_2\omega_1 g^{-1}\omega_1^{-1}\omega_2^{-1}|\omega_2|\omega_1])\hat{\alpha}([\omega_2|\omega_1|g^{-1}])},$$

which we recognize as $\tau_\pi^{\mathrm{ref}}(\hat{\alpha})([\omega_2|\omega_1]g)^{-1}$.

A similar calculation can be performed when $\pi(\omega_1) = -1$. The key difference is that, since both ω_1 and ω_2 are of degree -1, at the final stage of the calculation we produce a scalar multiple of the string diagram
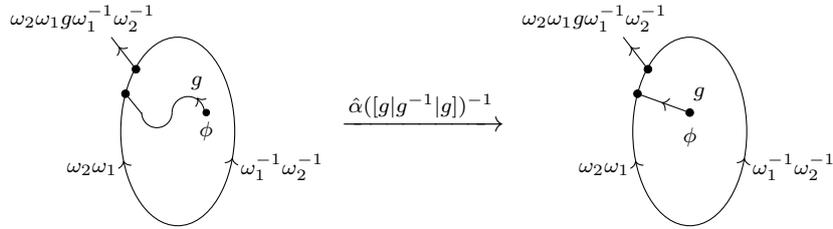
The last step used equation (17). This gives the additional factor of $\hat{\alpha}([g|g^{-1}|g])$ appearing in $\tau_\pi^{\mathrm{ref}}(\hat{\alpha})([\omega_2|\omega_1]g)$ when both ω_1 and ω_2 are of degree -1. This completes the proof. □

Let $\alpha \in Z^3(B\mathsf{G}, k^\times)$. In [43, §3.1] Willerton showed that the α-twisted Drinfeld double of G, as introduced in [9], is isomorphic (as an algebra) to the $\tau(\alpha)$-twisted groupoid algebra of $\Lambda B\mathsf{G}$:

$$D^\alpha(\mathsf{G}) \simeq k^{\tau(\alpha)}[\Lambda B\mathsf{G}].$$

Motivated by this, for a twisted 3-cocycle $\hat{\alpha} \in Z^3(B\hat{\mathsf{G}}, k_\pi^\times)$ the $\hat{\alpha}$-twisted thickened Drinfeld double of G was defined in [32, §4.1] to be the $\tau_\pi^{\mathrm{ref}}(\hat{\alpha})$-twisted groupoid algebra of $\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}$:

$$D^{\hat{\alpha}}(\mathsf{G}) = k^{\tau_\pi^{\mathrm{ref}}(\hat{\alpha})}[\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}].$$

The inclusion $\mathsf{G} \hookrightarrow \hat{\mathsf{G}}$ defines a faithful functor $\Lambda B\mathsf{G} \to \Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}$ under which $\tau_\pi^{\mathrm{ref}}(\hat{\alpha})$ restricts to

$\tau(\alpha)$. It follows that there is a k-algebra embedding $D^{\alpha}(\mathsf{G}) \hookrightarrow D^{\hat{\alpha}}(\mathsf{G})$, hence the terminology.

Corollary 6.10. *The Real categorical character of an $\hat{\alpha}$-twisted Real 2-representation of G is a module over the $\hat{\alpha}$-twisted thickened Drinfeld double of G.*

Proof. The category of vector bundles over $\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})\Lambda_{\pi}^{\mathrm{ref}}B\hat{\mathsf{G}}$ is equivalent to the category of $\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})$-twisted $k[\Lambda_{\pi}^{\mathrm{ref}}B\hat{\mathsf{G}}]$-modules, which is in turn equivalent to the category of $k^{\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})}[\Lambda_{\pi}^{\mathrm{ref}}B\hat{\mathsf{G}}]$-modules. The statement now follows from Theorem 6.9. $\qquad\square$

The next result describes the equivariance properties of Real 2-characters.

Theorem 6.11. *The Real 2-character of an $\hat{\alpha}$-twisted Real 2-representation ρ of G is a flat section of the line bundle $\tau\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})_k \to \Lambda\Lambda_{\pi}^{\mathrm{ref}}B\hat{\mathsf{G}}$. Equivalently, the equality*

$$\chi_{\rho}(\sigma g^{\pi(\sigma)}\sigma^{-1}, \sigma\omega\sigma^{-1}) = \frac{\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})([\sigma\omega\sigma^{-1}|\sigma]g)}{\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})([\sigma|\omega]g)} \cdot \chi_{\rho}(g, \omega)$$

holds for all $(g, \omega) \in \hat{\mathsf{G}}^{(2)}$ and $\sigma \in \hat{\mathsf{G}}$.

Proof. By Theorem 6.9, the Real categorical character $\mathbb{Tr}(\rho)$ is a $\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})$-twisted vector bundle over $\Lambda_{\pi}^{\mathrm{ref}}B\hat{\mathsf{G}}$. By the results of [43, §2.3.3], the holonomy of $\mathbb{Tr}(\rho)$ is a flat section of the transgressed line bundle $\tau\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})_k \to \Lambda\Lambda_{\pi}^{\mathrm{ref}}B\hat{\mathsf{G}}$. On the other hand, the holonomy of $\mathbb{Tr}(\rho)$ is by construction the Real 2-character of ρ. Combining these results gives the desired statement.

The explicit description of the $\hat{\mathsf{G}}$-equivariance of χ_{ρ} follows from Willerton's formula for the loop transgression of an untwisted 2-cocycle [43, §1.3.3]. $\qquad\square$

Flat sections of $\tau\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})_{\mathbb{C}} \to \Lambda\Lambda_{\pi}^{\mathrm{ref}}B\hat{\mathsf{G}}$ were first studied in [32], where they were shown to describe the complexified representation ring of $D^{\hat{\alpha}}(\mathsf{G})$. Theorem 6.11 gives a second interpretation of such sections, namely, as Real 2-class functions for $\hat{\alpha}$-twisted Real 2-representations of G. Corollary 6.10 explains the relationship between the two seemingly unrelated interpretations.

Upon substitution of the explicit expression for $\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})$ into the equality appearing in the statement of Theorem 6.11, the coefficient of $\chi_{\rho}(g, \omega)$ reproduces Sharpe's C-field discrete torsion phase factors for the three dimensional torus and the Klein bottle times S^1 in M-theory with orientifolds [37, §6.2]. This strongly suggests a role for Real 2-representation theory in M-theory. More precisely, consider an orientifold compatible C-field which is pulled back from the C-field $\hat{\alpha}$ on $B\hat{\mathsf{G}}$. We expect the 2-Hilbert spaces resulting from the higher geometric quantization of membranes in this background to be a Real representation of $\mathcal{G}(\mathsf{G}, \alpha)$.

6.3 Real 2-representations on $2\mathsf{Vect}_k$ We study twisted Real 2-representations on $2\mathsf{Vect}_k$. The ungraded case is treated in [12]; see also [17, §§5.1-2], [33].

Consider $2\mathsf{Vect}_k$ with its weak duality involution $(-)^{\vee}$ from Section 2.2. We begin with a cohomological classification of linear Real 2-representations on $2\mathsf{Vect}_k$. The underlying object $[n] \in \mathrm{Obj}(2\mathsf{Vect}_k)$ of such a representation is called its dimension. Denote by $\mathfrak{S}_n$ the symmetric group on n letters.

Theorem 6.12. *Equivalence classes of linear Real 2-representations of G on $2\mathsf{Vect}_k$ of dimension n are in bijection with equivalence classes of data consisting of*

(i) a group homomorphism $\rho_0 : \hat{\mathsf{G}} \to \mathfrak{S}_n$, and

(ii) *a class* $[\hat{\theta}] \in H^2(B\hat{G}, (k_\pi^\times)_{\rho_0}^n)$, *where* $(k_\pi^\times)_{\rho_0}^n$ *is the abelian group* $(k^\times)^n$ *with* $\hat{G}$-*action* $\omega \cdot (a_1, \ldots, a_n) = (a_{\rho_0(\omega)^{-1}(1)}^{\pi(\omega)}, \ldots, a_{\rho_0(\omega)^{-1}(n)}^{\pi(\omega)})$.

Two such data are equivalent if they differ by the action of $\mathfrak{S}_n$ *on* $\mathrm{Hom}_{\mathsf{Grp}}(\hat{G}, \mathfrak{S}_n)$.

Proof. The proof is a modification of the classification in the ungraded case [12, Theorem 5.5], [33, Proposition 4]. Let ρ be a linear Real 2-representation of dimension n. For each $\omega \in \hat{G}$, the 1-morphism $\rho(\omega) : [n] \to [n]$ is an equivalence and hence is a permutation 2-matrix (see [17, Lemma 5.3]). After noting that $(-)^\vee$ fixes the isomorphism class of a permutation 2-matrix, the existence of a 2-isomorphism $\rho(\omega_2) \circ {}^{\pi(\omega_2)}\rho(\omega_1) \simeq \rho(\omega_2\omega_1)$ implies that ρ defines a group homomorphism $\rho_0 : \hat{G} \to \mathfrak{S}_n$. Fix a basis of each vector space appearing in each 1-morphism $\rho(\omega)$. Then the 2-isomorphism ψ_{ω_2,ω_1} is given by an n-tuple $(\hat{\theta}_i([\omega_2|\omega_1]))_{i=1}^n \in (k^\times)^n$, the ith component being the isomorphism between the unique one dimensional vector spaces of the ith rows of $\rho(\omega_2) \circ {}^{\pi(\omega_2)}\rho(\omega_1)$ and $\rho(\omega_2\omega_1)$. By the associativity[3] constraint (10), the $\hat{\theta}_i$ assemble to a 2-cocycle $\hat{\theta} \in Z^2(B\hat{G}, (k_\pi^\times)_{\rho_0}^n)$. A different choice of basis of $\rho(\omega)$ defines a cohomologous 2-cocycle. Similarly, a contravariance respecting pseudonatural isomorphism $u : \rho \Rightarrow \rho'$ defines a twisted 1-cochain λ_u such that $\theta = \theta' \cdot d\lambda_u$. In this way, each equivalence class of linear Real 2-representations of dimension n defines a class in $H^2(B\hat{G}, (k_\pi^\times)_{\rho_0}^n)$.

Reversing the above construction associates to a 2-cocycle $\hat{\theta} \in Z^2(B\hat{G}, (k_\pi^\times)_{\rho_0}^n)$ an n-dimensional linear Real 2-representation, the entries of each 1-morphism $\rho(\omega)$ being either trivial or the trivialized k-line. This association is quasi-inverse to the construction of the previous paragraph. $\square$

The next result is a Real version of [33, Theorem 10].

Proposition 6.13. *The Real* 2-*character of the Real* 2-*representation* $\rho_{[\hat{\theta}]}$ *determined by* $[\hat{\theta}] \in H^2(B\hat{G}, (k_\pi^\times)_{\rho_0}^n)$ *is*

$$\chi_{\rho_{[\hat{\theta}]}}(g, \omega) = \sum_{\substack{i \in \{1,\ldots,n\} \\ \rho_0(g)(i)=\rho_0(\omega)(i)}} \hat{\theta}_i([g^{-1}|g])^{-\frac{\pi(\omega)-1}{2}} \frac{\hat{\theta}_i([\omega|g^{\pi(\omega)}])}{\hat{\theta}_i([g|\omega])}.$$

Proof. A short calculation shows that the right hand side of the claimed formula is independent of the choice of normalized cocycle representative of $[\hat{\theta}]$. Fix such a choice $\hat{\theta}$. A 2-morphism $\phi : 1_{[n]} \Rightarrow \rho_{\hat{\theta}}(g)$, $g \in G$, is an $n \times n$ matrix which has non-zero entries only at those diagonals for which $\rho_{\hat{\theta}}(g)$ is non-zero. Using this observation, the computation of $\chi_{\rho_{\hat{\theta}}}$ reduces to the one dimensional case. In this case, direct inspection of the string diagrams (19) and (20) shows that $\beta_{g,\omega} : k \to k$ is multiplication by

$$\hat{\theta}([g^{-1}|g])^{-\frac{\pi(\omega)-1}{2}} \frac{\hat{\theta}([\omega|g^{\pi(\omega)}])}{\hat{\theta}([\omega g^{\pi(\omega)}\omega^{-1}|\omega])}.$$

Upon restriction to graded commuting pairs, this gives the claimed formula. $\square$

A linear variation of the example from Section 5.3 can be used to give a geometric interpretation of Theorem 6.12. Instead of a matrix model of $2\mathsf{Vect}_k$, we work with the model given by k-linear additive finitely semisimple categories. The object $[n] \in \mathrm{Obj}(2\mathsf{Vect}_k)$ is modelled by the category of vector bundles $\mathsf{Vect}_k(X)$ over any set X of cardinality n. Suppose that $\hat{G}$ acts on

---

[3]As $2\mathsf{Vect}_k$ is a bicategory which is not a 2-category, a non-trivial associator 2-isomorphism must be incorporated in equation (10) in the obvious way.

X and fix a 2-cocycle $\hat{\theta} \in Z^2(X /\!\!/ \hat{\mathsf{G}}, k_\pi^\times)$ with coefficient system $k_\pi^\times$ twisted by the double cover $\pi : X /\!\!/ \mathsf{G} \to X /\!\!/ \hat{\mathsf{G}}$. Write $\hat{\theta}_x([\omega_2|\omega_1])$ for the value of $\hat{\theta}$ on the 2-chain $[\omega_2|\omega_1]x$. A linear Real 2-representation $\rho_{\hat{\theta}}$ is defined on $\mathsf{Vect}_k(X)$ as in the $\mathbb{F}_1$-linear case, where now the 2-isomorphism ψ_{ω_2,ω_1} multiplies the fibre over $x \in X$ by $\hat{\theta}_x([\omega_2|\omega_1])$, pre-composing with ev if $\omega_1, \omega_2 \in \hat{\mathsf{G}}\backslash\mathsf{G}$. In this language, Theorem 6.12 becomes the statement that all linear Real 2-representations of G in 2-vector spaces arise in this way while Proposition 6.13 becomes the joint fixed point formula

$$\chi_{\rho_{\hat{\theta}}}(g, \omega) = \sum_{x \in X^{g,\omega}} \tau_\pi^{\mathrm{ref}}(\hat{\theta}_x)([\omega]g)^{-1}.$$

To end this section we mention the twisted generalization of Theorem 6.12. An $\hat{\alpha}$-twisted Real 2-representation of G on $2\mathsf{Vect}_k$ determines an equivalence class of data consisting of

(i) a group homomorphism $\rho_0 : \hat{\mathsf{G}} \to \mathfrak{S}_n$,

(ii) a morphism $\rho_1 : k_\pi^\times \to (k_\pi^\times)_{\rho_0}^n$ of $\hat{\mathsf{G}}$-modules, and

(iii) a normalized 2-cochain $\hat{\theta} \in C^2(B\hat{\mathsf{G}}, (k_\pi^\times)_{\rho_0}^n)$

such that the equality $d\hat{\theta} = \rho_1(\hat{\alpha})$ holds in $Z^3(B\hat{\mathsf{G}}, (k_\pi^\times)_{\rho_0}^n)$. This data, up to equivalence as in Theorem 6.12, classifies equivalence classes of twisted Real 2-representations on $2\mathsf{Vect}_k$. We omit the proof, which is a straightforward combination of those of [12, Theorem 5.5] and Theorem 6.12. Proposition 6.13 is unchanged.

6.4 The antilinear theory We explain a categorification of the antilinear theory of Real representations of a finite group from Section 3.1.

Let k be a field which is a quadratic extension of a subfield k_0. Galois conjugation defines a strict k_0-linear involution $\overline{(-)} : \mathsf{Vect}_k \to \mathsf{Vect}_k$. Note that Section 3.1 could have been formulated in terms of this involution, in much the same way that Section 3.2 was formulated in terms of $(-)^\vee$.

Given a k-linear category $\mathcal{C}$, denote by $\overline{\mathcal{C}}$ the category with

$$\mathrm{Obj}(\overline{\mathcal{C}}) = \mathrm{Obj}(\mathcal{C}), \qquad \mathrm{Hom}_{\overline{\mathcal{C}}}(x, y) = \overline{\mathrm{Hom}_{\mathcal{C}}(x, y)}.$$

The assignment $\mathcal{C} \mapsto \overline{\mathcal{C}}$ extends to a strict involutive 2-functor $\overline{(-)} : \mathsf{Cat}_k \to \mathsf{Cat}_k$.

Let $\mathcal{V}$ be a k-linear bicategory. Define an involutive pseudofunctor $\overline{(-)} : \mathcal{V} \to \mathcal{V}$ by acting trivially on objects and by acting by conjugation on 1-morphism categories. In particular, the action of $\overline{(-)}$ on 2-morphisms is antilinear.

Definition 6.14. *A linear Real 2-representation of G on a k-linear 2-category $\mathcal{V}$ (in the antilinear approach) consists of data $V \in \mathrm{Obj}(\mathcal{V})$, $\rho(\omega)$, ψ_{ω_2,ω_1} and ψ_e as in Section 5.1, with the constraint (11) unchanged but with the constraint (10) replaced by the constraint*

$$\psi_{\omega_3\omega_2,\omega_1} \circ \left(\psi_{\omega_3,\omega_2} \circ {}^{\pi(\omega_3\omega_2)}\rho(\omega_1) \right) = \psi_{\omega_3,\omega_2\omega_1} \circ \left(\rho(\omega_3) \circ {}^{\pi(\omega_3)}\psi_{\omega_2,\omega_1} \right). \qquad (21)$$

Left superscripts now determine whether or not the 2-functor $\overline{(-)}$ is applied.

The above definition admits an obvious twisted generalization. All results of Section 6, and Section 8 below, continue to hold in the antilinear approach with essentially the same proofs, although Galois conjugation is used to define the coefficients of Real 2-Schur multipliers. The key point to keep in mind is that while ${}^{\pi(\omega_3)}\psi_{\omega_2,\omega_1}$ instead of ${}^{\pi(\omega_3)}\psi_{\omega_2,\omega_1}^{\pi(\omega_3)}$ appears in equation

(21), the pseudofunctor $\overline{(-)} : \mathcal{V} \to \mathcal{V}$ is antilinear on 2-morphism spaces, as opposed to linear and direction reversing.

7. Twisted Real induction

This section, which contains motivation and background material for that which follows, describes relevant aspects of the theory of twisted (Real) induction.

7.1 Induction Let G be a finite group. Fix a cocycle $\theta \in Z^2(B\mathsf{G}, k^\times)$. For later use, note that a θ-twisted representation ρ of G satisfies

$$\rho(g)^{-1} = \theta([g|g^{-1}])^{-1}\rho(g^{-1}), \qquad g \in \mathsf{G}. \tag{22}$$

The character of ρ satisfies (*cf.* equation (6))

$$\chi_\rho(hgh^{-1}) = \tau(\theta)([h]g)\chi_\rho(g), \qquad g, h \in \mathsf{G}. \tag{23}$$

Let $\mathsf{H} \leq \mathsf{G}$ be a subgroup. The induction functor $\mathrm{Ind}_\mathsf{H}^\mathsf{G} : \mathsf{Rep}_k^{\theta_{|\mathsf{H}}}(\mathsf{H}) \to \mathsf{Rep}_k^\theta(\mathsf{G})$ can be defined as follows. Fix a complete set of representatives $\{r_1, \ldots, r_p\}$ of G/H. Up to suitable equivalence, all constructions in this section, and the analogous constructions in those which follow, are independent of the choice of coset representatives. Given a $\theta_{|\mathsf{H}}$-twisted representation ρ of H on V, set

$$\mathrm{Ind}_\mathsf{H}^\mathsf{G}(\rho) = \bigoplus_{i=1}^p r_i \cdot V.$$

Here $r_i \cdot V$ is an isomorphic copy of V. For each $g \in \mathsf{G}$ and $i, j \in \{1, \ldots, p\}$, set

$$\mathrm{Ind}_\mathsf{H}^\mathsf{G}(\rho)(g)_{r_j r_i} = \begin{cases} \frac{\theta([g|r_i])}{\theta([r_j|r_j^{-1}gr_i])}\rho(r_j^{-1}gr_i) & \text{if } r_j^{-1}gr_i \in \mathsf{H}, \\ 0 & \text{else} \end{cases}$$

where we view $\mathrm{Ind}_\mathsf{H}^\mathsf{G}(\rho)(g)$ as a $p \times p$ block matrix.

Proposition 7.1. *The character of* $\mathrm{Ind}_\mathsf{H}^\mathsf{G}(\rho)$ *is*

$$\chi_{\mathrm{Ind}_\mathsf{H}^\mathsf{G}(\rho)}(g) = \frac{1}{|\mathsf{H}|} \sum_{\substack{r \in \mathsf{G} \\ rgr^{-1} \in \mathsf{H}}} \tau(\theta)([r]g)^{-1}\chi_\rho(rgr^{-1}).$$

Proof. After a short calculation using the 2-cocycle condition, the claimed equality becomes [28, Proposition 4.1]. □

Proposition 7.1 admits the following generalization to finite groupoids.

Proposition 7.2. *Let* $f : \mathfrak{H} \to \mathfrak{G}$ *be a faithful functor of finite groupoids. Fix* $\theta \in Z^2(\mathfrak{G}, k^\times)$. *For each* $\theta_{|\mathfrak{H}}$-*twisted representation* ρ *of* $\mathfrak{H}$ *and loop* (x, γ) *in* $\mathfrak{G}$, *the equality*

$$\chi_{\mathrm{Ind}_\mathfrak{H}^\mathfrak{G}(\rho)}(x \xrightarrow{\gamma} x) = \sum_{y \in \mathrm{Obj}(\mathfrak{H})} \frac{1}{|\mathcal{O}_\mathfrak{H}(y)||\mathrm{Aut}_\mathfrak{H}(y)|} \sum_{\substack{(x \xrightarrow{s} y) \in \mathfrak{G} \\ s\gamma s^{-1} \in \mathrm{Aut}_\mathfrak{H}(y)}} \tau(\theta)([s]\gamma)^{-1}\chi_\rho(y \xrightarrow{s\gamma s^{-1}} y)$$

holds, where $\mathcal{O}_{\mathfrak{H}}(y)$ denotes the orbit of y in $\mathfrak{H}$.

Proof. The proof is a twisted generalization of [17, Proposition 6.11]. Let $\{y_1, \ldots, y_n\}$ be a complete set of representatives for the $\mathfrak{H}$-isomorphism classes of objects which map to $\mathcal{O}_{\mathfrak{G}}(x)$ via f. For each $j \in \{1, \ldots, n\}$, fix a morphism $x \xrightarrow{s_j} f(y_j)$. We have

$$
\chi_{\mathrm{Ind}_{\mathfrak{H}}^{\mathfrak{G}}(\rho)}(x \xrightarrow{\gamma} x) = \sum_{j=1}^{n} \chi_{\mathrm{Ind}_{B\mathrm{Aut}_{\mathfrak{H}}(y_j)}^{\mathfrak{G}}(\rho)}(x \xrightarrow{\gamma} x)
$$

$$
\stackrel{\mathrm{Eq.\ (23)}}{=} \sum_{j=1}^{n} \tau(\theta)([s_j]\gamma)^{-1} \chi_{\mathrm{Ind}_{B\mathrm{Aut}_{\mathfrak{H}}(y_j)}^{\mathfrak{G}}(\rho)}\left(f(y_j) \xrightarrow{s_j \gamma s_j^{-1}} f(y_j)\right)
$$

$$
\stackrel{\mathrm{Prop.\ 7.1}}{=} \sum_{j=1}^{n} \frac{1}{|\mathrm{Aut}_{\mathfrak{H}}(y_j)|} \times
$$

$$
\sum_{\substack{x \xrightarrow{s} y_j \\ s\gamma s^{-1} \in \mathrm{Aut}_{\mathfrak{H}}(y_j)}} \tau(\theta)([s_j]\gamma)^{-1} \tau(\theta)([ss_j^{-1}]s_j \gamma s_j^{-1})^{-1} \chi_\rho(s\gamma s^{-1})
$$

$$
\stackrel{\mathrm{Eq.\ (5)}}{=} \sum_{j=1}^{n} \frac{1}{|\mathrm{Aut}_{\mathfrak{H}}(y_j)|} \sum_{\substack{x \xrightarrow{s} y_j \\ s\gamma s^{-1} \in \mathrm{Aut}_{\mathfrak{H}}(y_j)}} \tau(\theta)([s]\gamma)^{-1} \chi_\rho(s\gamma s^{-1}),
$$

which is easily seen to equal the desired expression. Note that Proposition 7.1 applies because of the assumption that f is faithful. $\qquad\square$

7.2 Real induction Let $\hat{\mathsf{H}} \leq \hat{\mathsf{G}}$ be a (non-trivially graded) $\mathbb{Z}_2$-graded subgroup. Fix $\hat{\theta} \in Z^2(B\hat{\mathsf{G}}, k_\pi^\times)$. In this section we interpret Real representations as generalized symmetric representations, as in Section 3.2. The Real induction functor $\mathrm{RInd}_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}} : \mathsf{RRep}_k^{\hat{\theta}_{|\hat{\mathsf{H}}}}(\mathsf{H}) \to \mathsf{RRep}_k^{\hat{\theta}}(\mathsf{G})$ is defined as follows. Fix a complete set $\mathcal{S} = \{\sigma_1, \ldots, \sigma_q\}$ of representatives of $\hat{\mathsf{G}}/\hat{\mathsf{H}}$. Let ρ be a $\hat{\theta}_{|\hat{\mathsf{H}}}$-twisted Real representation of H on V. As a vector space, put

$$
\mathrm{RInd}_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}}(\rho) = \bigoplus_{i=1}^{q} \sigma_i \cdot {}^{\pi(\sigma_i)}V.
$$

For each $\omega \in \hat{\mathsf{G}}$ and $i, j \in \{1, \ldots, q\}$, set

$$
\mathrm{RInd}_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}}(\rho)(\omega)_{\sigma_j \sigma_i} = \begin{cases} \frac{\hat{\theta}([\omega|\sigma_i])}{\hat{\theta}([\sigma_j|\sigma_j^{-1}\omega\sigma_i])} \cdot {}^{\pi(\sigma_j)}\rho(\sigma_j^{-1}\omega\sigma_i)^{\pi(\sigma_j)} & \text{if } \sigma_j^{-1}\omega\sigma_i \in \hat{\mathsf{H}}, \\ 0 & \text{else.} \end{cases}
$$

Given a morphism $\phi : \rho_1 \to \rho_2$ of twisted Real representations, the value of $\mathrm{RInd}_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}}(\phi)$ on the ith summand is ϕ if $\pi(\sigma_i) = 1$ and is $\rho_2(\sigma_i)^{-1} \circ \phi \circ \rho_1(\sigma_i)$ if $\pi(\sigma_i) = -1$.

Observe that there is a natural isomorphism

$$
\mathrm{Res}_{\mathsf{G}}^{\hat{\mathsf{G}}} \circ \mathrm{RInd}_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}} \simeq \mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}} \circ \mathrm{Res}_{\mathsf{H}}^{\hat{\mathsf{H}}}.
$$

This isomorphism can be used to compute $\chi_{\mathrm{RInd}_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}}(\rho)}$. However, for comparison with the case of Real 2-representations, it is instructive to compute $\chi_{\mathrm{RInd}_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}}(\rho)}$ directly.

Proposition 7.3. *The Real character of* $\mathrm{RInd}_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}}(\rho)$ *is*

$$\chi_{\mathrm{RInd}_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}}(\rho)}(g) = \frac{1}{2|\mathsf{H}|} \sum_{\substack{\omega \in \hat{\mathsf{G}} \\ \omega g^{\pi(\omega)}\omega^{-1} \in \mathsf{H}}} \tau_\pi^{\mathrm{ref}}(\hat{\theta})([\omega]g)^{-1}\chi_\rho(\omega g^{\pi(\omega)}\omega^{-1}).$$

Proof. For each $g \in \mathsf{G}$, we compute

$$\begin{aligned}
\chi_{\mathrm{RInd}_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}}(\rho)}(g) &= \sum_{\substack{i \in \{1,\dots,q\} \\ \sigma_i^{-1}g\sigma_i \in \mathsf{H}}} \frac{\hat{\theta}([g|\sigma_i])}{\hat{\theta}([\sigma_i|\sigma_i^{-1}g\sigma_i])} \mathrm{tr}_{\pi(\sigma_i)V}\left({}^{\pi(\sigma_i)}\rho(\sigma_i^{-1}g\sigma_i)^{\pi(\sigma_i)}\right) \\
&= \frac{1}{2|\mathsf{H}|} \sum_{\substack{\sigma \in \hat{\mathsf{G}} \\ \sigma g^{\pi(\sigma)}\sigma^{-1} \in \mathsf{H}}} \frac{\hat{\theta}([g|\sigma^{-1}])}{\hat{\theta}([\sigma^{-1}|\sigma g\sigma^{-1}])} \mathrm{tr}_V\left(\rho(\sigma^{-1}g\sigma)^{\pi(\sigma)}\right).
\end{aligned}$$

After using equation (22), this is seen to equal

$$\frac{1}{2|\mathsf{H}|} \sum_{\substack{\sigma \in \hat{\mathsf{G}} \\ \sigma g^{\pi(\sigma)}\sigma^{-1} \in \mathsf{H}}} \frac{\hat{\theta}([g|\sigma^{-1}])}{\hat{\theta}([\sigma^{-1}|\sigma g\sigma^{-1}])} \hat{\theta}([\sigma g\sigma^{-1}|\sigma g^{-1}\sigma^{-1}])^{\frac{\pi(\sigma)-1}{2}}\chi_\rho(\sigma g^{\pi(\sigma)}\sigma^{-1}).$$

A short calculation using equation (8) now completes the proof. $\qquad\square$

7.3 Hyperbolic induction A second form of Real induction, different from that of Section 7.2, is hyperbolic induction $\mathrm{HInd}_{\mathsf{G}}^{\hat{\mathsf{G}}} : \mathsf{Rep}_k^\theta(\mathsf{G})^{\simeq} \to \mathsf{RRep}_k^{\hat{\theta}}(\mathsf{G})$, where the domain is the maximal groupoid of $\mathsf{Rep}_k^\theta(\mathsf{G})$. This is simply the hyperbolic functor from Grothendieck–Witt theory. It admits the following explicit description. Fix an element $\varsigma \in \hat{\mathsf{G}}\backslash\mathsf{G}$. Given a θ-twisted representation ρ of G on V, the underlying vector space of $\mathrm{HInd}_{\mathsf{G}}^{\hat{\mathsf{G}}}(\rho)$ is $V \oplus V^\vee$. By Lemma 3.3 and Proposition 3.8, the $\hat{\mathsf{G}}$-action is given by

$$\mathrm{HInd}_{\mathsf{G}}^{\hat{\mathsf{G}}}(\rho)(g) = \begin{pmatrix} \rho(g) & 0 \\ 0 & \tau_\pi^{\mathrm{ref}}(\hat{\theta})([\varsigma^{-1}]g)^{-1}\rho(\varsigma^{-1}g^{-1}\varsigma)^\vee \end{pmatrix}, \qquad g \in \mathsf{G}$$

and

$$\mathrm{HInd}_{\mathsf{G}}^{\hat{\mathsf{G}}}(\rho)(\omega) = \begin{pmatrix} 0 & \frac{\hat{\theta}([\omega|\varsigma])}{\hat{\theta}([\varsigma^{-1}|\varsigma])}\rho(\omega\varsigma) \\ \frac{\hat{\theta}([\omega|\omega^{-1}])\hat{\theta}([\omega^{-1}|\varsigma])}{\hat{\theta}([\varsigma^{-1}|\varsigma])}\rho(\omega^{-1}\varsigma)^\vee & 0 \end{pmatrix}, \qquad \omega \in \hat{\mathsf{G}}\backslash\mathsf{G}.$$

More generally, given a subgroup $\mathsf{H} \leq \mathsf{G}$, the functor $\mathrm{HInd}_{\mathsf{H}}^{\hat{\mathsf{G}}} : \mathsf{Rep}_k^{\theta_{|\mathsf{H}}}(\mathsf{H})^{\simeq} \to \mathsf{RRep}_k^{\hat{\theta}}(\mathsf{G})$ is defined to be the composition $\mathrm{HInd}_{\mathsf{G}}^{\hat{\mathsf{G}}} \circ \mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}}$.

Proposition 7.4. *The Real character of* $\mathrm{HInd}_{\mathsf{H}}^{\hat{\mathsf{G}}}(\rho)$ *is*

$$\chi_{\mathrm{HInd}_{\mathsf{H}}^{\hat{\mathsf{G}}}(\rho)}(g) = \frac{1}{|\mathsf{H}|} \sum_{\substack{\omega \in \hat{\mathsf{G}} \\ \omega g^{\pi(\omega)}\omega^{-1} \in \mathsf{H}}} \tau_\pi^{\mathrm{ref}}(\hat{\theta})([\omega]g)^{-1}\chi_\rho(\omega g^{\pi(\omega)}\omega^{-1}).$$

Proof. Suppose first that $\mathsf{H} = \mathsf{G}$. For each $g \in \mathsf{G}$, we compute

$$
\begin{aligned}
\chi_{\mathrm{HInd}_{\mathsf{G}}^{\hat{\mathcal{G}}}(\rho)}(g) \quad &= \quad \chi_\rho(g) + \tau_\pi^{\mathrm{ref}}(\hat{\theta})([\varsigma^{-1}]g)^{-1}\chi_\rho(\varsigma^{-1}g^{-1}\varsigma) \\
&\overset{\text{Eq. } (23)}{=} \frac{1}{|\mathsf{G}|}\sum_{s\in\mathsf{G}}\tau(\theta)([s]g)^{-1}\chi_\rho(sgs^{-1}) + \\
&\qquad \frac{1}{|\mathsf{G}|}\sum_{s\in\mathsf{G}}\tau_\pi^{\mathrm{ref}}(\hat{\theta})([\varsigma^{-1}]g)^{-1}\tau(\theta)([s]\varsigma^{-1}g^{-1}\varsigma)^{-1}\chi_\rho(s\varsigma^{-1}g^{-1}\varsigma s^{-1}) \\
&\overset{\text{Eq. } (5)}{=} \frac{1}{|\mathsf{G}|}\sum_{\omega\in\hat{\mathsf{G}}}\tau_\pi^{\mathrm{ref}}(\hat{\theta})([\omega]g)^{-1}\chi_\rho(\omega g^{\pi(\omega)}\omega^{-1}).
\end{aligned}
$$

The case of an arbitrary subgroup follows by combining the previous case with equation (5) and Proposition 7.1. □

8. Twisted Real 2-induction

We define and study various forms of induction for linear Real representations of finite categorical groups. In particular, we compute the result of induction at the level of Real 2-characters.

8.1 Twisted 2-induction Let G be a finite group with subgroup H. Fix $\alpha \in Z^3(B\mathsf{G}, k^\times)$. Let ρ be a linear representation of $\mathcal{H} = \mathcal{G}(\mathsf{H}, \alpha_{|\mathsf{H}})$ on a category V. The induced linear representation $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\rho)$ of $\mathcal{G} = \mathcal{G}(\mathsf{G}, \alpha)$ was constructed in [19, Proposition 5.6]. An explicit construction, generalizing that of [17, §7.1] in the untwisted case, is as follows. Keeping the notation from Section 7.1, set

$$
\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\rho) = \prod_{i=1}^{p} r_i \cdot V.
$$

An element $g \in \mathsf{G}$ then acts via the $p \times p$ matrix whose $(j, i)^{\mathrm{th}}$ entry is

$$
\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\rho)(g)_{r_j r_i} = \begin{cases} \rho(h) & \text{if } gr_i = r_j h \text{ for some } h \in \mathsf{H}, \\ 0 & \text{else.} \end{cases}
$$

The $(k, i)^{\mathrm{th}}$ entry of $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\rho)(g_2) \circ \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\rho)(g_1)$ is $\rho(h_2) \circ \rho(h_1)$ if $g_1 r_i = r_j h_1$ and $g_2 r_j = r_k h_2$ for some $h_1, h_2 \in \mathsf{H}$ and is zero otherwise. If non-trivial, the corresponding entry of the 2-isomorphism $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\psi)_{g_2, g_1}$ is defined to be

$$
\frac{\alpha([g_2|r_j|h_1])}{\alpha([g_2|g_1|r_i])\alpha([r_k|h_2|h_1])}\psi_{h_2, h_1}.
$$

The coefficient of ψ_{h_2, h_1} ensures that this defines a Real representation of $\mathcal{G}$.

Theorem 8.1. *There is an isomorphism*

$$
\mathbb{Tr}(\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\rho)) \simeq \mathrm{Ind}_{\Lambda B\mathsf{H}}^{\Lambda B\mathsf{G}}(\mathbb{Tr}(\rho))
$$

of $\tau(\alpha)$-twisted representations of $\Lambda B\mathsf{G}$.

Here $\mathbb{Tr}(\rho)$ is viewed as a $\tau(\alpha_{|\mathsf{H}})$-twisted representation of $\Lambda B\mathsf{H}$ and $\mathrm{Ind}_{\Lambda B\mathsf{H}}^{\Lambda B\mathsf{G}}$ denotes twisted induction for groupoids. The untwisted version of Theorem 8.1 was proved by Ganter–Kapranov

[17, Theorem 7.5]. The twisted case can be handled by an elaboration of their argument. We omit this argument, as a further elaboration will be used to prove Theorem 8.3 below.

Corollary 8.2 (*cf.* [17, Corollary 7.6]). *The 2-character of* $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\rho)$ *is*

$$\chi_{\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\rho)}(g_1, g_2) = \frac{1}{|\mathsf{H}|} \sum_{\substack{s \in \mathsf{G} \\ s(g_1,g_2)s^{-1} \in \mathsf{H}^2}} \tau^2(\alpha)([s]g_1 \xrightarrow{g_2} g_1)^{-1} \cdot \chi_\rho(sg_1s^{-1}, sg_2s^{-1}).$$

Proof. By Theorem 8.1, it suffices to compute the character of $\mathrm{Ind}_{\Lambda B\mathsf{H}}^{\Lambda B\mathsf{G}}(\mathbb{Tr}(\rho))$. As the canonical functor $\Lambda B\mathsf{H} \to \Lambda B\mathsf{G}$ is faithful, this character can be computed using Proposition 7.2. Doing so gives the claimed result. $\square$

8.2 Real 2-induction Consider now a finite $\mathbb{Z}_2$-graded group $\hat{\mathsf{G}}$ with $\mathbb{Z}_2$-graded subgroup $\hat{\mathsf{H}}$. Let $\hat{\alpha} \in Z^3(B\hat{\mathsf{G}}, k_\pi^\times)$ and let ρ be a linear Real representation of $\mathcal{H} = \mathcal{G}(\mathsf{H}, \alpha_{|\mathsf{H}})$ on a category V, the Real structure being $\hat{\mathcal{H}} = \mathcal{G}(\hat{\mathsf{H}}, \hat{\alpha}_{|\hat{\mathsf{H}}})$. We define a Real 2-representation $\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho)$ of $\mathcal{G} = \mathcal{G}(\mathsf{G}, \alpha)$ with Real structure $\hat{\mathcal{G}} = \mathcal{G}(\hat{\mathsf{G}}, \hat{\alpha})$. As a category, set

$$\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho) = \prod_{i=1}^{q} \sigma_i \cdot {}^{\pi(\sigma_i)}V.$$

An element $\omega \in \hat{\mathsf{G}}$ acts by the $q \times q$ matrix whose $(j, i)^{\mathrm{th}}$ entry is

$$\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho)(\omega)_{\sigma_j \sigma_i} = \begin{cases} {}^{\pi(\sigma_j)}\rho(\eta) & \text{if } \omega\sigma_i = \sigma_j\eta \text{ for some } \eta \in \hat{\mathsf{H}}, \\ 0 & \text{else.} \end{cases}$$

The $(k, i)^{\mathrm{th}}$ entry of $\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho)(\omega_2) \circ {}^{\pi(\omega_2)}\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho)(\omega_1)$ is ${}^{\pi(\sigma_j)}\rho(\omega_2') \circ {}^{\pi(\eta_2\sigma_j)}\rho(\eta_1)$ if $\omega_1\sigma_i = \sigma_j\eta_1$ and $\omega_2\sigma_j = \sigma_k\eta_2$ for some $\eta_1, \eta_2 \in \hat{\mathsf{H}}$ and is zero otherwise. The component of $\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\psi)_{\omega_2,\omega_1}$ at this entry is defined to be

$$\frac{\hat{\alpha}([\omega_2|\sigma_j|\eta_1])}{\hat{\alpha}([\omega_2|\omega_1|\sigma_i])\hat{\alpha}([\sigma_k|\eta_2|\eta_1])} \cdot {}^{\pi(\sigma_k)}\psi_{\eta_2,\eta_1}^{\pi(\sigma_k)}.$$

It is straightforward to verify that this defines a linear Real representation of $\mathcal{G}$.

8.3 Induced Real categorical and 2-characters We generalize the work of Ganter–Kapranov in the untwisted ungraded case to compute the Real categorical and 2-characters of $\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho)$. We begin with the Real categorical character.

Theorem 8.3. *There is a canonical isomorphism*

$$\mathbb{Tr}(\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho)) \simeq \mathrm{Ind}_{\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{H}}}^{\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}}}(\mathbb{Tr}(\rho))$$

of $\tau_\pi^{\mathrm{ref}}(\hat{\alpha})$-*twisted representations of* $\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}}$.

Let $\mathsf{Z}_{\hat{\mathsf{G}}}^\varphi(g)$ be the stabilizer of $g \in \mathsf{G}$ under Real $\hat{\mathsf{G}}$-conjugation. Fix an equivalence

$$\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}} \simeq \bigsqcup_{g \in \pi_0(\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}})} B\mathsf{Z}_{\hat{\mathsf{G}}}^\varphi(g). \tag{24}$$

The set $\pi_0(\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}})$ is identified with the set of Real conjugacy classes of G. Denote by $[\![g]\!]_{\hat{\mathsf{G}}} \subset \mathsf{G}$ the Real conjugacy class of g. To prove Theorem 8.3 we first describe the action of $\mathsf{Z}_{\hat{\mathsf{G}}}^\varphi(g)$ on $\mathbb{Tr}_{\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho)}(g)$. We require some notation. The decomposition

$$[\![g]\!]_{\hat{\mathsf{G}}} \cap \mathsf{H} = \bigsqcup_{i=1}^{l} [\![h_i]\!]_{\hat{\mathsf{H}}} \tag{25}$$

induces a decomposition

$$\{\sigma \in \mathcal{S} \mid \sigma^{-1} g^{\pi(\sigma)} \sigma \in \mathsf{H}\} = \bigsqcup_{i=1}^{l} \mathcal{S}_i$$

where $\mathcal{S}_i = \{\sigma \in \mathcal{S} \mid \sigma^{-1} g^{\pi(\sigma)} \sigma \in [\![h_i]\!]_{\hat{\mathsf{H}}}\}$. For each $i \in \{1, \dots, l\}$, fix an element $\sigma_i \in \mathcal{S}_i$. Relabel the representatives of the Real $\hat{\mathsf{H}}$-conjugacy classes appearing in the decomposition (25) by $h_i = \sigma_i^{-1} g^{\pi(\sigma_i)} \sigma_i$.

Lemma 8.4 (*cf.* [17, Lemma 7.7]). *Elements of $\mathcal{S}_i$ can be chosen so that left multiplication by σ_i^{-1} induces a bijection from $\mathcal{S}_i$ to a complete system of representatives of $\mathsf{Z}_{\hat{\mathsf{G}}}^\varphi(h_i)/\mathsf{Z}_{\hat{\mathsf{H}}}^\varphi(h_i)$.*

Proof. The proof is nearly the same as that of [17, Lemma 7.7]; we include it for completeness. Let $\sigma \in \mathcal{S}_i$. Then $\sigma^{-1} g^{\pi(\sigma)} \sigma = \eta^{-1} h_i^{\pi(\eta)} \eta$ for some $\eta \in \hat{\mathsf{H}}$. It follows that $\sigma \eta^{-1} = \sigma$ in $\hat{\mathsf{G}}/\hat{\mathsf{H}}$ and $(\sigma \eta^{-1})^{-1} g^{\pi(\sigma \eta)} \sigma \eta^{-1} = h_i$ so that

$$(\sigma_i^{-1} \sigma \eta^{-1})^{-1} h_i^{\pi(\sigma_i^{-1} \sigma \eta^{-1})} \sigma_i^{-1} \sigma \eta^{-1} = h_i$$

and $\sigma_i^{-1} \sigma \eta^{-1} \in \mathsf{Z}_{\hat{\mathsf{G}}}^\varphi(h_i)$. Replacing σ with $\sigma \eta^{-1}$, we henceforth assume that $\sigma \in \mathcal{S}_i$ is such that $\sigma^{-1} g^{\pi(\sigma)} \sigma = h_i$ and $\sigma_i^{-1} \sigma \in \mathsf{Z}_{\hat{\mathsf{G}}}^\varphi(h_i)$.

Let $\sigma, \sigma' \in \mathcal{S}_i$ be distinct. Then $(\sigma_i^{-1} \sigma)^{-1}(\sigma_i^{-1} \sigma') = \sigma^{-1} \sigma'$ does not lie in $\hat{\mathsf{H}}$. It follows that $\sigma_i^{-1} \sigma \neq \sigma_i^{-1} \sigma'$ in $\mathsf{Z}_{\hat{\mathsf{G}}}^\varphi(h_i)/\mathsf{Z}_{\hat{\mathsf{H}}}^\varphi(h_i)$, proving injectivity of the map under consideration. To prove surjectivity, let $\mu \in \mathsf{Z}_{\hat{\mathsf{G}}}^\varphi(h_i)$. Then $\sigma_i \mu = \sigma \eta$ for some $\sigma \in \mathcal{S}$ and $\eta \in \hat{\mathsf{H}}$. We compute

$$\sigma^{-1} g^{\pi(\sigma)} \sigma = \eta h_i^{\pi(\sigma)} \eta^{-1},$$

whence $\sigma \in \mathcal{S}_i$. Since $\sigma^{-1} g^{\pi(\sigma)} \sigma = h_i$, we also find that $\eta \in \mathsf{Z}_{\hat{\mathsf{H}}}^\varphi(h_i)$. So $\sigma_i^{-1} \sigma = \mu \eta^{-1}$, showing that $\sigma_i^{-1} \sigma = \mu$ in $\mathsf{Z}_{\hat{\mathsf{G}}}^\varphi(h_i)/\mathsf{Z}_{\hat{\mathsf{H}}}^\varphi(h_i)$. $\qquad\square$

Remark 8.5. The representatives $\mathcal{S}$ of $\hat{\mathsf{G}}/\hat{\mathsf{H}}$ can be chosen to be a subset of G. Such a choice simplifies the description of induced Real 2-representations. However, it does not appear that there is a version of Lemma 8.4 which produces a set of representatives which is again a subset of G.

We henceforth assume that $\mathcal{S}$ is chosen as in Lemma 8.4. We have

$$\mathbb{Tr}_{\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho)}(g) = \bigoplus_{i=1}^{l} \bigoplus_{\{\sigma \in \mathcal{S}_i \mid \sigma^{-1} g \sigma \in \mathsf{H}\}} \sigma \cdot \mathbb{Tr}_{\pi(\sigma)\rho}(h_i^{\pi(\sigma)}).$$

If $\pi(\sigma) = -1$, then

$$\mathbb{Tr}_{\rho^{\mathrm{op}}}(h_i^{-1}) \simeq 2\mathrm{Hom}_{\mathsf{Cat}}(\rho(h_i^{-1}), 1_V).$$

Define k-linear a map $F_i : 2\mathrm{Hom}_{\mathsf{Cat}}(\rho(h_i^{-1}), 1_V) \to 2\mathrm{Hom}_{\mathsf{Cat}}(1_V, \rho(h_i))$ by

Using equation (17), the inverse $F_i^{-1} : 2\mathrm{Hom}_{\mathsf{Cat}}(1_V, \rho(h_i)) \to 2\mathrm{Hom}_{\mathsf{Cat}}(\rho(h_i^{-1}), 1_V)$ is seen to be

The maps F_i induce a vector space isomorphism

$$\mathbb{T}\mathrm{r}_{\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho)}(g) \simeq \bigoplus_{i=1}^{l} \bigoplus_{\{\sigma \in \mathcal{S}_i \mid \sigma^{-1} g \sigma \in \mathsf{H}\}} \sigma \cdot \mathbb{T}\mathrm{r}_\rho(h_i)$$

which we use below without mention.

Lemma 8.6 (*cf.* [17, Lemma 7.8]). *There is an isomorphism*

$$\mathbb{T}\mathrm{r}_{\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho)}(g) \simeq \bigoplus_{i=1}^{l} \mathrm{Ind}_{\mathsf{Z}_{\hat{\mathsf{H}}}^{\varphi}(h_i)}^{\mathsf{Z}_{\hat{\mathsf{G}}}^{\varphi}(g)}\left(\mathbb{T}\mathrm{r}_\rho(h_i)\right)$$

of $\tau_\pi^{\mathrm{ref}}(\hat{\alpha})$-twisted representations of $\mathsf{Z}_{\hat{\mathsf{G}}}^{\varphi}(g)$, the induction being along the composition $\mathsf{Z}_{\hat{\mathsf{H}}}^{\varphi}(h_i) \hookrightarrow$
$\mathsf{Z}_{\hat{\mathsf{G}}}^{\varphi}(h_i) \xrightarrow{l \mapsto \sigma_i l \sigma_i^{-1}} \mathsf{Z}_{\hat{\mathsf{G}}}^{\varphi}(g)$.

Proof. Let $\mu \in \mathsf{Z}_{\hat{\mathsf{G}}}^{\varphi}(g)$ and $\sigma \in \mathcal{S}_i$. Then $\mu\sigma = \tilde{\sigma}\eta$ for some $\tilde{\sigma} \in \mathcal{S}$ and $\eta \in \hat{\mathsf{H}}$. It is straightforward to verify that in fact $\tilde{\sigma} \in \mathcal{S}_i$ and $\eta \in \mathsf{Z}_{\hat{\mathsf{H}}}^{\varphi}(h_i)$. The equations

$$\mu^{-1}\tilde{\sigma} = \sigma\eta^{-1}, \qquad g\sigma = \sigma h_i^{\pi(\sigma)}, \qquad \mu\sigma = \tilde{\sigma}\eta$$

imply that μ acts on $\mathbb{T}\mathrm{r}_{\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\rho)}(g)$ by a linear map $\xi_1(\mu) : \sigma \cdot \mathbb{T}\mathrm{r}_\rho(h_i) \to \tilde{\sigma} \cdot \mathbb{T}\mathrm{r}_\rho(h_i)$. We claim that $\xi_1(\mu)$ is equal to $c_1 c_2 \cdot \hat{\beta}_{h_i, \eta}$, where

$$c_1 = \left(\frac{\hat{\alpha}([g|\mu|\sigma])\hat{\alpha}([\tilde{\sigma}|h_i^{\pi(\tilde{\sigma})}|\eta])}{\hat{\alpha}([g|\tilde{\sigma}|\eta])}\right)\left(\frac{\hat{\alpha}([\mu|\sigma|h_i^{\pi(\mu\sigma)}])}{\hat{\alpha}([\mu|g^{\pi(\mu)}|\sigma])\hat{\alpha}([\tilde{\sigma}|\eta|h_i^{\pi(\mu\sigma)}])}\right) \times$$
$$\left(\frac{\hat{\alpha}([g^{-1}|\sigma|h_i^{\pi(\sigma)}])}{\hat{\alpha}([g^{-1}|g|\sigma])\hat{\alpha}([\sigma|h_i^{-\pi(\sigma)}|h_i^{\pi(\sigma)}])}\right)^{-\frac{\pi(\mu)-1}{2}}$$

and

$$c_2 = \hat{\alpha}([h_i|h_i^{-1}|h_i])^{\delta_{\pi(\mu),\pi(\sigma),-1} - \delta_{\pi(\tilde{\sigma}),\pi(\eta),-1}}\left(\frac{\hat{\alpha}([h_i^{-1}|h_i|\eta])\hat{\alpha}([\eta|h_i^{-\pi(\eta)}|h_i^{\pi(\eta)}])}{\hat{\alpha}([h_i^{-1}|\eta|h_i^{\pi(\eta)}])}\right)^{-\frac{\pi(\tilde{\sigma})-1}{2}}.$$

Indeed, the factor c_1 is due to the scalars relating $\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\psi)_{g,\mu}^{-1}$, $\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\psi)_{\mu,g^{\pi(\mu)}}$ and $\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\psi)_{g^{-1},g}^{\mathrm{op}}$
to $\psi_{h_i^{\pi(\tilde{\sigma})},\eta}^{-1}$, $\psi_{\eta,h_i^{\pi(\mu\sigma)}}$ and $\psi_{h_i^{-\pi(\sigma)},h_i^{\pi(\sigma)}}$, respectively. Note that $\mathrm{RInd}_{\hat{\mathcal{H}}}^{\hat{\mathcal{G}}}(\psi)_{g^{-1},g}^{\mathrm{op}}$ appears only when

$\pi(\mu) = -1$. These maps contribute to $\xi_1(\mu)$ regardless of the degrees of σ and $\tilde{\sigma}$. The factor c_2 is due the maps $F_i^{\pm 1}$ and is best understood using string diagrams. For example, when μ and η are of degree $+1$ and σ and $\tilde{\sigma}$ are of degree -1, we have

$$\xi_1(\mu)(\phi) = c_1 \cdot$$

This string diagram can be evaluated as follows:

so that, after a short calculation, we arrive at

 $$= \frac{\hat{\alpha}([h_i^{-1}|h_i|\eta])\hat{\alpha}([\eta|h_i^{-1}|h_i])}{\hat{\alpha}([h_i^{-1}|\eta|h_i])} \beta_{h_i,\eta}(\phi) = c_2 \beta_{h_i,\eta}(\phi).$$

Similarly, when μ, σ are of degree $+1$ and $\tilde{\sigma}, \eta$ are of degree -1, we have

$$\xi_1(\mu)(\phi) = c_1 \cdot \quad \text{(string diagram)}$$

This string diagram can be evaluated in the same way as the previous diagram, the main difference being that the final step is not required. This leads to an additional factor of $\hat{\alpha}([h_i|h_i^{-1}|h_i])^{-1}$:

$$\text{(string diagram)} = \hat{\alpha}([h_i|h_i^{-1}|h_i])^{-1}\frac{\hat{\alpha}([h_i^{-1}|h_i|\eta])\hat{\alpha}([\eta|h_i|h_i^{-1}])}{\hat{\alpha}([h_i^{-1}|\eta|h_i^{-1}])}\beta_{h_i,\eta}(\phi).$$

The coefficient of $\beta_{h_i,\eta}(\phi)$ is again c_2. The remaining cases are dealt with similarly.

On the other hand, as $(\sigma_i^{-1}\mu\sigma_i)(\sigma_i^{-1}\sigma) = (\sigma_i^{-1}\tilde{\sigma})\eta$, the results of Section 7.1 imply that $\sigma_i^{-1}\mu\sigma_i \in \mathsf{Z}_{\mathsf{G}}^{\varphi}(h_i)$ acts on

$$\mathrm{Ind}_{\mathsf{Z}_{\mathsf{H}}^{\varphi}(h_i)}^{\mathsf{Z}_{\mathsf{G}}^{\varphi}(h_i)}\left(\mathrm{Tr}_\rho(h_i)\right) \simeq \bigoplus_{\sigma \in \mathcal{S}_i}\sigma_i^{-1}\sigma \cdot \mathrm{Tr}_\rho(h_i)$$

by the linear map $\xi_2(\sigma_i^{-1}\mu\sigma_i) : \sigma_i^{-1}\sigma \cdot \mathrm{Tr}_\rho(h_i) \to \sigma_i^{-1}\tilde{\sigma} \cdot \mathrm{Tr}_\rho(h_i)$ given by

$$\xi_2(\sigma_i^{-1}\mu\sigma_i) = \frac{\theta_{h_i}([\sigma_i^{-1}\mu\sigma_i|\sigma_i^{-1}\sigma])}{\theta_{h_i}([\sigma_i^{-1}\tilde{\sigma}|\eta])}\beta_{h_i,\eta}.$$

Here we have set $\theta_\gamma([\omega_2|\omega_1]) = \tau_\pi^{\mathrm{ref}}(\hat{\alpha})([\omega_2|\omega_1]\gamma)$. Noting that

$$\xi_2(\mu) = \frac{\theta_g([\sigma_i^{-1}|\mu])}{\theta_g([\sigma_i^{-1}\mu\sigma_i|\sigma_i^{-1}])}\xi_2(\sigma_i^{-1}\mu\sigma_i),$$

closedness of $\tau_\pi^{\mathrm{ref}}(\hat{\alpha})$ then gives

$$\xi_2(\mu) = \frac{\theta_{h_i}([\mu|\sigma])}{\theta_{h_i}([\tilde{\sigma}|\eta])}\frac{\theta_{h_i}([\sigma_i^{-1}|\tilde{\sigma}])}{\theta_{h_i}([\sigma_i^{-1}|\sigma])}\beta_{h_i,\eta}.$$

The explicit expression for $\tau_\pi^{\mathrm{ref}}(\hat{\alpha})$ shows that $\frac{\theta_{h_i}([\mu|\sigma])}{\theta_{h_i}([\tilde{\sigma}|\eta])}$ is equal to $c_1 c_2$ above. We therefore arrive at the equality

$$\xi_2(\mu) = \frac{\theta_{h_i}([\sigma_i^{-1}|\tilde{\sigma}])}{\theta_{h_i}([\sigma_i^{-1}|\sigma])}\xi_1(\mu).$$

In other words, the diagram

$$
\begin{array}{ccc}
\sigma \cdot \mathrm{Tr}_\rho(h_i) & \xrightarrow{\;\;\xi_1(\mu)\;\;} & \tilde\sigma \cdot \mathrm{Tr}_\rho(h_i) \\[2mm]
{\scriptstyle \tau_\pi^{\mathrm{ref}}(\hat\alpha)([\sigma_i^{-1}|\sigma]h_i)} \Big\downarrow & & \Big\downarrow {\scriptstyle \tau_\pi^{\mathrm{ref}}(\hat\alpha)([\sigma_i^{-1}|\tilde\sigma]h_i)} \\[2mm]
\sigma_i^{-1}\sigma \cdot \mathrm{Tr}_\rho(h_i) & \xrightarrow[\;\;\xi_2(\mu)\;\;]{} & \sigma_i^{-1}\tilde\sigma \cdot \mathrm{Tr}_\rho(h_i)
\end{array}
$$

commutes. The vertical (scalar multiplication) maps of this diagram assemble to define the desired isomorphism $\mathrm{Tr}_{\mathrm{RInd}^{\hat{\mathcal{G}}}_{\hat{\mathcal{H}}}(\rho)}(g) \xrightarrow{\;\sim\;} \bigoplus_{i=1}^l \mathrm{Ind}^{\mathsf{Z}^\varphi_{\hat{\mathsf{G}}}(g)}_{\mathsf{Z}^\varphi_{\hat{\mathsf{H}}}(h_i)}(\mathrm{Tr}_\rho(h_i))$ of twisted representations of $\mathsf{Z}^\varphi_{\hat{\mathsf{G}}}(g)$.   □

Theorem 8.3 follows at once from Lemma 8.6.

Theorem 8.7. *The Real 2-character of* $\mathrm{RInd}^{\hat{\mathcal{G}}}_{\hat{\mathcal{H}}}(\rho)$ *is*

$$
\chi_{\mathrm{RInd}^{\hat{\mathcal{G}}}_{\hat{\mathcal{H}}}(\rho)}(g,\omega) = \frac{1}{2|\mathsf{H}|} \sum_{\substack{\sigma \in \hat{\mathsf{G}} \\ \sigma(g,\omega)\sigma^{-1} \in \hat{\mathsf{H}}^2}} \tau\tau_\pi^{\mathrm{ref}}(\hat\alpha)([\sigma]g \xrightarrow{\omega} g)^{-1} \cdot \chi_\rho(\sigma g^{\pi(\sigma)}\sigma^{-1}, \sigma\omega\sigma^{-1}).
$$

Proof. By Theorem 8.3, it suffices to compute the character of $\mathrm{Ind}^{\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}}_{\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{H}}}(\mathrm{Tr}(\rho))$. As the canonical functor $\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{H}} \to \Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}$ is faithful, this character can be computed using Proposition 7.2.   □

8.4 Hyperbolic 2-induction We now turn to the categorification of hyperbolic induction. Let $\hat{\mathsf{G}}$ be a finite $\mathbb{Z}_2$-graded group. Fix $\hat\alpha \in Z^3(B\hat{\mathsf{G}}, k_\pi^\times)$ and $\varsigma \in \hat{\mathsf{G}} \backslash \mathsf{G}$. Let ρ be a linear representation of $\mathcal{G} = \mathcal{G}(\mathsf{G}, \alpha)$ on V. The underlying category of $\mathrm{HInd}^{\hat{\mathcal{G}}}_{\mathcal{G}}(\rho)$ is $V \times V^{\mathrm{op}}$. The required 1-morphisms are

$$
\mathrm{HInd}^{\hat{\mathcal{G}}}_{\mathcal{G}}(\rho)(g) = \begin{pmatrix} \rho(g) & 0 \\ 0 & \rho(\varsigma^{-1}g\varsigma)^{\mathrm{op}} \end{pmatrix}, \qquad g \in \mathsf{G}
$$

and

$$
\mathrm{HInd}^{\hat{\mathcal{G}}}_{\mathcal{G}}(\rho)(\omega) = \begin{pmatrix} 0 & \rho(\omega\varsigma) \\ \rho(\varsigma^{-1}\omega)^{\mathrm{op}} & 0 \end{pmatrix}, \qquad \omega \in \hat{\mathsf{G}} \backslash \mathsf{G}.
$$

The associativity 2-isomorphisms are

$$
\mathrm{HInd}^{\hat{\mathcal{G}}}_{\mathcal{G}}(\psi)_{g_2,g_1} = \begin{pmatrix} \psi_{g_2,g_1} & 0 \\ 0 & \frac{\hat\alpha([g_2|\varsigma|\varsigma^{-1}g_1\varsigma])}{\hat\alpha([g_2|g_1|\varsigma])\hat\alpha([\varsigma|\varsigma^{-1}g_2\varsigma|\varsigma^{-1}g_1\varsigma])}\psi^{-\mathrm{op}}_{\varsigma^{-1}g_2\varsigma,\varsigma^{-1}g_1\varsigma} \end{pmatrix}
$$

and

$$
\mathrm{HInd}^{\hat{\mathcal{G}}}_{\mathcal{G}}(\psi)_{\omega_2,g_1} = \begin{pmatrix} \frac{\hat\alpha([\omega_2|\varsigma|\varsigma^{-1}g_1\varsigma])}{\hat\alpha([\omega_2|g_1|\varsigma])}\psi_{\omega_2\varsigma,\varsigma^{-1}g_1\varsigma} & 0 \\ 0 & \hat\alpha([\varsigma|\varsigma^{-1}\omega_2|g_1])^{-1}\psi^{-\mathrm{op}}_{\varsigma^{-1}\omega_2,g_1} \end{pmatrix}
$$

and

$$
\mathrm{HInd}^{\hat{\mathcal{G}}}_{\mathcal{G}}(\psi)_{g_2,\omega_1} = \begin{pmatrix} \hat\alpha([g_2|\omega_1|\varsigma])^{-1}\psi_{g_2,\omega_1\varsigma} & 0 \\ 0 & \frac{\hat\alpha([g_2|\varsigma|\varsigma^{-1}\omega_1])}{\hat\alpha([\varsigma|\varsigma^{-1}g_2\varsigma|\varsigma^{-1}\omega_1])}\psi^{-\mathrm{op}}_{\varsigma^{-1}g_2\varsigma,\varsigma^{-1}\omega_1} \end{pmatrix}
$$

and

$$\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\psi)_{\omega_2,\omega_1} = \begin{pmatrix} \hat{\alpha}([\omega_2|\varsigma|\varsigma^{-1}\omega_1])\psi_{\omega_2\varsigma,\varsigma^{-1}\omega_1} & 0 \\ 0 & \frac{1}{\hat{\alpha}([\omega_2|\omega_1|\varsigma])\hat{\alpha}([\varsigma|\varsigma^{-1}\omega_2|\omega_2\varsigma])}\psi_{\varsigma^{-1}\omega_2,\omega_1\varsigma}^{-\mathrm{op}} \end{pmatrix}$$

where $g_1, g_2 \in \mathsf{G}$ and $\omega_1, \omega_2 \in \hat{\mathsf{G}} \backslash \mathsf{G}$. These expressions are most easily obtained by interpreting Real representations as homotopy fixed points of $\mathsf{Rep}_{\mathsf{Cat}_k,k}(\mathcal{G})$ and applying a categorified hyperbolic construction. In any case, it is straightforward to verify that this defines a linear Real representation of $\mathcal{G}$.

More generally, for a subgroup $\mathsf{H} \subset \mathsf{G}$, we define $\mathrm{HInd}_{\mathcal{H}}^{\hat{\mathcal{G}}} = \mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}} \circ \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}$.

8.5 Hyperbolically induced Real categorical and 2-characters We compute the Real categorical and 2-characters of $\mathrm{HInd}_{\mathcal{H}}^{\hat{\mathcal{G}}}(\rho)$. Since the method is similar to that of Section 8.3, we will at points be brief.

Theorem 8.8. *There is a canonical isomorphism*

$$\mathbb{Tr}(\mathrm{HInd}_{\mathcal{H}}^{\hat{\mathcal{G}}}(\rho)) \simeq \mathrm{Ind}_{\Lambda B\mathsf{H}}^{\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}}(\mathbb{Tr}(\rho))$$

of $\tau_\pi^{\mathrm{ref}}(\hat{\alpha})$-twisted representations of $\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}$.

Standard properties of induction for representations of groupoids yield an isomorphism

$$\mathrm{Ind}_{\Lambda B\mathsf{H}}^{\Lambda_\pi^{\mathrm{ref}} B\mathsf{G}}(\mathbb{Tr}_\rho(g)) \simeq \mathrm{Ind}_{\Lambda B\mathsf{G}}^{\Lambda_\pi^{\mathrm{ref}} B\mathsf{G}}\left(\mathrm{Ind}_{\Lambda B\mathsf{H}}^{\Lambda B\mathsf{G}}(\mathbb{Tr}_\rho(g))\right)$$

of $\tau_\pi^{\mathrm{ref}}(\hat{\alpha})$-twisted representations. Since $\mathrm{HInd}_{\mathcal{H}}^{\hat{\mathcal{G}}} = \mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}} \circ \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}$ and, by Theorem 8.3, we have

$$\mathbb{Tr}(\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\rho)) \simeq \mathrm{Ind}_{\Lambda B\mathsf{H}}^{\Lambda B\mathsf{G}}(\mathbb{Tr}(\rho)),$$

it suffices to prove Theorem 8.8 under the assumption that $\mathsf{H} = \mathsf{G}$.

We proceed as in the proof of Theorem 8.3. Fix again an equivalence of the form (24). Instead of (25) we consider a decomposition

$$[\![g]\!]_{\hat{\mathsf{G}}} = \bigsqcup_{i=1}^{l} [g_i]_{\mathsf{G}} \tag{26}$$

with $[g_i]_{\mathsf{G}} \subset \mathsf{G}$ the conjugacy class of g_i. We have $l \in \{1, 2\}$ according to whether or not the conjugacy class $[g_1]_{\mathsf{G}}$ is Real ($l = 1$) or non-Real ($l = 2$). The Real and non-Real cases have $\mathsf{Z}_{\mathsf{G}}(g) \subsetneq \mathsf{Z}_{\hat{\mathsf{G}}}^\varphi(g)$ and $\mathsf{Z}_{\mathsf{G}}(g) = \mathsf{Z}_{\hat{\mathsf{G}}}^\varphi(g)$, respectively. There is an induced decomposition $\mathcal{S} = \sqcup_{i=1}^{l} \mathcal{S}_i$ with $\mathcal{S}_i = \{\sigma \in \mathcal{S} \mid \sigma^{-1} g^{\pi(\sigma)} \sigma \in [g_i]_{\mathsf{G}}\}$. We set $\sigma_1 = e$ and, in the non-Real case, $\sigma_2 = \varsigma$. Relabel the representatives of the conjugacy classes appearing in the decomposition (26) by $g_i = \sigma_i^{-1} g^{\pi(\sigma_i)} \sigma_i$.

The obvious analogue of Lemma 8.4 holds by inspection. Explicitly, in the Real case we take ς to be any element of $\mathsf{Z}_{\hat{\mathsf{G}}}^\varphi(g) \backslash \mathsf{Z}_{\mathsf{G}}(g)$. The maps $F_i^\pm$ then yield an identification

$$\mathbb{Tr}_{\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\rho)}(g) \simeq \mathbb{Tr}_\rho(g) \oplus \varsigma \cdot \mathbb{Tr}_\rho(g_2)$$

where, by convention, $g_2 = g$ in the Real case.

Lemma 8.9. *There is an isomorphism*

$$\mathrm{Tr}_{\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\rho)}(g) \simeq \bigoplus_{i=1}^{l} \mathrm{Ind}_{\mathsf{Z}_{\mathsf{G}}(g_i)}^{\mathsf{Z}_{\hat{\mathsf{G}}}^{\varphi}(g)} (\mathrm{Tr}_{\rho}(g_i))$$

of $\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})$-twisted representations of $\mathsf{Z}_{\hat{\mathsf{G}}}^{\varphi}(g)$, the induction being along the composition $\mathsf{Z}_{\mathsf{G}}(g_i) \hookrightarrow \mathsf{Z}_{\hat{\mathsf{G}}}^{\varphi}(g_i) \xrightarrow{l \mapsto \sigma_i l \sigma_i^{-1}} \mathsf{Z}_{\hat{\mathsf{G}}}^{\varphi}(g).$

Proof. Consider first the non-Real case. Let $\mu \in \mathsf{Z}_{\hat{\mathsf{G}}}^{\varphi}(g) = \mathsf{Z}_{\mathsf{G}}(g)$ and $\sigma \in \mathcal{S}_i$. Then $\mu\sigma = \tilde{\sigma}p$ for some $\tilde{\sigma} \in \mathcal{S}_i$ and $p \in \mathsf{Z}_{\mathsf{G}}(g_i)$. For $i = 1$ we have $\sigma = e = \tilde{\sigma}$ and μ corresponds to the map $\xi_1(\mu) : \mathrm{Tr}_{\rho}(g) \to \mathrm{Tr}_{\rho}(g)$ given by $\beta_{g,p}$. For $i = 2$ we have $\sigma = \varsigma = \tilde{\sigma}$ and μ induces a map $\xi_1(\mu) : \varsigma \cdot \mathrm{Tr}_{\rho}(g_2) \to \varsigma \cdot \mathrm{Tr}_{\rho}(g_2)$ which is equal to $c_1 c_2 \cdot \beta_{g_2,p}$, where

$$c_1 = \left(\frac{\hat{\alpha}([g_2^{-1}|p|\varsigma^{-1}])\hat{\alpha}([\varsigma^{-1}|g|\mu])}{\hat{\alpha}([g_2^{-1}|\varsigma^{-1}|\mu])} \right) \left(\frac{\hat{\alpha}([p|\varsigma^{-1}|g])}{\hat{\alpha}([p|g_2^{-1}|\varsigma^{-1}])\hat{\alpha}([\varsigma^{-1}|\mu|g])} \right)$$

and

$$c_2 = \frac{\hat{\alpha}([g_2^{-1}|g_2|p])\hat{\alpha}([p|g_2^{-1}|g_2])}{\hat{\alpha}([g_2^{-1}|p|g_2])}.$$

The factor c_1 is due to the scalars relating $\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\psi)_{g,\mu}^{-1}$ and $\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\psi)_{\mu,g}$ to $\psi_{g_2^{-1},p}^{\mathrm{op}}$ and $\psi_{p,g_2^{-1}}^{-\mathrm{op}}$, respectively, while c_2 is due to the maps $F_i^{\pm 1}$, as in Lemma 8.6.

On the other hand, as $(\sigma_i^{-1}\mu\sigma_i)(\sigma_i^{-1}\sigma) = (\sigma_i^{-1}\tilde{\sigma})p$, the element $\sigma_i^{-1}\mu\sigma_i \in \mathsf{Z}_{\mathsf{G}}(g_i)$ acts on $\mathrm{Ind}_{\mathsf{Z}_{\mathsf{G}}(g_i)}^{\mathsf{Z}_{\hat{\mathsf{G}}}^{\varphi}(g_i)}(\mathrm{Tr}_{\rho}(g_i)) = \mathrm{Tr}_{\rho}(g_i)$ by the map

$$\xi_2(\sigma_i^{-1}\mu\sigma_i) = \frac{\theta_{g_i}([\sigma_i^{-1}\mu\sigma_i|\sigma_i^{-1}\sigma])}{\theta_{g_i}([\sigma_i^{-1}\tilde{\sigma}|p])} \beta_{g_i,p}.$$

Again, we have set $\theta_{\gamma}([\omega_2|\omega_1]) = \tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})([\omega_2|\omega_1]\gamma)$. As in Lemma 8.6, this leads to the expression

$$\xi_2(\mu) = \frac{\theta_{g_i}([\sigma_i^{-1}|\tilde{\sigma}])}{\theta_{g_i}([\sigma_i^{-1}|\sigma])} \frac{\theta_{g_i}([\mu|\sigma])}{\theta_{g_i}([\tilde{\sigma}|p])} \beta_{g_i,p}.$$

Since $\sigma = \tilde{\sigma}$, this becomes $\xi_2(\mu) = \beta_{g,p}$ for $i = 1$ and $\xi_2(\mu) = c_1 c_2 \cdot \beta_{g_2,p}$ for $i = 2$, where in the latter case we have used the explicit expression for $\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})$.

Consider now the Real case. Let $\mu \in \mathsf{Z}_{\hat{\mathsf{G}}}^{\varphi}(g)$ and $\sigma \in \mathcal{S} = \mathcal{S}_1$. Then $\mu\sigma = \tilde{\sigma}p$ for some $\tilde{\sigma} \in \mathcal{S}$ and $p \in \mathsf{Z}_{\mathsf{G}}(g)$. It follows that the action of μ on $\mathrm{Tr}_{\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\rho)}(g)$ induces a linear map $\xi_1(\mu) : \sigma \cdot \mathrm{Tr}_{\rho}(g) \to \tilde{\sigma} \cdot \mathrm{Tr}_{\rho}(g)$ of the form $c_1 c_2 \cdot \beta_{g,p}$. The factors c_1 and c_2, whose explicit forms we omit, arise in the same way as above. Note that c_1 may now receive contributions from $\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\psi)_{g^{-1},g}^{\mathrm{op}}$.

On the other hand, since $\mu\sigma = \tilde{\sigma}p$, the action of $\mu \in \mathsf{Z}_{\mathsf{G}}(g)$ on $\mathrm{Tr}_{\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\rho)}(g)$ induces a linear map $\xi_2(\mu) : \sigma \cdot \mathrm{Tr}_{\rho}(g) \to \tilde{\sigma} \cdot \mathrm{Tr}_{\rho}(g)$ given by

$$\xi_2(\mu) = \frac{\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})([\mu|\sigma]g)}{\tau_{\pi}^{\mathrm{ref}}(\hat{\alpha})([\tilde{\sigma}|p]g)} \beta_{g,p}.$$

For example, when $\pi(\mu) = -1$, $\sigma = \varsigma$ and $\tilde{\sigma} = e$, the coefficient of $\beta_{g,p}$ reads

$$\tau_\pi^{\mathrm{ref}}(\hat{\alpha})([\mu|\varsigma]g) = \hat{\alpha}([g|g^{-1}|g]) \frac{\hat{\alpha}([g^{-1}|\varsigma|g^{-1}])}{\hat{\alpha}([g^{-1}|g|\varsigma])\hat{\alpha}([\varsigma|g|g^{-1}])} \frac{\hat{\alpha}([\mu|\varsigma|g])\hat{\alpha}([g|\mu|\varsigma])}{\hat{\alpha}([\mu|g^{-1}|\varsigma])}.$$

The factor $\hat{\alpha}([g|g^{-1}|g])$ is equal to c_2 while the remaining terms multiply to c_1. Note that $\mathrm{HInd}_{\mathcal{G}}^{\hat{\mathcal{G}}}(\psi)_{g^{-1},g}^{\mathrm{op}}$ contributes a factor of 1 to c_1 in this case. The other cases are treated in the same way. $\square$

Theorem 8.10. *The Real 2-character of* $\mathrm{HInd}_{\mathcal{H}}^{\hat{\mathcal{G}}}(\rho)$ *is*

$$\chi_{\mathrm{HInd}_{\mathcal{H}}^{\hat{\mathcal{G}}}(\rho)}(g,\omega) = \frac{1}{|\mathsf{H}|} \sum_{\substack{\sigma \in \hat{\mathsf{G}} \\ \sigma(g,\omega)\sigma^{-1} \in \mathsf{H}^2}} \tau\tau_\pi^{\mathrm{ref}}(\hat{\alpha})([\sigma]g \xrightarrow{\omega} g)^{-1} \cdot \chi_\rho(\sigma g^{\pi(\sigma)}\sigma^{-1}, \sigma\omega\sigma^{-1}).$$

In particular, $\chi_{\mathrm{HInd}_{\mathcal{H}}^{\hat{\mathcal{G}}}(\rho)}$ *is supported on the subset* $\pi_0(\Lambda^2 B\mathsf{G}) \subset \pi_0(\Lambda\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}})$.

Proof. This is proved in the same way as Theorem 8.7, using Theorem 8.8 instead of Theorem 8.3. In the present case we apply Proposition 7.2 to the functor $\Lambda B\mathsf{H} \to \Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}$. This explains the coefficient $\frac{1}{|\mathsf{H}|}$, as opposed to $\frac{1}{2|\mathsf{H}|}$, in $\chi_{\mathrm{HInd}_{\mathcal{H}}^{\hat{\mathcal{G}}}(\rho)}$. $\square$

9. Conjectural applications to Real Hopkins–Kuhn–Ravenel characters

In this final section, motivated by the analogy between Borel equivariant Morava E-theory and the 2-representation theory of finite groups [17], [20], we speculate on a homotopy theoretic interpretation of our results.

Let G be a finite group. For each $n \geq 1$, denote by $\mathsf{G}^{(n)} \subset \mathsf{G}^n$ the subset of commuting n-tuples. The group G acts on $\mathsf{G}^{(n)}$ by simultaneous conjugation and the groupoid $\mathsf{G}^{(n)}/\!\!/\mathsf{G}$ is equivalent to the iterated loop groupoid $\Lambda^n B\mathsf{G}$. The space of locally constant functions on $\Lambda^n B\mathsf{G}$ valued in a ring S is denoted by $\mathrm{Cl}_n(\mathsf{G}, S)$ and is called the space of S-valued n-class functions on G.

Fix a prime p and let $E_n^\bullet$ be Morava E-theory at p. Let $\mathbf{B}\mathsf{G}$ be a classifying space of G. Hopkins, Kuhn and Ravenel proved in [20, Theorem C] that there is a generalized n-character map

$$E_n^\bullet(\mathbf{B}\mathsf{G}) \to \mathrm{Cl}_{n,p}(\mathsf{G}, C_\bullet) \tag{27}$$

which, after tensoring the source with $C_\bullet$, is an isomorphism. Here $C_\bullet$ is a $p^{-1}E_n^\bullet$-algebra, constructed in [20, §6.2], which is an $E_n^\bullet$-theoretic analogue of the field obtained from $\mathbb{Q}$ by adjoining all roots of unity, the latter field being that which one considers in classical character theory. The subscript p in $\mathrm{Cl}_{n,p}$ indicates that we restrict attention to functions defined on commuting n-tuples of G which have p^{th} power order. Given a subgroup $\mathsf{H} \hookrightarrow \mathsf{G}$, it was proved in [20, Theorem D] that the induced transfer map is

$$ind_{\mathsf{H}}^{\mathsf{G}}(\chi)(g_1, g_2, \ldots, g_n) = \frac{1}{|\mathsf{H}|} \sum_{\substack{g \in \mathsf{G} \\ g(g_1,g_2,\ldots,g_n)g^{-1} \in \mathsf{H}^n}} \chi(gg_1g^{-1}, \ldots, gg_ng^{-1}).$$

For $n \leq 3$ this is a p-completed version of the formula for induced n-characters; for $n = 1$ this is classical, while for $n = 2$ and $n = 3$ this is proved in [17, Corollary 7.6] and [42, Theorem 6.7],

respectively.

It is known from the work of Sati and Westerland [35, Theorem 2] that a cohomology class $\alpha \in H^{n+2}(-, \mathbb{Z})$ defines a twisted cohomology group $E_n^{\bullet+\alpha}(-)$. In particular, for each $\alpha \in Z^3(B\mathsf{G}, \mathsf{U}(1))$ there is a twisted form of transfer

$$E_n^{\bullet+\alpha_{|\mathsf{H}}}(\mathbf{B}\mathsf{H}) \to E_n^{\bullet+\alpha}(\mathbf{B}\mathsf{G}).$$

It is natural to expect that, after tensoring with $C_\bullet$, these cohomology groups are isomorphic to the p-completions of the twisted n-class function spaces $\Gamma_{\Lambda^n B\mathsf{H}}(\tau^n(\alpha_{|\mathsf{H}})_{C_\bullet})$ and $\Gamma_{\Lambda^n B\mathsf{G}}(\tau^n(\alpha)_{C_\bullet})$ and that under these identifications the transfer map is given by the obvious generalization of the formula appearing in Corollary 8.2.

Let now $\hat{\mathsf{G}}$ be a finite $\mathbb{Z}_2$-graded group. For each $n \geq 1$, define

$$\hat{\mathsf{G}}^{(n)} = \{(g, \omega_2, \ldots, \omega_n) \in \mathsf{G} \times \hat{\mathsf{G}}^{n-1} \mid \omega_i g = g^{\pi(\omega_i)} \omega_i, \ \omega_i \omega_j = \omega_j \omega_i, \ 2 \leq i, j \leq n\}.$$

The group $\hat{\mathsf{G}}$ acts on $\hat{\mathsf{G}}^{(n)}$ by Real conjugation on G and by ordinary conjugation on the remaining factors. The resulting groupoid $\hat{\mathsf{G}}^{(n)} /\!\!/_\varphi \hat{\mathsf{G}}$ is equivalent to $\Lambda^{n-1}\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}$. Denote by $\mathrm{Cl}\mathbb{R}_n(\mathsf{G}, S)$ the vector space of locally constant S-valued functions on $\Lambda^{n-1}\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}$, which we call S-valued Real n-class functions on G

Assume now that $p = 2$. In order to have a homotopy theoretic interpretation of our results, we would like to find a family of genuinely $\mathbb{Z}_2$-equivariant generalized cohomology theories $R_n^\bullet$, $n \geq 1$, which has the following properties:

(i) For each $n \geq 1$, there are maps of cohomology theories

$$c : R_n^\bullet \to E_n^\bullet, \qquad r : E_n^\bullet \to R_n^\bullet.$$

(ii) The assignment $\hat{\mathsf{G}} \mapsto R_n(\mathbf{B}\mathsf{G})$ extends to a $\mathbb{Z}_2$-graded generalization of a Mackey functor. In particular, associated to a $\mathbb{Z}_2$-graded subgroup $\hat{\mathsf{H}} \leq \hat{\mathsf{G}}$ are transfer and restriction maps,

$$rind_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}} : R_n^\bullet(\mathbf{B}\mathsf{H}) \to R_n^\bullet(\mathbf{B}\mathsf{G}), \qquad res_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}} : R_n^\bullet(\mathbf{B}\mathsf{G}) \to R_n^\bullet(\mathbf{B}\mathsf{H})$$

and associated to a subgroup $\mathsf{H} \leq \mathsf{G}$ is a map

$$h_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}} : E_n^\bullet(\mathbf{B}\mathsf{H}) \to R_n^\bullet(\mathbf{B}\mathsf{G}).$$

The map $res_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}}$ is induced by the map $\mathbf{B}\hat{\mathsf{H}} \to \mathbf{B}\hat{\mathsf{G}}$ over $\mathbf{B}\mathbb{Z}_2$ while $h_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}}$ is the composition of c with the $E_n^\bullet$-theory transfer map $ind_{\mathsf{H}}^{\mathsf{G}}$. The above maps should be compatible in the obvious sense.

(iii) There exists a $p^{-1}R_n^\bullet$-algebra $\widetilde{C}_\bullet$ such that for each finite $\mathbb{Z}_2$-graded group $\hat{\mathsf{G}}$ there is a Real n-character map

$$\chi : R_n^\bullet(\mathbf{B}\mathsf{G}) \to \mathrm{Cl}\mathbb{R}_{n,p}(\mathsf{G}, \widetilde{C}_\bullet) \tag{28}$$

which induces an isomorphism $E\mathbb{R}_n^\bullet(\mathbf{B}\mathsf{G}) \otimes_{R_n^\bullet} \widetilde{C}_\bullet \xrightarrow{\sim} \mathrm{Cl}\mathbb{R}_{n,p}(\mathsf{G}, \widetilde{C}_\bullet)$. Moreover, under the identifications (27) and (28), the maps $rind_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}}$ and $h_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}}$ agree with the natural generalizations of the maps $\mathrm{RInd}_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}}$ and $\mathrm{HInd}_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}}$, respectively, at the level of (Real) n-class functions.

(iv) There is an isomorphism $R_1^\bullet \simeq KR_{(2)}^\bullet$ under which r and c become the hyperbolic (realification) and forgetful (complexification) maps, respectively, of equivariant KR-theory [1].

(v) Let $\pi : X \to \hat{X}$ be a double cover. A class $\hat{\beta} \in H^{n+2}(\hat{X}, \mathbb{Z}_\pi)$ defines a twisted theory $R_n^{\bullet+\hat{\beta}}(X)$ and the obvious analogues of properties (i)-(iv) hold for the twisted theory. For example, when twisting by a cocycle $\hat{\beta} \in Z^{n+1}(B\hat{\mathsf{G}}, \mathsf{U}(1)_\pi)$ the group $\mathrm{Cl}\mathbb{R}_{n,p}(\mathsf{G}, \widetilde{C}_\bullet)$ is replaced by $\mathrm{Cl}\mathbb{R}_{n,p}^{\hat{\beta}}(\mathsf{G}, \widetilde{C}_\bullet)$, the space of flat sections of the line bundle $\tau^{n-1}\tau_\pi^{\mathrm{ref}}(\hat{\beta}) \to \Lambda^{n-1}\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}$ over $\widetilde{C}_\bullet$.

We briefly comment on the second part of (iii). Using the results of Sections 7.2, 7.3 and 8.3, 8.5, which should be viewed as being of height $n = 1$ and $n = 2$, respectively, we can write down explicit descriptions of the maps RInd and HInd at the level of Real n-class functions. For example, given $\chi \in \mathrm{Cl}\mathbb{R}_{n,p}^{\hat{\beta}}(\mathsf{H}, \widetilde{C}_\bullet)$, the requirement is that the value of $rind_{\hat{\mathsf{H}}}^{\hat{\mathsf{G}}}(\chi)$ at $(g, \omega_2, \ldots, \omega_n) \in \hat{\mathsf{G}}^{(n)}$ is

$$\frac{1}{2|\mathsf{H}|} \sum_{\substack{\sigma \in \hat{\mathsf{G}} \\ \sigma(g, \omega_2, \ldots, \omega_n)\sigma^{-1} \in \hat{\mathsf{H}}^n}} \tau^{n-1}\tau_\pi^{\mathrm{ref}}(\hat{\beta})([\sigma](g, \omega_2, \ldots, \omega_n))^{-1}$$

$$\times \chi(\sigma g^{\pi(\sigma)}\sigma^{-1}, \sigma\omega_2\sigma^{-1}, \ldots, \sigma\omega_n\sigma^{-1}).$$

A first guess is that $R_n^\bullet$ is the theory $E\mathbb{R}_n^\bullet$ constructed by Hu and Kriz [23]. This theory satisfies properties (i), (ii), (iv) and likely (v). However, as the following example shows, when $n \geq 2$ the group $E\mathbb{R}_n^\bullet(\mathbf{B}\mathsf{G})$ is too small to satisfy (iii).

Example 9.1. Let $\mathsf{G} = \mathbb{Z}_2$ with Real structure $\hat{\mathsf{G}} = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $\hat{\mathsf{G}}$ acts trivially on $\hat{\mathsf{G}}^{(2)} = \mathsf{G} \times \hat{\mathsf{G}}$ and $\mathrm{Cl}\mathbb{R}_{2,p}(\mathsf{G}, \widetilde{C}_\bullet)$ is a free $\widetilde{C}_\bullet$-module of rank $2|\mathsf{G}|^2 = 8$. On the other hand, [31, Theorem 3.2] gives an isomorphism

$$E\mathbb{R}_2^\bullet(\mathbf{B}\mathbb{Z}_2) \simeq E\mathbb{R}_2^\bullet[\![u]\!]/([2]_{\hat{F}}(u))$$

where $[2]_{\hat{F}}(u)$ a modified 2-series of the formal group law associated to $E_2^\bullet$,

$$[2]_{\hat{F}}(u) = 2u +_F \alpha u^2 +_F u^4.$$

Here $u \in E\mathbb{R}_2^{-16}(\mathbf{B}\mathbb{Z}_2)$ and $\alpha \in E\mathbb{R}_2^{16}(\mathbf{B}\mathbb{Z}_2)$. It follows that $E\mathbb{R}_2^\bullet(\mathbf{B}\mathbb{Z}_2)$ is a free $E\mathbb{R}_2^\bullet$-module of rank 4.

When $\mathsf{G} = \mathbb{Z}_{2^q}$ with trivial Real structure, the map $E\mathbb{R}_2^\bullet(\mathbf{B}\mathsf{G}) \to E_2^\bullet(\mathbf{B}\mathsf{G})$ has certain injectivity properties. See, for example, [30, Theorem 2.1] and [29, Theorem 1.5(iii)]. In contrast, the restriction $\mathrm{Cl}\mathbb{R}_2(\mathsf{G}, S) \to \mathrm{Cl}_2(\mathsf{G}, S)$ always has a non-trivial kernel, namely those Real 2-class functions which are supported on $\mathsf{G} \times (\hat{\mathsf{G}}\backslash\mathsf{G})$. In particular, $E\mathbb{R}_2^\bullet(\mathbf{B}\mathsf{G})$ seems to miss the 'Klein bottle sector' of $\mathrm{Cl}\mathbb{R}_2(\mathsf{G}, S)$.

Acknowledgements

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