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M. Kontsevich’s graph complexes and universal structures on graded symplectic manifolds

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Abstract

In the formulation of his celebrated *Formality conjecture*, M. Kontsevich introduced a universal version of the deformation theory for the Schouten algebra of polyvector fields on affine manifolds. This universal deformation complex takes the form of a differential graded Lie algebra of graphs, denoted \mathfrak{fGC}_2 , together with an injective morphism towards the Chevalley–Eilenberg complex associated with the Schouten algebra. The latter morphism is given by explicit local formulas making implicit use of the supergeometric interpretation of the Schouten algebra as the algebra of functions on a graded symplectic manifold of degree 1. The ambition of the present work is to generalise Kontsevich’s construction to graded symplectic manifolds of arbitrary degree $n \geq 1$. The corresponding graph model is given by the full Kontsevich graph complex \mathfrak{fGC}_d where $d = n + 1$ stands for the dimension of the associated AKSZ type σ -model. This generalisation is instrumental to classify universal structures on graded symplectic manifolds. In particular, the zeroth cohomology of the full graph complex \mathfrak{fGC}_d is shown to act via Lie_∞ -automorphisms on the algebra of functions on graded symplectic manifolds of degree n . This generalises the known action of the Grothendieck–Teichmüller algebra $\mathfrak{grt}_1 \simeq H^0(\mathfrak{fGC}_2)$ on the space of polyvector fields. This extended action can in turn be used to generate new universal deformations of Hamiltonian functions, generalising Kontsevich flows on the space of Poisson manifolds to differential graded manifolds of higher degrees. As an application of the general formalism, universal deformations of Courant algebroids via trivalent graphs are presented.

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1. Introduction

In a seminal 97’ preprint [85], M. Kontsevich proved his celebrated formality theorem by constructing an explicit Lie_∞ quasi-isomorphism

$$\mathcal{U}_\Phi : \mathcal{T}_{\text{poly}} \xrightarrow{\sim} \mathcal{D}_{\text{poly}} \tag{1}$$

between $\mathcal{T}_{\text{poly}}$, the graded Lie algebra of polyvector fields on the affine space \mathbb{R}^m , and $\mathcal{D}_{\text{poly}}$, the Hochschild differential graded Lie algebra (dg Lie algebra) of multidifferential operators on \mathbb{R}^m , and such that the first Taylor coefficient coincides with the Hochschild–Kostant–Rosenberg (HKR) quasi-isomorphism of complexes¹. An important corollary of the formality theorem is that it provides an explicit bijective map²:

$$\hat{\mathcal{U}}_\Phi : \text{FPoiss} \xrightarrow{\sim} \text{Star} \tag{2}$$

between the set FPoiss of (equivalence classes of) formal Poisson structures on \mathbb{R}^m and the set Star of (equivalence classes of) formal associative deformations of the algebra of functions on \mathbb{R}^m (also called *star products*). The bijection (2) straightforwardly³ induces a quantization map $\text{Poiss} \rightarrow \text{Star}$ assigning to any Poisson bivector $\pi \in \text{Poiss}$ on \mathbb{R}^m an equivalence class of star products $[\ast] \in \text{Star}$ quantizing π .

An important characteristic of Kontsevich’s formality morphism is that it is given by *universal* formulas *i.e.* formulas applying without distinction to all affine spaces of all finite dimensions and which are defined “graphically” via grafting of existing structures on $\mathcal{T}_{\text{poly}}$ without resorting to additional data. Such formality morphisms were called *stable* in [36]. Informally, these are Lie_∞ quasi-isomorphisms whose Taylor coefficients can be written as a sum over Kontsevich’s *admissible graphs* [85] where the coefficient in front of each graph is given by a weight function, *cf. e.g.* [121]. The master equation ensuring that the Taylor maps assemble to a Lie_∞ -morphism thus boils down to a series of identities on the weights. Although these equations are algebraic, the only known explicit solutions make use of transcendental methods⁴ involving integrals over (compactifications of) configuration spaces of points.

Kontsevich’s formality theorem indisputably constitutes the most remarkable result in the field of deformation quantization, providing a complete solution to the quantization problem formulated in [14, 13]. However, the transcendental methods involved in the construction are generically difficult to handle thus calling for more algebraic tools allowing to address issues in formality theory and deformation quantization. Such algebraic methods have in fact been introduced by M. Kontsevich prior to [85] in the formulation of his *Formality conjecture* around 93’-94’ [84] (*cf.* also [118]). More precisely, M. Kontsevich defined in [84] a universal version of the deformation theory for formality morphisms. Recall that, on very general grounds, any dg Lie algebra \mathfrak{g} is quasi-isomorphic (as a Lie_∞ -algebra) to its cohomology $H(\mathfrak{g})$ endowed with a certain Lie_∞ -structure obtained from the dg Lie algebra structure on \mathfrak{g} via the homotopy transfer theorem. This allows in particular to address formality questions by studying the space of Lie_∞ -structures on $H(\mathfrak{g})$. Going back to the case at hand, the relevant deformation theory is therefore

¹The subscript Φ in (1) will be hereafter interpreted as denoting a Drinfel’d associator.

²The proof that a Lie_∞ quasi-isomorphism between two dg Lie algebras induces a bijection between the associated Deligne groupoids [53] can be found in [85, 28] for the nilpotent case and in [125] for the pro-nilpotent case.

³Via composition of the bijective map (2) with the canonical “formalisation map” $\text{Poiss} \rightarrow \text{FPoiss} : \pi \mapsto \epsilon \pi$ where ϵ is a formal parameter.

⁴See [37, 42] for a recursive construction of formality morphisms over rationals.

controlled by the Chevalley–Eilenberg dg Lie algebra $\mathrm{CE}(\mathcal{T}_{\mathrm{poly}})$ associated with the Schouten algebra of polyvector fields. In [84], M. Kontsevich introduced a universal version of $\mathrm{CE}(\mathcal{T}_{\mathrm{poly}})$ in the guise of a dg Lie algebra of graphs, denoted fGC_2 , together with an injective morphism

$$\mathrm{fGC}_2 \hookrightarrow \mathrm{CE}(\mathcal{T}_{\mathrm{poly}}) \quad (3)$$

given by local formulas. The morphism (3) allows to reformulate questions regarding formality morphisms on affine manifolds (in the stable setting) into purely algebraic questions on the cohomology of the graph complex fGC_2 . In particular, obstructions to the existence of a stable formality morphism⁵ live in $H^1(\mathrm{fGC}_2)$ while such morphisms can be shown to be classified by $H^0(\mathrm{fGC}_2)$. More precisely, it was shown by V. A. Dolgushev in [36] that the exponentiation of the (pro-nilpotent) graded Lie algebra $H^0(\mathrm{fGC}_2)$ acts regularly on the space SQI of (homotopy classes of) stable Lie_∞ -quasi-isomorphisms of the form (1) so that SQI is a *torsor* (or principal homogeneous space) for the pro-unipotent group $\exp(H^0(\mathrm{fGC}_2))$. Furthermore, T. Willwacher constructed in [119] an explicit isomorphism of Lie algebras $H^0(\mathrm{GC}_2) \simeq \mathfrak{grt}_1$ where \mathfrak{grt}_1 stands for the Grothendieck–Teichmüller Lie algebra. Combining these two results leads to a full characterisation of the set SQI of stable formality maps as a GRT_1 -torsor⁶ where GRT_1 stands for the (pro-unipotent⁷) Grothendieck–Teichmüller group $\mathrm{GRT}_1 = \exp(\mathfrak{grt}_1)$. The Grothendieck–Teichmüller group was introduced by V. Drinfel’d⁸ in [44] in virtue of its relation to the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and the theory of quasi-Hopf algebras. Since then, the Grothendieck–Teichmüller group (together with the GRT_1 -torsor of Drinfel’d associators) have found a number of applications in various areas of mathematics including the Kashiwara–Vergne conjecture in Lie theory [2, 4, 1, 109], quantization of Lie bialgebras [46, 116], the study of multiple zeta values [88, 18, 50], rational homotopy automorphisms of the \mathbf{E}_2 -operad [48, 119], *etc.*

The action of the Grothendieck–Teichmüller group on formality morphisms can be traced back to an action of $\mathrm{GRT}_1 \simeq \exp(H^0(\mathrm{fGC}_2))$ on $\mathcal{T}_{\mathrm{poly}}$ via Lie_∞ -automorphisms⁹. Explicitly, to any graph cocycle $\gamma \in H^0(\mathrm{fGC}_2)$ one associates a (homotopy class of) Lie_∞ -automorphisms $\mathcal{U}^\Gamma : \mathcal{T}_{\mathrm{poly}} \xrightarrow{\sim} \mathcal{T}_{\mathrm{poly}}$ where $\Gamma := \exp(\gamma) \in \exp(H^0(\mathrm{fGC}_2))$. Composition with (1) leads to a new (homotopy class of) formality morphisms $\mathcal{U}_{\Phi, \Gamma} := \mathcal{U}_\Phi \circ \mathcal{U}^\Gamma : \mathcal{T}_{\mathrm{poly}} \xrightarrow{\sim} \mathcal{D}_{\mathrm{poly}}$. Furthermore, the bijection between Deligne groupoids derived from \mathcal{U}^Γ (*cf.* footnote 2) induces a stable deformation map $\hat{\mathcal{U}}^\Gamma : \mathrm{FPoiss} \xrightarrow{\sim} \mathrm{FPoiss}$. In particular, the latter can be used to map Poisson bivectors π (*cf.* footnote 3) towards stable formal Poisson structures deforming π . At first order, such stable deformations can be interpreted as maps from cocycles in $H^0(\mathrm{GC}_2)$ to stable flows on the space of Poisson bivectors. The first example¹⁰ of such flows is the so-called *tetrahedral* flow introduced by M. Kontsevich in [84, Section 5.3], *cf.* Section 6.1.

⁵Or equivalently non-trivial stable deformations of the Schouten graded Lie algebra as a Lie_∞ -algebra.

⁶This fact was conjectured by M. Kontsevich in [86] based on the relations between the transcendental formulas involved in his formality morphism and the theory of mixed Tate motives. The Grothendieck–Teichmüller group and Drinfel’d associators also appear in D. Tamarkin’s approach to formality [115, 61] via either the use of the Etingof–Kazhdan quantization of Lie bialgebras or the formality of little disks operad, *cf.* Section 6.1 for details.

⁷There are different versions of the Grothendieck–Teichmüller group, the most important ones being a profinite version $\widehat{\mathrm{GT}}$, a pro- l version GT_l and a pro-unipotent version GT . The latter is isomorphic to a graded version of the group, denoted GRT . We will only be concerned with the exponentiation $\mathrm{GRT}_1 = \exp(\mathfrak{grt}_1)$ related to GRT via $\mathrm{GRT} = \mathbb{K}^\times \ltimes \mathrm{GRT}_1$ where the action of the multiplicative group is via rescaling, *cf.* [123] for details.

⁸Inspired by A. Grothendieck’s suggestion in his *Esquisse d’un Programme* [57] of studying the combinatorial properties of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ via its natural action on the *tour* of Teichmüller groupoids.

⁹We refer to [119] (see also [94]) for the affine space case, [70] for a globalisation to any smooth manifolds and [41] for a generalisation to the sheaf of polyvector fields on any smooth algebraic variety.

¹⁰Various examples of $d = 2$ flows on the space of Poisson bivectors have recently been systematically investigated in a series of works by A. V. Kiselev and collaborators, see [15, 16, 20, 21, 22, 19, 77].

Remarkably, Kontsevich’s solution to the quantization problem of [14, 13] is inspired by ideas coming from string theory. Explicitly, Kontsevich’s quantization formula can be interpreted [85, 25] as the Feynman diagram expansion of a 2-dimensional topological field theory – the *Poisson σ -model* – introduced in [68, 65, 108]. As noted in [26], the quantization of the Poisson σ -model can be best interpreted within the AKSZ formalism [6]. The latter deals with theories living on the space of maps between a *source* manifold of dimension d and a *target* manifold classically endowed with a structure of differential graded symplectic manifold¹¹ of degree n and such that $d = n + 1$. The first and simplest example of such construction is provided by the Poisson σ -model where the source is of dimension $d = 2$ and the target is the (shifted) cotangent bundle of a (finite dimensional) Poisson manifold. More generally, we will refer to the geometrical structure necessary to define a AKSZ σ -model in dimension d as a *symplectic Lie n -algebroid*, with $d = n + 1$. While symplectic Lie 1-algebroids identify with Poisson manifolds, symplectic Lie 2-algebroids correspond to *Courant algebroids*. The latter first appeared implicitly in the study of integrable Dirac structures [43, 31, 32] before their precise geometric structure was abstracted and explicitly stated by the authors of [90] in their study of double of Lie bialgebroids. Courant algebroids play also a central rôle in the context of generalised complex geometry, see [62, 59]. Their graded geometrical interpretation was put forward by D. Roytenberg in [104, 105] and the corresponding *Courant σ -model* was constructed in [67, 106]. Higher examples of symplectic Lie n -algebroids can be found *e.g.* in [69, 89, 58].

An interesting open problem that arises from what precedes concerns the possibility of generalising the interplay between deformation quantization results (on the algebraic side) and quantization of AKSZ-type of models (on the field theoretic side). Motivated by this problem, the ambition of the present paper is to generalise some of the algebraic methods introduced by M. Kontsevich in [84] for Poisson manifolds to the case of higher symplectic Lie n -algebroids. Our main tool in this endeavour is given by a stable version of the Chevalley–Eilenberg dg Lie algebra associated with the deformation complex of symplectic Lie n -algebroids for arbitrary values of $n \geq 1$. Explicitly, this graph model takes the form of an injective morphism of dg Lie algebras:

$$\mathbf{fGC}_d \hookrightarrow \mathbf{CE}(\mathcal{T}_{\text{poly}}^{(n)}) \tag{4}$$

where $d = n + 1$, thus generalising (3) to any $d \geq 2$. Here, \mathbf{fGC}_d stands for the generalisation of Kontsevich’s graph complex to arbitrary dimension d (*cf. e.g.* [119]) and the dg Lie algebra $\mathcal{T}_{\text{poly}}^{(n)}$ – referred to hereafter as the *n -Schouten algebra* – controls the deformation theory of symplectic Lie n -algebroids. The morphism (4) will allow us to take advantage of the available results regarding the cohomology of \mathbf{fGC}_d in order to provide a classification of the stable structures on graded symplectic manifolds of arbitrary (positive) degree. In particular, we propose a classification of Lie_∞ -algebra structures deforming the n -Schouten algebra in a non-trivial way as well as of Lie_∞ -automorphisms of the n -Schouten algebra $\mathcal{T}_{\text{poly}}^{(n)}$. The latter yield in particular new stable flows on the space of symplectic Lie n -algebroids.

The present paper will focus on universal structures in the *stable* setting [see Definition 4.2 below] *i.e.* we consider cochains of the Chevalley–Eilenberg algebra obtained from graphs belonging to the Kontsevich graph complex of undirected graphs \mathbf{fGC}_d (or its directed analogue \mathbf{dfGC}_d). A direct consequence of this choice is that the only incarnation of the Grothendieck–Teichmüller Lie algebra as a universal structure occurs in dimension $d = 2$ where we recover

¹¹Also referred to as a NPQ-manifold of degree n , *cf.* Section 3.3 below.

the above mentioned action of GRT_1 on $\mathcal{T}_{\text{poly}}$ via Lie_∞ -automorphisms. In higher dimensions, the universal structures are insensitive to grt_1 and are in fact classified by loop cocycles. In order to obtain universal structures induced from the Grothendieck–Teichmüller Lie algebra in dimensions $d > 2$, one needs to move away from the stable setting to enter the *(multi)-oriented* regime. [We refer to [99] for a discussion of universal structures induced by (multi)-oriented graphs [122, 126, 127, 95, 96] allowing in particular to provide incarnations of the Grothendieck–Teichmüller algebra into the deformation theory of (quasi)-Lie bialgebroids.]

Summary and main results.

After displaying our conventions and notations in Section 2, we dedicate Sections 3 and 4 to a review – aimed at non-experts – of the principal tools and notions involved in the rest of the paper. In Section 3, we recall the basic concepts of graded geometry, detail the hierarchy of structures endowing graded manifolds (namely \mathbf{N} , \mathbf{NP} and \mathbf{NPQ} -manifolds) and discuss their associated (non-graded) geometric counterparts. In Section 4, we depart from the geometric to the algebraic realm and review the construction of the Kontsevich’s full graph complex fGC_d generalising fGC_2 to arbitrary dimension d . The differential graded Lie algebra structure on fGC_d is best introduced as a convolution Lie algebra from the graph operad \mathbf{Gra}_d whose construction we review. We conclude the section by recalling some known facts regarding the (even and odd) cohomology of fGC_d .

Building on the last two sections, we introduce our main results in Section 5. We start by displaying a tower of representations $\mathbf{Gra}_d \hookrightarrow \text{End}_{\mathcal{L}^\infty(\mathcal{V})}$ for all $d > 0$ where \mathcal{V} stands for an arbitrary \mathbf{NP} -manifold of degree n , such that $d = n + 1$. This tower of morphism of operads will in turn induce a tower of injective morphisms of dg Lie algebras $\text{fGC}_d \hookrightarrow \text{CE}(\mathcal{T}_{\text{poly}}^{(n)})$ thus providing a stable version of the Chevalley–Eilenberg complex for the n -Schouten algebra¹² $\mathcal{T}_{\text{poly}}^{(n)}$. Using this stable model, we show in particular [Corollary 5.7] that the pro-unipotent group $\exp(H^0(\text{fGC}_d))$ acts via Lie_∞ -automorphisms on $\mathcal{T}_{\text{poly}}^{(n)}$. More generally, stable structures on graded symplectic manifolds are classified in Proposition 5.9. We conclude the section by discussing Hamiltonian deformations and their linearisation, referred to as Hamiltonian flows. In particular, we present a canonical map from the zeroth cohomology $H^0(\text{fGC}_d)$ to stable Hamiltonian deformations [Proposition 5.13] and flows [Corollary 5.16] on the space of Hamiltonian functions thus generalising Kontsevich’s construction from Poisson bivectors to higher symplectic Lie n -algebroids. Furthermore, we describe a novel class of Hamiltonian deformations generated by *Weyl factors* induced by elements in $H^{-d}(\text{fGC}_d)$ [Corollary 5.19].

Finally, Section 6 is devoted to illustrate some of the machinery developed in Section 5 to the case of \mathbf{NPQ} -manifolds of degrees 1 and 2, respectively. After reviewing some known applications in the case $n = 1$ (corresponding to Poisson manifolds), we turn to the case $n = 2$ and present new results concerning deformations of Courant algebroids. In particular, we obtain an explicit expression for the unique deformation map for Courant algebroids induced by a loop graph and display a large class of Weyl deformations induced by trivalent graphs (modulo IHX relations). We conclude by a discussion regarding the implications of our results to the deformation quantization problem for Courant algebroids.

¹²Or equivalently for the graded Poisson algebra of functions $\mathcal{L}^\infty(\mathcal{V})$, being isomorphic to $\mathcal{T}_{\text{poly}}^{(n)}$ through degree suspension.

2. Conventions and notations

Suspension. We will work over a ground field \mathbb{K} of characteristic zero. Let $V = \bigoplus_{k \in \mathbb{Z}} V^k$ be a graded vector space over \mathbb{K} . The suspended graded vector space $V[k]$ is defined as $V[k]^n = V^{k+n}$ so that the suspension map $s : V[k] \rightarrow V$ is of degree k .

Invariants and coinvariants. Let G be a group and denote $\mathbb{K}\langle G \rangle$ the associated group ring over \mathbb{K} . A (right) representation of G is a (right) module M over the group ring $\mathbb{K}\langle G \rangle$. Letting M be a right $\mathbb{K}\langle G \rangle$ -module, we define the two following spaces:

- **Invariants:** $M^G := \{m \in M \mid m \cdot g = m \text{ for all } g \in G\}$
- **Coinvariants:** $M_G := M / \{m \cdot g - m \mid g \in G \text{ and } m \in M\}$

Note that while the space of invariants is a subspace of M , the space of coinvariants (or space of orbits) is defined as a quotient of M by the group action. In other words, there are natural maps $M^G \xrightarrow{i} M \xrightarrow{p} M_G$ where i is injective and p surjective. If M is a right $\mathbb{K}\langle G \rangle$ -module and N a left $\mathbb{K}\langle G \rangle$ -module, then $M \otimes N$ is a right $\mathbb{K}\langle G \rangle$ -module under the **diagonal right action** $(M \otimes N) \times G \rightarrow M \otimes N : (a, b) \times g \mapsto (a \cdot g, g^{-1} \cdot b)$.

The associated space of coinvariants is then denoted $M \otimes_G N$. Letting M, N be two right $\mathbb{K}\langle G \rangle$ -modules, a linear map $f : M \rightarrow N$ will be said **G -equivariant** if it is a morphism in the category of $\mathbb{K}\langle G \rangle$ -modules *i.e.* if $f(x \cdot g) = f(x) \cdot g$ for all $x \in M$ and $g \in G$. The space of G -equivariant maps will be denoted $\text{Hom}_G(M, N)$.

Symmetric group \mathbb{S}_N . The symmetric group \mathbb{S}_N is defined as the group of automorphisms of the set $\{1, 2, \dots, N\}$. An element $\sigma \in \mathbb{S}_N$ is called a **permutation** and is defined by its image $\{\sigma(1), \sigma(2), \dots, \sigma(N)\}$. The composition $\sigma \cdot \tau$ of two permutations $\sigma, \tau \in \mathbb{S}_N$ is given by $\{1, 2, \dots, N\} \xrightarrow{\tau} \{\tau(1), \tau(2), \dots, \tau(N)\} \xrightarrow{\sigma} \{\sigma(\tau(1)), \sigma(\tau(2)), \dots, \sigma(\tau(N))\}$. In the following, we will often represent a permutation σ by the $2 \times N$ matrix

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & N \\ \sigma(1) & \sigma(2) & \dots & \sigma(N) \end{pmatrix}.$$

We define right and left actions of the symmetric group \mathbb{S}_N on $V^{\otimes N}$:

$$\begin{aligned} V^{\otimes N} \times \mathbb{S}_N &\rightarrow V^{\otimes N} & (v_1, \dots, v_N) \cdot \sigma &= (v_{\sigma(1)}, \dots, v_{\sigma(N)}) \\ \mathbb{S}_N \times V^{\otimes N} &\rightarrow V^{\otimes N} & \sigma \cdot (v_1, \dots, v_N) &= (v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(N)}) \end{aligned}$$

Example 2.1. Let $\sigma, \tau \in \mathbb{S}_3$ be defined as follows

$$\sigma := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \tau := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \tau^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

admitting inverses as shown. We compute the following compositions:

$$\sigma \cdot \tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \tau \cdot \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Now, denoting $v := (v_1, v_2, v_3)$, one can check that:

$$(v \cdot \sigma) \cdot \tau = (v_1, v_3, v_2) \cdot \tau = (v_2, v_1, v_3) = v \cdot (\sigma \cdot \tau) \sigma \cdot (\tau \cdot v) = \sigma \cdot (v_2, v_3, v_1) = (v_2, v_1, v_3) = (\sigma \cdot \tau) \cdot v.$$

The previous actions on $V^{\otimes N}$ induce dual right and left actions of the symmetric group \mathbb{S}_N on $\text{Hom}(V^{\otimes N}, V)$:

$$\begin{aligned} \text{Hom}(V^{\otimes N}, V) \times \mathbb{S}_N &\rightarrow \text{Hom}(V^{\otimes N}, V) & (f \cdot \sigma)(v_1, \dots, v_N) &= f(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(N)}) \\ \mathbb{S}_N \times \text{Hom}(V^{\otimes N}, V) &\rightarrow \text{Hom}(V^{\otimes N}, V) & (\sigma \cdot f)(v_1, \dots, v_N) &= f(v_{\sigma(1)}, \dots, v_{\sigma(N)}) \end{aligned}$$

Example 2.2. Let $\sigma, \tau \in \mathbb{S}_3$ as in Example 2.1 and $f \in \text{Hom}(V^{\otimes 3}, V)$. Denoting $v := (v_1, v_2, v_3)$, one can check that:

$$\begin{aligned} ((f \cdot \sigma) \cdot \tau)v &= (f \cdot \sigma)(v_2, v_3, v_1) = (v_2, v_1, v_3) = (f \cdot (\sigma \cdot \tau))v \\ (\sigma \cdot (\tau \cdot f))v &= (\tau \cdot f)(v_1, v_3, v_2) = f(v_2, v_1, v_3) = ((\sigma \cdot \tau) \cdot f)v. \end{aligned}$$

In the following, we will denote sgn_N the **signature representation** of \mathbb{S}_N *i.e.* the one-dimensional $\mathbb{K}\langle \mathbb{S}_N \rangle$ -module associating to each permutation $\sigma \in \mathbb{S}_N$ its signature $|\sigma| \in \{-1, 1\}$. A collection of right $\mathbb{K}\langle \mathbb{S}_N \rangle$ -modules $M(N)$ for $N \geq 1$ will be referred to as a **S-module**.

(Un)shuffles. Let $p, q \in \mathbb{N}$. A **shuffle** of type (p, q) is a permutation $\sigma \in \mathbb{S}_{p+q}$ such that σ sends $\{1, \dots, p+q\}$ to $\{i_1, \dots, i_p \mid j_1, \dots, j_q\}$ where $i_1 < \dots < i_p$ and $j_1 < \dots < j_q$.

Example 2.3 (Shuffles).

- $\text{Sh}(1, 1) = \{(1|2), (2|1)\}$
- $\text{Sh}(1, 2) = \{(1|23), (2|13), (3|12)\}$
- $\text{Sh}(2, 1) = \{(12|3), (13|2), (23|1)\}$
- $\text{Sh}(1, 3) = \{(1|234), (2|134), (3|124), (4|123)\}$
- $\text{Sh}(2, 2) = \{(12|34), (13|24), (14|23), (23|14), (24|13), (34|12)\}$
- $\text{Sh}(3, 1) = \{(123|4), (124|3), (134|2), (234|1)\}$

The set of shuffles of type (p, q) is denoted $\text{Sh}(p, q)$. Since a shuffle $\sigma \in \text{Sh}(p, q)$ is completely determined by the set $\{i_1, \dots, i_p\}$, there are $\binom{p+q}{p}$ shuffles of type (p, q) . A **unshuffle** of type (p, q) is a permutation $\sigma \in \mathbb{S}_{p+q}$ such that the inverse permutation σ^{-1} is a shuffle of type (p, q) . The set of unshuffles of type (p, q) is denoted $\text{Sh}^{-1}(p, q)$.

Example 2.4 (Unshuffles).

- $\text{Sh}^{-1}(1, 1) = \{(12), (21)\}$
- $\text{Sh}^{-1}(1, 2) = \{(123), (213), (231)\}$
- $\text{Sh}^{-1}(2, 1) = \{(123), (132), (312)\}$
- $\text{Sh}^{-1}(1, 3) = \{(1234), (2134), (2314), (2341)\}$
- $\text{Sh}^{-1}(2, 2) = \{(1234), (1324), (1342), (3124), (3142), (3412)\}$
- $\text{Sh}^{-1}(3, 1) = \{(1234), (1243), (1423), (4123)\}$

Operads. We will consider operads in the category of (graded) vector spaces over \mathbb{K} . Our conventions will mostly follow the ones of the book [91]. We will denote **Ass**, **Lie** and **Com** the operads of (graded) vector spaces encoding (graded) associative, Lie and commutative associative algebras without unit, respectively. The cooperad governing cocommutative coassociative algebras without counit will be denoted **coCom**. The latter is defined explicitly as:

$$\text{coCom} = \begin{cases} \mathbf{0} & \text{for } n = 0 \\ \mathbb{K} & \text{for all } n > 0 \end{cases}$$

where \mathbb{K} stands for the trivial representation of \mathbb{S}_n .

Letting \mathcal{O} be an operad in the category of graded vector spaces, the set of graded vector spaces:

$$\mathcal{O}\{d\}(N) := \begin{cases} \mathcal{O}(N)[d(1 - N)] & \text{for } d \text{ even} \\ \mathcal{O}(N) \otimes \text{sgn}_N[d(1 - N)] & \text{for } d \text{ odd} \end{cases}$$

assemble to a \mathbb{S} -module. Endowing this \mathbb{S} -module with the partial composition maps, identity and right-actions of \mathcal{O} defines the **d -suspended operad** $\mathcal{O}\{d\}$. Alternatively, the d -suspended operad $\mathcal{O}\{d\}$ can be characterised as the unique operad for which the set of algebras of the operad \mathcal{O} on a graded vector space V are in one-to-one correspondence with the set of algebras of $\mathcal{O}\{d\}$ on the suspended graded vector space $V[d]$. In particular, $\text{End}_V\{d\} = \text{End}_{V[d]}$ where End_V denotes the endomorphism operad associated with the graded vector space V .

3. Graded geometry

The aim of the present section is to provide a short introduction to graded manifolds as well as their (non-graded) geometric counterparts. The latter objects are defined as the geometrical data associated with graded¹³ manifolds – understood as manifolds endowed with a grading of the corresponding structure sheaf [cf. [93] for precise definitions] – supplemented with some additional graded structures.

3.1 N-manifolds Letting \mathcal{V} be a \mathbb{N} -graded manifold, or **N-manifold**, of degree n (i.e. concentrated in degrees $0, \dots, n$) we will denote $\mathcal{C}^\infty(\mathcal{V})$ the associated algebra of functions. The subvector space of homogeneous functions of degree k will be denoted $\mathcal{C}^{\infty|k}(\mathcal{V}) \subset \mathcal{C}^\infty(\mathcal{V})$ so that $\mathcal{C}^\infty(\mathcal{V}) = \bigoplus_{k \geq 0} \mathcal{C}^{\infty|k}(\mathcal{V})$ is a graded algebra. Moreover, $\mathcal{C}^\infty(\mathcal{V})$ is a filtered algebra. Letting \mathcal{A}_k denote the (graded) subalgebra of $\mathcal{C}^\infty(\mathcal{V})$ locally generated by functions of degree $\leq k$, there is an increasing sequence:

$$\mathcal{C}^\infty(\mathcal{M}) = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \dots \subset \mathcal{A}_n = \mathcal{C}^\infty(\mathcal{V}) \tag{5}$$

where we have $\mathcal{C}^{\infty|k}(\mathcal{V}) = \mathcal{A}_k / \mathcal{A}_{k-1}$ so that $\mathcal{C}^\infty(\mathcal{V}) = \bigoplus_{k \geq 0} \mathcal{C}^{\infty|k}(\mathcal{V})$ is the graded algebra associated with the filtration (5). Corresponding to this filtration, there is a tower of fibrations:

$$\mathcal{M} = \mathcal{M}_0 \leftarrow \mathcal{M}_1 \leftarrow \dots \leftarrow \mathcal{M}_n = \mathcal{V} \tag{6}$$

where \mathcal{M} is an ordinary smooth manifold – referred to as the **base** – and such that $\mathcal{C}^\infty(\mathcal{M}) = \mathcal{C}^{\infty|0}(\mathcal{V}) = \mathcal{A}_0$. Furthermore, \mathcal{M}_1 is a vector bundle over \mathcal{M} and for all $k \geq 1$, $\mathcal{M}_k \leftarrow \mathcal{M}_{k+1}$ is an affine fibration, cf. [105] for details.

The geometry of the fibration underlying graded manifolds can be enriched by introducing additional (hierarchised) data on \mathcal{V} (cf. [105, 29]):

- A **NP-manifold** (\mathcal{V}, ω) **of degree** n is a \mathbb{N} -graded manifold \mathcal{V} endowed with a symplectic 2-form ω of intrinsic degree n .
- A **NPQ-manifold** $(\mathcal{V}, \omega, \mathbf{Q})$ **of degree** n is a \mathbb{N} -graded manifold \mathcal{V} endowed with a symplectic 2-form ω of intrinsic degree n and a homological vector field \mathbf{Q} (i.e. \mathbf{Q} is of degree 1 and satisfies $\mathbf{Q}^2 = 0$) such that $\mathcal{L}_{\mathbf{Q}}\omega = 0$.

¹³We will only deal with **\mathbb{N} -graded manifolds** for which the corresponding degree assigned to each local coordinate is a non-negative integer.

These additional data induce some extra geometric structures on the fibration (6). We will refer to the (non-graded) geometrical data associated with NPQ-manifolds of degree n as **symplectic Lie n -algebroids**.

3.2 NP-manifolds Endowing a graded manifold with a symplectic (*i.e.* non-degenerate and closed) 2-form has a number of consequences. First of all, the existence of a symplectic 2-form of degree n on a \mathbb{N} -graded manifold \mathcal{V} constrains the degree of \mathcal{V} to not exceed n [*cf.* [105] Lemma 2.4]. Secondly, it can be shown that any homogeneous symplectic 2-form of degree $n \geq 1$ is exact [*cf.* [105] Lemma 2.2]. These two properties can be used in order to provide a local presentation of NP-manifolds *à la* Darboux.

We distinguish between odd and even cases as follows. When n is odd, we introduce a set of homogeneous coordinates¹⁴:

$$\left\{ \begin{matrix} x_0^\mu, \psi_i^{\alpha_i}, \chi_{n-i}^{\alpha_i}, p_n^\mu \end{matrix} \right\} \quad \text{where} \quad i \in \left\{ 1, \dots, \frac{1}{2}(n-1) \right\}.$$

The symplectic 2-form of odd degree n can thus be written as:

$$\omega = dx^\mu \wedge dp_\mu + \sum_{i=1}^{\frac{1}{2}(n-1)} d\psi^{\alpha_i} \wedge d\chi_{\alpha_i}.$$

The associated Poisson bracket of degree $-n$ acts as follows:

$$\begin{aligned} \{f, g\}_\omega &= (-1)^k \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial p_\mu} + \frac{\partial f}{\partial p_\mu} \frac{\partial g}{\partial x^\mu} \\ &+ \sum_{i=1}^{\frac{1}{2}(n-1)} \left\{ (-1)^{k(i+1)} \frac{\partial f}{\partial \psi^{\alpha_i}} \frac{\partial g}{\partial \chi_{\alpha_i}} + (-1)^{ik} \frac{\partial f}{\partial \chi_{\alpha_i}} \frac{\partial g}{\partial \psi^{\alpha_i}} \right\} \end{aligned} \quad (7)$$

on homogeneous functions $f \in \mathcal{C}^{\infty|k}(\mathcal{V})$ and $g \in \mathcal{C}^{\infty|l}(\mathcal{V})$ of degree k and l , respectively.

When n is even, the corresponding set of homogeneous coordinates reads

$$\left\{ \begin{matrix} x_0^\mu, \psi_i^{\alpha_i}, \xi_{n/2}^a, \chi_{n-i}^{\alpha_i}, p_n^\mu \end{matrix} \right\} \quad \text{where} \quad i \in \left\{ 1, \dots, \frac{1}{2}n-1 \right\}.$$

The symplectic 2-form of even degree n is written as:

$$\omega = dx^\mu \wedge dp_\mu + \sum_{i=1}^{\frac{1}{2}n-1} d\psi^{\alpha_i} \wedge d\chi_{\alpha_i} + \frac{1}{2} \kappa_{ab} d\xi^a \wedge d\xi^b$$

where the bilinear form κ is non-degenerate and symmetric (*resp.* skewsymmetric¹⁵) for $n/2$ odd (*resp.* even) *i.e.* $\kappa_{ab} = -(-1)^{n/2} \kappa_{ba}$. The associated Poisson bracket thus takes the form:

$$\begin{aligned} \{f, g\}_\omega &= \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu} \frac{\partial g}{\partial x^\mu} \\ &+ \sum_{i=1}^{\frac{1}{2}n-1} \left\{ (-1)^{ik} \frac{\partial f}{\partial \psi^{\alpha_i}} \frac{\partial g}{\partial \chi_{\alpha_i}} - (-1)^{i(k+1)} \frac{\partial f}{\partial \chi_{\alpha_i}} \frac{\partial g}{\partial \psi^{\alpha_i}} \right\} + (-1)^{kn/2} \frac{\partial f}{\partial \xi^a} \kappa^{ab} \frac{\partial g}{\partial \xi^b}. \end{aligned} \quad (8)$$

It can be checked that the Poisson brackets (7) and (8) satisfy the following properties:

¹⁴The subscript denotes the corresponding degree.

¹⁵Note that, whenever $n = 4k$ (for some integer $k = 0, 1, 2, \dots$) the indices of type a, b, \dots should run over an even number of dimensions in order to ensure the existence of a skewsymmetric invertible bilinear form κ .

1. $\{f, g\}_\omega = -(-1)^n(-1)^{kl}\{g, f\}_\omega$
2. $\{f, g \cdot h\}_\omega = \{f, g\}_\omega \cdot h + (-1)^{l(k-n)}g \cdot \{f, h\}_\omega$
3. $\{\{f, g\}_\omega, h\}_\omega + (-1)^{k(l+m)}\{\{g, h\}_\omega, f\}_\omega + (-1)^{m(k+l)}\{\{h, f\}_\omega, g\}_\omega = 0$

for all homogeneous functions $f \in \mathcal{C}^{\infty|k}(\mathcal{V})$, $g \in \mathcal{C}^{\infty|l}(\mathcal{V})$ and $h \in \mathcal{C}^{\infty|m}(\mathcal{V})$ of degree k , l and m respectively.¹⁶

We conclude this quick survey of NP-manifolds by discussing the notion of gauge transformations.

Definition 3.1 (Gauge transformation). Let (\mathcal{V}, ω) be a NP-manifold. A gauge transformation on \mathcal{V} is a diffeomorphism $\phi : \mathcal{V} \rightarrow \mathcal{V}$ being

1. degree preserving
2. symplectomorphic *i.e.* $\phi^*\omega = \omega$.

In other words, the pullback map $\phi^* : \mathcal{C}^\infty(\mathcal{V}) \rightarrow \mathcal{C}^\infty(\mathcal{V})$ is:

1. of degree 0 *i.e.* $|\phi^*(f)| = |f|$
2. a morphism of Poisson algebras *i.e.* $\phi^*(f \cdot g) = \phi^*(f) \cdot \phi^*(g)$ and $\phi^*(\{f, g\}_\omega) = \{\phi^*(f), \phi^*(g)\}_\omega$ for all homogeneous functions $f, g \in \mathcal{C}^\infty(\mathcal{V})$.

Two functions $\mathcal{F}, \mathcal{F}' \in \mathcal{C}^\infty(\mathcal{V})$ will be said to be equivalent if there exists a gauge transformation $\phi : \mathcal{V} \rightarrow \mathcal{V}$ such that $\mathcal{F}' = \phi^*(\mathcal{F})$. Equivalence classes thereof will be denoted $[\mathcal{F}]$. Infinitesimal gauge transformations are symplectic vector fields $\mathcal{X} = \{f, \cdot\}_\omega$ of degree 0 generated by arbitrary functions $f \in \mathcal{C}^{\infty|n}(\mathcal{V})$ of degree n . Two equivalent functions $\mathcal{F}, \mathcal{F}' \in [\mathcal{F}]$ differ infinitesimally by a term of the form $\{\mathcal{F}, f\}_\omega$, with $f \in \mathcal{C}^{\infty|n}(\mathcal{V})$.

3.3 NPQ-manifolds We now turn to NPQ-manifolds and start by pointing out that the latter can be equivalently described in terms of a Poisson bracket together with a Hamiltonian function¹⁷ *i.e.* as a triplet $(\mathcal{V}, \{\cdot, \cdot\}_\omega, \mathcal{H})$ where:

1. \mathcal{V} is a \mathbb{N} -graded manifold.
2. $\{\cdot, \cdot\}_\omega$ is a non-degenerate Poisson bracket of degree $-n$ acting on the graded algebra of functions on \mathcal{V} .
3. \mathcal{H} is a Hamiltonian function *i.e.* a homogeneous function of degree $n + 1$ being nilpotent with respect to the graded Poisson bracket *i.e.* $\{\mathcal{H}, \mathcal{H}\}_\omega = 0$. The set of Hamiltonian functions will be denoted **Ham**.

Equivalence between the homological and Hamiltonian presentations of NPQ-manifolds of degree n is realised by identifying ω as the symplectic 2-form of degree n dual to $\{\cdot, \cdot\}_\omega$ and defining the privileged vector field $Q \in \Gamma^1(T\mathcal{V})$ of degree 1 on \mathcal{V} as $Q = \{\mathcal{H}, \cdot\}_\omega$. The nilpotency of \mathcal{H} ensures that Q is homological *i.e.* $[Q, Q] = 0$, with $[\cdot, \cdot]$ the graded Lie bracket on \mathcal{V} .

The importance of NPQ-manifolds (or equivalently symplectic Lie n -algebroids) stems from the fact that these naturally form the target space of the classical action associated with AKSZ-type σ -models [6] for which the source manifold has dimension $d = n + 1$.

Two Hamiltonian functions $\mathcal{H}, \mathcal{H}' \in \text{Ham}$ will be said to be equivalent if there exists a gauge transformation [see Definition 3.1] denoted $\phi : \mathcal{V} \rightarrow \mathcal{V}$ such that $\mathcal{H}' = \phi^*(\mathcal{H})$. Two equivalent Hamiltonians differ infinitesimally by a coboundary $\{\mathcal{H}, f\}_\omega$, with $f \in \mathcal{C}^{\infty|n}(\mathcal{V})$.

We conclude this brief survey by displaying examples of symplectic Lie n -algebroids in low degrees.

¹⁶In other words, the triplet $(\mathcal{C}^\infty(\mathcal{V}), \cdot, \{\cdot, \cdot\}_\omega)$ is a Ger_{n+1} -algebra, *cf.* Section 4.2.

¹⁷Indeed, it follows from Cartan's homotopy formula that the compatibility relation between the symplectic 2-form and the vector field ensures that the latter is Hamiltonian, *cf.* Lemma 2.2 in [105].

Example 3.2 (Symplectic Lie n -algebroids).

- $n = 0$ (Symplectic manifolds)

The manifold is coordinatised by a unique set of homogeneous coordinates ξ^a of degree 0, with $a \in \{1, \dots, D\}$ and D the (even) dimension of the manifold. The manifold is thus non-graded (or bosonic) *i.e.* \mathcal{V} identifies with its base \mathcal{M} . The symplectic 2-form of degree 0 takes the usual form $\omega = \frac{1}{2}\kappa_{ab} d\xi^a \wedge d\xi^b$ where the bilinear form κ is non-degenerate and skewsymmetric *i.e.* $\kappa_{ab} = -\kappa_{ba}$. The associated Poisson bracket thus takes the form:

$$\{f, g\}_\omega = \frac{\partial f}{\partial \xi^a} \kappa^{ab} \frac{\partial g}{\partial \xi^b}.$$

The absence of degree 1 coordinates prevents the existence of a Hamiltonian function \mathcal{H} (of would-be degree 1) in this case. Symplectic Lie 0-algebroids are thus in one-to-one correspondence with (ordinary) symplectic manifolds. Gauge transformations identify in this case with usual symplectomorphisms.

- $n = 1$ (Poisson manifolds)

The set of homogeneous coordinates takes the form $\left\{ x^\mu, p_\mu \right\}$. The symplectic 2-form of odd degree 1 can thus be written as $\omega = dx^\mu \wedge dp_\mu$ while the associated Poisson bracket of degree -1 acts as follows:

$$\{f, g\}_\omega = (-1)^k \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial p_\mu} + \frac{\partial f}{\partial p_\mu} \frac{\partial g}{\partial x^\mu}$$

on homogeneous functions $f \in \mathcal{C}^{\infty|k}(\mathcal{V})$ and $g \in \mathcal{C}^{\infty|l}(\mathcal{V})$ of degree k and l respectively. Up to degree suspension, $\{\cdot, \cdot\}_\omega$ identifies with the Schouten bracket acting on polyvector fields. The most general function of degree 2 reads $\mathcal{H} = \frac{1}{2}\pi^{\mu\nu}(x)p_\mu p_\nu$ with π a bivector, *i.e.* $\pi^{\mu\nu} = -\pi^{\nu\mu}$. It can be checked that¹⁸ $\{\mathcal{H}, \mathcal{H}\}_\omega = 0 \Leftrightarrow \pi^{\rho[\lambda} \partial_\rho \pi^{\mu\nu]} = 0$ *i.e.* \mathcal{H} is Hamiltonian if and only if π is a Poisson bivector. It follows that symplectic Lie 1-algebroids are in one-to-one correspondence with Poisson manifolds. Gauge transformations identify in this case with usual diffeomorphisms on the base manifold.

- $n = 2$ (Courant algebroids)

The set of homogeneous coordinates can be decomposed as $\left\{ x^\mu, \xi^a, p_\mu \right\}$. The symplectic 2-form of even degree 2 can thus be written as $\omega = dx^\mu \wedge dp_\mu + \frac{1}{2}\kappa_{ab} d\xi^a \wedge d\xi^b$ where the bilinear form κ is non-degenerate and symmetric *i.e.* $\kappa_{ab} = \kappa_{ba}$. The associated Poisson bracket of degree -2 acts as follows:

$$\{f, g\}_\omega = \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu} \frac{\partial g}{\partial x^\mu} + (-1)^k \frac{\partial f}{\partial \xi^a} \kappa^{ab} \frac{\partial g}{\partial \xi^b}$$

on homogeneous functions $f \in \mathcal{C}^{\infty|k}(\mathcal{V})$ and $g \in \mathcal{C}^{\infty|l}(\mathcal{V})$ of degree k and l respectively. The most general function of degree 3 reads $\mathcal{H} = \rho_a^\mu \xi^a p_\mu + \frac{1}{6} T_{abc} \xi^a \xi^b \xi^c$ where T_{abc} is totally skewsymmetric. It can be checked that the nilpotency condition $\{\mathcal{H}, \mathcal{H}\}_\omega = 0$ is equivalent to the three following constraints:

$$1. \mathcal{C}_1^{\mu\nu} := \rho_a^\mu \kappa^{ab} \rho_b^\nu = 0 \tag{9}$$

$$2. \mathcal{C}_2^{\mu} := \rho_c^\mu \kappa^{cd} T_{dab} + 2 \rho_{[a}^\lambda \partial_\lambda \rho_{b]}^\mu = 0 \tag{10}$$

$$3. \mathcal{C}_3^{abcd} := \frac{1}{4} T_{e[ab} \kappa^{ef} T_{cd]f} + \frac{1}{3} \rho_{[a}^\mu \partial_\mu T_{bcd]} = 0. \tag{11}$$

¹⁸Here and in the following, round (resp. square) brackets of indices will denote (skew)symmetrisation.

As will be reviewed in Section 6.2, symplectic Lie 2-algebroids are in one-to-one correspondence with Courant algebroids. Infinitesimal gauge transformations are generated by functions $f = X^\mu p_\mu + \Lambda_{ab} \xi^a \xi^b$ where the two terms on the right-hand side correspond to infinitesimal diffeomorphisms on the base manifold and infinitesimal rotations on the fiber, respectively.

4. Graph complexes

The aim of the present section is to review a particular family of graph complexes¹⁹ introduced by M. Kontsevich in [83, 82, 84]. The former is most clearly defined in terms of the convolution Lie algebra constructed from a suitable graph operad. We start by reviewing the construction of this graph operad – denoted \mathbf{Gra}_d hereafter²⁰ – from a combinatorial point of view before turning to the definition of the so-called full graph complex \mathbf{fGC}_d . After reviewing results regarding the cohomology of \mathbf{fGC}_d , we conclude by presenting a variant of the full graph complex whose elements are directed graphs. The material covered in this section is standard and can be found for example in [119, 39, 41].

4.1 The graph operad \mathbf{Gra}_d Our starting point towards a definition of the graph operad \mathbf{Gra}_d will be the set of **multidigraphs** (or quivers) *i.e.* directed graphs which are allowed to contain multiple edges and tadpoles²¹. The set of multidigraphs with N vertices and k directed edges will be denoted $\mathbf{gra}_{N,k}$. A typical example²² of multidigraph is given in Figure 1. There is a natural right-action of the semi-direct product $\mathbb{S}_k \ltimes \mathbb{S}_2^{\times k}$ on elements of $\mathbf{gra}_{N,k}$ by permutation of the ordering (\mathbb{S}_k) and flipping of the directions of the edges ($\mathbb{S}_2^{\times k}$). We will consider the 1-dimensional signature representation sgn_k (resp. $\text{sgn}_2^{\otimes k}$) as a left $\mathbb{K} \langle \mathbb{S}_k \ltimes \mathbb{S}_2^{\times k} \rangle$ -module with trivial action of $\mathbb{S}_2^{\times k}$ (resp. \mathbb{S}_k). For all $N \geq 1$ and $d \in \mathbb{N}$, we define the collection of graded vector spaces $\mathbf{Gra}_d(N)$ as:

¹⁹Graph complexes come in many variants. As shown in [52, 92, 30], to any cyclic operad \mathcal{O} one can associate a class of \mathcal{O} -graph complexes. In particular, $\mathcal{O} = \mathbf{Ass}$ corresponds to the class of ribbon graphs computing cohomology of moduli spaces of curves [101, 102] while the graph complex for $\mathcal{O} = \mathbf{Lie}$ computes cohomology of outer automorphisms of free groups [33]. We will solely be interested in the case $\mathcal{O} = \mathbf{Com}$. Also, graph complexes come in two dual versions: a *homological* version in which the boundary operator acts via “collapsing” of edges [83, 82] and a *cohomological* one in which the coboundary operator acts by “blowing up” edges [84, 119]. We will hereafter focus on the cohomological version.

²⁰In Section 5, we will relate the integer d (in the case when $d \geq 1$) to the dimension of the source of the relevant AKSZ σ -model on which \mathbf{Gra}_d will be shown to act. In other words, we will consider $d = n + 1$ where n is the degree of the corresponding NPQ-manifold, *cf.* Section 3.3.

²¹Formally, a multidigraph is defined as a four-tuple $\gamma = (V_\gamma, E_\gamma, s, t)$ where:

- V_γ is a set whose elements are called **vertices**.
- E_γ is a set whose elements are called **edges**.
- The map $s : E_\gamma \rightarrow V_\gamma$ assigns to each edge its **source**.
- The map $t : E_\gamma \rightarrow V_\gamma$ assigns to each edge its **target**.

An edge $e \in E_\gamma$ such that $s(e) = t(e)$ is called a **tadpole** while pairs of edges $e_1, e_2 \in E_\gamma$ such that $s(e_1) = s(e_2)$ and $t(e_1) = t(e_2)$ are called **double edges**. The set of edges connecting a given vertex $v \in V_\gamma$ will be denoted $E_\gamma(v)$. We will mostly deal with *labeled* multidigraphs *i.e.* multidigraphs endowed with two bijective maps $l_V : V_\gamma \rightarrow [|V_\gamma|]$ and $l_E : E_\gamma \rightarrow [|E_\gamma|]$ where $|V_\gamma|$ (resp. $|E_\gamma|$) denotes the number of vertices (resp. edges) of γ and $[n] := \{1, 2, \dots, n\}$. While depicting multidigraphs pictorially, we will represent edges by arrows from source to target vertices. To avoid ambiguity, labelling will be performed using Hindu-Arabic numerals for vertices and Roman numerals for edges. Note that we do not assume any compatibility between the labelling of vertices and edges *a priori*.

²²Note that the definition of a multidigraph does not assume connectedness.

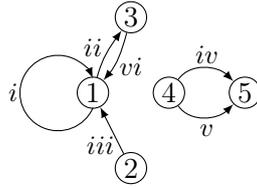


Figure 1: Example of graph in $\mathbf{gra}_{5,6}$

- d even: $\mathbf{Gra}_d(N) := \prod_{k \geq 0} (\mathbb{K} \langle \mathbf{gra}_{N,k} \rangle \otimes_{\mathbb{S}_k \times \mathbb{S}_2^{\times k}} \text{sgn}_k) [k(d-1)]$ (12)

- d odd: $\mathbf{Gra}_d(N) := \prod_{k \geq 0} (\mathbb{K} \langle \mathbf{gra}_{N,k} \rangle \otimes_{\mathbb{S}_k \times \mathbb{S}_2^{\times k}} \text{sgn}_2^{\otimes k}) [k(d-1)]$ (13)

where the subscript stands for taking coinvariants with respect to the diagonal right action of $\mathbb{S}_k \times \mathbb{S}_2^{\times k}$ and the term between brackets denotes degree suspension (*cf.* Section 2 for conventions). Elements of $\mathbf{Gra}_d(N)$ are linear combinations of equivalence classes of graphs in $\mathbf{gra}_{N,k}$, for arbitrary $k \geq 0$. Two graphs $\gamma, \gamma' \in \mathbf{gra}_{N,k}$ will be said equivalent (*i.e.* $\gamma \sim \gamma'$) if one of the two following condition holds:

1. There exists an element $\sigma \in \mathbb{S}_2^{\times k}$ such that $\Phi_\sigma^{\text{dir}}(\gamma) = (-1)^{d|\sigma|} \gamma'$ where Φ_σ^{dir} stands for the automorphism of $\mathbf{gra}_{N,k}$ that flips the direction of the edges according to σ ,

e.g. $\textcircled{1} \xrightarrow{i} \textcircled{2} \sim (-1)^d \textcircled{1} \xleftarrow{i} \textcircled{2}$.

2. There exists an element $\sigma \in \mathbb{S}_k$ such that $\Phi_\sigma^{\text{order}}(\gamma) = (-1)^{(d+1)|\sigma|} \gamma'$ where $\Phi_\sigma^{\text{order}}$ stands for the automorphism of $\mathbf{gra}_{N,k}$ that permutes the order of the edges according to σ ,

e.g. $\textcircled{1} \xrightarrow{i} \textcircled{2} \xrightarrow{ii} \textcircled{3} \sim (-1)^{d+1} \textcircled{1} \xrightarrow{ii} \textcircled{2} \xrightarrow{i} \textcircled{3}$.

According to the degree suspension in (12)-(13), each edge is assigned an intrinsic degree $1-d$, so that the degree of an element $\gamma \in \mathbf{gra}_{N,k}$ as seen in $\mathbf{Gra}_d(N)$ is given by $|\gamma| = k(1-d)$. It is also clear from their definition that graded vector spaces $\mathbf{Gra}_d(N)$ for different d of same parity only differ by their degree assignment and are thus isomorphic to each other. Following [107], we will call **zero graph** a graph $\gamma \in \mathbf{gra}_{N,k}$ which equals minus itself in $\mathbf{Gra}_d(N)$ and thus belongs to the zero class in $\mathbf{Gra}_d(N)$. It follows that a graph admitting an automorphism that permutes the edges ordering by an odd permutation is a zero graph whenever d is even. In particular, graphs admitting multiple edges are zero graphs for d even²³. On the other hand, a graph admitting an automorphism that flips an odd number of edges is automatically a zero graph whenever d is odd. In particular, graphs with tadpoles are zero graphs for d odd²⁴.

For all $N \geq 1$, the symmetric group \mathbb{S}_N acts naturally on the right on the graded vector space $\mathbf{Gra}_d(N)$ by permuting the label of vertices as $\{1, 2, \dots, N\} \xrightarrow{\sigma} \{\sigma^{-1}(1), \sigma^{-1}(2), \dots, \sigma^{-1}(N)\}$.

²³Whenever d is even, the double edges graph $\textcircled{1} \begin{matrix} \xrightarrow{i} \\ \xleftarrow{ii} \end{matrix} \textcircled{2}$ satisfies $\textcircled{1} \begin{matrix} \xrightarrow{i} \\ \xleftarrow{ii} \end{matrix} \textcircled{2} \sim -\textcircled{1} \begin{matrix} \xrightarrow{ii} \\ \xleftarrow{i} \end{matrix} \textcircled{2}$ and is thus a zero graph in $\mathbf{Gra}_d(2)$.

²⁴Whenever d is odd, the tadpole graph $\textcircled{1} \begin{matrix} \xrightarrow{i} \\ \xrightarrow{i} \end{matrix} \textcircled{1}$ satisfies $\textcircled{1} \begin{matrix} \xrightarrow{i} \\ \xrightarrow{i} \end{matrix} \textcircled{1} \sim -\textcircled{1} \begin{matrix} \xrightarrow{i} \\ \xrightarrow{i} \end{matrix} \textcircled{1}$ and is thus a zero graph in $\mathbf{Gra}_d(1)$.

We will denote $\Sigma_N : \mathbf{Gra}_d(N) \times \mathbb{S}_N \rightarrow \mathbf{Gra}_d(N)$ the corresponding right action²⁵. In other words, the set of graded vector spaces $\{\mathbf{Gra}_d(N)\}_{N \geq 1}$ assemble to a \mathbb{S} -module over \mathbb{K} . The \mathbb{S} -module $\{\mathbf{Gra}_d(N)\}_{N \geq 1}$ can further be given the structure of an operad by endowing it with partial composition operations. Explicitly, we define partial composition operations:

$$\circ_i : \mathbf{gra}_{M,j} \otimes \mathbf{gra}_{N,k} \rightarrow \mathbf{Gra}_d(M + N - 1) \text{ for all } 1 \leq i \leq M \text{ as } \gamma \circ_i \gamma' = \sum_{f \in \text{Hom}(E_\gamma(v_i), V_{\gamma'})} \gamma \circ_i^f \gamma' \quad (14)$$

where $\gamma \in \mathbf{gra}_{M,j}$ and $\gamma' \in \mathbf{gra}_{N,k}$. In the above formula, we let v_i be the i^{th} vertex of γ and the sum be performed over homomorphisms of sets between the set $E_\gamma(v_i)$ of edges of γ connecting v_i and the set $V_{\gamma'}$ of vertices of γ' . The operation \circ_i^f consists in first inserting the graph γ' in place of the vertex $v_i \in \gamma$ and then reconnecting the elements in $E_\gamma(v_i)$ to vertices of γ' along the map f . The output is a sum of graphs with $j + k$ edges in $\mathbf{Gra}_d(M + N - 1)$. As for labelling of vertices and edges, we follow the rules:

- The labels of the first $i - 1$ vertices of γ are left unchanged.
- The labels of the vertices of γ' are shifted up by $i - 1$.
- The last $M - i$ vertices of γ are shifted up by $N - 1$.
- All edges originating from γ are declared smaller than all edges originating from γ' .

The partial composition operations \circ_i can be checked to be equivariant with respect to the right-action of $\mathbb{S}_k \times \mathbb{S}_2^{\times k}$ on $\mathbf{gra}_{N,k}$ allowing to define partial composition operations $\circ_i : \mathbf{Gra}_d(M) \otimes \mathbf{Gra}_d(N) \rightarrow \mathbf{Gra}_d(M + N - 1)$.

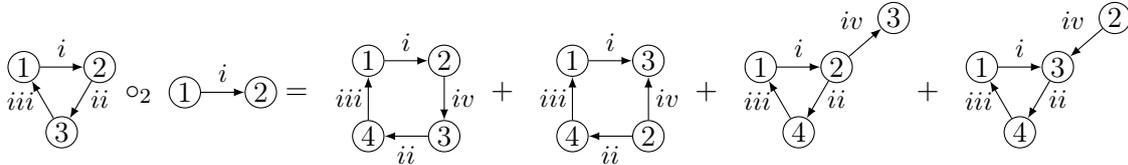


Figure 2: Example of partial composition $\mathbf{Gra}_d(3) \circ_2 \mathbf{Gra}_d(2) \rightarrow \mathbf{Gra}_d(4)$

The partial composition operations \circ_i on \mathbf{Gra}_d preserve the number of edges and thus have zero intrinsic degree. Further, they can be checked to satisfy the following properties for all $\gamma_m \in \mathbf{Gra}_d(m)$:

- **Sequential composition:**

$$(\gamma_m \circ_j \gamma_n) \circ_i \gamma_p = \gamma_m \circ_j (\gamma_n \circ_{i-j+1} \gamma_p) \text{ for all } j \leq i \leq j + n - 1.$$

²⁵For example, letting $\sigma \in \mathbb{S}_3$ be defined as

$$\sigma := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

the right-action of σ on the graph $\gamma \in \mathbf{Gra}_d(3)$ defined as:

$$\gamma := \textcircled{1} \xrightarrow{i} \textcircled{2} \xrightarrow{ii} \textcircled{3}$$

reads:

$$\Sigma_3(\gamma|\sigma) = \textcircled{2} \xrightarrow{i} \textcircled{3} \xrightarrow{ii} \textcircled{1}.$$

• **Parallel composition:**

$$(\gamma_m \circ_j \gamma_n) \circ_i \gamma_p = (-1)^{|\gamma_m||\gamma_p|} (\gamma_m \circ_{i-n+1} \gamma_p) \circ_j \gamma_n \text{ for all } i \geq j + n.$$

Finally, the partial composition operations are equivariant with respect to the right-action Σ_N of \mathbb{S}_N on $\text{Gra}_d(N)$.

The previous properties ensure that the \mathbb{S} -module $\{\text{Gra}_d(N)\}_{N \geq 1}$ is naturally endowed with a structure of operad [119]:

Proposition 4.1 (Operad Gra_d). *For all $d \in \mathbb{N}$, one can define an operad in the category of graded vector spaces as the quadruplet $(\text{Gra}_d, \Sigma, \circ_i, \text{id})$ where:*

- *The set of graded vector spaces $\{\text{Gra}_d(N)\}_{N \geq 1}$ endowed with the set of natural right-actions $\Sigma_N : \text{Gra}_d(N) \times \mathbb{S}_N \rightarrow \text{Gra}_d(N)$ is a \mathbb{S} -module.*
- *$\circ_i : \text{Gra}_d(M) \otimes \text{Gra}_d(N) \rightarrow \text{Gra}_d(M + N - 1)$ is the set of equivariant partial composition operations defined in eq.(14).*
- *The identity element $\text{id} \in \text{Gra}_d(1)$ is defined as the graph $\text{id} := \textcircled{1}$ of degree 0.*

As usual, representations of the graded operad Gra_d (or Gra_d -algebras) are ordered pairs (V, ρ) where V is a graded vector space and $\rho : \text{Gra}_d \rightarrow \text{End}_V$ is a morphism of operads, with End_V the endomorphism operad on V , see [91] for details.

4.2 Stable structures The notion of universal structures was first introduced in [84] to characterise a subclass of cochains in the Chevalley–Eilenberg algebra of polyvector fields $\text{CE}(\mathcal{T}_{\text{poly}})$. The terminology referred to the fact that such cochains are defined “graphically” via grafting of existing structures on $\mathcal{T}_{\text{poly}}$ without resorting to additional data and thus independently of the dimension of the underlying manifold. Such universal cochains were then argued to constitute natural candidate recipients for the possible obstructions to the existence of a formality morphism. The corresponding class of formality morphisms was then precisely defined in [36] in terms of the operads OC and KGra (cf. definitions therein). Informally, these are Lie_∞ -quasi-isomorphisms whose Taylor coefficients can be written as a sum over Kontsevich admissible graphs [85], independently of the dimension²⁶. More generally, universal structures can be loosely defined as originating from “graph operads”. In the present work, we will focus²⁷ on *stable* structures originating from the Kontsevich’s operads Gra_d and dGra_d of (un)directed graphs. The definition of stable structures adopted in the present work is adapted from [7, Definition 4.4.1.4]:

Definition 4.2 (Stable structure). Let \mathcal{P} be an operad in the category of graded vector spaces and V a graded vector space. A \mathcal{P} -algebra structure on V will be said *stable* if the action of the operad \mathcal{P} on V factors through the Kontsevich’s graph operad Gra_d (or its directed avatar dGra_d) as $\mathcal{P} \rightarrow (\text{d})\text{Gra}_d \rightarrow \text{End}_V$, for some $d \in \mathbb{N}$.

As will be recalled in Section 5.2, an important example of stable structures on symplectic Lie n -algebroids is given by the notion of Ger_d -algebras, namely a triplet $(\mathfrak{g}, \wedge, [\cdot, \cdot])$ such that:

1. (\mathfrak{g}, \wedge) is a Com -algebra.
2. $(\mathfrak{g}[d - 1], [\cdot, \cdot])$ is a Lie-algebra.

²⁶Two such morphisms thus only differ by their weight function, the latter depending on the choice of a Drinfel’d associator.

²⁷We refer to [114] and [99] for examples of universal structures induced from (multi)-oriented graphs on infinite-dimensional polyvector fields and (quasi)-Lie bialgebroids, respectively.

3. The bracket $[\cdot, \cdot]$ is a bi-derivation with respect to the product \wedge .²⁸

We will denote \mathbf{Ger}_d the operad whose associated representations are \mathbf{Ger}_d -algebras. The notion of \mathbf{Ger}_1 -algebra identifies with the one of Poisson algebra for which both binary operations are of zero degree on \mathfrak{g} . The case $d = 2$ was first introduced by M. Gerstenhaber in [51] in order to characterise the natural structure living on the Hochschild cohomology of an associative algebra. For this reason, \mathbf{Ger}_2 -algebras are usually referred to as **Gerstenhaber algebras**. Note that the definition of \mathbf{Ger}_d -algebras coincides with the one of \mathbf{e}_d -algebras [cf. e.g. [91, Section 13.3.16]] for $d \geq 2$ while \mathbf{e}_1 -algebras are conventionally chosen to be associative algebras. The following Proposition asserts that any \mathbf{Gra}_d -algebra (i.e. an algebra over Kontsevich's operad of undirected graphs) is endowed with a stable structure of \mathbf{Ger}_d -algebra.

Proposition 4.3 (T. Willwacher [119]). *For all $d \in \mathbb{N}$, there is a natural embedding of operads $i_d : \mathbf{Ger}_d \hookrightarrow \mathbf{Gra}_d$.*

Explicitly, the embedding of operads i_d is defined by the following action on generators $a_1 \wedge a_2, \{a_1, a_2\} \in \mathbf{Ger}_d(2)$:

- $i_d(a_1 \wedge a_2) = \Gamma_{\bullet\bullet}$ with \wedge the graded commutative associative product of degree 0
- $i_d(\{a_1, a_2\}) = \Gamma_{\bullet\bullet}$ with $\{\cdot, \cdot\}$ the graded Lie bracket of degree $1 - d$

where $\Gamma_{\bullet\bullet}$ and $\Gamma_{\bullet\bullet} \in \mathbf{Gra}_d(2)$ are respectively defined as:

$$\Gamma_{\bullet\bullet} := \textcircled{1} \textcircled{2} \quad , \quad \Gamma_{\bullet\bullet} := \textcircled{1} \xrightarrow{i} \textcircled{2}. \tag{15}$$

In particular, the previous embeddings provide a canonical morphism of operad $\mathbf{Lie}\{1-d\} \rightarrow \mathbf{Gra}_d$ so that any \mathbf{Gra}_d -algebra is naturally endowed with a Lie bracket of degree $1 - d$. Deformation complexes for these canonical morphisms will be shown to provide the definition of graph complexes in the next section.

We conclude by pointing out that in the case $d = 1$, there is a natural embedding of operads $\mathbf{Ass} \hookrightarrow \mathbf{Gra}_1$ mapping the generator $m_2 \in \mathbf{Ass}(2)$ (i.e. the associative binary product) of the associative operad \mathbf{Ass} to the element of $\mathbf{Gra}_1(2)$ being explicitly defined as the infinite sum of graphs [75]:

$$\textcircled{1} \textcircled{2} := \sum_{j \geq 0} \frac{1}{j!} \textcircled{1} \textcircled{2} \tag{16}$$

j edges

As a result, \mathbf{Gra}_1 -algebras are naturally endowed with a stable associative product [see eq.(36) below for an example].

4.3 The full graph complex \mathbf{fGC}_d We now turn to the definition of the full graph complex, denoted \mathbf{fGC}_d hereafter. The differential on \mathbf{fGC}_d stems from a richer structure – namely a pre-Lie structure – defined in terms of the graph operad \mathbf{Gra}_d using one of the following equivalent constructions:

²⁸Note that, in order to explicitly state the third compatibility relation, one needs first to pullback one of the defining maps along the suspension map $s : \mathfrak{g}[d-1] \rightarrow \mathfrak{g}$ of degree $d-1$ so that both products act on the same space. Explicitly, one can define the pushforward $\{\cdot, \cdot\}$ of the graded Lie bracket $[\cdot, \cdot]$ on \mathfrak{g} as $\{\cdot, \cdot\} := s \circ [\cdot, \cdot] \circ (s^{-1} \otimes s^{-1})$ so that $\{a, b\} = (-1)^{(d-1)a} s \circ [s^{-1}(a), s^{-1}(b)]$ for all $a, b \in \mathfrak{g}$. The pushforward bracket $\{\cdot, \cdot\}$ is of degree $1 - d$ and satisfies the following properties:

- graded-(skew)symmetric i.e. $\{a, b\} = (-1)^d (-1)^{ab} \{b, a\}$
- graded-Jacobi identity i.e. $\{\{a, b\}, c\} + (-1)^{a(b+c)} \{\{b, c\}, a\} + (-1)^{c(a+b)} \{\{c, a\}, b\} = 0$.

The graded Poisson identity on \mathfrak{g} thus reads $\{a, b \wedge c\} = \{a, b\} \wedge c + (-1)^{b(a+1-d)} b \wedge \{a, c\}$.

1. The pre-Lie algebra associated with the suspended operad $\mathbf{Gra}_d\{d\}$.
2. The convolution pre-Lie algebra $\mathbf{Hom}_{\mathbb{S}}(\mathbf{coCom}, \mathbf{Gra}_d\{d\})$.
3. The deformation complex of the trivial operad morphism $0 : \mathbf{Lie}\{1 - d\} \rightarrow \mathbf{Gra}_d$.

We pass on the explicit unfolding of these definitions²⁹ and merely present the final result:

Proposition 4.4 (Pre-Lie structure on \mathbf{fGC}_d). *For all $d \in \mathbb{N}$, the couple (\mathbf{fGC}_d, \circ) where:*

- *The graded vector space \mathbf{fGC}_d is defined as³⁰:*

$$\begin{aligned}
 - d \text{ even: } \mathbf{fGC}_d &:= \prod_{N \geq 1} (\mathbf{Gra}_d(N)[d(1 - N)])^{\mathbb{S}_N} \\
 - d \text{ odd: } \mathbf{fGC}_d &:= \prod_{N \geq 1} (\mathbf{Gra}_d(N) \otimes \text{sgn}_N[d(1 - N)])^{\mathbb{S}_N}
 \end{aligned}$$

where the superscript stands for taking invariants with respect to the right action of \mathbb{S}_N with sgn_N the 1-dimensional signature representation of \mathbb{S}_N . The terms between brackets denote degree suspension³¹.

- *The binary operation $\circ : \mathbf{fGC}_d \otimes \mathbf{fGC}_d \rightarrow \mathbf{fGC}_d$ is of degree 0 and defined via the formula*

$$\gamma \circ \gamma' = \sum_{\sigma \in \text{Sh}^{-1}(N', N-1)} (-1)^{d|\sigma|} \Sigma_{N+N'-1}(\gamma \circ_1 \gamma' | \sigma)$$

where $\Sigma_N : \mathbf{Gra}_d(N) \times \mathbb{S}_N \rightarrow \mathbf{Gra}_d(N)$ denotes the right action defined previously while N, N' stand for the number of vertices in the homogeneous graphs γ, γ' , respectively. The sum is performed over the unshuffles of type $(N', N - 1)$ and $|\sigma|$ denotes the signature of the permutation $\sigma \in \mathbb{S}_{N+N'-1}$.

is a graded pre-Lie algebra i.e. for all $\gamma_m \in \mathbf{fGC}_d$, the following relation holds:

$$(\gamma_1 \circ \gamma_2) \circ \gamma_3 - \gamma_1 \circ (\gamma_2 \circ \gamma_3) = (-1)^{|\gamma_2||\gamma_3|} ((\gamma_1 \circ \gamma_3) \circ \gamma_2 - \gamma_1 \circ (\gamma_3 \circ \gamma_2)).$$

Proposition 4.4 can be reformulated as the existence of a morphism of operads $\mathbf{preLie} \rightarrow \mathbf{End}_{\mathbf{fGC}_d}$. Composing with the morphism of operads $\mathbf{Lie} \rightarrow \mathbf{preLie}$ allows to endow \mathbf{fGC}_d with a structure of graded Lie algebra through the commutator (graded) Lie bracket $[\cdot, \cdot]$ defined as:

$$[\gamma_1, \gamma_2] = \gamma_1 \circ \gamma_2 - (-1)^{|\gamma_1||\gamma_2|} \gamma_2 \circ \gamma_1.$$

For all $d \in \mathbb{N}$, it can be checked that the element $\Gamma_{\bullet\bullet} \in \mathbf{fGC}_d^1$ [cf. (15)] is a Maurer–Cartan element for the graded Lie algebra $(\mathbf{fGC}_d, [\cdot, \cdot])$ i.e. $[\Gamma_{\bullet\bullet}, \Gamma_{\bullet\bullet}] = 0$. This property allows to define the differential operator $\delta := [\Gamma_{\bullet\bullet}, \cdot]$ acting through the adjoint action associated with the Maurer–Cartan element. The latter can be shown to square to zero³² as well as to be a derivation of the graded Lie bracket.

We sum up the previous discussion by the following proposition:

Proposition 4.5. *The triplet $(\mathbf{fGC}_d, \delta, [\cdot, \cdot])$ is a dg Lie algebra³³.*

Forgetting the Lie bracket, we refer to the couple (\mathbf{fGC}_d, δ) as the **full graph complex**.

²⁹We refer to [91, 97] for generic constructions and to [119, 39] for applications to the case at hand.

³⁰The sign conventions used relatively to the action of the various symmetry groups are summed up in Table 1.

³¹According to the suspension, the degree of an element $\gamma \in \mathbf{fGC}_d$ with N vertices and k edges is given by $|\gamma| = d(N - 1) + k(1 - d)$.

³²In retrospect, it can be checked that the choices made in Table 1 are the only ones ensuring that $\delta^2 \equiv 0$ [124].

³³Note that the dg Lie algebra $(\mathbf{fGC}_d, \delta, [\cdot, \cdot])$ can be defined from the onset as the deformation complex of the (non-trivial) operad morphism $\mathbf{Lie}\{1 - d\} \rightarrow \mathbf{Gra}_d$ defined in Section 4.1, cf. [119].

	$\mathbb{S}_2^{\times k}$	\mathbb{S}_k	\mathbb{S}_N
d even	+	-	+
d odd	-	+	-

Table 1: Symmetries of graphs in fGC_d

We conclude by displaying³⁴ distinguished examples of graphs in fGC_d :

$$\bullet \bullet \text{---} \bullet = \frac{1}{3} \left(\textcircled{1} \quad \textcircled{2} \xrightarrow{i} \textcircled{3} + \textcircled{2} \quad \textcircled{3} \xrightarrow{i} \textcircled{1} + \textcircled{3} \quad \textcircled{1} \xrightarrow{i} \textcircled{2} \right)$$

Figure 3: Example of graph in fGC_d

Example 4.6.

- The graph $\bullet \text{---} \bullet$ is a cocycle in the even and odd graph complexes.
- The tadpole graph $\textcircled{\bullet}$ is a cocycle in the even graph complex and a zero graph in the odd graph complex.
- The multi-arrows graph $\bullet \text{---} \bullet$ – sometimes referred to as the “ Θ -graph” – is a cocycle in the odd graph complex and a zero graph in the even graph complex.
- The Θ -graph cocycle can be promoted to a Maurer–Cartan element³⁵ in $(\text{fGC}_1, \delta, [\cdot, \cdot])$ as the sum of multi-arranged graphs [75]:

$$\textcircled{\bullet} \text{---} \textcircled{\bullet} := \sum_{k \geq 1} \frac{1}{(2k+1)!} \textcircled{\bullet} \text{---} \textcircled{\bullet} \quad (2k+1 \text{ edges}) \quad (17)$$

We conclude this section by introducing the **concatenation** of two graphs into a single (disconnected) graph, and denoted $\cup : \text{gra}_{M,j} \otimes \text{gra}_{N,k} \rightarrow \text{gra}_{M+N,j+k}$, as in the following example:

$$\begin{array}{c} \textcircled{1} \xrightarrow{i} \textcircled{2} \\ \textcircled{3} \xrightarrow{iii} \textcircled{1} \end{array} \cup \textcircled{1} \xrightarrow{i} \textcircled{2} = \begin{array}{c} \textcircled{1} \xrightarrow{i} \textcircled{2} \\ \textcircled{3} \xrightarrow{iii} \textcircled{1} \end{array} \quad \textcircled{4} \xrightarrow{iv} \textcircled{5} \quad (18)$$

³⁴As is customary, we will represent a given element of fGC_d as a linear combination of undirected graphs with black vertices since taking invariants with respect to \mathbb{S}_N makes the vertices undistinguishable. In order to obtain an explicit element of fGC_d from such a graph, one needs to go through the following steps (cf. Figure 3 for an example):

1. Choose an ordering of the edges.
2. Choose an orientation of the edges.
3. Sum over all possible ways of assigning labels to the vertices.
4. Divide by the order of the symmetry of the given graph.

Note that the overall sign is left ambiguous.

³⁵The obstruction to the prolongation of the Θ -graph to a full Maurer–Cartan element lies in $H^2(\text{fGC}_1^{\text{con}}) \simeq \mathbb{K} \langle L_3 \rangle$, cf. Section 4.4. Since the obstruction to the prolongation of the Θ -graph at order $k \geq 2$ has Betti number $k + 2$, it never hits the loop graph L_3 of Betti number 1. The prolongation of the Θ -graph to a Maurer–Cartan element in fGC_1 is thus unobstructed at all orders.

The concatenation product can be extended by linearity to yield a product $\cup : \text{Gra}_d(M) \otimes \text{Gra}_d(N) \rightarrow \text{Gra}_d(M + N)$, which can furthermore be checked to be associative and to satisfy the “commutation” relation:

$$\gamma \cup \gamma' = (-1)^{kk'(1-d)} \Sigma_{M+N}(\gamma' \cup \gamma | \sigma)$$

with $\gamma \in \text{gra}_{M,k}$, $\gamma' \in \text{gra}_{N,k'}$ and where the permutation $\sigma \in \mathbb{S}_{M+N}$ is defined as

$$\sigma := \begin{pmatrix} 1 & \cdots & M + N \\ N + 1 & \cdots & 1 \end{pmatrix}$$

so that:

$$\sigma^{-1} = \begin{pmatrix} 1 & \cdots & M + N \\ M + 1 & \cdots & 1 \end{pmatrix}.$$

We will denote with the same symbol \cup the corresponding concatenation operation of two graphs in fGC_d . The latter can be shown to be:

1. of degree d
2. graded commutative *i.e.* $\gamma \cup \gamma' = (-1)^{(\gamma+d)(\gamma'+d)} \gamma' \cup \gamma$
3. associative *i.e.* $(\gamma \cup \gamma') \cup \gamma'' = \gamma \cup (\gamma' \cup \gamma'')$.

Furthermore, the differential δ satisfies the Leibniz rule:

$$\delta(\gamma \cup \gamma') = \delta\gamma \cup \gamma' + (-1)^{|\gamma|+d} \gamma \cup \delta\gamma'. \tag{19}$$

Proposition 4.7. *The concatenation product endows $H^{-d}(\text{fGC}_d)$ with a structure of commutative algebra.*

Proof. Let γ, γ' be two non-trivial cocycles of degree $-d$ in fGC_d . The concatenation product being of degree d , the concatenation $\gamma \cup \gamma'$ is of degree $-d$. Furthermore, the Leibniz rule (19) ensures that $\gamma \cup \gamma'$ is a (necessarily non-trivial) cocycle hence $\gamma \cup \gamma' \in H^{-d}(\text{fGC}_d)$. Lastly, note that the concatenation product is associative and that its graded commutative property ensures that $\gamma \cup \gamma' = \gamma' \cup \gamma$ when restricted to graphs of degree $-d$. □

4.4 Cohomology of the full graph complex We now collect some known results regarding the cohomology of the full graph complex fGC_d . In the following, we will let $\text{fGC}_d^{\text{con}}$ denote the sub-dg Lie algebra of fGC_d spanned by *connected* graphs. Furthermore, we define GC_d as the subcomplex of $\text{fGC}_d^{\text{con}}$ spanned by graphs without tadpoles for which all vertices have valence at least 3. The latter subcomplex was introduced [in the case $d = 2$] in [84] and is sometimes referred to as the **Kontsevich graph complex**. As noted in [119], the full graph complex can be described in terms of its connected component³⁶ as $\text{fGC}_d = \widehat{S}(\text{fGC}_d^{\text{con}}[-d])[d]$.³⁷ In other words, computing the cohomology of fGC_d reduces to computing the cohomology of its connected component $\text{fGC}_d^{\text{con}}$. The latter admits the following decomposition:

³⁶For any graded vector space V , we will let $\widehat{S}(V)$ denote the (completed) symmetric product space of the graded vector space V defined as $\widehat{S}(V) := \prod_{j \geq 1} (V^{\otimes j})^{\mathbb{S}_j}$.

³⁷The degree shift by d reflects the degree of the concatenation product (18).

Theorem 4.8 (Kontsevich [83, 82], Willwacher [119]). *The connected part of the full graph complex satisfies:*

$$H^\bullet(\text{fGC}_d^{\text{con}}) = H^\bullet(\text{GC}_d) \oplus \bigoplus_{\substack{k=2d+1 \pmod 4 \\ k \geq 1}} \mathbb{K}[d-k]$$

where the class corresponding to $\mathbb{K}[d-k]$ is represented by a loop L_k with k edges, cf. Figure 4.

For symmetry reasons, the only non-zero loop classes are represented by:

- d even Loops L_k with $k = 4j + 1$ edges, $j \geq 0$
- d odd Loops L_k with $k = 4j + 3$ edges, $j \geq 0$.

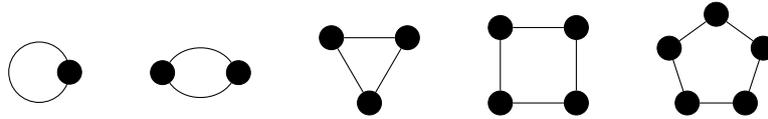


Figure 4: Loop graphs L_k for $k \in \{1, \dots, 5\}$

It follows from Proposition 4.8 that the cohomology of $\text{fGC}_d^{\text{con}}$ is located in GC_d , up to some known (loop) classes. We now focus on the cohomology of GC_d , for $d = 2, 3$ (see e.g. [75, 49] for a summary and [11, 75] for computer generated tables).

Cohomology of GC_2 . One of the major results of [119] is the following theorem:

Theorem 4.9 (Willwacher [119]). *The cohomology of the Kontsevich graph complex GC_2 satisfies:*

1. **Lower bound:** $H^{\leq -1}(\text{GC}_2) = \mathbf{0}$
2. **Dominant degree:** $H^0(\text{GC}_2) \simeq \mathfrak{grt}_1$ as Lie algebras where \mathfrak{grt}_1 stands for the Grothendieck–Teichmüller Lie algebra.
3. **Upper bound:** For graphs of Betti number³⁸ b , the cohomology $H^\bullet(\text{GC}_2)$ vanishes in degrees $\geq b - 2$.

Combining Theorems 4.8 and 4.9 leads to a complete characterisation of the connected part of the full graph complex for $d = 2$ in low degrees:

1. $H^{< -1}(\text{fGC}_2^{\text{con}}) = \mathbf{0}$
2. $H^{-1}(\text{fGC}_2^{\text{con}}) = \mathbb{K}\langle L_1 \rangle$
3. $H^0(\text{fGC}_2^{\text{con}}) = H^0(\text{GC}_2) \simeq \mathfrak{grt}_1$ as Lie algebras.

Explicit representatives of classes in the dominant degree $H^0(\text{GC}_2)$ can be constructed:

Theorem 4.10 (Willwacher [119]). *For every integer $j \geq 1$, there exists a non-trivial cocycle $\gamma_{2j+1} \in H^0(\text{GC}_2)$ admitting a non-zero coefficient in front of the wheel with $2j + 1$ spokes, cf. Figure 5.*

³⁸A statement of the third item can be found in [75]. Note that the first Betti number endows the dg Lie algebra fGC_d with an additional grading. It is defined explicitly as $b = k - N + c$ where k denotes the number of edges, N the number of vertices and c the number of connected components. Relatively to the bigrading given by both $|\gamma|$ and b , the graded Lie bracket is of bidegree $0|0$ while the differential is of bidegree $1|1$.

The Grothendieck–Teichmüller Lie algebra \mathfrak{grt}_1 is known to contain a series of non-trivial elements $\sigma_3, \sigma_5, \dots$ indexed by an odd integer³⁹. In fact, the Drinfel’d–Deligne–Ihara conjecture states that there is an isomorphism of Lie algebras between \mathfrak{grt}_1 and the (degree completion of) the free Lie algebra generated by the odd elements $\{\sigma_{2j+1}\}_{j \geq 1}$. Part of the conjecture has been proved by F. Brown in [18] who showed that these elements generate a free Lie subalgebra of \mathfrak{grt}_1 . In order to fully prove the conjecture, it remains to be shown that this free Lie subalgebra identifies with \mathfrak{grt}_1 .

In [119], T. Willwacher provides an explicit isomorphism of Lie algebras $H^0(\mathbf{GC}_2) \simeq \mathfrak{grt}_1$ under which the series of odd elements $\sigma_{2j+1} \in \mathfrak{grt}_1$ gets mapped to the series of graphs γ_{2j+1} in $H^0(\mathbf{GC}_2)$. An explicit transcendental formula for the cocycles γ_{2j+1} is given in [103] as a sum over $\mathbf{gra}_{2j+2, 4j+2}$ where the coefficients are provided by explicit converging integrals over the configuration space of $2n$ points in $\mathbb{C} \setminus \{0, 1\}$. However, a purely combinatorial construction of the γ_{2j+1} ’s is still missing.

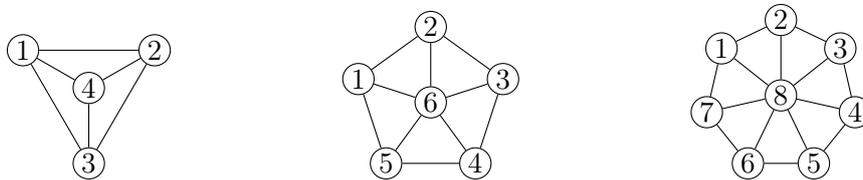


Figure 5: Wheel graphs for $j \in \{1, 2, 3\}$

Regarding higher degrees, computer experiments have exhibited sporadic classes in $H^{\geq 3}(\mathbf{GC}_2)$ while it remains a difficult open conjecture (Drinfel’d, Kontsevich) that $H^1(\mathbf{GC}_2) = \mathbf{0}$.

Cohomology of \mathbf{GC}_3 . The cohomology of the odd graph complex can be characterised in low degrees in a way similar to the even case (see *e.g.* [10, 75]):

1. **Upper bound:** $H^{\geq -2}(\mathbf{GC}_3) = \mathbf{0}$
2. **Dominant degree:** The dominant level of the odd graph complex \mathbf{GC}_3 is located in degree -3 . The corresponding cohomology space $H^{-3}(\mathbf{GC}_3)$ can be shown to be spanned by trivalent graphs (*cf.* Figure 6 for examples⁴⁰) modulo the so-called *IHX relation* reading (see *e.g.* [10]):

$$\text{I} = \text{H} + \text{X} . \tag{20}$$

The cohomology space $H^{-3}(\mathbf{GC}_3)$ is furthermore endowed with a structure of unital **Com-algebra**⁴¹ where the rôle of the unit is played by the Θ -graph \ominus . In fact, there is a

³⁹The odd elements $\{\sigma_{2j+1}\}_{j \geq 1}$ are the homogeneous components of odd degrees of the element $\psi \in \mathfrak{grt}_1$ defined such that $g = \exp(\psi)$ is the unique element of \mathbf{GRT}_1 sending the Knizhnik–Zamolodchikov associator $\Phi_{\mathbb{K}\mathbb{Z}}$ to the anti-Knizhnik–Zamolodchikov associator $\Phi_{\overline{\mathbb{K}\mathbb{Z}}}$, *cf.* *e.g.* [103].

⁴⁰ Trivalent graphs in the odd graph complex are usually depicted as *chord diagrams* where each intersection of three lines stands for a vertex. Note that modding by the IHX relation ensures that the trivalent graphs in Figure 6 satisfy the equivalence relations $A \sim 2B$ and $C \sim 4D \sim E \sim 2F$. The tetrahedron graph B is sometimes denoted t in the literature.

⁴¹The commutative product is defined as follows. Let γ, γ' be two trivalent graphs. Remove one (arbitrarily chosen) vertex of γ so that the resulting graph has now three dangling edges. Repeat the previous operation for γ' , then pick one dangling edge of the graph obtained from γ and connect it to one (arbitrarily chosen) dangling

morphism of commutative algebras:

$$\mathbb{K}\langle t, \omega_0, \omega_1, \dots \rangle / (\omega_p \omega_q - \omega_0 \omega_{p+q}, P) \rightarrow H^{-3}(\text{GC}_3) \tag{21}$$

for a certain (explicitly known) polynomial P , cf. [79, 80, 81, 117]. The map (21) is conjectured to be an isomorphism, up to a 1-dimensional class represented by the Θ -graph.

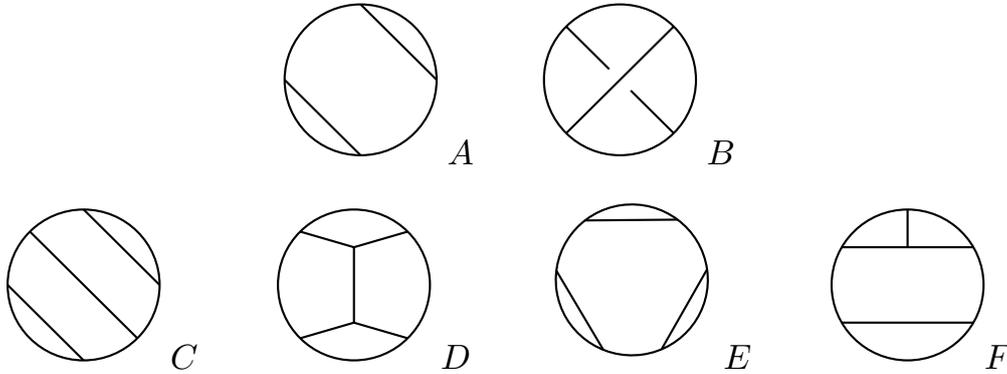


Figure 6: Non-trivial connected trivalent graphs in GC_3 for $N = 4$ (A, B) and $N = 6$ (C, D, E, F).

3. **Lower bound:** For graphs of Betti number b , the cohomology $H^\bullet(\text{GC}_3)$ vanishes in degrees $\leq -b - 2$.

Regarding lower degrees, computer experiments have shown that there exist sporadic classes in $H^{-6}(\text{GC}_3)$.

4.5 **The directed graph complex** We conclude this review of graph complexes by presenting an important variant of the full graph complex known as the **full directed graph complex** dfGC_d . Following similar steps as for fGC_d , we start by defining, for all $N \geq 1$, the graded vector space $\text{dGra}_d(N)$ as:

- d even: $\text{dGra}_d(N) := \prod_{k \geq 0} (\mathbb{K}\langle \text{gra}_{N,k} \rangle \otimes_{\mathbb{S}_k} \text{sgn}_k)[k(d-1)]$
- d odd: $\text{dGra}_d(N) := \prod_{k \geq 0} (\mathbb{K}\langle \text{gra}_{N,k} \rangle_{\mathbb{S}_k})[k(d-1)]$

where the subscript stands for taking coinvariants with respect to the diagonal right action of \mathbb{S}_k and the term between brackets denotes degree suspension.

In other words, the definition of $\text{dGra}_d(N)$ differs from the one of $\text{Gra}_d(N)$ by relaxing the modding out by $\mathbb{S}_2^{\otimes k}$. As a result, we deal with *directed graphs* i.e. whose edge orientation is fixed. Similarly to the undirected case, the set of graded vector spaces $\{\text{dGra}_d(N)\}_{N \geq 1}$ assemble to an operad dGra_d . There is an injective morphism of operads

$$\text{O}\vec{r} : \text{Gra}_d \hookrightarrow \text{dGra}_d \tag{22}$$

called the **orientation morphism** and defined by sending each *undirected* graph into a sum of *directed* graphs obtained by interpreting each undirected edge as a sum of directed edges in both directions, cf. Figure 7.

edge of the graph obtained from γ' . Repeat the operation by connecting the two remaining dangling edges of the graph obtained from γ to the ones of γ' . The outcome is a single trivalent graph. Modding out by the IHX relation ensures that the procedure is independent of both the choice of removed vertices and pairing of dangling edges and that the resulting product is commutative. For example, one can check that $A \cdot B = F$, cf. Figure 6.

Similarly to its undirected counterpart, the deformation complex dfGC_d of the trivial operad morphism $0 : \text{Lie}\{1 - d\} \rightarrow \text{dGra}_d$ is endowed with a pre-Lie structure [cf. Proposition 4.4]. The latter can in turn be extended to a dg Lie algebra structure where the differential is induced by the Maurer–Cartan element:

$$\bullet \longrightarrow \bullet := \textcircled{1} \longrightarrow \textcircled{2} + (-1)^d \textcircled{2} \longrightarrow \textcircled{1} .$$

We will pursue with the previously introduced notation and denote $\text{dfGC}_d^{\text{con}}$ the sub-dg Lie algebra spanned by connected graphs. The morphism of operads (22) induces a morphism of dg Lie algebras:

$$s^* \text{Or} : \text{fGC}_d^{\text{con}} \hookrightarrow \text{dfGC}_d^{\text{con}} . \tag{23}$$

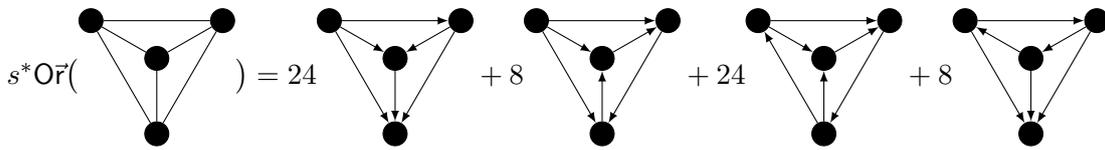


Figure 7: Orientation morphism

The following result was shown by T. Willwacher in [119], cf. also [40].

Theorem 4.11. *The morphism $s^* \text{Or} : \text{fGC}_d^{\text{con}} \hookrightarrow \text{dfGC}_d^{\text{con}}$ is a quasi-isomorphism of dg Lie algebras.*

Theorem 4.11 implies that the study of the cohomology of the directed graph complex boils down to the one of the full graph complex, so that essentially nothing new appears when going from undirected to directed graphs. However, the directed graph complex constitutes a useful intermediary when considering representations of the Kontsevich’s graph complex, cf. [19]. Furthermore, the directed graph complex possesses two interesting subcomplexes spanned by *oriented* and *sourced* graphs, respectively, which have recently been shown to provide some incarnations of the Grothendieck–Teichmüller Lie algebra \mathfrak{grt}_1 in higher dimensions⁴², see [122, 126, 127, 95, 96] for details and [99] for an application to representations of \mathfrak{grt}_1 on quasi-Lie bialgebroids.

5. Stable structures on graded manifolds

In the formulation of his “*Formality conjecture*” [84], M. Kontsevich introduced a stable version of the deformation complex of the Schouten algebra of polyvector fields, in the guise of an injective morphism $\text{fGC}_2 \hookrightarrow \text{CE}(\mathcal{T}_{\text{poly}})$. As shown in [119, 70], this morphism of dg Lie algebras can be best understood as originating from a morphism of operads $\text{Gra}_2 \hookrightarrow \text{End}_{\mathcal{C}^\infty(\mathcal{V})}$ where $\mathcal{V} := T^*[1]\mathcal{M}$ is a NP-manifold of degree 1 whose associated graded Poisson algebra of functions is isomorphic to $\mathcal{T}_{\text{poly}}$ endowed with the Schouten bracket (up to suspension). The aim of the present section is to generalise Kontsevich’s construction from $d = 2$ to arbitrary $d > 0$.

In other words, we will introduce a tower of representations $\text{Gra}_d \hookrightarrow \text{End}_{\mathcal{C}^\infty(\mathcal{V})}$ with \mathcal{V} an arbitrary NP-manifold of degree n , such that $d = n + 1$. This tower of morphism of operads will

⁴²That is, for values of $d > 2$.

in turn induce a tower of injective morphisms of dg Lie algebras $\mathbf{fGC}_d \hookrightarrow \mathbf{CE}(\mathcal{T}_{\text{poly}}^{(n)})$. Cochains in the image of this map will be called *stable* and results from the cohomology of \mathbf{fGC}_d (as recalled in Section 4.4) will allow to classify stable structures on NP-manifolds.

5.1 Representations of (un)directed graphs on NP-manifolds As for notation, we will let (\mathcal{V}, ω) be a NP-manifold of arbitrary degree $n \in \mathbb{N}$ with $d = n + 1$ and denote $\{\cdot, \cdot\}_\omega$ the associated Poisson bracket of degree $-n$. We will make use of the local presentation of NP-manifolds provided in Section 3.2. By analogy with the $n = 1$ case, we will denote $\mathcal{T}_{\text{poly}}^{(n)} := \mathcal{C}^\infty(\mathcal{V})[n]$ the n -suspension of the graded algebra of functions on \mathcal{V} along the suspension map $s : \mathcal{T}_{\text{poly}}^{(n)} \rightarrow \mathcal{C}^\infty(\mathcal{V})$ of degree $|s| = n$. We will also denote $\mathbf{CE}(\mathcal{T}_{\text{poly}}^{(n)})$ the cohomological Chevalley–Eilenberg bigraded vector space⁴³ associated with the graded vector space $\mathcal{T}_{\text{poly}}^{(n)}$ and $[\cdot, \cdot]_{\mathcal{S}}$ the pullback of the Poisson bracket $\{\cdot, \cdot\}_\omega$ by the suspension map $s : \mathcal{T}_{\text{poly}}^{(n)} \rightarrow \mathcal{C}^\infty(\mathcal{V})$ *i.e.* $[\cdot, \cdot]_{\mathcal{S}} = s^{-1} \circ \{\cdot, \cdot\}_\omega \circ (s \otimes s)$. It can be checked that $[\cdot, \cdot]_{\mathcal{S}}$ is a graded Lie bracket of degree 0, thus endowing $\mathcal{T}_{\text{poly}}^{(n)}$ with a (stable) structure of graded Lie algebra. Pursuing with the previous analogy, we will refer to $(\mathcal{T}_{\text{poly}}^{(n)}, [\cdot, \cdot]_{\mathcal{S}})$ as the n -**Schouten** algebra. This graded Lie structure on $\mathcal{T}_{\text{poly}}^{(n)}$ allows to endow $\mathbf{CE}(\mathcal{T}_{\text{poly}}^{(n)})$ with a structure of complex via the Chevalley–Eilenberg differential $\delta_{\mathcal{S}} := [[\cdot, \cdot]_{\mathcal{S}}, \cdot]_{\text{NR}}$ – where $[\cdot, \cdot]_{\text{NR}}$ is the Nijenhuis–Richardson bracket (24) – associated with the Schouten bracket.

Proposition 5.1. *The graded algebra of functions on \mathcal{V} is endowed with a structure of a \mathbf{dGra}_d -algebra.*

The corresponding morphism of operads of graded vector spaces will be denoted $\mathbf{dRep}^{(d)} : \mathbf{dGra}_d \hookrightarrow \mathbf{End}_{\mathcal{C}^\infty(\mathcal{V})}$ and defined explicitly as the sequence $\left\{ \mathbf{dRep}_N^{(d)} \right\}_{N \geq 1}$ of maps $\mathbf{dRep}_N^{(d)} : \mathbf{dGra}_d(N) \otimes \mathcal{C}^\infty(\mathcal{V})^{\otimes N} \rightarrow \mathcal{C}^\infty(\mathcal{V})$ reading, for all $\gamma \in \mathbf{dGra}_d(N)$:

$$\mathbf{dRep}_N^{(d)}(\gamma)(f_1 \otimes \cdots \otimes f_N) = \mu_N \left(\prod_{(i,j) \in E_\gamma} \bar{\Delta}_{ij}(f_1 \otimes \cdots \otimes f_N) \right) \tag{25}$$

⁴³Letting \mathfrak{g} be a graded vector space, the associated cohomological Chevalley–Eilenberg (bi)graded vector space (in the adjoint representation) is defined as

$$\mathbf{CE}(\mathfrak{g}) := \bigoplus_{n \in \mathbb{Z}} \mathbf{CE}^n(\mathfrak{g}) \text{ where } \mathbf{CE}^n(\mathfrak{g}) := \bigoplus_{i+j=n} \text{Hom}^i(\mathfrak{g}^{\wedge j+1}, \mathfrak{g}).$$

The latter is endowed with a bigrading: the first degree, denoted i , stems from the intrinsic degree of \mathfrak{g} while the second (shifted) degree, denoted j , stems from the number of inputs. The total degree is denoted $n = i + j$. The bigraded vector space $\mathbf{CE}(\mathfrak{g})$ carries a pre-Lie algebra structure through the **Nijenhuis–Richardson product** defined as

$$f \circ_{\text{NR}} g = \sum_{\sigma \in \text{Sh}^{-1}(q, p-1)} (-1)^{|\sigma|} (-1)^{(p-1)l} \Sigma_{p+q-1}(f \circ_1 g | \sigma)$$

for all bihomogeneous functions $f \in \text{Hom}^k(\mathfrak{g}^{\wedge p}, \mathfrak{g})$ and $g \in \text{Hom}^l(\mathfrak{g}^{\wedge q}, \mathfrak{g})$.

The commutator

$$[f, g]_{\text{NR}} := f \circ_{\text{NR}} g - (-1)^{kl} g \circ_{\text{NR}} f \tag{24}$$

is a graded Lie bracket referred to as the **Nijenhuis–Richardson bracket**. Maurer–Cartan elements thereof (*i.e.* elements $\mathfrak{m} \in \mathbf{CE}^1(\mathfrak{g})$ satisfying $[\mathfrak{m}, \mathfrak{m}]_{\text{NR}} = 0$) are in one-to-one correspondence with Lie_∞ -algebra structures on \mathfrak{g} . Given a Lie_∞ -algebra \mathfrak{m} , we can define a differential $\delta_{\mathfrak{m}}$ on $\mathbf{CE}(\mathfrak{g})$ as $\delta_{\mathfrak{m}} := [\mathfrak{m}, \cdot]_{\text{NR}}$ (which is a derivation of $[\cdot, \cdot]_{\text{NR}}$ as a consequence of the graded Jacobi identity and squares to zero as a consequence of the graded Jacobi identity and $[\mathfrak{m}, \mathfrak{m}]_{\text{NR}} = 0$) so that $(\mathbf{CE}(\mathfrak{g}), \delta_{\mathfrak{m}}, [\cdot, \cdot]_{\text{NR}})$ is a dg Lie algebra.

where:

- The f_i 's are functions on \mathcal{V} .
- The symbol μ_N denotes the multiplication map on N elements:

$$\begin{aligned} \mu_N &: \mathcal{C}^\infty(\mathcal{V})^{\otimes N} \rightarrow \mathcal{C}^\infty(\mathcal{V}) \\ &: f_1 \otimes f_2 \otimes \cdots \otimes f_N \mapsto f_1 \cdot f_2 \cdots f_N \end{aligned} \tag{26}$$

- The product is performed over the set of edges E_γ . For each edge $(i, j) \in E_\gamma$ connecting vertices labeled by integers i and j , the derivative operator $\bar{\Delta}_{ij}$ is defined as:

$$- d \text{ even: } \bar{\Delta}_{ij} = \frac{\partial}{\partial x_{(i)}^\mu} \frac{\partial}{\partial p_{\mu}^{(j)}} + \frac{\partial}{\partial \psi_{(i)}^{\alpha_k}} \frac{\partial}{\partial \chi_{\alpha_k}^{(j)}} \tag{27}$$

$$- d \text{ odd: } \bar{\Delta}_{ij} = \frac{\partial}{\partial x_{(i)}^\mu} \frac{\partial}{\partial p_{\mu}^{(j)}} + \frac{\partial}{\partial \psi_{(i)}^{\alpha_k}} \frac{\partial}{\partial \chi_{\alpha_k}^{(j)}} + \frac{1}{2} \frac{\partial}{\partial \xi_{(i)}^a} \kappa^{ab} \frac{\partial}{\partial \xi_{(j)}^b} \tag{28}$$

where the sub(super)scripts (i) or (j) indicate that the derivative acts on the i -th or j -th factor in the tensor product.

Proof. The maps $\mathbf{dRep}_N^{(d)}$ can be checked to satisfy the three following properties:

1. $\mathbf{dRep}_1^{(d)}(\textcircled{1}) = \text{id}_{\mathcal{C}^\infty(\mathcal{V})}$
2. $\mathbf{dRep}_{M+N-1}^{(d)}(\gamma \circ_i^{\mathbf{dGra}} \gamma') = \mathbf{dRep}_M^{(d)}(\gamma) \circ_i^{\mathbf{End}} \mathbf{dRep}_N^{(d)}(\gamma')$ for all $\gamma \in \mathbf{dGra}_d(M)$ and $\gamma' \in \mathbf{dGra}_d(N)$ where the partial composition maps of the endomorphism operad $\mathbf{End}_{\mathcal{C}^\infty(\mathcal{V})}$ take the form:

$$\theta \circ_i^{\mathbf{End}} \theta' = \theta \circ (1^{\otimes i-1} \otimes \theta' \otimes 1^{\otimes M-i})$$

for all $\theta \in \text{Hom}(\mathcal{C}^\infty(\mathcal{V})^{\otimes M}, \mathcal{C}^\infty(\mathcal{V}))$ and $\theta' \in \text{Hom}(\mathcal{C}^\infty(\mathcal{V})^{\otimes N}, \mathcal{C}^\infty(\mathcal{V}))$.

3. $\mathbf{dRep}_N^{(d)}(\Sigma_N^{\mathbf{dGra}}(\gamma|\sigma)) = \Sigma_N^{\mathbf{End}}(\mathbf{dRep}_N^{(d)}(\gamma)|\sigma)$ where the endomorphism operad right action reads:

$$\Sigma_N^{\mathbf{End}}(\theta|\sigma)(f_1, \dots, f_N) := \theta(f_{\sigma_{(1)}^{-1}}, \dots, f_{\sigma_{(N)}^{-1}}) \tag{29}$$

for all $f_i \in \mathcal{C}^\infty(\mathcal{V})$, $\theta \in \text{Hom}(\mathcal{C}^\infty(\mathcal{V})^{\otimes N}, \mathcal{C}^\infty(\mathcal{V}))$ and $\sigma \in \mathbb{S}_N$.

The three above properties ensure that the maps $\left\{ \mathbf{dRep}_N^{(d)} \right\}_{N \geq 1}$ assemble to form a morphism of operads. \square

Example 5.2. Let us exemplify the second item of the previous proof on the following partial composition of graphs for odd d :

$$\textcircled{1} \xrightarrow{i} \textcircled{2} \circ_2 (\textcircled{1} \textcircled{2}) = \textcircled{1} \xrightarrow{i} \textcircled{2} \textcircled{3} + \textcircled{1} \xrightarrow{i} \textcircled{3} \textcircled{2}. \tag{30}$$

Applying the map (25) on both graphs appearing on the right-hand side of (30) yields:

$$\begin{aligned} \mathbf{dRep}_2^{(d)}(\textcircled{1} \xrightarrow{i} \textcircled{2} \textcircled{3})(f_1 \otimes f_2 \otimes f_3) &= \frac{\partial f_1}{\partial x^\mu} \frac{\partial f_2}{\partial p_\mu} f_3 \\ &+ \sum_{i=1}^{\frac{1}{2}n-1} (-1)^{ik} \frac{\partial f_1}{\partial \psi^{\alpha_i}} \frac{\partial f_2}{\partial \chi_{\alpha_i}} f_3 + (-1)^{kn/2} \frac{\partial f_1}{\partial \xi^a} \kappa^{ab} \frac{\partial f_2}{\partial \xi^b} f_3 \end{aligned}$$

$$\begin{aligned} \text{dRep}_2^{(d)}(\textcircled{1} \xrightarrow{i} \textcircled{3}) \quad \textcircled{2})(f_1 \otimes f_2 \otimes f_3) &= \frac{\partial f_1}{\partial x^\mu} f_2 \frac{\partial f_3}{\partial p_\mu} \\ &+ \sum_{i=1}^{\frac{1}{2}n-1} (-1)^{i(k+l)} \frac{\partial f_1}{\partial \psi^{\alpha_i}} f_2 \frac{\partial f_3}{\partial \chi_{\alpha_i}} + (-1)^{n/2(k+l)} \frac{\partial f_1}{\partial \xi^a} f_2 \frac{\partial f_3}{\partial \xi_a} \end{aligned}$$

for all homogeneous functions $f_1 \in \mathcal{C}^{\infty|k}(\mathcal{V})$, $f_2 \in \mathcal{C}^{\infty|l}(\mathcal{V})$ and $f_3 \in \mathcal{C}^{\infty|m}(\mathcal{V})$ of degree k , l and m respectively. By comparison, applying the map (25) on both graphs appearing on the left-hand side of (30) yields:

$$\begin{aligned} \text{dRep}_2^{(d)}(\textcircled{1} \xrightarrow{i} \textcircled{2})(f_1 \otimes f_2) &= \frac{\partial f_1}{\partial x^\mu} \frac{\partial f_2}{\partial p_\mu} + \sum_{i=1}^{\frac{1}{2}n-1} (-1)^{ik} \frac{\partial f_1}{\partial \psi^{\alpha_i}} \frac{\partial f_2}{\partial \chi_{\alpha_i}} + (-1)^{kn/2} \frac{\partial f_1}{\partial \xi^a} \kappa^{ab} \frac{\partial f_2}{\partial \xi^b} \\ \text{dRep}_2^{(d)}(\textcircled{1} \quad \textcircled{2})(f_1 \otimes f_2) &= f_1 \cdot f_2. \end{aligned}$$

Partial composition of the corresponding operators in the endomorphism operad $\text{End}_{\mathcal{C}^\infty(\mathcal{V})}$ yields:

$$\begin{aligned} \text{dRep}_2^{(d)}(\textcircled{1} \xrightarrow{i} \textcircled{2}) \circ_i^{\text{End}} \text{dRep}_2^{(d)}(\textcircled{1} \quad \textcircled{2})(f_1 \otimes f_2 \otimes f_3) \\ &= \text{dRep}_2^{(d)}(\textcircled{1} \xrightarrow{i} \textcircled{2})(f_1 \otimes \text{dRep}_2^{(d)}(\textcircled{1} \quad \textcircled{2})(f_2 \otimes f_3)) \\ &= \frac{\partial f_1}{\partial x^\mu} \frac{\partial(f_2 \cdot f_3)}{\partial p_\mu} + \sum_{i=1}^{\frac{1}{2}n-1} (-1)^{ik} \frac{\partial f_1}{\partial \psi^{\alpha_i}} \frac{\partial(f_2 \cdot f_3)}{\partial \chi_{\alpha_i}} + (-1)^{kn/2} \frac{\partial f_1}{\partial \xi^a} \kappa^{ab} \frac{\partial(f_2 \cdot f_3)}{\partial \xi^b} \end{aligned}$$

Applying the graded Leibniz rule⁴⁴ yields the equality.

Composing the representation morphism $\text{dRep}^{(d)} : \text{dGra}_d \hookrightarrow \text{End}_{\mathcal{C}^\infty(\mathcal{V})}$ with the orientation morphism (22) endows the algebra of functions $\mathcal{C}^\infty(\mathcal{V})$ with a structure of Gra_d -algebra through the morphism $\text{Rep}^{(d)} : \text{Gra}_d \xrightarrow{\text{Or}} \text{dGra}_d \xrightarrow{\text{dRep}^{(d)}} \text{End}_{\mathcal{C}^\infty(\mathcal{V})}$.

The maps $\left\{ \text{Rep}_N^{(d)} \right\}_{N \geq 1}$ can be defined explicitly in a form similar to (25) as:

$$\text{Rep}_N^{(d)}(\gamma)(f_1 \otimes \cdots \otimes f_N) = \mu_N \left(\prod_{(i,j) \in E_\gamma} \Delta_{ij}(f_1 \otimes \cdots \otimes f_N) \right) \quad (31)$$

where one traded the $\bar{\Delta}$ operators with:

$$\Delta_{ij} = \frac{\partial}{\partial x_{(i)}^\mu} \frac{\partial}{\partial p_{(j)}^\mu} + \frac{\partial}{\partial p_{(i)}^\mu} \frac{\partial}{\partial x_{(j)}^\mu} + \frac{\partial}{\partial \psi_{(i)}^{\alpha_k}} \frac{\partial}{\partial \chi_{(j)}^{\alpha_k}} + \frac{\partial}{\partial \chi_{(i)}^{\alpha_k}} \frac{\partial}{\partial \psi_{(j)}^{\alpha_k}} \quad (32)$$

$$\Delta_{ij} = \frac{\partial}{\partial x_{(i)}^\mu} \frac{\partial}{\partial p_{(j)}^\mu} - \frac{\partial}{\partial p_{(i)}^\mu} \frac{\partial}{\partial x_{(j)}^\mu} + \frac{\partial}{\partial \psi_{(i)}^{\alpha_k}} \frac{\partial}{\partial \chi_{(j)}^{\alpha_k}} - (-1)^k \frac{\partial}{\partial \chi_{(i)}^{\alpha_k}} \frac{\partial}{\partial \psi_{(j)}^{\alpha_k}} + \frac{\partial}{\partial \xi_{(i)}^a} \kappa^{ab} \frac{\partial}{\partial \xi_{(j)}^b} \quad (33)$$

where d is even or odd respectively. The differential operator Δ_{ij} enjoys the following properties⁴⁵:

⁴⁴Letting ψ be a coordinate of degree i and $f \in \mathcal{C}^{\infty|k}(\mathcal{V})$, $g \in \mathcal{C}^{\infty|l}(\mathcal{V})$ be homogeneous functions of degree k and l respectively, the graded Leibniz rule reads:

$$\frac{\partial(f \cdot g)}{\partial \psi} = \frac{\partial f}{\partial \psi} \cdot g + (-1)^{ik} f \cdot \frac{\partial g}{\partial \psi}.$$

⁴⁵In contrast, the differential operator $\bar{\Delta}_{ij}$ only satisfies properties 1-2 consistently with the definition of dGra_d which omits to mod out by $\mathbb{S}_2^{\otimes k}$, cf. Section 4.5.

1. $|\Delta_{ij}| = 1 - d$ consistently with the grading of an edge in Gra_d .
2. $\Delta_{ij} \Delta_{kl} = -(-1)^d \Delta_{kl} \Delta_{ij}$ consistently with the fact that permuting two edges in graphs in Gra_d brings a sign only for even d .
3. $\Delta_{ij} = (-1)^d \Delta_{ji}$ consistently with the fact that flipping the orientation of an edge in graphs in Gra_d brings a sign only for odd d .

The tower of morphisms $\text{Rep}^{(d)} : \text{Gra}_d \hookrightarrow \text{End}_{\mathcal{C}^\infty(\mathcal{V})}$ generalises to all d the Kontsevich morphism for $d = 2$, cf. [84, 119, 70] and more recently [21, 19, 107, 77]. The morphism $\text{Rep}^{(d)}$ preserves the concatenation product as:

$$\text{Rep}_{N+N'}^{(d)}(\gamma \cup \gamma')(f_1 \otimes \cdots \otimes f_{N+N'}) = \text{Rep}_N^{(d)}(\gamma)(f_1 \otimes \cdots \otimes f_N) \cdot \text{Rep}_{N'}^{(d)}(\gamma')(f_{N+1} \otimes \cdots \otimes f_{N+N'}) \quad (34)$$

where the product on the right-hand side is the pointwise product of functions on \mathcal{V} .

5.2 Stable structures on $\mathcal{C}^\infty(\mathcal{V})$ The aim of the present section is to make use of the morphism $\text{Rep}^{(d)}$ in order to define stable structures (in the sense of Definition 4.2) on the algebra of functions $\mathcal{C}^\infty(\mathcal{V})$. In particular, it was noted earlier (cf. Section 3.2) that the algebra of functions on \mathcal{V} was naturally endowed with a structure of Ger_d -algebra. This statement can be refined as follows:

Proposition 5.3. *The Ger_d -algebra structure on $\mathcal{C}^\infty(\mathcal{V})$ is stable i.e. the action of the operad Ger_d factors through:*

$$\text{Ger}_d \xrightarrow{i_d} \text{Gra}_d \xrightarrow{\text{Rep}^{(d)}} \text{End}_{\mathcal{C}^\infty(\mathcal{V})}$$

where $i_d : \text{Ger}_d \hookrightarrow \text{Gra}_d$ is the natural embedding of operads defined in Proposition 4.3.

Proof. The statement follows straightforwardly from:

$$\text{Rep}_2^{(d)}(\Gamma_{\bullet\bullet})(f \otimes g) = f \cdot g \quad , \quad \text{Rep}_2^{(d)}(\Gamma_{\bullet\bullet})(f \otimes g) = \{f, g\}_\omega .$$

□

In the case $d = 2$, Proposition 5.3 can be completed⁴⁶ by stating that the graded algebra of functions $\mathcal{C}^\infty(\mathcal{V})$ (isomorphic to the graded algebra of polyvector fields $\mathcal{T}_{\text{poly}}$ on \mathcal{M}) is locally⁴⁷ endowed with a stable structure of BV-algebra⁴⁸, cf. [98]. The BV Laplacian Δ is then defined

⁴⁶For $d = 2$, T. Willwacher showed that $\mathcal{T}_{\text{poly}}$ (together with a choice of Poisson bivector) is in fact endowed with a much larger stable structure, namely a Br_∞ -algebra structure of homotopy braces [120].

⁴⁷This local definition can be made global whenever the underlying manifold \mathcal{M} possesses a volume form. Letting \mathcal{M} be a manifold of dimension n , a volume form on \mathcal{M} provides an isomorphism $i : \mathcal{T}_{\text{poly}}^\bullet \rightarrow \Omega^{n-\bullet}(\mathcal{M})$ between polyvector fields and differential forms on \mathcal{M} . The divergence operator $\Delta = i^{-1} \circ d_{\text{dR}} \circ i$ defined as the pullback of the de Rham differential along i allows to upgrade the Ger_2 -algebra structure on polyvector fields of \mathcal{M} to that of a BV-algebra. In a local chart, the expression of Δ coincides with the one arising from the tadpole graph due to the fact that a volume form has no local structure. An extension of Kontsevich’s formality morphism relating the BV-algebra of polyvector fields and the BV_∞ -algebra of multidifferential operators on manifolds endowed with a volume form has been worked out in [24].

⁴⁸A **Batalin–Vilkovisky algebra** (or BV-algebra for short) is a graded commutative algebra (\mathfrak{g}, \wedge) endowed with a unary operator $\Delta : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying:

1. Δ is of degree -1
2. $\Delta^2 = 0$

as the image of the tadpole:

$$L_1 = \text{tadpole} \quad i.e. \quad \Delta := \text{Rep}_1^{(2)}(L_1). \quad (35)$$

In the case $d = 1$, the classification recalled in Section 3.3 ensures that a NP-manifold of degree 0 is in fact a (non-graded) symplectic manifold (\mathcal{M}, κ) . In that case, the chain of morphisms of operads $\text{Ass} \xrightarrow{i_1} \text{Gra}_1 \xrightarrow{\text{Rep}^{(1)}} \text{End}_{\mathcal{G}^\infty(\mathcal{M})}$ endows the algebra of functions on the symplectic manifold \mathcal{M} with a stable associative structure:

$$f *_{\text{GM}} g = \text{Rep}_2^{(1)}(\text{tadpole})(f \otimes g),$$

where the graph $\text{tadpole} \in \text{Gra}_1(2)$ is defined in (16), cf. [75]. The induced associative product is the **Groenewold–Moyal** product [56, 100] reading explicitly as:

$$(f *_{\text{GM}} g)(\xi) := \exp\left(\epsilon \kappa^{ab} \frac{\partial}{\partial \zeta^a} \frac{\partial}{\partial \eta^b}\right) f(\zeta) g(\eta) \Big|_{\zeta=\eta=\xi} \quad (36)$$

where ϵ is a formal parameter.

5.2.1 Stable cochains of the Chevalley–Eilenberg algebra The following proposition can be seen as a corollary of our main Proposition 5.1 and defines a stable graph model for $\text{CE}(\mathcal{T}_{\text{poly}}^{(n)})$ which generalises Kontsevich's model (4) in $d = 2$ to arbitrary $d \geq 1$:

Proposition 5.4. *The morphism of operads $\text{Rep}^{(d)} : \text{Gra}_d \hookrightarrow \text{End}_{\mathcal{G}^\infty(\mathcal{V})}$ induces a morphism of dg Lie algebras:*

$$s^* \text{Rep}^{(d)} : (\text{fGC}_d, \delta, [\cdot, \cdot]) \hookrightarrow (\text{CE}(\mathcal{T}_{\text{poly}}^{(n)}), \delta_S, [\cdot, \cdot]_{\text{NR}}). \quad (37)$$

Proof. The proof follows straightforwardly from the equivariance (29) of the morphism $\text{Rep}^{(d)}$ and from the equality $s^* \text{Rep}^{(d)}(\Gamma_{\bullet\bullet}) = [\cdot, \cdot]_S$. \square

Pursuing with the terminology introduced in Definition 4.2, Chevalley–Eilenberg cochains in the image of (37) will be referred to as *stable*. In other words, the dg Lie algebra of graphs $(\text{fGC}_d, \delta, [\cdot, \cdot])$ provides a stable version of the Chevalley–Eilenberg dg Lie algebra $(\text{CE}(\mathcal{T}_{\text{poly}}^{(n)}), \delta_S, [\cdot, \cdot]_{\text{NR}})$. The former thus controls the deformation theory – in the stable setting⁴⁹ – of the n -Schouten Lie algebra as a Lie_∞ -algebra. The two following corollaries make this fact explicit:

$$3. \quad \Delta(a \wedge b \wedge c) - \Delta(a \wedge b) \wedge c + \Delta a \wedge b \wedge c - (-1)^{|a|} a \wedge \Delta(b \wedge c) - (-1)^{|b|(|a|-1)} b \wedge \Delta(a \wedge c) + (-1)^{|a|} a \wedge \Delta b \wedge c + (-1)^{|a|+|b|} a \wedge b \wedge \Delta c = 0.$$

A BV-algebra is in particular a Ger_2 -algebra where the graded Lie bracket is defined as the obstruction for Δ to be a derivation of \wedge i.e. $\{a, b\} = \Delta(a \wedge b) - \Delta a \wedge b - (-1)^{|a|} a \wedge \Delta b$. The induced bracket can be shown to satisfy the axioms of a Ger_2 -algebra as a consequence of the axioms of Δ . Alternatively, a BV-algebra can be defined as a Ger_2 -algebra endowed with a unary operator Δ of degree -1 and satisfying $\Delta(a \wedge b) - \Delta a \wedge b - (-1)^{|a|} a \wedge \Delta b = \{a, b\}$.

⁴⁹Note that *not* all conceivable deformations of $\mathcal{T}_{\text{poly}}^{(n)}$ are universal, i.e. defined in terms of graphs. For example, letting $H \in \Omega^3(\mathcal{M})$ be a closed 3-form on a manifold \mathcal{M} , one can define a *non-stable* deformation of $\mathcal{T}_{\text{poly}}^{(1)} = \mathcal{T}_{\text{poly}}$ by defining a higher bracket of arity 3 as $l_3 \in \text{Hom}^{-1}(\mathcal{T}_{\text{poly}}^{\wedge 3}, \mathcal{T}_{\text{poly}}) \in \text{CE}^1(\mathcal{T}_{\text{poly}})$ as $l_3(X_1, X_2, X_3) = H(X_1, X_2, X_3)$, where the X_i 's are polyvector fields. Denoting l_2 the usual Schouten bracket, the triplet $(\mathcal{T}_{\text{poly}}(\mathcal{M}), l_2, l_3)$ forms a Lie_∞ -algebra. Associated Maurer–Cartan elements are so-called **twisted Poisson structures** [78, 111] i.e. bivectors $\pi \in \Gamma(\wedge^2 T\mathcal{M})$ satisfying $[\pi, \pi]_S = \frac{1}{3} H(\pi, \pi, \pi)$. The latter can be interpreted as Dirac structures for the standard Courant algebroid on \mathcal{M} twisted by H , cf. [110].

Corollary 5.5. Maurer–Cartan elements for the dg Lie algebra $(\mathfrak{fGC}_d, \delta, [\cdot, \cdot])$ are mapped via $s^*\text{Rep}^{(d)}$ to stable deformations of the graded Lie algebra $(\mathcal{T}_{\text{poly}}^{(n)}, [\cdot, \cdot]_{\mathfrak{S}})$ as a Lie_{∞} -algebra.

Example 5.6 (Groenewold–Moyal commutator). Let (\mathcal{M}, κ) be a symplectic manifold. The Maurer–Cartan element (17) prolongating the Θ -graph is mapped via $s^*\text{Rep}^{(1)}$ to the (essentially unique) stable deformation of $(\mathcal{C}^{\infty}(\mathcal{M}), \{\cdot, \cdot\}_{\kappa})$ as a Lie algebra⁵⁰ where the Poisson bracket is deformed into the **Groenewold–Moyal commutator** $[f, g]_{\text{GM}} := f *_{\text{GM}} g - g *_{\text{GM}} f$ on $\mathcal{C}^{\infty}(\mathcal{M})$ constructed from (36), cf. [75].

We refer to Example 5.10 for an example of deformation of the 3-Schouten algebra as a genuine Lie_{∞} -algebra. The Lie algebra $H^0(\mathfrak{fGC}_d)$ being pro-nilpotent⁵¹, one defines the pro-unipotent group $\exp(H^0(\mathfrak{fGC}_d))$ as [123]:

- Group elements are elements of $H^0(\mathfrak{fGC}_d)$, viewed as a set.
- The unit is $\mathbf{0} \in H^0(\mathfrak{fGC}_d)$.
- The inverse map sends γ to $-\gamma$.
- The group operation is defined as $\gamma_1 \cdot \gamma_2 = \text{BCH}(\gamma_1, \gamma_2)$ where BCH stands for the Baker–Campbell–Hausdorff formula.

Corollary 5.7. The pro-unipotent group $\exp(H^0(\mathfrak{fGC}_d))$ acts via Lie_{∞} -automorphisms defined up to equivalence on the n -Schouten algebra. Such Lie_{∞} -automorphisms will be referred to as stable.

Proof. The proof is identical to the one of the case $d = 2$ (cf. e.g. Theorem 1. in [70], based on [119]) that we review for completeness. Let $\gamma \in H^0(\mathfrak{fGC}_d)$ be a non-trivial cocycle in \mathfrak{fGC}_d . The morphism $s^*\text{Rep}^{(d)}$ of dg Lie algebras introduced in Proposition 5.4 maps γ to a zero degree Chevalley–Eilenberg cocycle for the $(d - 1)$ -Schouten algebra denoted $s^*\text{Rep}^{(d)}(\gamma)$. In other words, $s^*\text{Rep}^{(d)}(\gamma) \in H^0(\text{CE}(\mathcal{T}_{\text{poly}}^{(n)}))$ is a Lie_{∞} -derivation of $(\mathcal{T}_{\text{poly}}^{(n)}, [\cdot, \cdot]_{\mathfrak{S}})$. This ensures that $\exp(s^*\text{Rep}^{(d)}(\gamma))$ is a Lie_{∞} -automorphism of $(\mathcal{T}_{\text{poly}}^{(n)}, [\cdot, \cdot]_{\mathfrak{S}})$. Furthermore, since $\exp(H^0(\mathfrak{fGC}_d))$ is pro-unipotent, to any element $\Gamma \in \exp(H^0(\mathfrak{fGC}_d))$ one can associate a unique element $\gamma \in H^0(\mathfrak{fGC}_d)$ so that $\Gamma = \exp(\gamma)$. We can thus define a Lie_{∞} -action via its Taylor coefficients:

$$\begin{aligned} \mathcal{U}_N &: \exp(H^0(\mathfrak{fGC}_d)) \times \mathcal{T}_{\text{poly}}^{(n) \wedge N} \rightarrow \mathcal{T}_{\text{poly}}^{(n)} \\ &: (\Gamma, X_1, \dots, X_N) \mapsto \exp(s^*\text{Rep}^{(d)}(\gamma))(X_1, \dots, X_N) \end{aligned}$$

for all $N \geq 1$. Finally, we note that two equivalent cocycles γ and γ' induce homotopic Lie_{∞} -automorphisms. □

5.2.2 Classification of stable structures We now make use of the results regarding cohomology of the full graph complex as reviewed in Section 4.4 in order to provide a classification of stable structures on the $(d - 1)$ -Schouten algebra for all $d \geq 1$ – where the term stable structures will refer to⁵²:

1. Stable Lie_{∞} -automorphisms of the $(d - 1)$ -Schouten algebra
2. Stable deformations of the $(d - 1)$ -Schouten algebra as a Lie_{∞} -algebra.

⁵⁰Note that $n = 0$ so that the ordinary (i.e. non-graded) Lie algebra $(\mathcal{C}^{\infty}(\mathcal{M}), \{\cdot, \cdot\}_{\kappa})$ identifies with the 0-Schouten algebra $(\mathcal{T}_{\text{poly}}^{(0)}, [\cdot, \cdot]_{\mathfrak{S}})$. In this case, the deformation complex $(\text{CE}(\mathcal{T}_{\text{poly}}^{(n)}), \delta_{\mathfrak{S}}, [\cdot, \cdot]_{\text{NR}})$ controls the deformation theory of $(\mathcal{C}^{\infty}(\mathcal{M}), \{\cdot, \cdot\}_{\kappa})$ as an ordinary Lie algebra.

⁵¹See [36] for a proof of the $d = 2$ case. The proof for all d is identical.

⁵²We will focus on stable structures obtained from *connected* graphs.

Since the case $d = 2$ has already been addressed in the literature, we treat it separately:

1. Recall from Theorem 4.9 that there exists an isomorphism of Lie algebras $H^0(\mathfrak{fGC}_2^{\text{con}}) \simeq \mathfrak{grt}_1$. By Corollary 5.7, it follows that the Grothendieck–Teichmüller group $\text{GRT}_1 := \exp(\mathfrak{grt}_1)$ acts via Lie_∞ -automorphisms on the Schouten algebra $\mathcal{T}_{\text{poly}}$, see [119, 70].
2. As noted earlier, it is a difficult open conjecture (Drinfel'd, Kontsevich) that $H^1(\mathfrak{fGC}_2^{\text{con}}) = \mathbf{0}$.⁵³ If the conjecture holds, then there are no stable deformations of the Schouten algebra as a Lie_∞ -algebra⁵⁴ i.e. $(\mathcal{T}_{\text{poly}}, [\cdot, \cdot]_{\mathcal{S}})$ is rigid as a stable Lie_∞ -algebra.

We now turn to the case $d \neq 2$. The following classification of the cohomology of the connected part of the full graph complex in low degrees is obtained from the various bounds collected in Section 4.4:

Lemma 5.8. *Let us denote L_k the loop graph with k edges. The cohomology of the (connected part of) the full graph complex $\mathfrak{fGC}_d^{\text{con}}$ in low degrees for all $d \neq 2$ is given by:*

- Degree 0: $H^0(\mathfrak{fGC}_{4j+3}^{\text{con}}) = \mathbb{K} \langle L_{4j+3} \rangle$ for all $j \geq 0$ and trivial otherwise.
- Degree 1: $H^1(\mathfrak{fGC}_1^{\text{con}}) = \mathbb{K} \langle \Theta \rangle$, $H^1(\mathfrak{fGC}_{4j+4}^{\text{con}}) = \mathbb{K} \langle L_{4j+5} \rangle$ for all $j \geq -1$ and trivial otherwise.
- Degree 2: $H^2(\mathfrak{fGC}_{4j+1}^{\text{con}}) = \mathbb{K} \langle L_{4j+3} \rangle$ for all $j \geq 0$ and trivial otherwise.

Note that the only non-loop cocycle in this classification is given by the Θ -graph. The stable Lie_∞ -structure induced by the Maurer–Cartan element (17) prolongating the latter is given by the Groenewold–Moyal bracket on symplectic manifolds –cf. Example 5.6 – which constitutes the unique⁵⁵ stable structure in dimension $d = 1$. It follows that the only stable structures in dimension $d > 2$ are induced by loop classes⁵⁶:

Proposition 5.9 (Loop induced stable structures).

1. Let (\mathcal{V}, ω) be a NP-manifold of (even) degree $n = 4j + 2$, $j \geq 0$. The loop cocycle L_{4j+3} induces a 1-dimensional family of stable Lie_∞ -automorphisms of the n -Schouten algebra $(\mathcal{T}_{\text{poly}}^{(n)}, [\cdot, \cdot]_{\mathcal{S}})$.
2. Let (\mathcal{V}, ω) be a NP-manifold of (odd) degree $n = 4j + 3$, $j \geq 0$. The loop cocycle L_{4j+5} induces a 1-dimensional family of stable deformations of the n -Schouten algebra $(\mathcal{T}_{\text{poly}}^{(n)}, [\cdot, \cdot]_{\mathcal{S}})$ as a Lie_∞ -algebra.

Proof. The first statement is merely a rephrasing of the first item of Lemma 5.8, given the definition of stable automorphism provided in Corollary 5.7. As for the second statement, since $H^2(\mathfrak{fGC}_{4j+4}^{\text{con}})$ vanishes for all $j \geq 0$, there is no obstruction to the prolongation of the loop

⁵³As noted in [96], although the cohomology of \mathfrak{GC}_2 in degree 1 is conjectured to be trivial, a choice of Drinfel'd associator is necessary in order to convert cocycles of degree 1 in \mathfrak{GC}_2 into coboundaries of degree 0 so that an iterative procedure can exist but cannot be trivial.

⁵⁴If true, the statement only holds in the stable setting, cf. footnote 55 for a statement in the oriented setting.

⁵⁵Departing from the stable regime to the oriented regime, we note that the incarnation in $d = 2$ of the Θ -graph induces the Kontsevich–Shoikhet cocycle in $H^1(\mathfrak{GC}_2^{\text{or}})$ whose prolongation to a Maurer–Cartan element is mapped to the Kontsevich–Shoikhet Lie_∞ -algebra structure deforming the Schouten algebra of infinite dimensional polyvector fields, cf. [114, 122]. Further, the incarnation of the Θ -graph in $d = 3$ yields a potential obstruction to the quantization of Lie bialgebroids [99].

⁵⁶Stable structures induced from the Grothendieck–Teichmüller algebra \mathfrak{grt}_1 only occur in dimension $d = 2$. However, departing from the stable to the (multi)-oriented setting allows to generate stable structures from \mathfrak{grt}_1 in dimension $d > 2$, see e.g. [99] for an example on (quasi)-Lie bialgebroids in $d = 3$.

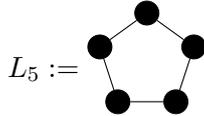
cocycle L_{4j+5} into a formal Maurer–Cartan element⁵⁷ $\mathfrak{m}_{4j+5} \in \text{MC}(\text{fGC}_{4j+4}[[\epsilon]])$. Corollary 5.5 then ensures that the formal Maurer–Cartan element \mathfrak{m}_{4j+5} is mapped via $s^*\text{Rep}^{(d)}$ to a stable deformation of the graded Lie algebra $(\mathcal{T}_{\text{poly}}^{(n)}, [\cdot, \cdot]_{\mathfrak{S}})$ as a Lie_{∞} -algebra⁵⁸. \square

Proposition 5.9 thus provides two mechanisms for generating stable structures on graded manifolds of specific degrees. The following example illustrates the procedure in the odd degree case (for $j = 0$):

Example 5.10. Let (\mathcal{V}, ω) be a NP-manifold of degree 3 coordinatised by, cf. e.g. [69, 89, 58]:

$$\left\{ x_0^\mu, \psi_1^\alpha, \chi_2^\alpha, p_3^\mu \right\} .$$

The pentagon graph:



can be promoted to a formal Maurer–Cartan element $\mathfrak{m}_5 \in \text{MC}(\text{fGC}_4[[\epsilon]])$ reading:

$$\mathfrak{m}_5 := \epsilon^5 L_5 + \epsilon^8 \mathfrak{m}_5^{(8)} + \dots + \epsilon^{3p+2} \mathfrak{m}_5^{(3p+2)} + \dots$$

which induces a Lie_{∞} -algebra structure on the shifted graded algebra of functions $\mathcal{T}_{\text{poly}}^{(3)} := \mathcal{C}^{\infty}(\mathcal{V})[3]$ with non-vanishing brackets $l_2, l_5, l_8, \dots, l_{3p+2}$, $p \geq 0$ such that, for all $X_i \in \mathcal{T}_{\text{poly}}^{(3)}$:

$$\begin{aligned} l_2(X_1, X_2) &= s^*\text{Rep}^{(4)}(\Gamma_{\bullet\bullet})(X_1, X_2) = [X_1, X_2]_{\mathfrak{S}} \\ l_5(X_1, \dots, X_5) &= s^*\text{Rep}^{(4)}(L_5)(X_1, \dots, X_5) \\ &= s^{-1}\mu_5\left(\Delta_{12} \Delta_{23} \Delta_{34} \Delta_{45} \Delta_{51}(s(X_1), \dots, s(X_5))\right) \\ &\vdots \\ l_{3p+2}(X_1, \dots, X_{3p+2}) &= s^*\text{Rep}^{(4)}(\mathfrak{m}_5^{(3p+2)})(X_1, \dots, X_{3p+2}) \end{aligned}$$

where:

$$\Delta_{ij} = \frac{\partial}{\partial x_{(i)}^\mu} \frac{\partial}{\partial p_\mu^{(j)}} + \frac{\partial}{\partial p_\mu^{(i)}} \frac{\partial}{\partial x_{(j)}^\mu} + \frac{\partial}{\partial \psi_{(i)}^\alpha} \frac{\partial}{\partial \chi_\alpha^{(j)}} + \frac{\partial}{\partial \chi_\alpha^{(i)}} \frac{\partial}{\partial \psi_{(j)}^\alpha} .$$

5.3 Stable Hamiltonian deformations Up to now, our attention has been focused on NP-manifolds. We will now consider additional structures on graded manifolds by focusing on NPQ-manifolds *i.e.* symplectic graded manifolds endowed with a Hamiltonian structure – cf. Section 3.3 – and discuss how the previously developed machinery can be used in order to generate Hamiltonian deformations from graph cocycles. More precisely, we discuss two separate mechanisms that allow to map graph cocycles to Hamiltonian deformations. The first one is a generalisation to any d of the mechanism first identified by Kontsevich for $d = 2$ [84] mapping elements from $H^0(\text{fGC}_d)$ to *stable* deformations (hence quantizations) of Hamiltonian functions [see Proposition 5.13 below]. The second mechanism is novel and allows to map graphs in

⁵⁷For any (graded) vector space V , we will denote $V[[\epsilon]]$ the (graded) vector space of formal series in the formal parameter ϵ with coefficients in V . Consistently, whenever $(V, [\cdot, \cdot])$ is a graded Lie algebra (or more generally a Lie_{∞} -algebra), we will denote $\text{MC}(V[[\epsilon]])$ the set of *formal* Maurer–Cartan elements *i.e.* elements $\mathfrak{m} \in \epsilon V^1[[\epsilon]]$ satisfying $[\mathfrak{m}, \mathfrak{m}] = 0$.

⁵⁸The only non-trivial higher brackets l_m of the Lie_{∞} -algebra induced by the loop cocycle L_{4j+5} have arities $m = p(4j + 3) + 2$ for all $p \geq 0$ with $l_2 = [\cdot, \cdot]_{\mathfrak{S}}$.

$H^{-d}(\mathrm{fGC}_d)$ to Weyl Hamiltonian deformations [see Corollary 5.19 below]. This new mechanism will prove to be particularly relevant for $d = 3$, mapping trivalent graphs to Weyl deformations of Courant algebroids.

We will consider a NP-manifold (\mathcal{V}, ω) of arbitrary degree $n \in \mathbb{N}$ with $d = n + 1$. We will use the notation $[\mathcal{F}]$ to designate equivalence classes of functions $\mathcal{F}' \sim \mathcal{F}$ of \mathcal{V} modulo gauge transformations [see Definition 3.1]. The set of Hamiltonian functions⁵⁹ on (\mathcal{V}, ω) will be denoted \mathbf{Ham} . We start by defining the notion of **Hamiltonian deformation**:

Definition 5.11 (Hamiltonian deformation). Let $\mathcal{H} \in \mathbf{Ham}$ be a Hamiltonian function on \mathcal{V} .

A Hamiltonian deformation of \mathcal{H} is a formal power series $\mathcal{H}_* \in \epsilon \mathcal{C}^{\infty|d}(\mathcal{V})[[\epsilon]]$ such that

1. \mathcal{H}_* is nilpotent with respect to the graded Poisson bracket *i.e.* $\{\mathcal{H}_*, \mathcal{H}_*\}_\omega = 0$.
2. The first order of the expansion of \mathcal{H}_* in terms of the formal parameter coincides with \mathcal{H} or equivalently $\mathcal{H} = \frac{1}{\epsilon} \mathcal{H}_*|_{\epsilon=0}$.

The set of Hamiltonian deformations of a given Hamiltonian function \mathcal{H} will be denoted $\mathbf{FHam}_{\mathcal{H}}$. A map $[\mathbf{Ham}] \rightarrow [\mathbf{FHam}]$ which assigns to each equivalence class [see Section 3.3] $[\mathcal{H}] \in \mathbf{Ham}$ an equivalence class $[\mathbf{FHam}_{\mathcal{H}}]$ will be referred to as a **Hamiltonian deformation map**.

Example 5.12 (Formal Poisson structures). Recall from Example 3.2 that Hamiltonian functions on NP-manifolds of degree 1 are in one-to-one correspondence with Poisson structures on the corresponding base manifold. In this context, a Hamiltonian deformation is a map $[\pi] \mapsto [\pi_*]$ sending each Poisson bivector π in the equivalence class $[\pi]$ (under diffeomorphism of the base manifold) to a formal Poisson structure $\pi_* = \epsilon\pi + \epsilon^2\pi_{(2)} + \cdots + \epsilon^k\pi_{(k)} + \cdots$ where the $\pi_{(i)}$'s are bivectors such that $[\pi_*, \pi_*]_{\mathcal{S}} = 0$ and the equivalence relation on $[\pi_*]$ is given by formal diffeomorphisms.

The first defining condition of a Hamiltonian deformation ensures that the pullback $\mathbf{m} := s^{-1}(\mathcal{H}_*)$ is a formal Maurer–Cartan element of the graded Lie algebra $(\mathcal{T}_{\mathrm{poly}}^{(n)}[[\epsilon]], [\cdot, \cdot]_{\mathcal{S}})$ *i.e.* $\mathbf{m} \in \mathbf{MC}(\mathcal{T}_{\mathrm{poly}}^{(n)}[[\epsilon]])$. This fact, combined with Corollary 5.7, yields the following Proposition:

Proposition 5.13. *There is a canonical map $H^0(\mathrm{fGC}_d) \rightarrow ([\mathbf{Ham}] \rightarrow [\mathbf{FHam}])$ mapping cocycles in the zeroth graph cohomology to Hamiltonian deformation maps. Such Hamiltonian deformation maps will be referred to as stable.*

Proof. Recall from Corollary 5.7 that to each cocycle $\gamma \in H^0(\mathrm{fGC}_d)$, one can associate a homotopy class of Lie_{∞} -automorphisms $\mathcal{U}^{\Gamma} = \exp(s^*\mathrm{Rep}^{(d)}(\gamma))$ of $(\mathcal{T}_{\mathrm{poly}}^{(n)}, [\cdot, \cdot]_{\mathcal{S}})$, where we denoted $\Gamma := \exp(\gamma)$. The latter induces a bijective map between equivalence classes of formal Maurer–Cartan elements of $(\mathcal{T}_{\mathrm{poly}}^{(n)}[[\epsilon]], [\cdot, \cdot]_{\mathcal{S}})$ as $\hat{\mathcal{U}}^{\Gamma} : \mathbf{MC}(\mathcal{T}_{\mathrm{poly}}^{(n)}[[\epsilon]]) / \sim \xrightarrow{\sim} \mathbf{MC}(\mathcal{T}_{\mathrm{poly}}^{(n)}[[\epsilon]]) / \sim : [\mathbf{m}] \mapsto [\hat{\mathcal{U}}^{\Gamma}(\mathbf{m})]$ where $\hat{\mathcal{U}}^{\Gamma}(\mathbf{m}) := \sum_{k=1}^{\infty} \frac{1}{k!} \mathcal{U}_k^{\Gamma}(\mathbf{m}^{\otimes k})$. Let $\mathcal{H} \in \mathbf{Ham}$ be a Hamiltonian function. We will denote $\mathbf{m}_{\mathcal{H}} \in \mathbf{MC}(\mathcal{T}_{\mathrm{poly}}^{(n)}[[\epsilon]])$ the canonical formal Maurer–Cartan element defined as $\mathbf{m}_{\mathcal{H}} := \epsilon s^{-1}(\mathcal{H})$. The latter is mapped via $\hat{\mathcal{U}}^{\Gamma}$ to $\hat{\mathcal{U}}^{\Gamma}(\mathbf{m}_{\mathcal{H}}) = \sum_{k=1}^{\infty} \frac{\epsilon^k}{k!} \mathcal{U}_k^{\Gamma}(s^{-1}(\mathcal{H})^{\otimes k})$. It follows from the above reasoning that $\hat{\mathcal{U}}^{\Gamma}(\mathbf{m}_{\mathcal{H}})$ is a formal Maurer–Cartan element. Finally, we define $\mathcal{H}_* := s(\hat{\mathcal{U}}^{\Gamma}(\mathbf{m}_{\mathcal{H}}))$. Since $\hat{\mathcal{U}}_1^{\Gamma}$ is the identity of $\mathcal{T}_{\mathrm{poly}}^{(n)}$, then $\mathcal{H} = \frac{1}{\epsilon} \mathcal{H}_*|_{\epsilon=0}$ and hence \mathcal{H}_* is a Hamiltonian deformation of \mathcal{H} . We conclude that the map $[\mathbf{Ham}] \rightarrow [\mathbf{FHam}] : [\mathcal{H}] \mapsto [\mathcal{H}_*]$ is a Hamiltonian deformation map. Two equivalent cocycles γ and γ' (*i.e.* such that there exists a degree -1 cochain χ such that $\gamma' - \gamma = \delta\chi$) induce homotopic Lie_{∞} -automorphisms thus yielding the same Hamiltonian deformation map. \square

⁵⁹Recall from Section 3.3 that a Hamiltonian function is a function $\mathcal{H} \in \mathcal{C}^{\infty|d}(\mathcal{V})$ such that $\{\mathcal{H}, \mathcal{H}\}_{\omega} = 0$.

The previous Proposition takes full advantage of the previously recalled Kontsevich machinery mapping graph cocycles to Lie_∞ -automorphisms. As we will see, there exists an alternative way to map graph cocycles to Hamiltonian deformations, in a way that does not lift to a Lie_∞ -automorphism. Before addressing them in the full case, we make a detour by considering infinitesimal deformations (or flows).

Infinitesimal deformations. Let (\mathcal{V}, ω) be a NP-manifold of arbitrary degree $n \in \mathbb{N}$ and denote $d = n + 1$.

Proposition 5.14. *There is a canonical map $H^\bullet(\text{fGC}_d) \rightarrow ([\text{Ham}] \rightarrow [H^{\bullet+d}(\mathcal{C}^\infty(\mathcal{V})|\mathbb{Q})])$.*

Explicitly, cocycles of degree $p \in \mathbb{Z}$ in fGC_d with N vertices are sent to applications mapping any Hamiltonian function $\mathcal{H} \in \text{Ham}$ to a degree $p + d$ cocycle:

$$\mathcal{C}_\gamma(\mathcal{H}) := \text{Rep}_N^{(d)}(\gamma)(\mathcal{H}^{\otimes N}) \tag{38}$$

in the cohomological complex induced on $\mathcal{C}^{\infty|\bullet}(\mathcal{V})$ by the associated⁶⁰ homological vector field $\mathbb{Q} := \{\mathcal{H}, \cdot\}_\omega$.

Proof. Let γ be a non-trivial cocycle of degree p in fGC_d with N vertices and k edges. The cocycle condition $\delta\gamma = 0$ ensures that $s^*\text{Rep}^{(d)}(\gamma)$ is a degree p Chevalley–Eilenberg cocycle for the graded Lie algebra $(\mathcal{T}_{\text{poly}}^{(n)}, [\cdot, \cdot]_\mathbb{S})$. Explicitly, denoting \mathcal{U}_N^Γ the component of $s^*\text{Rep}^{(d)}(\gamma)$ acting on N inputs⁶¹, the Chevalley–Eilenberg cocycle condition can be expressed as:

$$\begin{aligned} [[\cdot, \cdot]_\mathbb{S}, \mathcal{U}_N^\Gamma]_{\text{NR}} &= \sum_{\sigma \in \text{Sh}^{-1}(N, 1)} (-1)^{|\sigma|} (-1)^{p \Sigma_{N+1}} ([\cdot, \cdot]_\mathbb{S} \circ_1 \mathcal{U}_N^\Gamma | \sigma) \\ &- \sum_{\sigma \in \text{Sh}^{-1}(2, N-1)} (-1)^{|\sigma|} \Sigma_{N+1} (\mathcal{U}_N^\Gamma \circ_1 [\cdot, \cdot]_\mathbb{S} | \sigma) = 0. \end{aligned} \tag{39}$$

Acting on $s^{-1}(\mathcal{H})^{\otimes N+1}$, the second term vanishes due to $\{\mathcal{H}, \mathcal{H}\}_\omega = 0$, thus yielding:

$$\left[s^{-1}(\mathcal{H}), \mathcal{U}_N^\Gamma(s^{-1}(\mathcal{H})^{\otimes N}) \right]_\mathbb{S} = 0.$$

Denoting $\mathcal{C}_\gamma(\mathcal{H}) := s(\mathcal{U}_N^\Gamma(s^{-1}(\mathcal{H})^{\otimes N})) = \text{Rep}_N^{(d)}(\gamma)(\mathcal{H}^{\otimes N})$ leads to the cocycle equation $\{\mathcal{H}, \mathcal{C}_\gamma(\mathcal{H})\}_\omega = 0$. The latter function is of degree $|\mathcal{C}_\gamma(\mathcal{H})| = |\gamma| + dN$ in $\mathcal{C}^\infty(\mathcal{V})$ where $|\gamma|$ stands for the degree of γ in Gra_d *i.e.* $|\gamma| = k(1 - d)$. Since γ is assumed to be of degree p in fGC_d , then $d(N - 1) + k(1 - d) = p$ and thus $|\gamma| = k(1 - d) = p + d(1 - N)$ so that $|\mathcal{C}_\gamma(\mathcal{H})| = p + d$. A reasoning similar to the above shows that two equivalent cocycles $\gamma' - \gamma = \delta\chi$ yield cohomologically equivalent functions $\mathcal{C}_{\gamma'}(\mathcal{H}) - \mathcal{C}_\gamma(\mathcal{H}) = \{\mathcal{H}, \mathcal{C}_\chi(\mathcal{H})\}$ where we denoted $\mathcal{C}_\chi(\mathcal{H}) := \text{Rep}_{N-1}^{(d)}(\chi)(\mathcal{H}^{\otimes N-1})$. Finally, two gauge equivalent Hamiltonians, *i.e.* differing infinitesimally by a coboundary $\{\mathcal{H}, f\}_\omega$, yield two equivalent functions, differing by a coboundary $\{\mathcal{H}, \text{Rep}_N^{(d)}(\gamma)(f \otimes \mathcal{H}^{\otimes N-1})\}_\omega$ and a gauge term $\{f, \mathcal{C}_\gamma(\mathcal{H})\}_\omega$, as can be shown by letting (39) act on $f \otimes \mathcal{H}^{\otimes N}$. \square

Of particular interest for us will be the two subcases $p = 0$ and $p = -d$ allowing to map graph cocycles to flows on the space of Hamiltonian functions and Weyl rescalings of Hamiltonian functions, respectively.

⁶⁰Note that the class $[H^{\bullet+d}(\mathcal{C}^\infty(\mathcal{V})|\mathbb{Q})]$ is independent of the choice of representative in the gauge class $[\mathcal{H}]$ used to define \mathbb{Q} .

⁶¹Or equivalently the first non-trivial Taylor coefficient, beside the identity, of the Lie_∞ -automorphism \mathcal{U}^Γ , with $\Gamma := \exp(\gamma)$, *cf.* the proof of Proposition 5.13.

Stable Hamiltonian flows ($p = 0$).

Definition 5.15 (Hamiltonian flow). A map $[\text{Ham}] \rightarrow [H^d(\mathcal{C}^\infty(\mathcal{V})|\mathbf{Q})]$ mapping (gauge classes of) Hamiltonian functions $\mathcal{H} \in \text{Ham}$ to (gauge classes of) degree d elements in the cohomology of the complex induced on $\mathcal{C}^{\infty|\bullet}(\mathcal{V})$ by the homological vector field $\mathbf{Q} := \{\mathcal{H}, \cdot\}_\omega$ will be called a Hamiltonian flow.

The set of Hamiltonian flows will be denoted Hflow . The following corollary of Proposition 5.14 can be seen as a linearisation of Proposition 5.13:

Corollary 5.16. *There is a canonical map $H^0(\text{fGC}_d) \rightarrow \text{Hflow}$. Hamiltonian flows in the image of this map are called stable.*

In plain words, the above corollary allows to map graph cocycles of degree 0 to infinitesimal deformations of Hamiltonian functions – or equivalently to flows on the space of Hamiltonian functions – thus generalising to all d the Kontsevich's construction of stable flows on the space of Poisson manifolds from cocycles in $H^0(\text{GC}_2)$ [cf. Section 5.3 in [84] and Section 6.1 below].

As noted in Section 4.4, the zeroth cohomology is the dominant degree of the cohomology of GC_2 , being isomorphic to the infinite dimensional Grothendieck–Teichmüller Lie algebra grt_1 . The previous construction for $d = 2$ thus allows to generate infinitely many Hamiltonian flows on the space of Poisson bivectors. Explicit examples of stable Hamiltonian flows for $d = 2, 3$ will be exhibited in Section 6.

Stable Weyl flows ($p = -d$).

Definition 5.17 (Weyl flow). A map $[\text{Ham}] \rightarrow [H^0(\mathcal{C}^\infty(\mathcal{V})|\mathbf{Q})]$ mapping (gauge equivalence classes of) Hamiltonian functions $\mathcal{H} \in \text{Ham}$ to (gauge equivalence classes of) degree 0 elements in the cohomology of the complex induced on $\mathcal{C}^{\infty|\bullet}(\mathcal{V})$ by the homological vector field $\mathbf{Q} := \{\mathcal{H}, \cdot\}_\omega$ will be called a Weyl factor.

Explicitly, Weyl flows map Hamiltonian functions $\mathcal{H} \in \text{Ham}$ to functions⁶² $\Omega \in \mathcal{C}^\infty(\mathcal{M})$ – called **Weyl factors** – satisfying $\mathbf{Q}[\Omega] = 0$ [or equivalently $\{\mathcal{H}, \Omega\}_\omega = 0$ with $\mathcal{H} \in \text{Ham}$ the Hamiltonian function associated to \mathbf{Q}]. The set $H^0(\mathcal{C}^\infty(\mathcal{V})|\mathbf{Q})$ of Weyl factors is a commutative algebra with product the pointwise product of functions on the base manifold.

We now show that cocycle graphs of degree $-d$ induce Weyl factors as follows:

Proposition 5.18. *Let $(\mathcal{V}, \{\cdot, \cdot\}_\omega, \mathbf{Q})$ be a NPQ-manifold. There is a canonical morphism of commutative algebras $H^{-d}(\text{fGC}_d) \rightarrow H^0(\mathcal{C}^\infty(\mathcal{V})|\mathbf{Q})$ mapping cocycles in the degree $-d$ graph cohomology to Weyl factors. Such Weyl factors will be referred to as stable.*

Indeed, it follows from Proposition 5.14 for $p = -d$ that there exists a canonical map $H^{-d}(\text{fGC}_d) \rightarrow ([\text{Ham}] \rightarrow [H^0(\mathcal{C}^\infty(\mathcal{V})|\mathbf{Q})])$, reading, for any particular Hamiltonian function $\mathcal{H} \in \text{Ham}$ as:

$$\begin{aligned} H^{-d}(\text{fGC}_d) &\rightarrow H^0(\mathcal{C}^\infty(\mathcal{V})|\mathbf{Q}) \\ \gamma &\mapsto \Omega_\gamma(\mathcal{H}) := \text{Rep}_N^{(d)}(\gamma)(\mathcal{H}^{\otimes N}) \end{aligned} \tag{40}$$

where $\Omega_\gamma(\mathcal{H}) := \mathcal{C}_\gamma(\mathcal{H})$ is defined as in eq.(38), with N the number of vertices of the cocycle γ .

⁶²Note that there are no functions of degree -1 in $\mathcal{C}^\infty(\mathcal{V})$ hence the definition of Weyl factors does not involve equivalence classes.

Proof. It remains to be shown that the above map $\gamma \mapsto \Omega_\gamma(\mathcal{H})$ is a morphism of commutative algebras which is ensured by the fact that $\text{Rep}^{(d)}$ preserves the concatenation product (34). \square

5.4 Weyl Hamiltonian deformations

Corollary 5.19. *There are infinitely many maps $H^{-d}(\text{fGC}_d) \rightarrow ([\text{Ham}] \rightarrow [\text{FHam}])$ mapping cocycles in the graph cohomology of degree $-d$ to Hamiltonian deformation maps. Such Hamiltonian deformation maps will be referred to as Weyl.*

Let γ be a non-trivial cocycle of degree $-d$ in fGC_d with $N > 0$ vertices. Given a Hamiltonian function $\mathcal{H} \in \text{Ham}$, we can define the Weyl factor $\Omega_\gamma(\mathcal{H})$ as in (40). The latter can be used to define a Hamiltonian deformation map $[\text{Ham}] \rightarrow [\text{FHam}] : [\mathcal{H}] \mapsto [\mathcal{H}_*]$ where \mathcal{H}_* reads as:

$$\mathcal{H}_* = \epsilon \mathcal{H} + \sum_{k=1}^{\infty} a_k \epsilon^{kN+1} \Omega_\gamma(\mathcal{H})^k \cdot \mathcal{H}$$

where the coefficients $a_k \in \mathbb{K}$ are arbitrary.

Proof. We need to check that \mathcal{H}_* is a Hamiltonian deformation of \mathcal{H} . The condition $\{\mathcal{H}_*, \mathcal{H}_*\}_\omega = 0$ follows straightforwardly from $\{\mathcal{H}, \mathcal{H}\}_\omega = 0$ (\mathcal{H} being Hamiltonian), $\{\mathcal{H}, \Omega_\gamma(\mathcal{H})\}_\omega = 0$ (Proposition 5.14) and $\{\Omega_\gamma(\mathcal{H}), \Omega_\gamma(\mathcal{H})\}_\omega = 0$ (holds identically from degree consideration). The second condition $\mathcal{H} = \frac{1}{\epsilon} \mathcal{H}_*|_{\epsilon=0}$ follows directly from $N > 0$. \square

Contradistinctly to stable Hamiltonian deformation maps [Proposition 5.13], Weyl Hamiltonian deformation maps do not descend from Lie_∞ -automorphisms of $\mathcal{T}_{\text{poly}}^{(n)}$ and thus constitute a novel way to define consistent deformations from graph cocycles.

Similarly as before, one can make use of the results reviewed in Section 4.4 in order to classify stable Weyl factors. In fact one can check that stable Weyl factors originating from connected graphs only occur in $d = 3$:

Proposition 5.20. $H^{-d}(\text{fGC}_d^{\text{con}}) = \mathbf{0}$ for all $d \neq 3$.

Proof. Let γ be a graph of $\text{fGC}_d^{\text{con}}$ with N vertices and k edges. The corresponding degree is given by $|\gamma|_d = d(N - 1) + k(1 - d)$. Imposing $|\gamma|_d = -d$ thus yields the constraint: $dN = k(d - 1)$. Setting $d = 0$ in the latter constraint imposes $k = 0$, hence the corresponding graph is either disconnected or the trivial graph or the graph with one unique vertex, which is never a cocycle. We will then assume $d > 0$. The above constraint thus yields $N = k - \frac{k}{d}$ which is only integer for either $k = 0$ (ruled out by the above reasoning) or $k \geq d$. Hence we will be focusing on the cases $0 < d \leq k$. We first observe that graphs satisfying the above constraint cannot be (non-trivial) loops as setting $k = N$ would yield $N = 0$ for all d . Hence Theorem 4.8 ensures that $H^{-d}(\text{fGC}_d^{\text{con}}) = H^{-d}(\text{GC}_d)$. We now distinguish between even and odd d .

- *Even d :* As noted earlier, the cohomology $H^\bullet(\text{GC}_d)$ for even d is isomorphic to $H^\bullet(\text{GC}_2)$, although the isomorphism is not degree preserving. According to the bounds on cohomology degree displayed in Theorem 4.9, for γ to be a non-trivial cocycle, one needs $1 < |\gamma|_2 < b - 2$, where b stands for the first Betti number associated to γ . In the case $d = 2$, we have $|\gamma|_2 = -2$ hence the first inequality ensures that $H^{-2}(\text{GC}_2)$ is empty. For $d \geq 4$, the second part of the inequality reads $|\gamma|_2 < b - 2 \Leftrightarrow 3N - 2k - 1 < 0$. Multiplying by $d > 0$ on both sides and using the constraint yields $k < \frac{d}{d-3}$. Combining with $0 < d \leq k$ yields $d < \frac{d}{d-3}$ which admits no solution for $d > 0$. Hence $H^{-d}(\text{GC}_d)$ is empty for even d .

- *Odd d :* In the odd case, the cohomology $H^\bullet(\text{GC}_d)$ is isomorphic to $H^\bullet(\text{GC}_3)$. According to the bounds on cohomology degree displayed in Section 4.4, the inequality $-b-2 < |\gamma|_3 < -2$ needs to hold for γ to be a non-trivial cocycle. The second inequality reads more explicitly $3N - 2k - 1 < 0$. Multiplying by $d > 0$ on both sides and using the constraint yields $k(d - 3) < d$. Assuming $d \neq 3$, and using the previous reasoning ensures that $H^{-d}(\text{GC}_d)$ is empty for odd $d \neq 3$. \square

It follows that stable Weyl factors induced by connected graphs only occur in dimension $d = 3$. The corresponding cohomology space is then $H^{-3}(\text{GC}_3)$ which happens to coincide with the dominant degree of GC_3 [cf. Section 4.4], thus allowing to generate infinitely many stable Weyl factors for Courant algebroids from trivalent graphs modulo IHX relations (see Figure 6 and eq.(20)). Explicit examples thereof will be displayed in Section 6.2.

6. Stable deformations of symplectic Lie n -algebroids

The aim of the present section is to illustrate some of the machinery developed in Section 5 to the case of NPQ-manifolds of degrees 1 and 2. As recalled in Example 3.2, the associated geometric notions (*i.e.* symplectic Lie 1, 2-algebroids) identify with the one of Poisson manifolds and Courant algebroids, respectively.

6.1 Poisson manifolds ($n = 1$) As shown in [105], NP-manifolds \mathcal{V} of degree 1 are in bijective correspondence with ordinary smooth manifolds \mathcal{M} via the identification of \mathcal{V} with the shifted cotangent bundle $T^*[1]\mathcal{M}$. The tower of fibrations (6) thus reduces to the vector bundle structure $T^*[1]\mathcal{M} \rightarrow \mathcal{M}$. The graded Poisson algebra of functions on $T^*[1]\mathcal{M}$ is isomorphic to the Gerstenhaber algebra of polyvector fields $\mathcal{T}_{\text{poly}}$ and Hamiltonian functions are in bijection with Poisson bivectors on \mathcal{M} .

The representation morphism $\text{Rep}^{(2)} : \text{Gra}_2 \hookrightarrow \text{End}_{\mathcal{G}^\infty(T^*[1]\mathcal{M})}$ of the 2-dimensional graph operad Gra_2 on the space of functions of the shifted cotangent bundle was first introduced by M. Kontsevich in [84, Section 5.2] and reads as (31) with Δ given by [cf. eq.(32)]:

$$\Delta_{ij} = \frac{\partial}{\partial x_{(i)}^\mu} \frac{\partial}{\partial p_\mu^{(j)}} + \frac{\partial}{\partial p_\mu^{(i)}} \frac{\partial}{\partial x_{(j)}^\mu}.$$

Following the leitmotiv of Section 5, the representation morphism $\text{Rep}^{(2)}$ can be used in order to induce stable structures on \mathcal{M} . In particular, using the isomorphism $H^0(\text{GC}_2) \simeq \mathfrak{grt}_1$, it follows from Corollary 5.7 that the Grothendieck–Teichmüller group $\text{GRT}_1 := \exp(\mathfrak{grt}_1)$ acts via Lie_∞ -automorphisms on the Schouten algebra $\mathcal{T}_{\text{poly}}$, see [119, 70].

At the linear level, Corollary 5.16 ensures that cocycles in $H^0(\text{GC}_2)$ yield stable flows on the space of Poisson bivectors. In other words, given a manifold \mathcal{M} and a cocycle $\gamma \in H^0(\text{GC}_2)$ with N vertices, one can define a map $\pi \mapsto \dot{\pi}$ mapping Poisson bivectors $\pi \in \Gamma(\wedge^2 T\mathcal{M})$ (thus satisfying $[\pi, \pi]_{\mathfrak{S}} = 0$) on \mathcal{M} to stable *Lichnerowicz cocycles i.e.* bivectors $\dot{\pi} \in \Gamma(\wedge^2 T\mathcal{M})$ satisfying $\delta_\pi \dot{\pi} := [\pi, \dot{\pi}]_{\mathfrak{S}} = 0$. Concretely, this is done by first defining the function $\mathcal{H} = \frac{1}{2}\pi^{\mu\nu}(x)p_\mu p_\nu$ – which can be checked to be Hamiltonian (*i.e.* $\{\mathcal{H}, \mathcal{H}\}_\omega = 0$) as a consequence of the fact that π is Poisson – and then define the function $\dot{\mathcal{H}} := \text{Rep}_N^{(2)}(\gamma)(\mathcal{H}^{\otimes N})$ – satisfying $\{\mathcal{H}, \dot{\mathcal{H}}\}_\omega = 0$ as a consequence of $\delta\gamma = 0$ and $\{\mathcal{H}, \mathcal{H}\}_\omega = 0$. Finally, one defines $\dot{\pi}$ as the principal symbol of the function $\dot{\mathcal{H}}$ *i.e.* $\dot{\mathcal{H}} = \frac{1}{2}\dot{\pi}^{\mu\nu}p_\mu p_\nu$.

The simplest example of the previous construction is given by the *tetrahedral flow* introduced in [84, Section 5.3] and further studied in [16, 15]. The latter is induced by the tetrahedron graph $\gamma_3 \in H^0(\text{GC}_2)$ [cf. Proposition 4.10 and Figure 5]. An explicit expression for the map $\pi \mapsto \dot{\pi}$ can be obtained by first using the orientation morphism (23) on γ_3 as in Figure 7, yielding a linear combination of four directed graphs. Decorating vertices with copies of the Hamiltonian function \mathcal{H} and interpreting edges as differential operators $\bar{\Delta}$ [see eq.(27)], the first two graphs vanish since they include vertices with more than two outgoing edges. The two remaining graphs yield the following local expression⁶³ for the Lichnerowicz cocycle $\dot{\pi}$ associated with the Poisson bivector:

$$\dot{\pi}^{\mu\nu} = \partial_\epsilon \pi^{\alpha\beta} \partial_\alpha \pi^{\gamma\delta} \partial_\gamma \pi^{\epsilon\lambda} \partial_{\beta\delta\lambda} \pi^{\mu\nu} + 6 \partial_\epsilon \pi^{\alpha\beta} \partial_\alpha \pi^{\gamma\delta} \partial_{\gamma\lambda} \pi^{\epsilon[\mu} \partial_{\beta\delta} \pi^{\nu]\lambda}. \quad (41)$$

Furthermore, it follows from Proposition 5.13 that the cocycle $\dot{\pi}$ can be promoted to a full formal Maurer-Cartan element in $(\mathcal{T}_{\text{poly}}[[\epsilon]], \delta_\pi, [\cdot, \cdot]_{\mathfrak{S}})$ thus yielding a stable formal Poisson structure⁶⁴ $\pi_* = \pi + \epsilon^4 \dot{\pi} + \dots$ such that $[\pi_*, \pi_*]_{\mathfrak{S}} = 0$.

Note that the tetrahedral flow is only the first and simplest example of an infinite set of stable flows on the space of Poisson bivectors provided by elements in the Grothendieck–Teichmüller algebra grt_1 . We refer in particular to [22] and [20] for results regarding the flows associated with the pentagon γ_5 and heptagon graphs γ_7 , respectively.

Relation to quantization. Before concluding with the $n = 1$ case, we recall known results regarding the deformation quantization problem for Poisson manifolds. Our emphasis will be on the classification problem for formality morphisms and how the above results regarding stable deformations of Poisson structures can be used to shed light on the matter. Such considerations will hopefully provide guiding lines in order to address cases for which the deformation quantization problem is less well understood [cf. Section 6.2 for a related discussion on Courant algebroids]. First, recall from the Introduction that Kontsevich’s solution to the deformation quantization problem for Poisson manifolds involves a formality morphism (1) *i.e.* a Lie_∞ quasi-isomorphism between the Schouten algebra on $\mathcal{T}_{\text{poly}}$ and the Hochschild dg Lie algebra of multidifferential operators $\mathcal{D}_{\text{poly}}$. As emphasised earlier, Kontsevich’s formality morphism is *stable* in a precise sense introduced in [36]. The set of (homotopy classes⁶⁵ of) stable formality morphisms of the form (1) will be denoted **SQI**. A first incarnation of the Grothendieck–Teichmüller group as playing a classification rôle for **SQI** stems from a construction due to D. Tamarkin in his formulation of an alternative proof to Kontsevich’s formality theorem [115, 61]. The latter provides a bijective map $\mathcal{U} : \text{DAss} \xrightarrow{\sim} \text{SQI}$ where **DAss** stands for the set of Drinfel’d associators. As mentioned earlier, the set **DAss** is a GRT_1 -torsor thus providing an (implicit) action of the Grothendieck–Teichmüller group on **SQI**. However, Tamarkin’s map is far from being explicit making it difficult to precisely characterise the corresponding GRT_1 -action on quantization procedures. The situation has been clarified by V. A. Dolgushev who showed in [36] that the set **SQI** is naturally endowed with a regular action of the pro-unipotent group $\exp(H^0(\text{GC}_2))$. This result, combined with T. Willwacher’s isomorphism $H^0(\text{GC}_2) \simeq \text{grt}_1$ [119] defines a regular GRT_1 -action on **SQI**, so that both sides of Tamarkin’s map $\mathcal{U} : \text{DAss} \xrightarrow{\sim} \text{SQI}$ are GRT_1 -torsors. It has furthermore

⁶³Although the two terms of eq.(41) already appeared in [84], the relative factor $1 : 6$ was only recently obtained in [16, 15] where it was also shown to constitute the unique choice allowing for the cocycle property to hold.

⁶⁴We refer to [8] for results regarding the Lichnerowicz cohomology associated with stable deformations π_* of Poisson manifolds.

⁶⁵We refer to [35] for a definition of the notion of homotopy equivalence between Lie_∞ -morphisms.

been shown in [38] that Tamarkin's map is equivariant with respect to the action of GRT_1 i.e. \mathcal{U} is a bijection of GRT_1 -torsors. Under this bijection, the (homotopy class of) Kontsevich's morphism with standard (or harmonic) propagator [85] is mapped to the Alekseev–Torossian associator [5, 112] (see [38]) while the (homotopy class of) Kontsevich's morphism with logarithmic propagator [86, 3] is mapped [103] to the Knizhnik–Zamolodchikov associator [44].

In this picture, one can argue that the map assigning to each group element in $\exp(H^0(\text{GC}_2)) \simeq \text{GRT}_1$ a homotopy class of Lie_∞ -automorphisms of $\mathcal{T}_{\text{poly}}$ [cf. [119, 70] and Corollary 5.7 for its generalisation to all d] constitutes a useful intermediate step allowing a precise characterisation of the arbitrariness in quantization procedures. In order to illustrate this, we let $\Phi \in \text{DAss}$ be a Drinfel'd associator and denote $[\mathcal{U}_\Phi] : \mathcal{T}_{\text{poly}} \xrightarrow{\sim} \mathcal{D}_{\text{poly}}$ the homotopy class of formality morphisms associated with Φ through Tamarkin's procedure. Let furthermore $\Gamma \in \exp(H^0(\text{GC}_2)) \simeq \text{GRT}_1$. The following diagram commutes (in the category of Lie_∞ -algebras with homotopy classes of Lie_∞ -quasi-isomorphisms as morphisms):

$$\begin{array}{ccc}
 \mathcal{T}_{\text{poly}} & \xrightarrow{[\mathcal{U}_\Phi]} & \mathcal{D}_{\text{poly}} \\
 [\mathcal{U}^\Gamma] \downarrow & \searrow [\mathcal{U}_{\Phi \cdot \Gamma}] & \downarrow [\mathcal{U}_{\mathcal{D}}(\Phi, \Gamma)] \\
 \mathcal{T}_{\text{poly}} & \xrightarrow{[\mathcal{U}_\Phi]} & \mathcal{D}_{\text{poly}}
 \end{array} \tag{42}$$

where:

- $[\mathcal{U}_{\Phi \cdot \Gamma}] : \mathcal{T}_{\text{poly}} \xrightarrow{\sim} \mathcal{D}_{\text{poly}}$ denotes the homotopy class of formality morphisms associated with the Drinfel'd associator $\Phi \cdot \Gamma$.
- $[\mathcal{U}^\Gamma] : \mathcal{T}_{\text{poly}} \xrightarrow{\sim} \mathcal{T}_{\text{poly}}$ denotes the homotopy class of Lie_∞ -automorphisms of $\mathcal{T}_{\text{poly}}$ associated with the element $\Gamma \in \text{GRT}_1$ through Corollary 5.7 (for $d = 2$).
- $[\mathcal{U}_{\mathcal{D}}(\Phi, \Gamma)] : \mathcal{D}_{\text{poly}} \xrightarrow{\sim} \mathcal{D}_{\text{poly}}$ denotes the homotopy class of Lie_∞ -quasi-isomorphisms from $\mathcal{D}_{\text{poly}}$ to itself associated with the pair (Φ, Γ) and defined through $[\mathcal{U}_{\mathcal{D}}(\Phi, \Gamma)] = [\mathcal{U}_\Phi \circ \mathcal{U}^\Gamma \circ \mathcal{U}_\Phi^{-1}]$, where \mathcal{U}_Φ^{-1} is a homotopy inverse⁶⁶ of the representative \mathcal{U}_Φ .

The regularity of the action of GRT_1 on SQI can be restated as follows: for any pair of homotopy classes of stable formality morphisms $[\mathcal{U}_\Phi]$ and $[\mathcal{U}_{\Phi'}]$, there exists a unique element $\Gamma \in \text{GRT}_1$ such that $[\mathcal{U}_{\Phi'}] = [\mathcal{U}_\Phi \circ \mathcal{U}^\Gamma]$, for any representatives \mathcal{U}_Φ and \mathcal{U}^Γ . The space SQI of homotopy classes of stable formality morphisms can then be fully explored by composition with Lie_∞ -automorphisms of $\mathcal{T}_{\text{poly}}$ induced from GRT_1 . Such a reasoning can also be shown to hold at the level of quantization maps. Indeed, each arrow appearing in Diagram 42 is a (homotopy class of) Lie_∞ -quasi-isomorphisms and thus induces a bijection between (equivalence classes of) Maurer–Cartan sets [cf. footnote 2] as:

$$\begin{array}{ccc}
 \text{FPoiss} & \xrightarrow{[\hat{\mathcal{U}}_\Phi]} & \text{Star} \\
 [\hat{\mathcal{U}}^\Gamma] \downarrow & \searrow [\hat{\mathcal{U}}_{\Phi \cdot \Gamma}] & \downarrow [\hat{\mathcal{U}}_{\mathcal{D}}(\Phi, \Gamma)] \\
 \text{FPoiss} & \xrightarrow{[\hat{\mathcal{U}}_\Phi]} & \text{Star}
 \end{array}
 \qquad
 \begin{array}{ccc}
 [\pi] & \xrightarrow{[\hat{\mathcal{U}}_\Phi]} & [*] \\
 [\hat{\mathcal{U}}^\Gamma] \downarrow & \searrow [\hat{\mathcal{U}}_{\Phi \cdot \Gamma}] & \downarrow [\hat{\mathcal{U}}_{\mathcal{D}}(\Phi, \Gamma)] \\
 [\pi'] & \xrightarrow{[\hat{\mathcal{U}}_\Phi]} & [*']
 \end{array}$$

⁶⁶Here, we use the fact that a Lie_∞ -morphism is a quasi-isomorphism if and only if it is a homotopy equivalence [Section 3.7 of [73] for a statement as well as references therein]. This implies that there exists a (non-canonical) Lie_∞ -quasi-isomorphism $\mathcal{U}_\Phi^{-1} : \mathcal{D}_{\text{poly}} \rightarrow \mathcal{T}_{\text{poly}}$ such that $\mathcal{U}_\Phi \circ \mathcal{U}_\Phi^{-1} \sim \text{id}_{\mathcal{D}_{\text{poly}}}$ and $\mathcal{U}_\Phi^{-1} \circ \mathcal{U}_\Phi \sim \text{id}_{\mathcal{T}_{\text{poly}}}$. Contrarily to its counterpart \mathcal{U}^Γ , the family of Lie_∞ -automorphisms $\mathcal{U}_{\mathcal{D}}$ does depend on the existence of a Drinfel'd associator (although the explicit choice does not matter due to the equivariance relation $\mathcal{U}_{\mathcal{D}}(\Phi \cdot \Gamma', \Gamma) = \mathcal{U}_{\mathcal{D}}(\Phi, \text{Ad}_{\Gamma'} \Gamma)$ with $\text{Ad}_{\Gamma'} \Gamma = \Gamma' \cdot \Gamma \cdot \Gamma'^{-1}$).

Mimicking the above reasoning allows to span the whole space of stable quantization maps $\hat{\mathcal{U}}_\Phi$ by composition with stable deformation maps $\hat{\mathcal{U}}^\Gamma$ induced from GRT_1 [cf. Proposition 5.13]. The resulting characterisation of the action of GRT_1 on stable quantization maps in terms of stable deformations has the merit to make certain features relatively explicit. As an example, it follows from the previous reasoning that formal Poisson structures $[\pi]$ which are insensitive to stable deformations⁶⁷ admit canonical quantizations *i.e.* their quantum class is unique⁶⁸. Straightforward reasoning on the number of derivatives involved in stable deformations [see *e.g.* eq.(41)] entails that Poisson bivectors whose local description is at most quadratic in coordinates admit a unique local quantization. This is in particular the case for constant Poisson bivectors (and in particular for symplectic manifolds in Darboux coordinates) which are uniquely (locally) quantized by the Groenewold–Moyal star product [56, 100], *cf.* eq.(36). Slightly less trivial is the Kostant–Souriau–Kirillov Poisson bracket – defined on the dual of any Lie algebra – which is linear in coordinates. The latter admits two known quantizations, namely the Gutt [60, 45] and Kontsevich [85] star products. According to the previous reasoning, these two star products must belong to the same equivalence class. However, they do not coincide, as shown for example in [85, 72, 113, 34, 9]. Rather, they are related via an isomorphism given by the Duflo map (*cf.* *e.g.* Theorem 14 in [47]).

6.2 Courant algebroids ($n = 2$) The present section applies the results of Section 5 to symplectic Lie 2-algebroids. The latter notion identifies with the one of Courant algebroids that we now review, following the presentation *à la* Dorfman (*cf.* *e.g.* [71] for details and [87] for a historical account).

Definition 6.1 (Courant algebroid). A Courant algebroid is a quadruplet $(E, \langle \cdot, \cdot \rangle_E, \mathcal{D}, [\cdot, \cdot]_E)$ where:

- The pair $(E, \langle \cdot, \cdot \rangle_E)$ is a **pseudo-Euclidean vector bundle** *i.e.*
 - $E \rightarrow \mathcal{M}$ is a vector bundle over the smooth manifold \mathcal{M} . We will denote $(\mathcal{C}^\infty(\mathcal{M}), \cdot)$ the commutative associative algebra of functions on \mathcal{M} and $*$: $\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(E) \rightarrow \Gamma(E)$ the module structure on fibers of E . The latter satisfies the associativity relation $f * (g * X) = (f \cdot g) * X$ for all $f, g \in \mathcal{C}^\infty(\mathcal{M})$ and $X \in \Gamma(E)$.
 - The map $\langle \cdot, \cdot \rangle_E : \Gamma(E) \otimes \Gamma(E) \rightarrow \mathcal{C}^\infty(\mathcal{M})$ satisfies the following conditions:
 1. $\mathcal{C}^\infty(\mathcal{M})$ -bilinear *i.e.* $\langle f * X, Y \rangle_E = \langle X, f * Y \rangle_E = f \cdot \langle X, Y \rangle_E$
 2. symmetric *i.e.* $\langle X, Y \rangle_E = \langle Y, X \rangle_E$
 3. non-degenerate *i.e.* $\langle X, Y \rangle_E = 0$ for all $Y \in \Gamma(E) \Leftrightarrow X = 0$.
 A bilinear form satisfying these conditions will be referred to as a **fiber-wise metric**.
- The pair $(\mathcal{D}, [\cdot, \cdot]_E)$ is a **Courant–Dorfman structure** on $(E, \langle \cdot, \cdot \rangle_E)$ *i.e.*
 - $[\cdot, \cdot]_E : \Gamma(E) \otimes \Gamma(E) \rightarrow \Gamma(E)$ is a \mathbb{K} -bilinear form on the fibers of E called the **Dorfman bracket**.
 - $\mathcal{D} : \mathcal{C}^\infty(\mathcal{M}) \rightarrow \Gamma(E)$ is a \mathbb{K} -linear derivation *i.e.* $\mathcal{D}(f \cdot g) = f * \mathcal{D}g + g * \mathcal{D}f$ for all $f, g \in \mathcal{C}^\infty(\mathcal{M})$. The derivation \mathcal{D} defines a $\mathcal{C}^\infty(\mathcal{M})$ -linear map $\rho : \Gamma(E) \rightarrow \Gamma(T\mathcal{M})$ called the **anchor** as $\rho_X[f] = \langle X, \mathcal{D}f \rangle_E$ for all $f \in \mathcal{C}^\infty(\mathcal{M})$, $X \in \Gamma(E)$.

such that the following conditions are satisfied:

⁶⁷That is, such that $\hat{\mathcal{U}}^\Gamma([\pi]) = [\pi]$ for all $\Gamma \in \text{GRT}_1$.

⁶⁸In other words, the associated (class of) star products $[\ast] = \hat{\mathcal{U}}_\Phi([\pi])$ does not depend on the choice of Drinfel’d associator Φ .

1. The Dorfman bracket satisfies the Jacobi identity in its Leibniz form:

$$[X, [Y, Z]_E]_E = [[X, Y]_E, Z]_E + [Y, [X, Z]_E]_E \text{ for all } X, Y, Z \in \Gamma(E)$$

so that the pair $(\Gamma(E), [\cdot, \cdot]_E)$ is a \mathbb{K} -Leibniz algebra.

2. The symmetric part of the Dorfman bracket is controlled by the derivation \mathcal{D} as:

$$[X, Y]_E + [Y, X]_E = \mathcal{D} \langle X, Y \rangle_E \text{ for all } X, Y \in \Gamma(E) .$$

3. The fiber-wise metric $\langle \cdot, \cdot \rangle_E$ is compatible with the Courant–Dorfman structure $(\mathcal{D}, [\cdot, \cdot]_E)$, *i.e.*

$$\langle X, \mathcal{D} \langle Y, Z \rangle_E \rangle_E = \langle [X, Y]_E, Z \rangle_E + \langle Y, [X, Z]_E \rangle_E = 0 \text{ for all } X, Y, Z \in \Gamma(E) .$$

Introducing a basis $\{e_a\}_{a=1, \dots, \dim E}$ of the space of sections $\Gamma(E)$ allows to provide a component expression of the Courant algebroid maps as follows:

- In components, the fiber wise metric reads $\langle X, Y \rangle_E = \kappa_{ab} X^a Y^b$ where the constant matrix κ satisfies:
 1. κ is symmetric *i.e.* $\kappa_{ab} = \kappa_{ba}$.
 2. κ admits an inverse κ^{-1} such that $\kappa^{ac} \kappa_{cb} = \delta_b^a$ with δ the Kronecker delta.
- The component expression for the Courant–Dorfman structure $(\mathcal{D}, [\cdot, \cdot]_E)$ is captured by a pair (ρ_a^μ, T_{abc}) , where T_{abc} is totally skewsymmetric. Explicitly, we have:
 - \mathcal{D} -map: $\mathcal{D}f = \kappa^{ab} \rho_b^\mu \partial_\mu f e_a$
 - Anchor: $\rho_X[f] = X^a \rho_a^\mu \partial_\mu f$
 - Dorfman bracket: $[X, Y]_E = (\rho_X[Y^a] - \rho_Y[X^a] - T_{bc}^a X^b Y^c + \kappa^{ab} \rho_b^\mu \partial_\mu X^c \kappa_{cd} Y^d) e_a$

where indices are raised and lowered with κ .

It can be checked that the defining conditions of a Courant algebroid are satisfied if and only if the pair (ρ_a^μ, T_{abc}) satisfies the set of conditions:

1. $\mathcal{C}_1^{\mu\nu} := \rho_a^\mu \kappa^{ab} \rho_b^\nu = 0$
2. $\mathcal{C}_2^{ab} := \rho_c^\mu \kappa^{cd} T_{dab} + 2 \rho_{[a}^\lambda \partial_\lambda \rho_b]^\mu = 0$
3. $\mathcal{C}_3^{abcd} := \frac{1}{4} T_e[ab \kappa^{ef} T_{cd}]_f + \frac{1}{3} \rho_{[a}^\mu \partial_\mu T_{bcd]} = 0$.

Comparing this set of constraints with (9)-(11) allows to relate Courant algebroids with symplectic Lie 2-algebroids (or NPQ-manifolds of degree 2). The precise nature of this relation is articulated in the following theorem:

Theorem 6.2 (D. Roytenberg [105]).

- NP-manifolds of degree 2 are in bijective correspondence with pseudo-Euclidean vector bundles.
- NPQ-manifolds of degree 2 are in bijective correspondence with Courant algebroids.

The Poisson algebra of functions associated with a given NP-manifold \mathcal{V} of degree 2 (or equivalently the 2-Schouten algebra $\mathcal{T}_{\text{poly}}^{(2)} = \mathcal{C}^\infty(\mathcal{V}) [2]$) was referred to as the *Rothstein algebra* in [74]. The latter can be interpreted as the deformation complex of Hamiltonian functions on \mathcal{V} . Via the second point of Theorem 6.2, this can be rephrased as saying that the Rothstein algebra controls the deformation theory of Courant–Dorfman structures $(\mathcal{D}, [\cdot, \cdot]_E)$ on the pseudo-Euclidean vector bundle $(E, \langle \cdot, \cdot \rangle_E)$ – where E is defined by the fibration (6) as $\mathcal{M} \leftarrow E[1] \leftarrow \mathcal{V}$ – according to the following sequence of bijective correspondences:

$$(\mathcal{D}, [\cdot, \cdot]_E) \Leftrightarrow (\rho_a^\mu, T_{abc}) \Leftrightarrow \mathcal{H} = \rho_a^\mu \xi^a p_\mu + \frac{1}{6} T_{abc} \xi^a \xi^b \xi^c \tag{43}$$

where the right-hand side makes use of the local set of coordinates $\{x_0^\mu, \xi_1^a, p_2^\mu\}$ [Example 3.2].

The supergeometric interpretation of Courant algebroids provided by Theorem 6.2 will allow us to apply the results of Section 5.3 in order to generate new stable deformation formulas for Courant–Dorfman structures $(\mathcal{D}, [\cdot, \cdot]_E)$ on a given pseudo-Euclidean vector bundle $(E, \langle \cdot, \cdot \rangle_E)$.⁶⁹ As noted in Lemma 5.8, the zeroth cohomology of the connected part of the full Kontsevich graph complex in $d = 3$ is one dimensional and spanned by the triangle class *i.e.* $H^0(\mathfrak{fGC}_3^{\text{con}}) = \mathbb{K}\langle L_3 \rangle$, *cf.* Figure 4. This result ensures (*cf.* Proposition 5.9) that there exists a *unique* stable deformation of Courant algebroids that we now explicitly characterise.

Letting $(E, \langle \cdot, \cdot \rangle_E)$ be a pseudo-Euclidean vector bundle, we use the bijective correspondence (43) in order to associate to each Courant–Dorfman structure $(\mathcal{D}, [\cdot, \cdot]_E)$ on $(E, \langle \cdot, \cdot \rangle_E)$ the corresponding Hamiltonian function $\mathcal{H} = \rho_a^\mu \xi^a p_\mu + \frac{1}{6} T_{abc} \xi^a \xi^b \xi^c$ with associated homological vector field $Q := \{\mathcal{H}, \cdot\}_\omega$. Via Corollary 5.16, the stable Hamiltonian flow associated with L_3 is defined as:

$$\dot{\mathcal{H}} = \text{Rep}_3^{(3)}(L_3)(\mathcal{H}^{\otimes 3}) = \mu_3(\Delta_{12} \Delta_{23} \Delta_{31}(\mathcal{H}^{\otimes 3}))$$

where the expression of the operator $\text{Rep}_3^{(3)}$ follows (31) with Δ given by [*cf.* eq.(33)]:

$$\Delta_{ij} = \frac{\partial}{\partial x_{(i)}^\mu} \frac{\partial}{\partial p_\mu^{(j)}} - \frac{\partial}{\partial p_\mu^{(i)}} \frac{\partial}{\partial x_{(j)}^\mu} + \frac{\partial}{\partial \xi_{(i)}^a} \kappa^{ab} \frac{\partial}{\partial \xi_{(j)}^b}.$$

Explicitly, the triangle Hamiltonian flow maps any Hamiltonian function \mathcal{H} towards the associated Rothstein cocycle $\dot{\mathcal{H}} \in H^3(\mathcal{C}^\infty(\mathcal{V}) | Q)$ defined as $\dot{\mathcal{H}} = \dot{\rho}_a^\mu \xi^a p_\mu + \frac{1}{6} \dot{T}_{abc} \xi^a \xi^b \xi^c$ where:

$$\begin{aligned} \dot{\rho}_a^\mu &= \text{Rep}_3^{(3)}(L_3)(\mathcal{H}^{\otimes 3})_{a^\mu} = \begin{array}{c} \rho_\bullet - \rho_\bullet^\mu \\ \searrow \nearrow \\ \rho_a \end{array} + \begin{array}{c} \rho_\bullet \rightarrow \rho_\bullet^\mu \\ \searrow \nearrow \\ T_{a\bullet\bullet} \end{array} \\ \dot{T}_{abc} &= 6 \text{Rep}_3^{(3)}(L_3)(\mathcal{H}^{\otimes 3})_{abc} = \begin{array}{c} \rho_a \rightarrow \rho_b \\ \searrow \nearrow \\ \rho_c \end{array} - \begin{array}{c} \rho_a \leftarrow \rho_b \\ \searrow \nearrow \\ \rho_c \end{array} - \begin{array}{c} T_{a\bullet\bullet} - T_{b\bullet\bullet} \\ \searrow \nearrow \\ T_{c\bullet\bullet} \end{array} \\ &\quad + 3 \begin{array}{c} \rho_\bullet \rightarrow \rho_a \\ \searrow \nearrow \\ T_{b\bullet\bullet} \end{array} + 3 \begin{array}{c} \rho_\bullet - T_{a\bullet\bullet} \\ \searrow \nearrow \\ T_{bc\bullet} \end{array} + \text{skewsym. } (a - b - c) \end{aligned}$$

Here, the directed arrows stand for space-time derivatives while undirected arrows represent contractions of fiber indices with the non-degenerate symmetric bilinear form κ . The local expression of the Hamiltonian flow induced by the triangle cocycle can be equivalently expressed in components as:

$$\begin{aligned} \dot{\rho}_a^\mu &= \rho_b^\lambda \partial_\lambda \rho_a^\nu \partial_\nu \rho^{b\mu} + \rho_b^\lambda \partial_\lambda \rho_c^\mu T_a^{bc} \\ \dot{T}_{abc} &= \partial_\mu \rho_a^\nu \partial_\nu \rho_b^\lambda \partial_\lambda \rho_c^\mu - \partial_\mu \rho_a^\lambda \partial_\nu \rho_b^\mu \partial_\lambda \rho_c^\nu - T_a^{de} T_{bdf} T_{ce}^f \\ &\quad + 3 \rho_d^\mu \partial_\mu \rho_{[a}^\nu \partial_\nu T_{bc]}^d + 3 \rho_d^\mu T_{[a}^{de} \partial_\mu T_{bc]e} \end{aligned} \tag{44}$$

where indices are raised and lowered with κ . Consistently with Proposition 5.14, it can be checked that $\{\mathcal{H}, \dot{\mathcal{H}}\}_\omega = 0$ modulo the relations (9)-(11) coming from $\{\mathcal{H}, \mathcal{H}\}_\omega = 0$.

⁶⁹In this sense, the procedure does not deform the *full* Courant algebroid structure since the bilinear form $\langle \cdot, \cdot \rangle_E$ remains undeformed.

It should be emphasised that the situations corresponding to $d = 2$ and $d = 3$ are drastically different. In the case $d = 2$, the zeroth cohomology is the “dominant” degree *i.e.* contains an infinite number of non-trivial classes leading to infinitely many stable deformations of Poisson manifolds, *cf.* Section 6.1. On the contrary, for $d = 3$, the zeroth cohomology is one-dimensional and thus yields a unique stable deformation of Courant–Dorfman structures given by (44). The “dominant” degree of \mathfrak{fGC}_3 being -3 (*cf.* Section 4.4), it would be desirable to find a construction mapping elements of $H^{-3}(\mathfrak{fGC}_3)$ to Courant–Dorfman deformations. This is achieved through Weyl factors, as we now show.

Weyl factors for Courant algebroids. Recall from Section 4.4 that $H^{-3}(\mathfrak{fGC}_3)$ is spanned by trivalent graphs modulo IHX relations [see Figure 6 and eq.(20)]. Proposition 5.18 ensures that each element $\gamma \in H^{-3}(\mathfrak{fGC}_3)$ is mapped to a stable Weyl factor for Courant algebroids. Explicitly, given a trivalent graph γ , Hamiltonian functions \mathcal{H} can be mapped to Weyl factors $\Omega_\gamma(\mathcal{H}) \in H^0(\mathcal{C}^\infty(\mathcal{V})|\mathbb{Q}) \simeq \text{Ker } \mathcal{D}$. The explicit local expression of the Weyl factor $\Omega_\gamma(\mathcal{H})$ associated to a given graph $\gamma \in H^{-3}(\mathfrak{fGC}_3)$ with N vertices is given by $\Omega_\gamma(\mathcal{H}) := \text{Rep}_N^{(3)}(\gamma)(\mathcal{H}^{\otimes N})$. We now exemplify this construction by displaying the Weyl factors associated to the simplest trivalent graphs. The simplest example of trivalent graph is given by the “ Θ ” graph \bigoplus being the only connected trivalent graph with $N = 2$ vertices. The latter yields the following Weyl factor:

$$\begin{aligned} \Omega_\Theta(\mathcal{H}) &:= \text{Rep}_2^{(3)}(\Theta)(\mathcal{H}^{\otimes 2}) = T_{\bullet\bullet\bullet} \text{---} T_{\bullet\bullet\bullet} + 6 \rho_\bullet \text{---} \rho_\bullet \\ &= T_{abc} T^{abc} + 6 \partial_\nu \rho_a^\lambda \partial_\lambda \rho^{a\nu} \end{aligned} \tag{45}$$

and the equality $\mathcal{D} \Omega_\Theta = 0$ follows from $\{\mathcal{H}, \mathcal{H}\}_\omega = 0$. In particular, this ensures that $\mathcal{H}_\Theta := \Omega_\Theta(\mathcal{H}) \cdot \mathcal{H}$ is a Rothstein cocycle.⁷⁰

The next to simplest case is given by the graphs A and B from Figure 6 for $N = 4$ yielding:

$$\begin{aligned} \Omega_A(\mathcal{H}) &:= \text{Rep}_4^{(3)}(A)(\mathcal{H}^{\otimes 4}) = T_{abc} T^{abd} T^{cef} T_{def} + 4 \partial_\mu \rho_a^\nu \partial_\nu \rho_b^\mu \partial_\lambda \rho^{a|\rho} \partial_\rho \rho^{b|\lambda} \\ &\quad - 8 \partial_\mu \rho_a^\nu \partial_\nu \rho^{a|\lambda} \partial_\lambda \rho_b^\rho \partial_\rho \rho^{b|\mu} + 4 \partial_\mu \rho^{a|\nu} \partial_\nu \rho_d^\mu T_{abc} T^{dbc} \\ \Omega_B(\mathcal{H}) &:= \text{Rep}_4^{(3)}(B)(\mathcal{H}^{\otimes 4}) \\ &= T_{abc} T^a_{de} T^{bdf} T^{ce}_f - 8 \partial_\mu \rho_a^\nu \partial_\nu \rho_b^\lambda \partial_\lambda \rho_c^\mu T^{abc} - 6 \partial_\mu \rho_a^\nu \partial_\nu \rho_b^\lambda \partial_\lambda \rho^{a|\rho} \partial_\rho \rho^{b|\mu}. \end{aligned}$$

Together with $\Omega_\Theta(\mathcal{H})^2$ (corresponding to the disconnected graph $\gamma = \Theta \cup \Theta$), these are the only Weyl factors available for $N = 4$. Note however that the trivalent graphs A and B can be related through the IHX relation (20) as $A \sim 2B$, *cf.* footnote 40. This ensures that their respective Weyl factors are related via $\Omega_A(\mathcal{H}) = 2\Omega_B(\mathcal{H})$ where the correspondence can be shown by making use of the constraints (9)-(11) coming from $\{\mathcal{H}, \mathcal{H}\}_\omega = 0$.

Relation to quantization. Remarkably, Kontsevich’s original quantization formula (*i.e.* with standard propagator) can be interpreted [85, 25, 26, 27] as a 3-point function in the path integral quantization of a 2-dimensional topological field theory – the **Poisson σ -model**, introduced in [65, 68, 108] – whose source is of dimension $d = 2$ and whose target is the (shifted) cotangent bundle associated to the Poisson manifold. The graphs appearing in Kontsevich’s stable formula

⁷⁰It can be checked by brute-force computation that the vector space of universal Hamiltonian flows for $N = 3$ is of dimension 2 and spanned by the triangle flow (44) and the Weyl Θ -flow defined from (45).

can therefore be interpreted from the point of view of quantum field theory as Feynman diagrams associated with the quantification of the Poisson σ -model.

As mentioned previously, the Poisson σ -model constitutes the first rung of an infinite ladder of AKSZ σ -models [6] associating to any symplectic Lie n -algebroid a topological field theory of dimension $d = n + 1$. An interesting open problem concerns the possibility of generalising such interplay between deformation quantization results (on the algebraic side) and quantization of AKSZ-type of models (on the field theoretic side) to higher values of n .

For $n = 2$, the relevant AKSZ σ -model was constructed by D. Roytenberg in [106] (*cf.* [67] for an earlier derivation from consistent deformations of a Chern–Simons gauge theory coupled with a 0-dimensional BF theory). Such model associates to any Courant algebroid a canonical 3-dimensional topological field theory – the **Courant σ -model**. From the field theory side, quantization of the Courant σ -model within the Batalin–Vilkovisky formalism [12] has been considered in [64, 63] (*cf.* also [66] for a discussion of observables in general AKSZ σ -models).

On the algebraic side, a possible candidate for the quantum notion associated with Courant algebroids is given by **vertex algebroids**, as introduced in [54] from truncation of vertex algebras [55]. Indeed, it was shown in [17] that the semi-classicalisation of (commutative) vertex algebroids yields a Courant algebroid. This suggests a formulation of a deformation quantization problem for Courant algebroids, similar to the one formulated in [14, 13] for Poisson manifolds.

Although it is outside of the scope of the present paper to address the quantization problem for Courant algebroids, we note that some insights can be gained⁷¹ from the classification of graph cocycles in $H^\bullet(\mathfrak{fGC}_3)$:

1. $H^1(\mathfrak{fGC}_3) = \mathbf{0}$: *The existence of stable formality morphisms for Courant algebroids is unobstructed.*
2. $H^0(\mathfrak{fGC}_3) = \mathbb{K}$: *The space of stable formality morphisms for Courant algebroids is of dimension 1.*

The first statement asserts that the 2-Schouten algebra $(\mathcal{T}_{\text{poly}}^{(2)}, [\cdot, \cdot]_{\mathfrak{S}})$ is rigid – at least in the “stable setting” – *i.e.* it does not admit non-trivial deformations as a Lie_∞ -algebra. In other words, the graded Lie algebra $\mathcal{T}_{\text{poly}}^{(2)}$ is intrinsically formal⁷² in the stable setting. The homotopy transfer theorem thus ensures that, given a dg Lie algebra (or Lie_∞ -algebra) $\mathcal{D}_{\text{poly}}^{(2)}$ such that $H^\bullet(\mathcal{D}_{\text{poly}}^{(2)})$ is isomorphic to $\mathcal{T}_{\text{poly}}^{(2)}$ as a graded Lie algebra and a quasi-isomorphism of complexes $\mathcal{U}_1 : \mathcal{T}_{\text{poly}}^{(2)} \xrightarrow{\sim} \mathcal{D}_{\text{poly}}^{(2)}$ given by stable formulæ, the “HKR-type” map \mathcal{U}_1 can always be prolonged to a full stable Lie_∞ -quasi-isomorphism $\mathcal{U} : \mathcal{T}_{\text{poly}}^{(2)} \xrightarrow{\sim} \mathcal{D}_{\text{poly}}^{(2)}$, which in turn would provide a quantization map for Courant algebroids.

Recall that such reasoning⁷³ constituted the initial rationale behind the introduction of the graph complex \mathfrak{fGC}_2 in [84]. However, in this case, it is a hard open conjecture that $H^1(\mathfrak{fGC}_2) = \mathbf{0}$ so that M. Kontsevich had to rely on different methods in order to prove his formality theorem for Poisson manifolds. On the contrary, for Courant algebroids, it is straightforward to show the rigidity of the 2-Schouten algebra (see above) so that one can use the original Kontsevich approach to prove a formality theorem for Courant algebroids.

From the second statement, we learn that such morphism is not unique, but rather that

⁷¹We refer to [99] for more details on the partition of deformation quantization problems according to graph cohomology.

⁷²A graded Lie algebra $(\mathfrak{h}, [\cdot, \cdot])$ is said to be **intrinsically formal** if any Lie_∞ -algebra (\mathfrak{g}, l) restricting to the dg Lie algebra $(\mathfrak{h}, 0, [\cdot, \cdot])$ (*i.e.* with vanishing differential) in cohomology (*i.e.* $H(\mathfrak{g}, l) = (\mathfrak{h}, 0, [\cdot, \cdot])$) is formal.

⁷³We refer to Section 4.1 of [99] for more details.

formality morphisms form a 1-dimensional space. Consequently, there should exist a one-parameter family of stable quantization maps for Courant algebroids⁷⁴. This is again in sharp contrast with the Poisson case for which $H^0(\mathfrak{fGC}_2)$ is infinite-dimensional (being isomorphic to the Grothendieck–Teichmüller Lie algebra \mathfrak{grt}_1) and thus the space of formality morphisms (and consequently also the one of stable quantization maps for Poisson manifolds) forms an infinite-dimensional space (in bijective correspondence with the space of Drinfel'd associators).

However, as shown in Corollary 5.19, there is a way to consistently deform Courant algebroids via trivalent vertices (modulo IHX relations) using Weyl deformation maps. Composing such deformations with a given quantization map yields an infinite-dimensional space of quantizations for Courant algebroids. In other words, despite the fact that the space of stable formality morphisms is finite-dimensional (of dimension 1), the space of universal quantizations of a given Courant algebroid is infinite-dimensional, the arbitrariness being encoded into trivalent graphs (on top of the triangle graph).

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⁷⁴As in the Poisson case, some Courant algebroids are insensitive to deformations so that their associated quantum class is unique *i.e.* independent of the parameter. This can be shown to be the case of *exact* Courant algebroids [see *e.g.* [71] for a definition] which are insensitive to the triangular deformation defined by (44).

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