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# Higher algebra of $A_\infty$ and $\Omega BAs$ -algebras in Morse theory I

Thibaut Mazuir<sup>a</sup>

<sup>a</sup>*Institut für Mathematik, Humboldt Universität zu Berlin, Germany*

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## Abstract

Elaborating on works by Abouzaid and Mescher, we prove that for a Morse function on a smooth compact manifold, its Morse cochain complex can be endowed with an  $\Omega BAs$ -algebra structure through counts of perturbed Morse gradient trees. This rich structure descends to its already known  $A_\infty$ -algebra structure. We then introduce the notion of  $\Omega BAs$ -morphism between two  $\Omega BAs$ -algebras and prove that given two Morse functions, one can construct an  $\Omega BAs$ -morphism between their associated  $\Omega BAs$ -algebras through counts of 2-colored perturbed Morse gradient trees. This continuation morphism is a quasi-isomorphism and induces a standard  $A_\infty$ -morphism between the induced  $A_\infty$ -algebras. We work with integer coefficients, and provide to this extent a detailed account on the sign conventions for  $A_\infty$ -algebras,  $\Omega BAs$ -algebras,  $A_\infty$ -morphisms and  $\Omega BAs$ -morphisms, using polytopes and moduli spaces of metric trees which explicitly realize the dg operadic objects encoding them. Our proofs also involve showing at the level of polytopes that an  $\Omega BAs$ -morphism between  $\Omega BAs$ -algebras naturally induces an  $A_\infty$ -morphism between  $A_\infty$ -algebras. This paper is addressed to people acquainted with either differential topology or algebraic operads, and written in a way to be hopefully understood by both communities. It comes in particular with a short survey on operads,  $A_\infty$ -algebras and  $A_\infty$ -morphisms, the associahedra and the multiplihedra. All the details on transversality, gluing maps, signs and orientations for the moduli spaces defining the algebraic structures on the Morse cochains are thoroughly carried out. It moreover lays the basis for a second article in which we solve the problem of finding a satisfactory notion of higher morphisms between  $A_\infty$ -algebras and between  $\Omega BAs$ -algebras, and show how this higher algebra of  $A_\infty$  and  $\Omega BAs$ -algebras provides a natural framework to give a higher categorical meaning to the fact that continuation morphisms in Morse theory are well-defined up to homotopy at the chain level.

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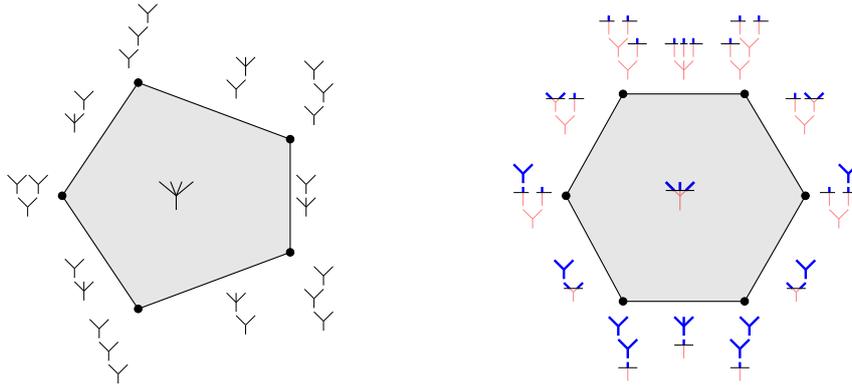
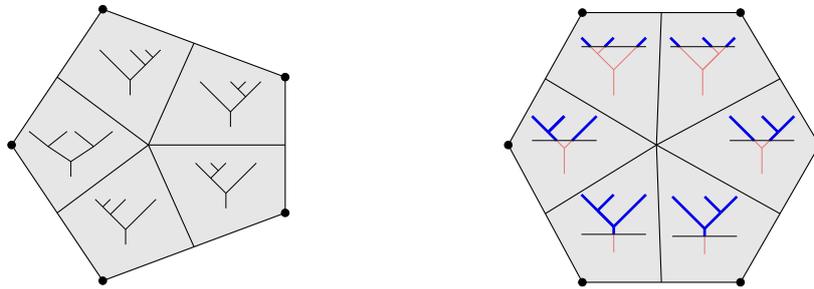
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Email address: [thibaut.mazuir@hotmail.fr](mailto:thibaut.mazuir@hotmail.fr) (Thibaut Mazuir)

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The associahedron  $K_4$  and the multiplihedron  $J_3$  ...... and their  $\Omega$ BAAs-cell decompositions

## INTRODUCTION

**Context and main goal** — The notion of  $A_\infty$ -algebra was first introduced in the seminal work of Stasheff on iterated loop spaces [39] to describe *strongly homotopy associative algebras*: it corresponds to the datum of a cochain complex  $A$  endowed with operations  $m_n : A^{\otimes n} \rightarrow A$  for  $n \geq 2$ , such that  $m_2$  represents the multiplication on  $A$  and the operations  $m_n$  are the higher homotopies encoding the lack of associativity of  $m_2$ . This structure is in fact encoded by an operad: the operad  $A_\infty$ , which is the minimal model of the operad  $As$  encoding associative algebras, as explained in [25] for instance.

*Morse theory* corresponds to the study of manifolds endowed with a *Morse function*, i.e. a function whose critical points are non-degenerate (see [36]). Given a smooth compact manifold  $M$ , Fukaya constructed in [13] an  $A_\infty$ -category whose objects are functions  $f_i$  on the manifold  $M$ , whose spaces of morphisms between two functions  $f_i$  and  $f_j$  (such that  $f_i - f_j$  is Morse) are the Morse cochain complexes  $C^*(f_i - f_j)$ , and whose higher multiplications are defined by counts of Morse ribbon trees. Adapting this construction to the case of a single Morse function  $f$  on the manifold  $M$ , Abouzaid defines in [3] an  $A_\infty$ -algebra structure on the Morse cochains  $C^*(f)$  through counts of *perturbed Morse gradient ribbon trees*. His work was subsequently continued by Mescher in [35]. See also [11] and [1].

The goal of this work in three articles is to study the unicity up to homotopy of the strongly homotopy associative algebra structure on the Morse cochains  $C^*(f)$ . The main objectives of this first paper are as follows:

1. We provide a quick and self-contained survey on operads and strongly homotopy associative structures for the use of geometers. We give special attention to the definition of the polytopes and moduli spaces of metric trees encoding these structures, and also prove some folklore theorems in the process.
2. We prove that the Morse cochains can in fact be endowed with an  $\Omega BAs$ -algebra structure (an alternative and refined notion of strongly homotopy associative algebra) and construct *continuation morphisms* between the Morse cochains of two different Morse functions: these continuation morphisms are  $\Omega BAs$ -morphisms (a newly defined notion of morphisms between  $\Omega BAs$ -algebras which preserve the product up to homotopy) and induce isomorphisms in homology. This is a first step towards the formulation of the unicity up to homotopy of the  $\Omega BAs$ -algebra structure on the Morse cochains at the chain-level.
3. We carry out a detailed study of the analysis, and the signs and orientations involved in the definition of these structures, and construct explicit gluing maps for the moduli spaces of perturbed Morse trees used therein. This completes in particular some of the proofs of Abouzaid in [3] which were only sketched in his paper. Our proofs are moreover written in a way to be hopefully accessible to people from the algebraic operads community.

**Outline of the paper** — Our first part begins with concise and self-contained recollections on the theory of algebraic (nonsymmetric) operads, that we subsequently specialize to the case of  $A_\infty$ -algebras,  $A_\infty$ -morphisms between them and their homotopy theory. We resort in particular to the convenient setting of operadic bimodules to define the operadic bimodule  $M_\infty$  encoding  $A_\infty$ -morphisms between  $A_\infty$ -algebras. We then recall how the operad  $A_\infty$  and the operadic bimodule  $M_\infty$  can be realized using families of polytopes, respectively known as the associahedra and the multiplihedra. The associahedra can themselves be realized as geometric moduli spaces: the compactified moduli spaces of stable metric ribbon trees  $\overline{\mathcal{T}}_n$ . These moduli spaces come with a refined cell decomposition encoding the operad  $\Omega BAs$ . Likewise, the multiplihedra can be realized as the compactified moduli spaces of 2-colored metric stable ribbon trees  $\overline{\mathcal{CT}}_n$ . Endowing these moduli spaces with a refined cell decomposition, we introduce a new operadic bimodule: the *operadic bimodule*  $M_{\Omega BAs}$ , encoding  $\Omega BAs$ -morphisms between  $\Omega BAs$ -algebras.

*The operadic bimodule  $M_{\Omega BAs}$  is the quasi-free  $(\Omega BAs, \Omega BAs)$ -operadic bimodule generated by the set of 2-colored stable ribbon trees*

$$M_{\Omega BAs} := \mathcal{F}^{\Omega BAs, \Omega BAs}(\uparrow, \downarrow, \downarrow, \downarrow, \dots),$$

*where a 2-colored stable ribbon tree  $t_c$  with  $e(t)$  internal edges and whose gauge crosses  $j$  vertices has degree  $|t_c| := j - e(t) - 1$ . The differential of a 2-colored stable ribbon tree  $t_c$  is given by the signed sum of all 2-colored stable ribbon trees obtained from  $t_c$  under four families of tree transformations prescribed by the top dimensional strata in the boundary of the compactified moduli space  $\overline{\mathcal{CT}}_n(t_c)$ . (Definition 3.2.13 p.109 and Definition 3.2.19 p.111)*

The  $\Omega BAs$  framework provides another template to study algebras which are associative up to homotopy, together with morphisms between them which preserve the product up to homotopy. These sections are followed by a comprehensive study on the  $A_\infty$  and  $\Omega BAs$  sign conventions. In the  $A_\infty$  case, we show how the two usual sign conventions for  $A_\infty$ -algebras and  $A_\infty$ -morphisms are naturally induced by the shifted bar construction viewpoint. Using the Loday realizations of the associahedra [29] and the Forcey–Loday realizations of the multiplihedra [20], we give a complete proof of the following two folklore theorems:

*The codimension 1 boundaries of the Loday realizations of the associahedra and of the Forcey–Loday realizations of the multiplihedra determine the usual sign conventions for  $A_\infty$ -algebras and  $A_\infty$ -morphisms between them. (Theorem 2.2.1 p.101 and Theorem 2.3.1 p.102)*

In the  $\Omega BAs$  case, we start by recalling the formulation of the operad  $\Omega BAs$  by Markl and Shnider in [28]. We then proceed to study the moduli spaces of stable 2-colored metric ribbon trees  $\mathcal{CT}_n(t_c)$  and compute the signs arising in the top dimensional strata of their boundary in Propositions 5.2.14 to 5.2.18 (p.130 to 134). This allows us to complete our definition of the operadic bimodule  $M_{\Omega BAs}$  by making explicit the signs for the action-composition maps and the differential. We subsequently prove the following two propositions:

*We give an alternative and more geometric construction of the morphism of operads  $A_\infty \rightarrow \Omega BAs$  defined in [28], using the realizations of the associahedra as geometric moduli spaces. (Proposition 3.1.15 p.106) We build an explicit morphism of operadic bimodules  $M_\infty \rightarrow M_{\Omega BAs}$  applying the same ideas to the moduli spaces realizing the multiplihedra. (Proposition 3.2.25 p.112)*

In the second part of this paper, we adapt the constructions of Abouzaid [3], using the terminology of Mescher [35], to perform two constructions on the Morse cochains  $C^*(f)$ . Firstly, we introduce the notion of smooth choices of perturbation data  $\mathbb{X}_n$  on the moduli spaces  $\mathcal{T}_n$  that we use to define the moduli spaces of perturbed Morse gradient trees  $\mathcal{T}_t^{\mathbb{X}_t}(y; x_1, \dots, x_n)$  modeled on a stable ribbon tree type  $t$ .

*Under some generic assumptions on the choice of perturbation data  $\{\mathbb{X}_n\}_{n \geq 2}$ , the moduli spaces  $\mathcal{T}_t^{\mathbb{X}_t}(y; x_1, \dots, x_n)$  are orientable manifolds. If they have dimension 0, they are compact. If they have dimension 1, they can be compactified to compact manifolds with boundary, whose boundary is modeled on the boundary of the moduli spaces  $\mathcal{T}_n(t)$ . (Theorem 6.4.5 p.145 and Theorem 6.4.6 p.145)*

In this context, *generic* means that the set of perturbation data for which the moduli spaces  $\mathcal{T}_t^{\mathbb{X}_t}(y; x_1, \dots, x_n)$  are orientable manifolds is residual in the sense of Baire in the space of all perturbation data. We then show that under a generic choice of perturbation data  $\{\mathbb{X}_n\}_{n \geq 2}$  the Morse cochains  $C^*(f)$  can be endowed with an  $\Omega BAs$ -algebra structure, through counts of perturbed Morse ribbon trees:

*Defining for every  $n$  and every stable ribbon tree type  $t$  of arity  $n$  the operation  $m_t$  as*

$$m_t : C^*(f) \otimes \cdots \otimes C^*(f) \longrightarrow C^*(f)$$

$$x_1 \otimes \cdots \otimes x_n \longmapsto \sum_{|y| = \sum_{i=1}^n |x_i| - e(t)} \# \mathcal{T}_t^{\mathbb{X}_t}(y; x_1, \dots, x_n) \cdot y ,$$

*these operations endow the Morse cochains  $C^*(f)$  with an  $\Omega BAs$ -algebra structure. (Theorem 6.5.1 p.146)*

The  $\Omega BAs$  formalism is the natural and intrinsic combinatorial viewpoint arising when realizing the moduli spaces of stable metric ribbon trees in Morse theory. We recover the  $A_\infty$ -algebra structure of [3] using the morphism of operads  $A_\infty \rightarrow \Omega BAs$  of Proposition 3.1.15: it corresponds to forgetting about all operations of the  $\Omega BAs$ -algebra structure but the ones labeled by binary ribbon trees, and defining each operations  $m_n$  of the  $A_\infty$ -algebra structure as a signed sum of all operations associated to binary ribbon trees of arity  $n$ . We point out that this viewpoint is also more ad hoc to resort to when defining *smooth* choices of perturbation data, as we work in

that case with the natural smooth structures on the moduli spaces  $\mathcal{T}_n(t)$ , rather than proving that the moduli spaces  $\mathcal{T}_n$  can be endowed with a smooth structure compatible with the operadic composition and inducing the natural smooth structure on each cell  $\mathcal{T}_n(t)$ .

Given now two Morse functions  $f$  and  $g$ , we can perform the same constructions in Morse theory using this time the moduli spaces  $\mathcal{CT}_n$  as blueprints. The counterparts of Theorems 6.4.5 and 6.4.6 (Theorem 7.3.3 p.152 and Theorem 7.3.4 p.153) still hold. Moreover, given two generic choices of perturbation data  $\mathbb{X}^f$  and  $\mathbb{X}^g$ , we construct an  $\Omega BAs$ -morphism between the  $\Omega BAs$ -algebras  $C^*(f)$  and  $C^*(g)$  through counts of 2-colored perturbed Morse gradient trees. This construction provides a first geometric and explicit instance of the newly defined notion of  $\Omega BAs$ -morphism:

Let  $(\mathbb{Y}_n)_{n \geq 1}$  be a generic choice of perturbation data on the moduli spaces  $\mathcal{CT}_n$ . Defining for every  $n$  and every 2-colored stable ribbon tree type  $t_c$  of arity  $n$  the operations  $\mu_{t_c}$  as

$$\begin{aligned} \mu_{t_c}^{\mathbb{Y}} : C^*(f) \otimes \cdots \otimes C^*(f) &\longrightarrow C^*(g) \\ x_1 \otimes \cdots \otimes x_n &\longmapsto \sum_{|y|=\sum_{i=1}^n |x_i|+|t_c|} \#\mathcal{CT}_{t_c}^{\mathbb{Y}}(y; x_1, \dots, x_n) \cdot y . \end{aligned}$$

these operations fit into an  $\Omega BAs$ -morphism  $\mu^{\mathbb{Y}} : (C^*(f), m_t^{\mathbb{X}^f}) \rightarrow (C^*(g), m_t^{\mathbb{X}^g})$ . (Theorem 7.4.1 p.154)

We moreover show that:

The  $\Omega BAs$ -morphism  $\mu^{\mathbb{Y}}$  is a quasi-isomorphism. (Theorem 7.4.5 p.154)

This  $\Omega BAs$ -morphism yields in particular an  $A_\infty$ -morphism between two  $A_\infty$ -algebras, using the morphism of operadic bimodules of Proposition 3.2.25. These constructions are followed by a section dedicated to a comprehensive proof of Theorem 6.4.5, which clarifies and completes the arguments in [3, Section 7]. We resort in particular to an argument commonly attributed to Taubes to prove the existence of generic choices of *smooth* perturbation data. Our last section on signs and orientations is dedicated to a thorough sign check for Theorems 6.5.1 and 7.4.1. We introduce for this purpose the notion of *twisted  $A_\infty$ -algebra*:

A twisted  $A_\infty$ -algebra is a dg module  $A$  endowed with two different differentials  $\partial_1$  and  $\partial_2$ , and a sequence of degree  $2 - n$  operations  $m_n : A^{\otimes n} \rightarrow A$  such that

$$[\partial, m_n] = - \sum_{\substack{i_1+i_2+i_3=n \\ 2 \leq i_2 \leq n-1}} (-1)^{i_1+i_2i_3} m_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3}) ,$$

where  $[\partial, \cdot]$  denotes the bracket for the maps  $(A^{\otimes n}, \partial_1) \rightarrow (A, \partial_2)$ . A twisted  $\Omega BAs$ -algebra and a twisted  $\Omega BAs$ -morphism are defined similarly. (Definition 9.3.1 p.163)

We show that we have in fact defined a twisted  $\Omega BAs$ -algebra structure on the Morse cochains in Theorem 6.5.1, and a twisted  $\Omega BAs$ -morphism between two Morse cochains complexes in Theorem 7.4.1. When the manifold  $M$  is odd-dimensional, the word *twisted* can moreover be dropped. Our computations are performed using the convenient viewpoint of signed short exact sequences of vector bundles. We finally construct explicit gluing maps for the 1-dimensional moduli spaces of perturbed Morse gradient trees (Section 9.4.3 p.168 and Section 9.5.4 p.173), and prove a key technical lemma on orientation and transversality (Lemma 9.4.4 p.168) which was missing from the proof by Abouzaid of the signs in the  $A_\infty$  case in [2, Appendix C].

This article lays the ground for a second article [32] tackling two main problems. Firstly, understand and define a suitable homotopic notion of higher morphisms between  $A_\infty$ -algebras,

which would give a satisfactory description of the *higher algebra of  $A_\infty$ -algebras*. Secondly, realize these higher morphisms through counts of perturbed Morse trees in order to give a higher categorical meaning to the fact that continuation morphisms in Morse theory are well-defined up to homotopy at the chain level. We explain this problem in Section 7.5 p.154.

## PART I: ALGEBRA

### 1. Operadic algebra

This first section is mostly derived from [24] and [41], and gives a gentle introduction on operadic algebra and the particular case of the operad  $A_\infty$ . The only original viewpoint that we introduce is to see  $A_\infty$ -morphisms as being encoded by the operadic bimodule  $M_\infty$  (Definition 1.5.6). All the signs will be worked out in Section 4.2, and will temporarily be written  $\pm$  in this section.

**Notations and terminology** We let  $\mathcal{C}$  be one of the following two monoidal categories: the category of differential graded  $\mathbb{Z}$ -modules with cohomological convention ( $\mathbf{dg}\text{-mod}, \otimes$ ) or the category ( $\mathbf{Poly}, \times$ ) whose objects are polytopes, which is defined in Definition 2.1.2. We will write  $\otimes$  for the tensor product on  $\mathcal{C}$ , and  $I$  for its identity element. We will also use the abbreviation *dg* for "differential graded  $\mathbb{Z}$ " in the rest of this paper. With this terminology, a *dg module* will in particular exactly be a *cochain complex*.

#### 1.1 Operads

**Definition 1.1.1.** (i) A (*non-symmetric*)  $\mathcal{C}$ -operad  $P$  consists in the data of a collection of objects  $\{P_n\}_{n \geq 1}$  of  $\mathcal{C}$  together with a unit element  $e \in P_1$  and with compositions

$$P_k \otimes P_{i_1} \otimes \cdots \otimes P_{i_k} \xrightarrow{c_{i_1, \dots, i_k}} P_{i_1 + \dots + i_k}$$

which are unital and associative.

(ii) Equivalently, an operad is the data of a collection of objects  $\{P_n\}_{n \geq 1}$  together with a unit element  $e \in P_1$  and with partial composition maps

$$\circ_i : P_k \otimes P_h \longrightarrow P_{h+k-1}, \quad 1 \leq i \leq k$$

which are unital and associative.

The objects  $P_n$  are to be thought as spaces encoding arity  $n$  operations while the compositions  $c_{i_1, \dots, i_k}$  define how to compose these operations together.

**Definition 1.1.2.** A *morphism of operads*  $P \rightarrow Q$  is a sequence of maps  $P_n \rightarrow Q_n$  compatible with the compositions and preserving the identity.

There is a third equivalent definition of operads using the notion of Schur functors. Call any collection  $P = \{P_n\}_{n \geq 1}$  of objects of  $\mathcal{C}$  a  $\mathbb{N}$ -module. To each  $\mathbb{N}$ -module one can associate its *Schur functor*, which is the endofunctor  $S_P : \mathcal{C} \rightarrow \mathcal{C}$  defined as

$$C \longmapsto \bigoplus_{n=1}^{\infty} P_n \otimes C^{\otimes n}.$$

Given two  $\mathbb{N}$ -modules  $P$  and  $Q$ , composing their Schur functors gives the following formula

$$S_P \circ S_Q : \mathcal{C} \longrightarrow \bigoplus_{n=1}^{\infty} \left( \bigoplus_{i_1 + \dots + i_k = n} P_k \otimes Q_{i_1} \otimes \dots \otimes Q_{i_k} \right) \otimes \mathcal{C}^{\otimes n} .$$

In other words, there is a  $\mathbb{N}$ -module associated to the composition of the Schur functors of two  $\mathbb{N}$ -modules, and it is given by

$$P \circ Q = \left\{ \bigoplus_{i_1 + \dots + i_k = n} P_k \otimes Q_{i_1} \otimes \dots \otimes Q_{i_k} \right\}_{n \geq 1} .$$

The category  $(\text{End}(\mathcal{C}), \circ, Id_{\mathcal{C}})$ , endowed with composition of endofunctors, is a monoidal category. In particular, there is a well-defined notion of monoid in  $\text{End}(\mathcal{C})$ . A monoid structure on an endofunctor  $F : \mathcal{C} \rightarrow \mathcal{C}$  is the data of natural transformations  $\mu_F : F \circ F \rightarrow F$  and  $e : Id_{\mathcal{C}} \rightarrow F$ , which satisfy the usual commutative diagrams for monoids. This yields the third equivalent definition for the notion of operad:

**Definition 1.1.3.** A  $\mathcal{C}$ -operad is the data of a  $\mathbb{N}$ -module  $P = \{P_n\}$  of  $\mathcal{C}$  together with a monoid structure on its Schur functor  $S_P$ .

**1.2  $P$ -algebras** Let  $A$  be a dg module and  $n \geq 1$ . Define the graded module  $\text{Hom}(A^{\otimes n}, A)^i$  of  $i$ -graded maps  $A^{\otimes n} \rightarrow A$ , and endow it with the differential  $[\partial, f] = \partial f - (-1)^{|f|} f \partial$ . The  $\mathbb{N}$ -module  $\text{End}_A(n) := \text{Hom}(A^{\otimes n}, A)$  in dg modules can then naturally be endowed with an operad structure, where composition maps are defined as one expects. Let  $P$  be a dg operad. A *structure of  $P$ -algebra* on  $A$  is defined to be the datum of a morphism of operads

$$P \longrightarrow \text{End}_A ,$$

in other words the datum of a way to interpret each operation of  $P_n$  in  $\text{Hom}(A^{\otimes n}, A)$ , such that abstract composition in  $P$  coincides with actual composition in  $\text{End}_A$ .

**Definition 1.2.1.** A morphism of  $P$ -algebras between  $A$  and  $B$  is a chain map  $f : A \rightarrow B$  such that for every  $m_n \in P_n$ ,

$$m_n^B \circ f^{\otimes n} = f \circ m_n^A .$$

**1.3 Operadic bimodules** Let now  $(\mathcal{D}, \otimes_{\mathcal{D}}, I)$  be any monoidal category, and  $(A, \mu_A)$  and  $(B, \mu_B)$  be two monoids in  $\mathcal{D}$ . Reproducing the diagrams of usual algebra, one can define the notion of an  $(A, B)$ -bimodule in  $\mathcal{D}$ . It is simply the data of an object  $R$  of  $\mathcal{D}$ , together with action maps  $\lambda : A \otimes R \rightarrow R$  and  $\mu : R \otimes B \rightarrow R$  which are compatible with the product on  $A$  and  $B$ , act trivially under their identity elements and satisfy the obvious associativity conditions. A monoid in  $\text{dg-mod}$  is then for instance a unital associative differential graded algebra, and the notion of bimodules in the previous paragraph then coincides with the usual notion of bimodules over dg algebras.

**Definition 1.3.1.** Given  $P$  and  $Q$  two operads seen as their Schur functors  $S_P$  and  $S_Q$ , let  $R = \{R_n\}$  be a  $\mathbb{N}$ -module of  $\mathcal{C}$  seen as its Schur functor  $S_R$ . A  $(P, Q)$ -operadic bimodule structure on  $R$  is a  $(S_P, S_Q)$ -bimodule structure  $\lambda : S_P \circ S_R \rightarrow S_R$  and  $\mu : S_R \circ S_Q \rightarrow S_R$  on  $S_R$  in  $(\text{End}(\mathcal{C}), \circ, Id_{\mathcal{C}})$ .

This definition is of course of no use for actual computations. Unraveling the definitions, we get an equivalent definition for  $(P, Q)$ -operadic bimodules.

**Definition 1.3.2.** (i) A  $(P, Q)$ -operadic bimodule structure on  $R$  corresponds to the data of action-composition maps

$$\begin{aligned} R_k \otimes Q_{i_1} \otimes \cdots \otimes Q_{i_k} &\xrightarrow{\mu_{i_1, \dots, i_k}} R_{i_1 + \dots + i_k}, \\ P_h \otimes R_{j_1} \otimes \cdots \otimes R_{j_h} &\xrightarrow{\lambda_{j_1, \dots, j_h}} R_{j_1 + \dots + j_h}, \end{aligned}$$

which are compatible with one another, with identities, and with compositions in  $P$  and  $Q$ .

(ii) Equivalently, the action of  $Q$  on  $R$  can be reduced to partial action-composition maps

$$\circ_i : R_k \otimes Q_h \longrightarrow R_{h+k-1} \quad 1 \leq i \leq k.$$

We point out that the action of  $P$  on  $R$  cannot be reduced to partial action-composition maps, as  $R$  does not necessarily have an identity.

Let  $A$  and  $B$  be two dg modules. We have seen that they each determine an operad,  $\text{End}_A$  and  $\text{End}_B$  respectively:

**Definition 1.3.3.** The operadic bimodule  $\text{Hom}(A, B)$  is defined to be the  $\mathbb{N}$ -module in dg modules  $\text{Hom}(A, B) := \{\text{Hom}(A^{\otimes n}, B)\}_{n \geq 1}$  endowed with its  $(\text{End}_B, \text{End}_A)$ -operadic bimodule structure, where the action-composition maps are defined as one could expect.

## 1.4 The operad $A_\infty$

### 1.4.1 $A_\infty$ -algebras

**Definition 1.4.1.** Let  $A$  be a graded module. We define  $sA$  to be the graded module  $(sA)^i := A^{i-1}$ . In other words,  $|sa| = |a| - 1$ .

**Definition 1.4.2.** Let  $A$  be a dg module with differential  $m_1$ . A structure of  $A_\infty$ -algebra on  $A$  is the data of a collection of degree  $2 - n$  maps

$$m_n : A^{\otimes n} \longrightarrow A, \quad n \geq 1,$$

extending  $m_1$  and which satisfy the following equations, called the  $A_\infty$ -equations

$$[m_1, m_n] = \sum_{\substack{i_1+i_2+i_3=n \\ 2 \leq i_2 \leq n-1}} \pm m_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3}).$$

We refer to Section 4.2.4 for the signs  $\pm$ . Representing  $m_n$  as  $\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}$  a corolla of arity  $n$ , these equations can be written as

$$[m_1, \begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}] = \sum_{\substack{i_1+i_2+i_3=n \\ 2 \leq i_2 \leq n-1}} \pm \begin{array}{c} \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \diagdown \quad \diagup \end{array} \end{array}.$$

We have in particular that

$$\begin{aligned} [m_1, m_2] &= 0, \\ [m_1, m_3] &= m_2(\text{id} \otimes m_2 - m_2 \otimes \text{id}). \end{aligned}$$

Defining  $H^*(A)$  to be the cohomology of  $A$  relative to  $m_1$ , the last two equations show that  $m_2$  descends to an associative product on  $H^*(A)$ . An  $A_\infty$ -algebra is simply a correct notion of a dg algebra whose product is associative up to homotopy, where the operations  $m_n$  for  $n \geq 4$  are the higher homotopies keeping track of the homotopy associativity of  $m_2$ .

### 1.4.2 The operad $A_\infty$

The  $A_\infty$ -algebra structure defined previously is actually governed by the following operad:

**Definition 1.4.3.** The operad  $A_\infty$  is the quasi-free dg operad generated in arity  $n \geq 2$  by one operation  $m_n$  of degree  $2 - n$  and whose differential is defined by

$$\partial(m_n) = \sum_{\substack{i_1+i_2+i_3=n \\ 2 \leq i_2 \leq n-1}} \pm m_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3}).$$

This is often written as  $A_\infty = \mathcal{F}(\Upsilon, \Psi, \Psi, \dots)$  where

$$\partial(\Psi) = \sum_{\substack{i_1+i_2+i_3=n \\ 2 \leq i_2 \leq n-1}} \pm \begin{array}{c} \overbrace{\Psi}^{i_2} \\ \underbrace{\Psi}_{i_3} \\ \underbrace{\Psi}_{i_1} \end{array}.$$

Recall that quasi-free means that the operad is freely generated by the operations  $\Psi$  as a graded object, with the additional datum of a differential on its generating operations that is non-canonical. We then check that an  $A_\infty$ -algebra structure on a dg module  $A$  amounts simply to a morphism of operads  $A_\infty \rightarrow \text{End}_A$ .

### 1.4.3 The bar construction viewpoint

**Definition 1.4.4.** The (reduced) *bar construction* of a graded module  $V$  is defined to be the graded module

$$\overline{TV} := V \oplus V^{\otimes 2} \oplus \dots$$

endowed with the coassociative comultiplication

$$\Delta_{\overline{TV}}(v_1 \dots v_n) := \sum_{i=1}^{n-1} v_1 \dots v_i \otimes v_{i+1} \dots v_n.$$

**Lemma 1.4.5.** *There is a correspondence*

$$\left\{ \begin{array}{l} \text{collections of morphisms of degree } 2 - n \\ m_n : A^{\otimes n} \rightarrow A, n \geq 1 \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{collections of morphisms of degree } +1 \\ b_n : (sA)^{\otimes n} \rightarrow sA, n \geq 1 \end{array} \right\},$$

$$\updownarrow$$

$$\left\{ \text{coderivations } D \text{ of degree } +1 \text{ of } \overline{T}(sA) \right\}$$

and a correspondence

$$\left\{ \begin{array}{l} \text{collections of morphisms of degree } 2 - n \\ m_n : A^{\otimes n} \rightarrow A, n \geq 1, \\ \text{satisfying the } A_\infty\text{-equations} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{coderivations } D \text{ of degree } +1 \text{ of } \\ \overline{T}(sA) \text{ such that } D^2 = 0 \end{array} \right\}.$$

*Proof.* The first correspondence results from the universal property of the bar construction and the observation that the datum of a degree  $2 - n$  map  $A^{\otimes n} \rightarrow A$  is equivalent to the datum of a degree  $+1$  map  $(sA)^{\otimes n} \rightarrow sA$ . The second correspondence results from the fact that the coderivation  $D : \overline{T}(sA) \rightarrow \overline{T}(sA)$  associated to the family of maps  $b_n : (sA)^{\otimes n} \rightarrow sA$ , has restriction to the summand  $(sA)^{\otimes n}$  of  $\overline{T}(sA)$  given by

$$\sum_{i_1+i_2+i_3=n} \pm \text{id}^{\otimes i_1} \otimes b_{i_2} \otimes \text{id}^{\otimes i_3} .$$

The  $A_\infty$ -equations are then easily seen to be equivalent to the equation  $D^2 = 0$ .  $\square$

Hence, the following equivalent definition for the notion of  $A_\infty$ -algebra:

**Definition 1.4.6.** An  $A_\infty$ -algebra structure on a graded module  $A$  is a coderivation  $D : \overline{T}(sA) \rightarrow \overline{T}(sA)$  of degree  $+1$  which squares to 0.

**1.5  $A_\infty$ -morphisms** Using Definition 1.2.1, a morphism between two  $A_\infty$ -algebras  $A$  and  $B$  is simply a chain map  $f : A \rightarrow B$  which is compatible with all the  $m_n$ . This notion of morphism is however not satisfactory from an homotopy-theoretic point of view. Indeed, an  $A_\infty$ -algebra being an algebra whose product is associative up to homotopy, the correct homotopy notion of a morphism between two  $A_\infty$ -algebras would be that of a map which preserves the product  $m_2$  up to homotopy, i.e. of a chain map  $f_1 : A \rightarrow B$  together with higher coherent homotopies, the first one satisfying

$$[\partial, f_2] = f_1 m_2^A - m_2^B(f_1 \otimes f_1) .$$

### 1.5.1 $A_\infty$ -morphisms

**Definition 1.5.1.** An  $A_\infty$ -morphism between two  $A_\infty$ -algebras  $A$  and  $B$  is a dg coalgebra morphism  $F : (\overline{T}(sA), D_A) \rightarrow (\overline{T}(sB), D_B)$  between their bar constructions.

**Lemma 1.5.2.** *There is a one-to-one correspondence*

$$\left\{ \begin{array}{l} \text{collections of morphisms of degree } 1 - n \\ f_n : A^{\otimes n} \rightarrow B, \quad n \geq 1, \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{morphisms of graded coalgebras} \\ F : \overline{T}(sA) \rightarrow \overline{T}(sB) \end{array} \right\} .$$

*Proof.* The proof is similar to the proof of Lemma 1.4.5. The component of  $F$  mapping  $(sA)^{\otimes n}$  to  $(sB)^{\otimes s}$  is given by

$$\sum_{i_1+\dots+i_s=n} \pm f_{i_1} \otimes \dots \otimes f_{i_s} . \quad \square$$

A coalgebra morphism preserves the differentials if and only if for all  $n \geq 1$ ,

$$\sum_{i_1+i_2+i_3=n} \pm f_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2}^A \otimes \text{id}^{\otimes i_3}) = \sum_{i_1+\dots+i_s=n} \pm m_s^B(f_{i_1} \otimes \dots \otimes f_{i_s}) . \quad (\star)$$

These equations can be rewritten as

$$[m_1, f_n] = \sum_{\substack{i_1+i_2+i_3=n \\ i_2 \geq 2}} \pm f_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2}^A \otimes \text{id}^{\otimes i_3}) + \sum_{\substack{i_1+\dots+i_s=n \\ s \geq 2}} \pm m_s^B(f_{i_1} \otimes \dots \otimes f_{i_s}) . \quad (\star)$$

This yields the following equivalent definition:

**Definition 1.5.3.** An  $A_\infty$ -morphism between two  $A_\infty$ -algebras  $A$  and  $B$  is a family of maps  $f_n : A^{\otimes n} \rightarrow B$  of degree  $1 - n$  satisfying Equation  $(\star)$ .

The signs  $\pm$  are made explicit in Section 4.2.4. We check that we recover in particular

$$\begin{aligned} [m_1, f_1] &= 0, \\ [m_1, f_2] &= f_1 m_2^A - m_2^B(f_1 \otimes f_1). \end{aligned}$$

As a result, an  $A_\infty$ -morphism of  $A_\infty$ -algebras induces a morphism of associative algebras on the level of cohomology.

**Definition 1.5.4.** An  $A_\infty$ -quasi-isomorphism is defined to be an  $A_\infty$ -morphism inducing an isomorphism in cohomology.

### 1.5.2 Composing $A_\infty$ -morphisms

Given two coalgebra morphisms  $F : \overline{TV} \rightarrow \overline{TW}$  and  $G : \overline{TW} \rightarrow \overline{TZ}$ , the family of morphisms associated to  $G \circ F$  is

$$(G \circ F)_n := \sum_{i_1 + \dots + i_s = n} \pm g_s(f_{i_1} \otimes \dots \otimes f_{i_s}),$$

where the signs  $\pm$  are given in Section 4.2.4.

**Definition 1.5.5.** (i) The composition of two  $A_\infty$ -morphisms  $f : A \rightarrow B$  and  $g : B \rightarrow C$  is defined to be

$$(g \circ f)_n := \sum_{i_1 + \dots + i_s = n} \pm g_s(f_{i_1} \otimes \dots \otimes f_{i_s}).$$

(ii) The category  $\mathbf{A}_\infty - \mathbf{alg}$  is defined to be the category whose objects are  $A_\infty$ -algebras and morphisms the  $A_\infty$ -morphisms between them, where composition is defined by the previous formula.

### 1.5.3 The $(A_\infty, A_\infty)$ -operadic bimodule encoding $A_\infty$ -morphisms

**Definition 1.5.6.** The operadic bimodule  $M_\infty$  is the quasi-free  $(A_\infty, A_\infty)$ -operadic bimodule generated in arity  $n \geq 1$  by one operation  $f_n$  of degree  $1 - n$  and whose differential is defined by

$$\partial(f_n) = \sum_{\substack{i_1 + i_2 + i_3 = n \\ i_2 \geq 2}} \pm f_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3}) + \sum_{\substack{i_1 + \dots + i_s = n \\ s \geq 2}} \pm m_s(f_{i_1} \otimes \dots \otimes f_{i_s}).$$

Representing the generating operations of the operad  $A_\infty$  acting on the right in blue (thick line)  and the ones of the operad  $A_\infty$  acting on the left in red (thin line) , we represent  $f_n$  by . This operadic bimodule can then be written as

$$M_\infty = \mathcal{F}^{A_\infty, A_\infty}(\uparrow, \downarrow, \downarrow, \downarrow, \dots),$$

with differential defined as

$$\partial(\downarrow) = \sum_{\substack{i_1 + i_2 + i_3 = n \\ i_2 \geq 2}} \pm \begin{array}{c} \overbrace{\downarrow}^{i_2} \\ \downarrow \downarrow \\ \underbrace{\uparrow \uparrow}_{i_1} \quad \underbrace{\uparrow \uparrow}_{i_3} \\ \downarrow \end{array} + \sum_{\substack{i_1 + \dots + i_s = n \\ s \geq 2}} \pm \begin{array}{c} \overbrace{\downarrow}^{i_1} \quad \dots \quad \overbrace{\downarrow}^{i_s} \\ \downarrow \dots \downarrow \\ \downarrow \end{array}.$$

Consider  $A$  and  $B$  two  $A_\infty$ -algebras, which we can see as two morphisms of operads  $A_\infty \rightarrow \text{End}_A$  and  $A_\infty \rightarrow \text{End}_B$ . Recall from Definition 1.3.3 that  $\text{Hom}(A, B)$  is a  $(\text{End}_B, \text{End}_A)$ -operadic bimodule. The previous two morphisms of operads make  $\text{Hom}(A, B)$  into an  $(A_\infty, A_\infty)$ -operadic bimodule. An  $A_\infty$ -morphism between  $A$  and  $B$  is then simply a morphism of  $(A_\infty, A_\infty)$ -operadic bimodules

$$M_\infty \longrightarrow \text{Hom}(A, B) .$$

It is in that sense that  $M_\infty$  is the  $(A_\infty, A_\infty)$ -operadic bimodule encoding the notion of  $A_\infty$ -morphisms of  $A_\infty$ -algebras.

#### 1.5.4 The framework of 2-colored operads

In fact, our choice of notation  $\color{blue}{\curvearrowright}$  reveals that the operad  $A_\infty$  and the operadic bimodule  $M_\infty$  naturally define a 2-colored operad:

**Definition 1.5.7.** The 2-colored operad  $A_\infty^2$  is the quasi-free 2-colored operad

$$A_\infty^2 := \mathcal{F}(\color{red}{\curvearrowright}, \color{red}{\curvearrowleft}, \color{red}{\curvearrowright}, \dots, \color{blue}{\curvearrowright}, \color{blue}{\curvearrowleft}, \color{blue}{\curvearrowright}, \dots, \color{red}{\curvearrowright}, \color{blue}{\curvearrowright}, \color{blue}{\curvearrowright}, \color{blue}{\curvearrowright}, \dots) ,$$

whose differential on the generating operations is given by the previous formulae for the operad  $A_\infty$  and the operadic bimodule  $M_\infty$ .

A 2-colored operad can be roughly defined as an operad whose operations have entries and output labeled either in red or in blue, and whose operations can only be composed along the same color. See [43] for a complete definition.

**1.6 Homotopy theory of  $A_\infty$ -algebras**  $A_\infty$ -algebras with  $A_\infty$ -morphisms between them provide a suitable framework to study homotopy theory of dg algebras. Following [26], this stems from the fact that the 2-colored operad  $A_\infty^2$  is a resolution

$$A_\infty^2 \xrightarrow{\sim} \text{As}^2 ,$$

of the 2-colored operad encoding associative algebras with morphisms of algebras, and a fibrant-cofibrant object in the model category of 2-colored operads in dg modules. We illustrate these statements with two fundamental theorems. We refer moreover to [27] for a more general version of Theorem 1.6.1.

**Theorem 1.6.1** (Homotopy transfer theorem, [19]). *Let  $(A, \partial_A)$  and  $(H, \partial_H)$  be two dg modules. Suppose that  $H$  is a deformation retract of  $A$ , that is that they fit into a diagram*

$$h \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} (A, \partial_A) \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{i} \end{array} (H, \partial_H) ,$$

where  $\text{id}_A - ip = [\partial, h]$ . Then if  $(A, \partial_A)$  is endowed with an associative algebra structure,  $H$  can be made into an  $A_\infty$ -algebra such that  $i$  and  $p$  extend to  $A_\infty$ -morphisms.

**Theorem 1.6.2** (Fundamental theorem of  $A_\infty$ -quasi-isomorphisms, [22]). *Let  $f : A \rightarrow B$  be an  $A_\infty$ -quasi-isomorphism. Then there exists an  $A_\infty$ -quasi-isomorphism  $B \rightarrow A$  which inverts  $f$  on the level of cohomology.*

## 2. Associahedra and multiplihedra

We recall in the first section the monoidal category  $\mathbf{Poly}$  defined in [29], which yields a good framework to handle operadic calculus in a category whose objects are polytopes. We then introduce in Sections 2.2 and 2.3 the two main combinatorial objects of this article: the *associahedra* and the *multiplihedra*, which are polytopes that respectively encode the notions of  $A_\infty$ -algebras and  $A_\infty$ -morphisms between them.

### 2.1 Three monoidal categories and their operadic algebra

#### 2.1.1 The monoidal categories $\mathbf{dg-mod}$ , $\mathbf{CW}$ and $\mathbf{Poly}$

- Definition 2.1.1.** (i) We define  $\mathbf{dg-mod}$  to be the category whose objects are differential graded  $\mathbb{Z}$ -modules with cohomological convention, and morphisms the morphisms of dg modules. It is a monoidal category with the classical tensor product of dg modules and unit the ring  $\mathbb{Z}$  seen as a dg module concentrated in degree 0.
- (ii) We define  $\mathbf{CW}$  to be the category whose objects are finite CW-complexes and whose morphisms are CW-maps between CW-complexes. This category is again a monoidal category with product the usual cartesian product and unit the point  $*$ .

The cellular chain functor  $C_*^{cell} : \mathbf{CW} \rightarrow \mathbf{dg-mod}$  is then strong monoidal. To be consistent with the cohomological degree convention on  $A_\infty$ -algebras, we will actually work with the strong monoidal functor  $C_{-*}^{cell} : \mathbf{CW} \rightarrow \mathbf{dg-mod}$ , where  $C_{-*}^{cell}(P)$  is simply the dg module  $C_*^{cell}(P)$  taken with its opposite grading.

**Definition 2.1.2.** The category  $\mathbf{Poly}$  is the category whose objects are polytopes and whose morphisms are continuous maps  $f : P \rightarrow Q$  which are homeomorphisms  $P \rightarrow |\mathcal{D}|$  where  $\mathcal{D}$  is a polytopal subcomplex of  $Q$  and  $f^{-1}(\mathcal{D})$  is a polytopal subdivision of  $P$ . Its morphisms will be called *polytopal maps*. It is a monoidal category with product the usual cartesian product and unit the polytope reduced to a point  $*$ . It is moreover a monoidal subcategory of  $\mathbf{CW}$ .

A *polytope* is here simply defined to be the convex hull of a finite number of points in a Euclidean space  $\mathbb{R}^n$ , while we refer to [29, Section 1.3] for more details on the notions of *polytopal complex* and *polytopal subdivision*. We also refer to Remark 2.2.2 for an explanation on the definition of the morphisms of the category  $\mathbf{Poly}$ .

#### 2.1.2 From operadic algebra in $\mathbf{Poly}$ to operadic algebra in $\mathbf{dg-mod}$

Let  $\{X_n\}$  be a  $\mathbf{Poly}$ -operad, that is a collection of polytopes  $X_n$  together with polytopal maps

$$\circ_i : X_k \times X_h \longrightarrow X_{h+k-1} ,$$

satisfying the compatibility conditions of partial compositions. The functor  $C_{-*}^{cell}$  being strong monoidal, it yields a new dg operad  $\{P_n\}$  defined by  $P_n := C_{-*}^{cell}(X_n)$  and whose partial compositions are

$$\circ_i : C_{-*}^{cell}(X_k) \otimes C_{-*}^{cell}(X_h) \xrightarrow{\sim} C_{-*}^{cell}(X_k \times X_h) \xrightarrow{C_{-*}^{cell}(\circ_i)} C_{-*}^{cell}(X_{h+k-1}) .$$

Similarly, an operadic bimodule in  $\mathbf{Poly}$  is sent to a dg operadic bimodule under the strong monoidal functor  $C_{-*}^{cell}$ .

**2.2 The associahedra**  $A_\infty$ -structures were introduced for the first time in two seminal papers by Stasheff on homotopy associative H-spaces [39]. In the first paper of the series, he defined cell complexes  $K_n \subset I^{n-2}$  called *associahedra*, which govern  $A_n$ -structures on topological spaces. The associahedra were later realized as polytopes by Haiman in [16], Lee in [21] or Loday in [23]. They were recently endowed with an operad structure in the category  $\mathbf{Poly}$  by Masuda, Thomas, Tonks and Vallette in [29], using the notion of weighted Loday realizations.

**Theorem 2.2.1** ([29]). *There exists realizations of the associahedra as polytopes, which can be endowed with a structure of operad in the category  $\mathbf{Poly}$  and whose image under the functor  $C_{-*}^{cell}$  yields the operad  $A_\infty$ .*

We refer to Section 4.3 in the appendix for a complete description of the associahedra of [29] as well as a proof that  $A_\infty(n) = C_{-*}^{cell}(K_n)$ . The fact that the Loday associahedra form an operad in  $\mathbf{Poly}$  is moreover already proven in [29]. The first three associahedra  $K_2$ ,  $K_3$  and  $K_4$  are represented in Figure 2.1, labeling their cells by the operations they define in  $A_\infty$  when seen in  $C_{-*}^{cell}(K_n)$ .

REMARK 2.2.2. The assumptions in the definition of the morphisms of the category  $\mathbf{Poly}$  was motivated in [29] by Theorem 2.2.1: they are the minimal assumptions to be required in order for the realizations  $K_n$  to carry the structure of an operad in  $\mathbf{Poly}$ .

These polytopes are in fact constructed such that the boundary of  $K_n$  is exactly

$$\partial K_n = \bigcup_{\substack{i_1+i_2+i_3=n \\ 2 \leq i_2 \leq n-1}} K_{i_1+1+i_3} \times K_{i_2} ,$$

and such that partial compositions are then simply polytopal inclusions of  $K_k \times K_h$  in the boundary of  $K_{h+k-1}$ .

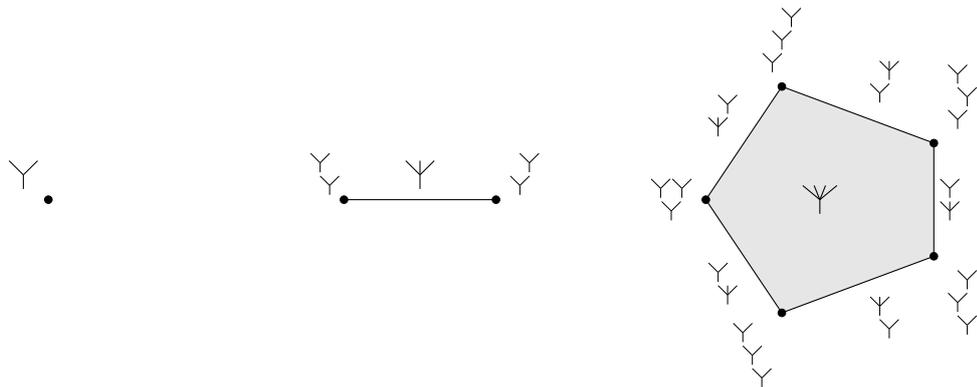


Figure 2.1: The associahedra  $K_2$ ,  $K_3$  and  $K_4$

**2.3 The multiplihedra** Just like the operad  $A_\infty$ , the dg operadic bimodule  $M_\infty$  is the image under the functor  $C_{-*}^{cell}$  of an operadic bimodule in  $\mathbf{Poly}$ , called the *multiplihedra*. Iwase and Mimura realized the multiplihedra as cell complexes in [18] following the hints of Stasheff in [39]. The multiplihedra were later realized as polytopes in [12]. They were finally adapted by Laplante-Anfossi and the author in [20], by using again the notion of weighted Loday realizations in order to prove the following theorem:

**Theorem 2.3.1** ([20]). *There exists realizations of the multiplihedra as polytopes, which can be endowed with an operadic bimodule structure over the associahedra of Theorem 2.2.1, i.e. with polytopal action-composition maps*

$$K_s \times J_{i_1} \times \cdots \times J_{i_s} \xrightarrow{\mu} J_{i_1+\cdots+i_s} ,$$

$$J_k \times K_h \xrightarrow[\circ_i]{} J_{h+k-1} ,$$

and whose image under the functor  $C_{-*}^{cell}$  yields the dg operadic bimodule  $M_\infty$ .

We refer this time to Section 4.4 for a complete definition of the realizations  $J_n$  as well as a proof that  $M_\infty(n) = C_{-*}^{cell}(J_n)$ . We simply point out that these realizations have the following properties

(i) the boundary of  $J_n$  is exactly

$$\partial J_n = \bigcup_{\substack{i_1+i_2+i_3=n \\ i_2 \geq 2}} J_{i_1+1+i_3} \times K_{i_2} \cup \bigcup_{\substack{i_1+\cdots+i_s=n \\ s \geq 2}} K_s \times J_{i_1} \times \cdots \times J_{i_s} ,$$

(ii) action-compositions are polytopal inclusions of faces in the boundary of  $J_n$ .

The first three polytopes  $J_1$ ,  $J_2$  and  $J_3$  are represented in Figure 2.2, labeling their cells by the operations they define in  $M_\infty$ .

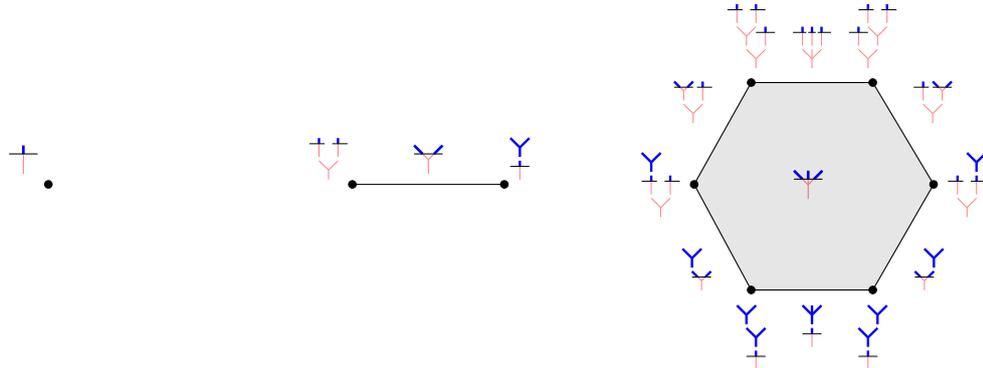


Figure 2.2: The multiplihedra  $J_1$ ,  $J_2$  and  $J_3$

### 3. Moduli spaces of metric trees

**3.1 The associahedra and metric ribbon trees** Sections 3.1.1 and 3.1.2 are inspired from [3, Section 7].

#### 3.1.1 Stable metric ribbon trees

**Definition 3.1.1.** (i) A (rooted) ribbon tree, is the data of a tree together with a cyclic ordering on the edges at each vertex of the tree and a distinguished vertex adjacent to an external edge called the *root*. This external edge is then called the *outgoing edge*, while all the other external edges are called the *incoming edges*. For a ribbon tree  $t$ , we will write  $E(t)$  for the set of its internal edges,  $\overline{E}(t)$  for the set of all its edges, and  $e(t)$  for its number of internal edges.

- (ii) A *metric ribbon tree* is the data of a ribbon tree, together with a length  $l_e \in ]0, +\infty[$  for each of its internal edge  $e$ . The external edges are thought as having length equal to  $+\infty$ .
- (iii) A ribbon tree is called *stable* if all its inner vertices are at least trivalent. It is called *binary* if all its inner vertices are trivalent. We denote  $SRT_n$  the set of all stable ribbon trees of arity  $n$ , and  $BRT_n$  the set of all binary ribbon trees. Note in particular that for a binary tree  $t \in BRT_n$  we have that  $e(t) = n - 2$ .

The best way to understand this definition is with the examples depicted in Figure 3.1.

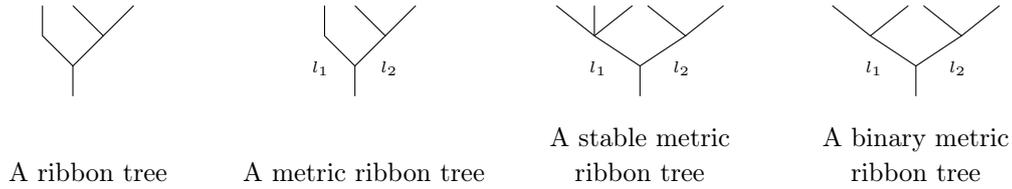


Figure 3.1

- Definition 3.1.2.**
- (i) A *broken ribbon tree* corresponds to the data of a ribbon tree together with a distinguished subset of internal edges which are labeled as *broken*.
  - (ii) A broken ribbon tree is said to be *stable* if its underlying ribbon tree is stable.
  - (iii) A *broken metric ribbon tree* corresponds to the data of a broken ribbon tree together with a length  $l_e \in ]0, +\infty[$  for each of its unbroken internal edge  $e$ . A broken internal edge  $e$  is moreover considered to have length  $l_e = +\infty$ .

The best way to understand this definition is again with the examples depicted in Figure 3.2.



Figure 3.2

### 3.1.2 Moduli spaces of stable metric ribbon trees

**Definition 3.1.3.** We define  $\mathcal{T}_n$  to be *moduli space of stable metric ribbon trees* with  $n$  incoming edges. For each stable ribbon tree type  $t$ , we define moreover  $\mathcal{T}_n(t) \subset \mathcal{T}_n$  to be the moduli space

$$\mathcal{T}_n(t) := \{\text{stable metric ribbon trees of type } t\} .$$

We then have that

$$\mathcal{T}_n = \bigcup_{t \in SRT_n} \mathcal{T}_n(t) .$$

Recalling that  $e(t)$  denotes the number of internal edges for a ribbon tree of type  $t$ , each  $\mathcal{T}_n(t)$  is naturally topologized as  $]0, +\infty[^{e(t)}$ , and they form a stratification of  $\mathcal{T}_n$ . This is illustrated in Figures 3.3 and 3.4. Interpreting a length in  $]0, +\infty[^{e(t)}$  which goes towards 0 as the contraction of the corresponding edge of  $t$ , the strata  $\mathcal{T}_n(t)$  can in fact be consistently glued together. With this observation, one can prove that the space  $\mathcal{T}_n$  is in fact itself homeomorphic to  $\mathbb{R}^{n-2}$ .

**Definition 3.1.4.** We define  $\overline{\mathcal{T}}_n$  to be the compactification of the moduli space  $\mathcal{T}_n$ , by allowing lengths of internal edges to go to  $+\infty$ .

The compactified moduli space  $\overline{\mathcal{T}}_n$  can then be seen a  $(n - 2)$ -dimensional CW-complex , where  $\mathcal{T}_n$  is seen as its unique  $(n - 2)$ -dimensional stratum and whose codimension 1 strata are given by

$$\bigcup_{\substack{i_1+i_2+i_3=n \\ 2 \leq i_2 \leq n-1}} \mathcal{T}_{i_1+1+i_3} \times \mathcal{T}_{i_2} .$$

They correspond to metric trees with one broken edge. More generally, the codimension  $m$  strata are given by metric trees with  $m$  broken edges.

**Definition 3.1.5.** This cell decomposition of  $\overline{\mathcal{T}}_n$  will be called its  $A_\infty$ -cell decomposition. We will denote it as  $(\overline{\mathcal{T}}_n)_{A_\infty}$

**Theorem 3.1.6** ([6, Section 1.4], [24, Appendix C.2]). *The compactified moduli space  $(\overline{\mathcal{T}}_n)_{A_\infty}$  is isomorphic as a CW-complex to the associahedron  $K_n$ .*

Theorem 3.1.6 is illustrated in Figure 3.4.

### 3.1.3 The $\Omega BAs$ -cell decomposition of $\overline{\mathcal{T}}_n$

The compactifications of the moduli spaces  $\mathcal{T}_n$  of Definition 3.1.4 can in fact be obtained by first compactifying each stratum  $\mathcal{T}_n(t)$  individually and then gluing consistently all compactifications together. For  $t \in SRT_n$ , the stratum  $\mathcal{T}_n(t)$  is homeomorphic to  $]0, +\infty[^{e(t)}$  and its compactification  $\overline{\mathcal{T}}_n(t) \subset \overline{\mathcal{T}}_n$  is homeomorphic to  $[0, +\infty]^{e(t)}$ . A length equal to 0 simply corresponds to collapsing one edge of  $t$  and a length equal to  $+\infty$  is interpreted as breaking this edge. This is illustrated in the instance of a cell of the moduli space  $\mathcal{T}_4$  in Figure 3.3.

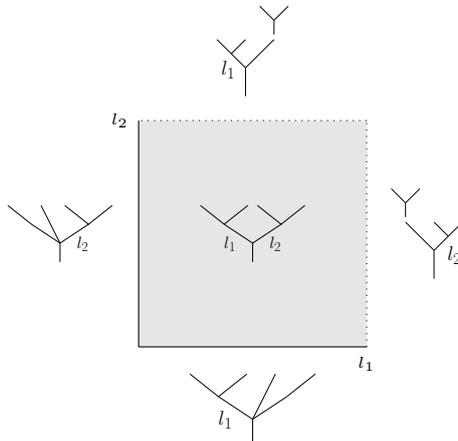


Figure 3.3: Compactification of a stratum of  $\mathcal{T}_4$ . The solid edges are inner strata of  $\mathcal{T}_4$  labeled by stable ribbon trees, while the dotted edges are the outer strata obtained by allowing lengths to go to  $+\infty$  and are labeled by broken stable ribbon trees.

The viewpoint introduced in the previous paragraph yields a new cell decomposition of  $\overline{\mathcal{T}}_n$ , two examples of which are given in Figure 3.4. Its strata are indexed by broken stable ribbon trees, a broken stable ribbon tree with  $i$  finite internal edges labeling an  $i$ -dimensional stratum.

**Definition 3.1.7.** The cell decomposition of  $\overline{\mathcal{T}}_n$  by broken stable ribbon tree type will be called its  $\Omega BAs$ -cell decomposition. We will denote it as  $(\overline{\mathcal{T}}_n)_{\Omega BAs}$ .

Strata of  $(\overline{\mathcal{T}}_n)_{\Omega BAs}$  labeled by unbroken trees are then called *internal* i.e. are lying in the interior of  $(\overline{\mathcal{T}}_n)_{A_\infty}$ , while strata labeled by broken trees are called *external* i.e. lie in the boundary of  $(\overline{\mathcal{T}}_n)_{A_\infty}$ . It is moreover clear that the  $\Omega BAs$ -cell decomposition on  $\overline{\mathcal{T}}_n$  refines its  $A_\infty$ -cell decomposition.

**Proposition 3.1.8.** *The compactified moduli spaces  $(\overline{\mathcal{T}}_n)_{\Omega BAs}$  form an operad in CW.*

*Proof.* Endowing the  $\overline{\mathcal{T}}_n$  with their  $\Omega BAs$ -cell decomposition, it is clear that the obvious maps

$$(\overline{\mathcal{T}}_k)_{\Omega BAs} \times (\overline{\mathcal{T}}_h)_{\Omega BAs} \xrightarrow{\circ_i} (\overline{\mathcal{T}}_{h+k-1})_{\Omega BAs}$$

are then cellular maps and satisfy the axioms of the partial compositions of an operad in CW.  $\square$

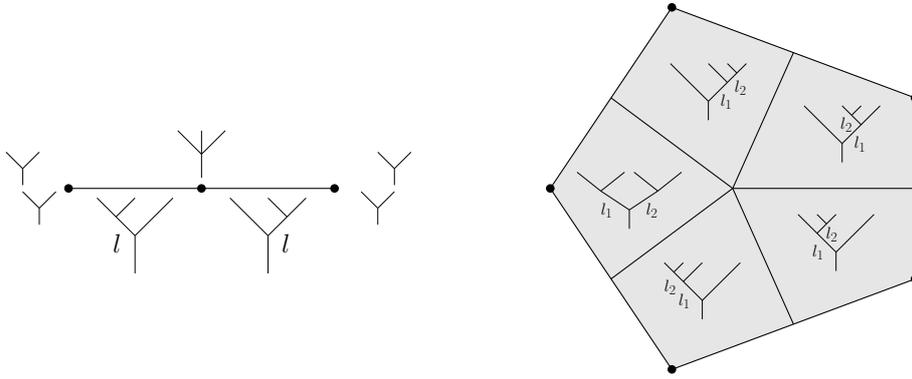


Figure 3.4: The compactified moduli spaces  $(\overline{\mathcal{T}}_3)_{\Omega BAs}$  and  $(\overline{\mathcal{T}}_4)_{\Omega BAs}$

### 3.1.4 The operad $\Omega BAs$

**Definition 3.1.9** (Definition 5.1.2). The *operad  $\Omega BAs$*  is the quasi-free operad generated by the set of stable ribbon trees, where a stable ribbon tree  $t$  has degree  $|t| := -e(t)$ . Its differential on a stable ribbon tree  $t$  is given by the signed sum of all stable ribbon trees obtained from  $t$  by breaking or collapsing exactly one of its internal edges.

We will denote this operad as

$$\Omega BAs := \mathcal{F}(\nabla, \Psi, \Upsilon, \Upsilon, \dots, SRT_n, \dots),$$

and refer to Definition 5.1.2 for a complete description of this operad and its sign conventions.

EXAMPLE 3.1.10. We have for instance that

$$\begin{aligned} |\nabla| &= -2, \\ \partial(\nabla) &= \pm \Upsilon \pm \Upsilon \pm \Psi \pm \Psi. \end{aligned}$$

**Proposition 3.1.11.** *The functor  $C_{-*}^{cell}$  maps the operad  $(\overline{\mathcal{T}}_n)_{\Omega BAs}$  to the operad  $\Omega BAs$ .*

*Proof.* See Section 5.1.3.  $\square$

REMARK 3.1.12. As explained in [24, Section 6.5], the dg operad  $\Omega BAs$  is in fact the bar-cobar construction of the operad  $As$  encoding associative algebras.

### 3.1.5 From the operad $A_\infty$ to the operad $\Omega BAs$

The  $\Omega BAs$ -cell decomposition on the compactified moduli spaces  $\overline{\mathcal{T}}_n$  can in fact be described explicitly under the isomorphism of Theorem 3.1.6:

**Lemma 3.1.13** ([24, Appendix C.2],[31, Section 2]). *The compactified moduli space  $(\overline{\mathcal{T}}_n)_{\Omega BAs}$  is isomorphic as a CW complex to the associahedron  $K_n$  endowed with its dual subdivision.*

REMARK 3.1.14. The associahedron  $K_n$  is a simple polytope: its dual subdivision is thereby often referred to as its *cubical subdivision*. The open cubes forming this cubical subdivision are then exactly the strata of  $(\overline{\mathcal{T}}_n)_{\Omega BAs}$ .

This is illustrated in Figure 3.4. We will prove that Lemma 3.1.13 implies that the dg operads  $A_\infty$  and  $\Omega BAs$  are related by an explicit morphism of operads:

**Proposition 3.1.15** ([28]). *There exists a morphism of operads  $A_\infty \rightarrow \Omega BAs$  given on the generating operations of  $A_\infty$  by*

$$m_n \longmapsto \sum_{t \in BRT_n} \pm m_t .$$

*Proof.* We compute the explicit signs in Section 5.1.4 and prove that this morphism between dg operads stems from the image under the functor  $C_{-*}^{cell}$  of the identity map  $\text{id} : (\overline{\mathcal{T}}_n)_{A_\infty} \rightarrow (\overline{\mathcal{T}}_n)_{\Omega BAs}$  refining the cell decomposition on  $\overline{\mathcal{T}}_n$ . The formula for  $m_n$  then simply corresponds to associating to the  $(n - 2)$ -dimensional cell of  $(\overline{\mathcal{T}}_n)_{A_\infty}$ , the signed sum of all  $n - 2$ -dimensional cells of  $(\overline{\mathcal{T}}_n)_{\Omega BAs}$ .  $\square$

This geometric construction of the morphism  $A_\infty \rightarrow \Omega BAs$  is an adaptation of the algebraic construction by Markl and Shnider in [28]. Proposition 3.1.15 dates in fact back to [15], and is built in the theory of Koszul duality, as explained in [24, Sections 7 and 9]. It implies moreover in particular that in order to construct a structure of  $A_\infty$ -algebra on a dg module, it is enough to endow it with a structure of  $\Omega BAs$ -algebra.

## 3.2 The multiplihedra and 2-colored metric ribbon trees

### 3.2.1 2-colored metric ribbon trees

**Definition 3.2.1.** (i) A *2-colored ribbon tree* is defined to be a ribbon tree together with a distinguished subset of inner vertices  $E_{col}(T)$  called the *colored vertices*. This set is such that, either there is exactly one colored vertex in every non-self crossing path from an incoming edge to the root and none in the path from the outgoing edge to the root, or there is no colored vertex in any non-self crossing path from an incoming edge to the root and exactly one in the path from the outgoing edge to the root.

(ii) A *2-colored metric ribbon tree* is the data of a length for all internal edges  $l_e \in ]0, +\infty[$ , such that the lengths of all non self-crossing paths from a colored vertex to the root are equal.

**Definition 3.2.2.** A *gauged metric ribbon tree* is defined to be a metric ribbon tree together with a length  $\lambda \in \mathbb{R}$ . This length is to be thought of as a *gauge* (a dividing line) drawn over the metric tree, at distance  $\lambda$  from its root, where the positive direction is pointing down.

We illustrate the following lemma in Figure 3.5.

**Lemma 3.2.3.** *The datum of a 2-colored metric ribbon tree is equivalent to the datum of a gauged metric ribbon tree.*

*Proof.* Given a 2-colored metric ribbon tree  $T_c := (t_c, \{l_e\}_{e \in E(t_c)})$ , denote  $L$  the length of any non-self crossing path from a colored vertex to the root. We form a metric ribbon tree  $T$  from  $T_c$  by forgetting the colored vertices as follows:

- (i) If the colored vertex  $v$  is bivalent, we delete  $v$  and form a new edge connecting the two non-colored vertices adjacent to  $v$ . The length of this edge is set to be the sum of the lengths of the two edges adjacent to  $v$ . If  $v$  is adjacent to an external edge, this newly obtained edge has length  $+\infty$ , i.e. is an external edge.
- (ii) If the colored vertex  $v$  is at least trivalent, we do not delete it and simply forget the fact that it is colored.

The gauged metric tree associated to  $T_c$  is then the metric tree  $T$  endowed with a gauge:

- (i) At distance  $\lambda := L$  from its root if the 2-colored tree  $t_c$  has a unique colored vertex which is bivalent and located below the root.
- (ii) At distance  $\lambda := -L$  from its root otherwise.

Conversely, consider a gauged metric ribbon tree  $(T, \lambda)$ . We form a 2-colored ribbon tree  $t_c$  from the ribbon tree  $t$ , by defining the set of colored vertices to be the set of intersection points between the gauge and the tree  $t$ . This set is made of the vertices of  $t$  that are intersected by the gauge  $\lambda$ , as well as new bivalent colored vertices, corresponding to the intersection of the gauge with the edges of  $t$ . The lengths of the internal edges of the 2-colored tree  $t_c$  are then defined to be the unique lengths such that the method of the previous paragraph recovers exactly the metric gauged tree  $(T, \lambda)$ .  $\square$



Figure 3.5: An example of a stable 2-colored metric ribbon tree with the two definitions: here  $0 < -\lambda < l$ ,  $l_1 = l_3 = -\lambda$  and  $l = l_1 + l_2$

Following Lemma 3.2.3, a 2-colored ribbon tree can thereby equivalently be seen as a ribbon tree together with a gauge drawn over it, where the intersection points between the gauge and the tree are exactly the 2-colored vertices. The gauge divides the tree into two parts, each of which we think of as being colored in a different color (*colored vertices* should then be thought as being *2-colored*, as they mark the limit between the two colors).

REMARK 3.2.4. The *gauge* of a 2-colored ribbon tree is called a *cut* in [20, Section 1.1.1].

**Definition 3.2.5.** (i) A 2-colored ribbon tree  $t_c$  is *stable* if all its inner non-colored vertices are at least trivalent and all its colored vertices are at least bivalent. We denote  $SCRT_n$  the set of all stable 2-colored ribbon trees.

- (ii) We also denote  $CBRT_n$  to be the set of all *2-colored binary ribbon trees*, i.e. of 2-colored ribbon trees all of whose non-colored vertices are trivalent and all of whose colored vertices are bivalent.

For a stable 2-colored ribbon tree  $t_c$ , we will denote  $t$  the underlying stable ribbon tree obtained by forgetting the colored vertices as in the proof of Lemma 3.2.3.

EXAMPLE 3.2.6. The underlying stable ribbon tree of  is .

### 3.2.2 Moduli spaces of stable 2-colored metric ribbon trees

This section is inspired from [31, Section 7] (in which the two authors refer to stable 2-colored metric ribbon trees as *stable colored rooted metric ribbon trees*).

**Definition 3.2.7.** For  $n \geq 2$ , we define  $\mathcal{CT}_n$  to be the *moduli space of stable 2-colored metric ribbon trees* of arity  $n$ . We also denote  $\mathcal{CT}_1 := \{+\}$  the singleton space whose only element is the unique 2-colored ribbon tree of arity 1.

The space  $\mathcal{CT}_n$  is homeomorphic to  $\mathbb{R}^{n-1}$ : the moduli space  $\mathcal{T}_n$  is homeomorphic to  $\mathbb{R}^{n-2}$  and using Lemma 3.2.3 we have that  $\mathcal{CT}_n \simeq \mathbb{R} \times \mathcal{T}_n$ , as the datum of a gauge on a stable metric ribbon tree adds a factor  $\mathbb{R}$ . Allowing internal edges of 2-colored metric trees to go to  $+\infty$ , the moduli space  $\mathcal{CT}_n$  can be compactified into a  $(n - 1)$ -dimensional CW-complex whose  $n - 1$  dimensional stratum is given by  $\mathcal{CT}_n$ .

**Definition 3.2.8.** We define  $\overline{\mathcal{CT}}_n$  to be the compactification of the moduli space  $\mathcal{CT}_n$  under the previous rule.

Two sequences of stable 2-colored metric ribbon trees converging in the compactification  $\overline{\mathcal{CT}}_3$  are represented in Figure 3.6. The codimension 1 strata of the compactification  $\overline{\mathcal{CT}}_n$  are moreover given by the union

$$\bigcup_{i_1 + \dots + i_s = n} \mathcal{T}_s \times \mathcal{CT}_{i_1} \times \dots \times \mathcal{CT}_{i_s} \cup \bigcup_{i_1 + i_2 + i_3 = n} \mathcal{CT}_{i_1 + 1 + i_3} \times \mathcal{T}_{i_2} .$$

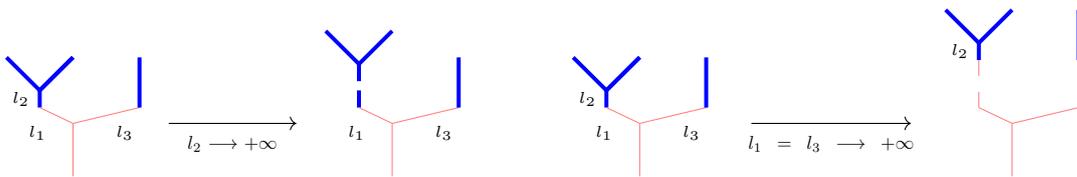


Figure 3.6: Two sequences of stable 2-colored metric ribbon trees converging in the compactification  $\overline{\mathcal{CT}}_3$

**Definition 3.2.9.** This cell decomposition of  $\overline{\mathcal{CT}}_n$  will be called its  $A_\infty$ -cell decomposition. We will denote it as  $(\overline{\mathcal{CT}}_n)_{A_\infty}$ .

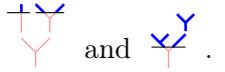
**Theorem 3.2.10** ([31]). *The compactified moduli space  $(\overline{\mathcal{CT}}_n)_{A_\infty}$  is isomorphic as a CW-complex to the multiplicated  $J_n$ .*

Theorem 3.2.10 is illustrated in Figure 3.8.

### 3.2.3 Broken 2-colored ribbon trees

- Definition 3.2.11.** (i) A *broken 2-colored ribbon tree* corresponds to the data of a 2-colored ribbon tree together with a distinguished subset of internal edges which are labeled as *broken*. This subset is such that if the gauge of the 2-colored ribbon tree is above the root, either no internal edge below the gauge is broken or there is at least one internal edge below the gauge which is broken in each non-self crossing path from an incoming edge to the root.
- (ii) A broken 2-colored ribbon tree is said to be *stable* if its underlying 2-colored ribbon tree is stable.

We will write  $t_{br,c}$  for a broken 2-colored stable ribbon tree and  $t_c$  for an (unbroken) 2-colored stable ribbon tree.

EXAMPLE 3.2.12. The following two broken 2-colored ribbon trees are stable: .

**Definition 3.2.13.** Let  $t_c$  be a stable 2-colored ribbon tree. We introduce four tree transformations:

- (i) The gauge moves to cross exactly one additional vertex of the underlying stable ribbon tree  $t$  of  $t_c$  (gauge-vertex).
- (ii) An internal edge located above the gauge or intersecting it breaks or, when the gauge is below the root, the outgoing edge breaks between the gauge and the root (above-break).
- (iii) Edges (internal or incoming) that are possibly intersecting the gauge of  $t_c$ , break below it, such that there is exactly one edge breaking in each non-self crossing path from an incoming edge to the root (below-break).
- (iv) An internal edge that does not intersect the gauge collapses (int-collapse).

EXAMPLE 3.2.14. The broken 2-colored trees resulting from the transformations of Definition 3.2.13 for the stable 2-colored ribbon tree  read as

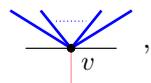
- (i) The gauge moves to cross exactly one vertex of  : ,  and .
- (ii) An internal edge breaks above the gauge:  and .
- (iii) Both internal edges break below the gauge: .

We point out that no internal edge can collapse in this example.

### 3.2.4 The moduli spaces $\mathcal{CT}_n(t_c)$

**Definition 3.2.15.** Given a stable 2-colored ribbon tree  $t_c$  of arity  $n$ , we define  $\mathcal{CT}_n(t_c)$  to be the moduli space of stable 2-colored metric ribbon trees modeled on  $t_c$ .

We refer to Definition 5.2.5 for an explicit description of the moduli spaces  $\mathcal{CT}_n(t_c)$  using the viewpoint of Lemma 3.2.3. We will prove in Section 5.2.2 that for a 2-colored stable ribbon tree  $t_c$ , writing again  $e(t)$  for the number of internal edges of the underlying stable ribbon tree  $t$ , the stratum  $\mathcal{CT}_n(t_c)$  is a polyhedral cone in  $\mathbb{R}^{e(t)+1}$ : denoting  $j$  the number of vertices  $v$  of  $t$  crossed by the gauge as depicted below



the polyhedral cone  $\mathcal{CT}_n(t_c)$  has dimension  $e(t) + 1 - j$ .

EXAMPLE 3.2.16. Applying Definition 5.2.5, we have for instance that

$$\mathcal{CT}_4(\text{diagram}) = \{(\lambda, l_1, l_2), 0 < -\lambda < l_1, l_2\} \subset \mathbb{R} \times ]0, +\infty[^2 .$$

The moduli space  $\mathcal{CT}_n$  then has a cell decomposition by stable 2-colored ribbon tree type,

$$\mathcal{CT}_n = \bigcup_{t_c \in SCRT_n} \mathcal{CT}_n(t_c) .$$

See also Remark 5.2.13.

### 3.2.5 The $\Omega BAs$ -cell decomposition of $\overline{\mathcal{CT}}_n$

The stratum  $\mathcal{CT}_n(t_c)$  can be compactified by allowing lengths of internal edges to go towards 0 or  $+\infty$ , with combinatorics induced by the equalities defined by the colored vertices. The codimension 1 strata of its compactification are then labeled by the broken 2-colored trees obtained under the tree transformations of Definition 3.2.13 (see also Section 5.2.3). The compactification  $\overline{\mathcal{CT}}_n$  is simply obtained by gluing these compactifications. This yields a new cell decomposition of  $\overline{\mathcal{CT}}_n$ , where each stratum is labeled by a broken 2-colored stable ribbon tree.

**Definition 3.2.17.** The cell decomposition of  $\overline{\mathcal{CT}}_n$  by broken stable 2-colored ribbon tree type will be called its  $\Omega BAs$ -cell decomposition. We will denote it as  $(\overline{\mathcal{CT}}_n)_{\Omega BAs}$ .

Again, strata of  $(\overline{\mathcal{CT}}_n)_{\Omega BAs}$  labeled by unbroken 2-colored trees are called *internal*, while strata labeled by broken 2-colored trees are called *external*. The cell decompositions for  $(\overline{\mathcal{CT}}_2)_{\Omega BAs}$  and  $(\overline{\mathcal{CT}}_3)_{\Omega BAs}$  are represented in Figure 3.8. The compactification of

$$\mathcal{CT}_3(\text{diagram}) = \{(\lambda, l) \text{ such that } l > 0 ; -\lambda > l\}$$

is moreover illustrated in Figure 3.7. The solide edges are inner strata of  $\mathcal{CT}_3$  labeled by stable 2-colored trees, while the dotted edges are outer strata obtained by allowing lengths to go to  $+\infty$  and are labeled by broken stable 2-colored trees.

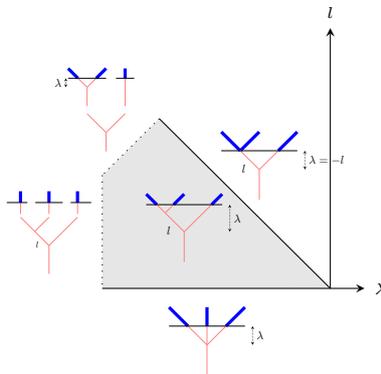


Figure 3.7: Compactification of a stratum of  $\mathcal{CT}_3$

**Proposition 3.2.18.** The compactified moduli spaces  $(\overline{\mathcal{CT}}_n)_{\Omega BAs}$  form an operadic bimodule over the operad  $(\overline{\mathcal{T}}_n)_{\Omega BAs}$  in CW.

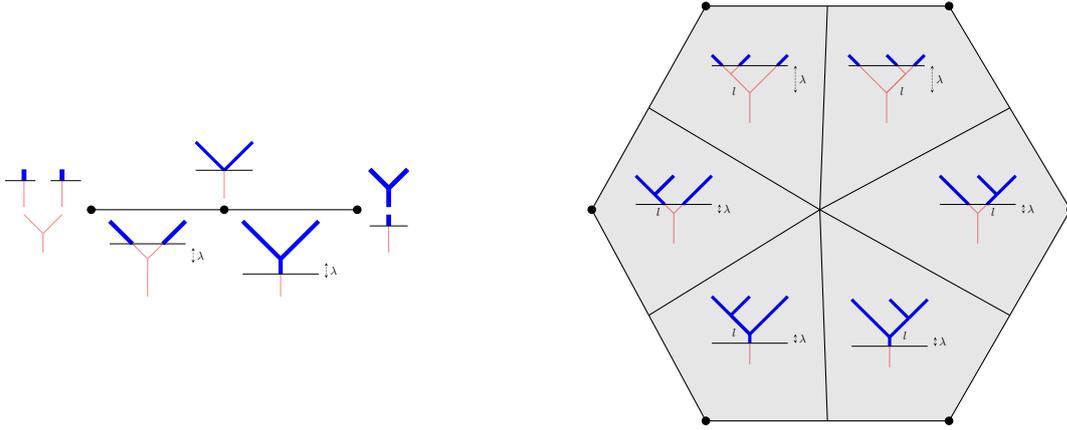


Figure 3.8: The compactified moduli spaces  $\overline{\mathcal{CT}}_2$  and  $\overline{\mathcal{CT}}_3$  with their cell decomposition by broken stable 2-colored ribbon tree type

*Proof.* Endowing the compactified moduli spaces  $\overline{\mathcal{T}}_n$  and  $\overline{\mathcal{CT}}_n$  with their  $\Omega BAs$ -cell decomposition, it is again straightforward that the obvious maps

$$\begin{aligned} \overline{\mathcal{T}}_s \times \overline{\mathcal{CT}}_{i_1} \times \cdots \times \overline{\mathcal{CT}}_{i_s} &\longrightarrow \overline{\mathcal{CT}}_{i_1+\cdots+i_s} , \\ \overline{\mathcal{CT}}_k \times \overline{\mathcal{T}}_h &\xrightarrow{o_i} \overline{\mathcal{CT}}_{h+k-1} , \end{aligned}$$

are cellular and satisfy the axioms of Item (ii) in Definition 1.3.2 for the action-composition maps of an operadic bimodule structure on  $\overline{\mathcal{CT}}_n$ .  $\square$

### 3.2.6 The operadic bimodule $M_{\Omega BAs}$

**Definition 3.2.19.** We define  $M_{\Omega BAs}$  to be the  $(\Omega BAs, \Omega BAs)$ -operadic bimodule obtained by applying the functor  $C_{-*}^{cell}$  to the operadic bimodule  $(\overline{\mathcal{CT}}_n)_{\Omega BAs}$ .

We point out that we used Proposition 3.1.11 to define the operadic bimodule  $M_{\Omega BAs}$ .

**Proposition 3.2.20.** *The operadic bimodule  $M_{\Omega BAs}$  is the quasi-free  $(\Omega BAs, \Omega BAs)$ -operadic bimodule generated by the set of stable 2-colored ribbon trees. A 2-colored stable ribbon tree  $t_c$  with  $e(t)$  internal edges and whose gauge crosses  $j$  vertices has degree  $|t_c| := j - e(t) - 1$ . The differential of a stable 2-colored ribbon tree  $t_c$  is given by the signed sum of all stable 2-colored ribbon trees obtained from  $t_c$  under the four tree transformations of Definition 3.2.13.*

*Proof.* The description of  $M_{\Omega BAs}$  as the quasi-free operadic bimodule generated by the set of stable 2-colored ribbon trees is straightforward from Section 3.2.5. We refer to Lemmas 5.3.1 and 5.3.3 for a complete proof of this proposition and explicit sign computations.  $\square$

In other words, we defined the quasi-free  $(\Omega BAs, \Omega BAs)$ -operadic bimodule

$$M_{\Omega BAs} := \mathcal{F}^{\Omega BAs, \Omega BAs}(\uparrow, \downarrow, \downarrow, \downarrow, \dots, SCRT_n, \dots) .$$

We point out that the symbol  $\downarrow$  used here is the same as the one used for the only arity 2 generating operation of  $M_\infty$ . It will however be clear from the context what  $\downarrow$  stands for in the rest of this paper.

EXAMPLE 3.2.21. We compute for instance that

$$| \begin{array}{c} \color{blue}{\swarrow} \color{blue}{\searrow} \\ \color{red}{\downarrow} \end{array} | = -3 ,$$

$$\partial( \begin{array}{c} \color{blue}{\swarrow} \color{blue}{\searrow} \\ \color{red}{\downarrow} \end{array} ) = \pm \begin{array}{c} \color{blue}{\swarrow} \color{blue}{\searrow} \\ \color{red}{\downarrow} \end{array} .$$

REMARK 3.2.22. We expect in fact that the operadic bimodule  $M_{\Omega BA_s}$  should appear as a bar-cobar construction of the  $(As, As)$ -operadic bimodule  $M_{As}$  encoding morphisms of associative algebras. See for instance [9] for notions of bar and cobar constructions of operadic bimodules.

### 3.2.7 From the operadic bimodule $M_\infty$ to the operadic bimodule $M_{\Omega BA_s}$

**Lemma 3.2.23** ([12, Section 5],[31, Section 7]). *The compactified moduli space  $(\overline{\mathcal{CT}}_n)_{\Omega BA_s}$  is isomorphic as a CW complex to the multiplihedron endowed with its dual subdivision.*

Lemma 3.2.23 is illustrated in Figure 3.8.

REMARK 3.2.24. The multiplihedron  $J_n$  is not a simple polytope in general: see for instance  $J_4$  in Figure 4.2, which has four edges connecting at a vertex. We can thereby not speak about a cubical subdivision of the multiplihedron, unlike in Remark 3.1.14.

We now point out that the morphism of operads  $A_\infty \rightarrow \Omega BA_s$  makes the  $(\Omega BA_s, \Omega BA_s)$ -operadic bimodule  $M_{\Omega BA_s}$  into an  $(A_\infty, A_\infty)$ -operadic bimodule. Lemma 3.2.23 then implies the following result:

**Proposition 3.2.25.** *There exists a morphism of  $(A_\infty, A_\infty)$ -operadic bimodules  $M_\infty \rightarrow M_{\Omega BA_s}$  given on the generating operations of  $M_\infty$  by*

$$f_n \mapsto \sum_{t_c \in CBRT_n} \pm f_{t_c} .$$

*Proof.* We refer to Section 5.3.3 for a complete proof as well as the explicit signs. As for Proposition 3.1.15, this morphism will stem from the image under the functor  $C_{-*}^{cell}$  of the identity  $id : (\overline{\mathcal{CT}}_n)_{A_\infty} \rightarrow (\overline{\mathcal{CT}}_n)_{\Omega BA_s}$  refining the cell decomposition on  $\overline{\mathcal{CT}}_n$ .  $\square$

As a result, in order to construct an  $A_\infty$ -morphism between two  $A_\infty$ -algebras whose  $A_\infty$ -algebra structure comes from an  $\Omega BA_s$ -algebra structure, it is enough to construct an  $\Omega BA_s$ -morphism between them.

### 3.2.8 The 2-colored operad $\Omega BA_s^2$

As explained in Section 1.6, it follows from [26] that since the 2-colored operad  $A_\infty^2$  is a fibrant-cofibrant replacement of  $As^2$  in the model category of 2-colored dg operads, the category of  $A_\infty$ -algebras with  $A_\infty$ -morphisms between them yields a nice homotopic framework to study the notion of homotopy associative dg algebras. In fact, most classical theorems for  $A_\infty$ -algebras can be proven using the machinery of model categories on the model category of 2-colored dg operads. We can thus similarly introduce the 2-colored operad  $\Omega BA_s^2$ , which is again a fibrant-cofibrant replacement of  $As^2$  in the model category of 2-colored operads. The notions of  $\Omega BA_s$ -algebras with  $\Omega BA_s$ -morphisms between them then yield another satisfactory homotopic framework to study homotopy associative dg algebras, in which most classical theorems for  $A_\infty$ -algebras still hold. We point out however that we did not define a *category* of  $\Omega BA_s$ -algebras, as we did not define a way to compose  $\Omega BA_s$ -morphisms. See [33, Section III.1.1.1] for a discussion of that matter.

## 4. Signs and polytopes for $A_\infty$ -algebras and $A_\infty$ -morphisms

The goal of this section is twofold: work out all the signs written as  $\pm$  in the  $A_\infty$ -equations in Section 1 and complete the proofs of Theorems 2.2.1 and 2.3.1 by proving that the Loday realizations of the associahedra of [29] and the Forcey–Loday realizations of the multiplihedra of [20] determine indeed our sign conventions for  $A_\infty$ -algebras and  $A_\infty$ -morphisms.

### 4.1 Basic conventions for signs and orientations

#### 4.1.1 Koszul sign rule

All the formulae in this section will be written using the Koszul sign rule that we briefly recall. We will work exclusively with cohomological conventions.

Given  $A$  and  $B$  two dg modules, the differential on  $A \otimes B$  is defined as

$$\partial_{A \otimes B}(a \otimes b) = \partial_A a \otimes b + (-1)^{|a|} a \otimes \partial_B b .$$

Given  $A$  and  $B$  two dg modules, we consider the graded module  $\text{Hom}(A, B)$  whose degree  $r$  component is given by all maps  $A \rightarrow B$  of degree  $r$ . We endow it with the differential

$$\partial_{\text{Hom}(A, B)}(f) := \partial_B \circ f - (-1)^{|f|} f \circ \partial_A =: [\partial, f] .$$

Given  $f : A \rightarrow A'$  and  $g : B \rightarrow B'$  two graded maps between dg modules, we set

$$(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b) .$$

Finally, given  $f : A \rightarrow A'$ ,  $f' : A' \rightarrow A''$ ,  $g : B \rightarrow B'$  and  $g' : B' \rightarrow B''$ , we define

$$(f' \otimes g') \circ (f \otimes g) = (-1)^{|g'| |f|} (f' \circ f) \otimes (g' \circ g) .$$

We check in particular that with this sign rule, the differential on a tensor product  $A_1 \otimes \cdots \otimes A_n$  is given by

$$\partial_{A_1 \otimes \cdots \otimes A_n} = \sum_{i=1}^n \text{id}_{A_1} \otimes \cdots \otimes \partial_{A_i} \otimes \cdots \otimes \text{id}_{A_n} .$$

#### 4.1.2 Orientation of the boundary of a manifold with boundary

Let  $(M, \partial M)$  be an oriented  $n$ -manifold with boundary. We choose to orient its boundary  $\partial M$  as follows: given  $x \in \partial M$ , a basis  $e_1, \dots, e_{n-1}$  of  $T_x(\partial M)$ , and an outward pointing vector  $\nu \in T_x M$ , the basis  $e_1, \dots, e_{n-1}$  is positively oriented if and only if the basis  $\nu, e_1, \dots, e_{n-1}$  is a positively oriented basis of  $T_x M$ . Note that in the particular case when the manifold with boundary is a half-space inside the Euclidean space  $\mathbb{R}^n$ , defined by an inequality

$$\sum_{i=1}^n a_i x_i \leq C ,$$

the vector  $(a_1, \dots, a_n)$  is outward-pointing.

EXAMPLE 4.1.1. We recover for instance the classical singular differential under this convention. Take  $X$  a topological space. Given a singular simplex  $\sigma : \Delta^n \rightarrow X$ , its differential is classically defined as

$$\partial_{\text{sing}}(\sigma) := \sum_{i=0}^n (-1)^i \sigma_i ,$$

where  $\sigma_i$  stands for the restriction  $[0 < \dots < \hat{i} < \dots < n] \hookrightarrow \Delta^n \rightarrow X$ . Realizing  $\Delta^n$  as a polytope in  $\mathbb{R}^n$  and orienting it with the canonical orientation of  $\mathbb{R}^n$ , we check that its boundary reads exactly as

$$\partial\Delta^n = \bigcup_{i=0}^n (-1)^i \Delta_i^{n-1},$$

where  $\Delta_i^{n-1}$  is the  $(n-1)$ -simplex corresponding to the face  $[0 < \dots < \hat{i} < \dots < n]$ . The sign  $(-1)^i$  means that the orientation of  $\Delta_i^{n-1}$  induced by its canonical identification with  $\Delta^{n-1}$  and its orientation as the boundary of  $\Delta^n$ , differ by a  $(-1)^i$  sign.

### 4.1.3 Coorientations

Our convention for orienting the boundary of an oriented manifold with boundary  $(M, \partial M)$  can in fact be rephrased as follows: *the boundary  $\partial M$  is cooriented by the outward pointing vector field  $\nu$* . More generally consider an oriented manifold  $N$  and a submanifold  $S \subset N$ .

**Definition 4.1.2.** A *coorientation* of  $S$  is defined to be an orientation of the normal bundle to  $S$ .

Given any complement bundle  $\nu_S$  to  $TS$  in  $TN|_S$ ,

$$TN|_S = \nu_S \oplus TS,$$

this orientation induces in turn an orientation on  $\nu_S$ , the normal bundle being canonically isomorphic to  $\nu_S$ . The manifold  $S$  is then orientable if and only if it is coorientable. This can be proven using the first Stiefel-Whitney class for instance.

**Definition 4.1.3.** Given a coorientation for  $S$ , *the induced orientation on  $S$*  is set to be the one whose concatenation with that of  $\nu_S$ , in the order  $(\nu_S, TS)$ , gives the orientation on  $TN|_S$ .

**4.2 Signs for  $A_\infty$ -algebras and  $A_\infty$ -morphisms using the bar construction** There exist various conventions on signs for  $A_\infty$ -algebras and  $A_\infty$ -morphisms between them, which can seem inexplicable when met out of context. The goal of this section is to give a comprehensive account of the two sign conventions coming from the bar construction, and to state our choice of signs for the rest of the paper (Section 4.2.4).

#### 4.2.1 $A_\infty$ -algebras

We will first be interested in the following two sign conventions for  $A_\infty$ -algebras:

$$[m_1, m_n] = - \sum_{\substack{i_1+i_2+i_3=n \\ 2 \leq i_2 \leq n-1}} (-1)^{i_1 i_2 + i_3} m_{i_1+1+i_3} (\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3}), \quad (\text{A})$$

$$[m_1, m_n] = - \sum_{\substack{i_1+i_2+i_3=n \\ 2 \leq i_2 \leq n-1}} (-1)^{i_1+i_2 i_3} m_{i_1+1+i_3} (\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3}), \quad (\text{B})$$

which can be rewritten as

$$\sum_{i_1+i_2+i_3=n} (-1)^{i_1 i_2 + i_3} m_{i_1+1+i_3} (\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3}) = 0, \quad (\text{A})$$

$$\sum_{i_1+i_2+i_3=n} (-1)^{i_1+i_2 i_3} m_{i_1+1+i_3} (\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3}) = 0. \quad (\text{B})$$

REMARK 4.2.1. Conventions (A) are for instance used in [39], while conventions (B) are used in [22].

First, note that these two sign conventions are equivalent in the following sense: given a sequence of operations  $m_n : A^{\otimes n} \rightarrow A$  satisfying equations (A), we check that the operations  $m'_n := (-1)^{\binom{n}{2}} m_n$  satisfy equations (B). This sign change does not come out of the blue, and appears in the following proof that these equations come from the bar construction. We introduce the suspension and desuspension maps

$$\begin{array}{ll} s : A \longrightarrow sA & w : sA \rightarrow A \\ a \longmapsto sa & sa \longmapsto a , \end{array}$$

which are respectively of degree  $-1$  and  $+1$ . We check that with the Koszul sign rule,

$$w^{\otimes n} \circ s^{\otimes n} = (-1)^{\binom{n}{2}} \text{id}_{A^{\otimes n}} .$$

We note that a degree  $2 - n$  map  $m_n : A^{\otimes n} \rightarrow A$  yields a degree  $+1$  map  $b_n := sm_n w^{\otimes n} : (sA)^{\otimes n} \rightarrow sA$ . Consider now a collection of degree  $2 - n$  maps  $m_n : A^{\otimes n} \rightarrow A$ , and the associated degree  $+1$  maps  $b_n : (sA)^{\otimes n} \rightarrow sA$ . Denoting  $D$  the unique coderivation on  $\overline{T}(sA)$  associated to the  $b_n$ , the equation  $D^2 = 0$  is then equivalent to the equations

$$\sum_{i_1+i_2+i_3=n} b_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes b_{i_2} \otimes \text{id}^{\otimes i_3}) = 0 .$$

There are now two ways to unravel the signs from these equations.

The first way consists in simply replacing the  $b_i$  by their definition. It leads to the (A) sign conventions:

$$\begin{aligned} & \sum_{i_1+i_2+i_3=n} b_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes b_{i_2} \otimes \text{id}^{\otimes i_3}) \\ = & \sum_{i_1+i_2+i_3=n} sm_{i_1+1+i_3}(w^{\otimes i_1} \otimes w \otimes w^{\otimes i_3})(\text{id}^{\otimes i_1} \otimes sm_{i_2} w^{\otimes i_2} \otimes \text{id}^{\otimes i_3}) \\ = & \sum_{i_1+i_2+i_3=n} (-1)^{i_3} sm_{i_1+1+i_3}(w^{\otimes i_1} \otimes m_{i_2} w^{\otimes i_2} \otimes w^{\otimes i_3}) \\ = & \sum_{i_1+i_2+i_3=n} (-1)^{i_3+i_1 i_2} sm_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3})(w^{\otimes i_1} \otimes w^{\otimes i_2} \otimes w^{\otimes i_3}) \\ = & s \left( \sum_{i_1+i_2+i_3=n} (-1)^{i_1 i_2+i_3} m_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3}) \right) w^{\otimes n} . \end{aligned}$$

The second way consists in first composing and post-composing by  $w$  and  $s^{\otimes n}$  and then replacing the  $b_i$  by their definition. It leads to the (B) sign conventions and makes the  $(-1)^{\binom{n}{2}}$  sign change

appear:

$$\begin{aligned}
& \sum_{i_1+i_2+i_3=n} wb_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes b_{i_2} \otimes \text{id}^{\otimes i_3})s^{\otimes n} \\
&= \sum_{i_1+i_2+i_3=n} wb_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes b_{i_2} \otimes \text{id}^{\otimes i_3})(s^{\otimes i_1} \otimes s^{\otimes i_2} \otimes s^{\otimes i_3}) \\
&= \sum_{i_1+i_2+i_3=n} (-1)^{i_1} wb_{i_1+1+i_3}(s^{\otimes i_1} \otimes b_{i_2} s^{\otimes i_2} \otimes s^{\otimes i_3}) \\
&= \sum_{i_1+i_2+i_3=n} (-1)^{i_1} wsm_{i_1+1+i_3} w^{\otimes i_1+1+i_3} (s^{\otimes i_1} \otimes sm_{i_2} w^{\otimes i_2} s^{\otimes i_2} \otimes s^{\otimes i_3}) \\
&= \sum_{i_1+i_2+i_3=n} (-1)^{i_1} m_{i_1+1+i_3} w^{\otimes i_1+1+i_3} (s^{\otimes i_1} \otimes (-1)^{\binom{i_2}{2}} sm_{i_2} \otimes s^{\otimes i_3}) \\
&= \sum_{i_1+i_2+i_3=n} (-1)^{i_1+i_2i_3} m_{i_1+1+i_3} w^{\otimes i_1+1+i_3} s^{\otimes i_1+1+i_3} (\text{id}^{\otimes i_1} \otimes (-1)^{\binom{i_2}{2}} m_{i_2} \otimes \text{id}^{\otimes i_3}) \\
&= \sum_{i_1+i_2+i_3=n} (-1)^{i_1+i_2i_3} (-1)^{\binom{i_1+1+i_3}{2}} m_{i_1+1+i_3} (\text{id}^{\otimes i_1} \otimes (-1)^{\binom{i_2}{2}} m_{i_2} \otimes \text{id}^{\otimes i_3}) .
\end{aligned}$$

#### 4.2.2 $A_\infty$ -morphisms

We now delve into the two sign conventions for  $A_\infty$ -morphisms that are coming with the bar construction viewpoint. They are as follows:

$$\begin{aligned}
[m_1, f_n] &= \sum_{\substack{i_1+i_2+i_3=n \\ i_2 \geq 2}} (-1)^{i_1i_2+i_3} f_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3}) \\
&\quad - \sum_{\substack{i_1+\dots+i_s=n \\ s \geq 2}} (-1)^{\epsilon_A} m_s(f_{i_1} \otimes \dots \otimes f_{i_s}) ,
\end{aligned} \tag{A}$$

$$\begin{aligned}
[m_1, f_n] &= \sum_{\substack{i_1+i_2+i_3=n \\ i_2 \geq 2}} (-1)^{i_1+i_2i_3} f_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3}) \\
&\quad - \sum_{\substack{i_1+\dots+i_s=n \\ s \geq 2}} (-1)^{\epsilon_B} m_s(f_{i_1} \otimes \dots \otimes f_{i_s}) ,
\end{aligned} \tag{B}$$

which can be rewritten as

$$\begin{aligned}
\sum_{i_1+i_2+i_3=n} (-1)^{i_1i_2+i_3} f_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3}) &= \sum_{i_1+\dots+i_s=n} (-1)^{\epsilon_A} m_s(f_{i_1} \otimes \dots \otimes f_{i_s}) , \tag{A} \\
\sum_{i_1+i_2+i_3=n} (-1)^{i_1+i_2i_3} f_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3}) &= \sum_{i_1+\dots+i_s=n} (-1)^{\epsilon_B} m_s(f_{i_1} \otimes \dots \otimes f_{i_s}) , \tag{B}
\end{aligned}$$

where

$$\epsilon_A = \sum_{u=1}^s i_u \left( \sum_{u < t \leq s} (1 - i_t) \right) , \quad \epsilon_B = \sum_{u=1}^s (s - u)(1 - i_u) .$$

These two sign conventions are again equivalent: given a sequence of operations  $m_n$  and  $f_n$  satisfying equations (A), we check that the operations  $m'_n := (-1)^{\binom{n}{2}} m_n$  and  $f'_n := (-1)^{\binom{n}{2}} f_n$  satisfy equations (B). The  $(-1)^{\binom{n}{2}}$  twist comes again from the formula  $w^{\otimes n} \circ s^{\otimes n} = (-1)^{\binom{n}{2}} \text{id}_{A^{\otimes n}}$ .

Consider now two dg modules  $A$  and  $B$ , together with a collection of degree  $2 - n$  maps  $m_n : A^{\otimes n} \rightarrow A$  and  $m_n : B^{\otimes n} \rightarrow B$  (we use the same notation for sake of readability), and a collection

of degree  $1 - n$  maps  $f_n : A^{\otimes n} \rightarrow B$ . We associate again to the  $m_n$  the degree  $+1$  maps  $b_n$ , and also associate to the  $f_n$  the degree  $0$  maps  $F_n := s f_n w^{\otimes n} : (sA)^{\otimes n} \rightarrow sB$ . We denote  $D_A$  and  $D_B$  the unique coderivations acting respectively on  $\overline{T}(sA)$  and  $\overline{T}(sB)$ , and  $F : \overline{T}(sA) \rightarrow \overline{T}(sB)$  the unique coalgebra morphism associated to the  $F_n$ . The equation  $FD_A = D_B F$  is then equivalent to the equations

$$\sum_{i_1+i_2+i_3=n} F_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes b_{i_2} \otimes \text{id}^{\otimes i_3}) = \sum_{i_1+\dots+i_s=n} b_s(F_{i_1} \otimes \dots \otimes F_{i_s}).$$

There are again two ways to unravel the signs from these equations, which will lead to conventions (A) and (B). The proofs proceed exactly as in Section 4.2.1.

### 4.2.3 Composition of $A_\infty$ -morphisms

Let  $f_n : A^{\otimes n} \rightarrow B$  and  $g_n : B^{\otimes n} \rightarrow C$  be two  $A_\infty$ -morphisms under conventions (A). The arity  $n$  component of their composition  $g \circ f$  is defined as

$$\sum_{i_1+\dots+i_s=n} (-1)^{\epsilon_A} g_s(f_{i_1} \otimes \dots \otimes f_{i_s}), \quad (\text{A})$$

where  $\epsilon_A$  is as previously.

Let  $f_n : A^{\otimes n} \rightarrow B$  and  $g_n : B^{\otimes n} \rightarrow C$  be two  $A_\infty$ -morphisms under conventions (B). The arity  $n$  component of their composition  $g \circ f$  is this time defined as

$$\sum_{i_1+\dots+i_s=n} (-1)^{\epsilon_B} g_s(f_{i_1} \otimes \dots \otimes f_{i_s}), \quad (\text{B})$$

where  $\epsilon_B$  is as previously.

We check that in each case, this newly defined morphism satisfies the  $A_\infty$ -equations, respectively under the sign conventions (A) and (B). This can again be proven using the bar construction and applying the previous transformations.

### 4.2.4 Choice of convention in this paper

We will work in the rest of this paper under the set of conventions (B). This choice of conventions will be accounted for in Sections 4.3 and 4.4: the signs are the ones which arise naturally from the realizations of the associahedra and the multiplihedra à la Loday. The operations  $m_n$  of an  $A_\infty$ -algebra will satisfy equations

$$[m_1, m_n] = - \sum_{\substack{i_1+i_2+i_3=n \\ 2 \leq i_2 \leq n-1}} (-1)^{i_1+i_2 i_3} m_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3}),$$

an  $A_\infty$ -morphism between two  $A_\infty$ -algebras will satisfy equations

$$[m_1, f_n] = \sum_{\substack{i_1+i_2+i_3=n \\ i_2 \geq 2}} (-1)^{i_1+i_2 i_3} f_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3}) - \sum_{\substack{i_1+\dots+i_s=n \\ s \geq 2}} (-1)^{\epsilon_B} m_s(f_{i_1} \otimes \dots \otimes f_{i_s}),$$

and two  $A_\infty$ -morphisms will be composed as

$$\sum_{i_1+\dots+i_s=n} (-1)^{\epsilon_B} g_s(f_{i_1} \otimes \dots \otimes f_{i_s}),$$

where  $\epsilon_B = \sum_{u=1}^s (s-u)(1-i_u)$ .

**4.3 Loday associahedra and signs** We recall now the definition of the weighted Loday realizations of the associahedra in [29], and use them to prove the second part of Theorem 2.2.1.

**Definition 4.3.1** ([29]). Given  $n \geq 1$ , define a weight  $\omega$  to be a list of  $n$  positive integers  $(\omega_1, \dots, \omega_n)$ . The *Loday realization of weight  $\omega$*  of  $K_n$  is defined to be the intersection in  $\mathbb{R}^{n-1}$  of the hyperplane of equation

$$H_\omega : \sum_{i=1}^{n-1} x_i = \sum_{1 \leq k < l \leq n} \omega_k \omega_l$$

and of the half-spaces of equation

$$D_{i_1, i_2, i_3} : x_{i_1+1} + \dots + x_{i_1+i_2-1} \geq \sum_{i_1+1 \leq k < l \leq i_1+i_2} \omega_k \omega_l ,$$

for all  $i_1 + i_2 + i_3 = n$  and  $2 \leq i_2 \leq n - 1$ . This polytope is denoted  $K_\omega$ .

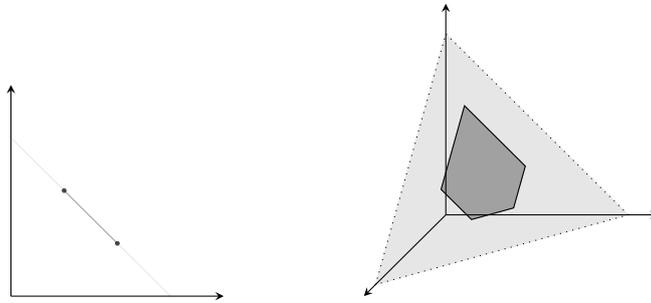


Figure 4.1: The Loday realizations  $K_3$  and  $K_4$ : the lighter grey depicts  $H_\omega$ , while the darker grey stands for  $K_\omega$ .

For the weight  $\mathbf{1}_n$  of length  $n$  whose entries are all equal to 1, we will denote  $K_n := K_{\mathbf{1}_n}$ . The Loday realizations  $K_3$  and  $K_4$  are represented in Figure 4.1. Our goal is now to prove the second part of Theorem 2.2.1:

**Theorem 2.2.1.** *The Loday associahedra  $K_n$  form an operad in Poly whose image under the functor  $C_{-*}^{cell}$  is the operad  $A_\infty$ .*

*Proof.* We have to prove that after choosing an orientation for each polytope  $K_n$ , their boundary reads as

$$\partial K_n = - \bigcup_{\substack{i_1+i_2+i_3=n \\ 2 \leq i_2 \leq n-1}} (-1)^{i_1+i_2i_3} K_{i_1+1+i_3} \times K_{i_2} ,$$

where  $K_{i_1+1+i_3} \times K_{i_2}$  is sent to  $m_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3})$  under the functor  $C_{-*}^{cell}$ . The signs mean that after comparing the product orientation on  $K_{i_1+1+i_3} \times K_{i_2}$  induced by the orientations of  $K_{i_1+1+i_3}$  and  $K_{i_2}$ , to the orientation of the boundary of  $K_n$ , they differ by the sign  $-(-1)^{i_1+i_2i_3}$ .

**Step 1** We begin by explaining how to obtain the set-theoretic decomposition of the boundary

$$\partial K_n = \bigcup_{\substack{i_1+i_2+i_3=n \\ 2 \leq i_2 \leq n-1}} K_{i_1+1+i_3} \times K_{i_2} .$$

The top dimensional strata in the boundary of some  $K_\omega$  are obtained by allowing exactly one of the inequalities

$$x_{i_1+1} + \cdots + x_{i_1+i_2-1} \geq \sum_{i_1+1 \leq k < l \leq i_1+i_2} \omega_k \omega_l ,$$

to become an equality. We write  $H_{i_1, i_2, i_3}$  for these hyperplanes. Defining two new weights

$$\begin{aligned} \bar{\omega} &:= (\omega_1, \dots, \omega_{i_1}, \omega_{i_1+1} + \cdots + \omega_{i_1+i_2}, \omega_{i_1+i_2+1}, \dots, \omega_n) , \\ \tilde{\omega} &:= (\omega_{i_1+1}, \dots, \omega_{i_1+i_2}) , \end{aligned}$$

the map

$$\begin{aligned} \theta : \mathbb{R}^{i_1+i_3} \times \mathbb{R}^{i_2-1} &\longrightarrow \mathbb{R}^{n-1} \\ (x_1, \dots, x_{i_1+i_3}) \times (y_1, \dots, y_{i_2-1}) &\longmapsto (x_1, \dots, x_{i_1}, y_1, \dots, y_{i_2-1}, x_{i_1+1}, \dots, x_{i_1+i_3}) \end{aligned}$$

induces a bijection between  $K_{\bar{\omega}} \times K_{\tilde{\omega}}$  and the codimension 1 face of  $K_\omega$  corresponding to the intersection with  $H_{i_1, i_2, i_3}$ .

**Step 2** The directing hyperplane  $\bar{H}_\omega$  of the affine hyperplane  $H_\omega$  has basis

$$e_j^\omega = (1, 0, \dots, 0, -1_{j+1}, 0, \dots, 0) ,$$

where  $-1$  is in the  $j+1$ -th spot, and we add a superscript  $\omega$  for later use. We choose this basis as a positively oriented basis for  $\bar{H}_\omega$ : this defines our orientation of  $K_\omega$ . Choosing any  $(a_1, \dots, a_{n-1}) \in H_\omega$ , the basis  $e_j^\omega$  parametrizes  $H_\omega$  under the map

$$(y_1, \dots, y_{n-2}) \longmapsto \left( \sum_{j=1}^{n-2} y_j + a_1, -y_1 + a_2, \dots, -y_{n-2} + a_{n-1} \right) .$$

Hence in the coordinates of the basis  $e_j^\omega$ , the half-space  $H_\omega \cap D_{i_1, i_2, i_3}$  reads as

$$\begin{aligned} \text{when } i_1 = 0: & \quad -y_{i_2-1} - \cdots - y_{n-2} \leq C , \\ \text{when } i_1 \geq 1: & \quad y_{i_1} + \cdots + y_{i_1+i_2-2} \leq C , \end{aligned}$$

where  $C$  denotes some constant that we are not interested in. Hence, in the basis  $e_j^\omega$ , an outward pointing vector for the boundary  $H_\omega \cap H_{i_1, i_2, i_3}$  is

$$\begin{aligned} \text{when } i_1 = 0: & \quad \nu := (0, \dots, 0, -1_{i_2-1}, \dots, -1_{n-2}) , \\ \text{when } i_1 \geq 1: & \quad \nu := (0, \dots, 0, 1_{i_1}, \dots, 1_{i_1+i_2-2}, 0, \dots, 0) . \end{aligned}$$

We have chosen orienting bases for the directing hyperplanes  $\bar{H}_\omega$ , and computed all outward pointing vectors for the boundaries in these bases. It only remains to study the image of these bases under the maps  $\theta$ . We write  $e_j^{\bar{\omega}}$  for the orienting basis of  $K_{\bar{\omega}}$  and  $e_j^{\tilde{\omega}}$  for the one of  $K_{\tilde{\omega}}$ . We distinguish two cases.

When  $i_1 = 0$ , the map  $\theta$  reads as

$$\theta(x_1, \dots, x_{i_3}, y_1, \dots, y_{i_2-1}) = (y_1, \dots, y_{i_2-1}, x_1, \dots, x_{i_3}) ,$$

and we compute that:

$$\theta(e_j^{\bar{\omega}}) = -e_{i_2-1}^\omega + e_{j+i_2-1}^\omega \qquad \theta(e_j^{\tilde{\omega}}) = e_j^\omega .$$

The determinant then has value

$$\det_{e_j^\omega} \left( \nu, \theta(e_j^{\bar{\omega}}), \theta(e_j^{\tilde{\omega}}) \right) = -i_3(-1)^{i_2 i_3} .$$

Thus, we recover the  $-(-1)^{i_1+i_2 i_3} K_{i_1+1+i_3} \times K_{i_2}$  oriented component of the boundary.

When  $i_1 \geq 1$ , the map  $\theta$  now reads as

$$\theta(x_1, \dots, x_{i_3}, y_1, \dots, y_{i_2-1}) = (x_1, \dots, x_{i_1}, y_1, \dots, y_{i_2-1}, x_{i_1+1}, \dots, x_{i_1+i_3}) ,$$

and we compute that:

$$j \leq i_1 - 1 , \theta(e_j^{\bar{\omega}}) = e_j^\omega \qquad j \geq i_1 , \theta(e_j^{\bar{\omega}}) = e_{j+i_2-1}^\omega \qquad \theta(e_j^{\tilde{\omega}}) = e_{j+i_1}^\omega - e_{i_1}^\omega .$$

This time,

$$\det_{e_j^\omega} \left( \nu, \theta(e_j^{\bar{\omega}}), \theta(e_j^{\tilde{\omega}}) \right) = -(i_2 - 1)(-1)^{i_1+i_2 i_3} .$$

We find again the  $-(-1)^{i_1+i_2 i_3} K_{i_1+1+i_3} \times K_{i_2}$  oriented component of the boundary, which concludes the proof of Theorem 2.2.1.  $\square$

**4.4 Forcey–Loday multiplihedra and signs** We now define the weighted Forcey–Loday realizations of the multiplihedra of [20] and prove the second part of Theorem 2.3.1.

**Definition 4.4.1** ([20]). Given  $n \geq 1$ , choose a weight  $\omega = (\omega_1, \dots, \omega_n)$ . The *Forcey–Loday realization* of weight  $\omega$  of  $J_n$  is defined as the intersection in  $\mathbb{R}^{n-1}$  of the half-spaces of equation

$$D_{i_1, i_2, i_3} : x_{i_1+1} + \dots + x_{i_1+i_2-1} \geq \sum_{i_1+1 \leq k < l \leq i_1+i_2} \omega_k \omega_l ,$$

for all  $i_1 + i_2 + i_3 = n$  and  $i_2 \geq 2$ , with the half-spaces of equation

$$D^{i_1, \dots, i_s} : x_{i_1} + x_{i_1+i_2} + \dots + x_{i_1+\dots+i_{s-1}} \leq 2 \sum_{1 \leq t < u \leq s} \Omega_t \Omega_u$$

for all  $i_1 + \dots + i_s = n$ , with each  $i_t \geq 1$  and  $s \geq 2$ , and where  $\Omega_t := \sum_{a=1}^{i_t} \omega_{i_1+\dots+i_{t-1}+a}$ . This polytope is denoted  $J_\omega$ .

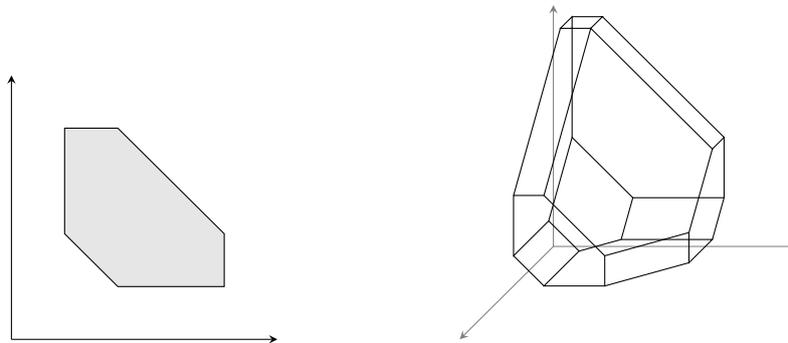


Figure 4.2: The Forcey–Loday realizations  $J_3$  and  $J_4$

For the weight  $\mathbf{1}_n$  of length  $n$  whose entries are all equal to 1, we write again  $J_n := J_{\mathbf{1}_n}$ . The Forcey–Loday realizations  $J_3$  and  $J_4$  are represented in Figure 4.2. We can now prove the second part of Theorem 2.3.1:

**Theorem 2.3.1.** *The Forcey–Loday multiplihedra  $J_n$  form an operadic bimodule in  $\text{Poly}$  whose image under the functor  $C_{-*}^{\text{cell}}$  is the operadic bimodule  $M_\infty$ .*

*Proof.* Our goal is to prove that, after orienting the  $K_n$  as before and choosing an orientation for the  $J_n$ , the boundary of  $J_n$  reads as

$$\partial J_n = \bigcup_{\substack{i_1+i_2+i_3=n \\ i_2 \geq 2}} (-1)^{i_1+i_2i_3} J_{i_1+1+i_3} \times K_{i_2} \cup - \bigcup_{\substack{i_1+\dots+i_s=n \\ s \geq 2}} (-1)^{\epsilon_B} K_s \times J_{i_1} \times \dots \times J_{i_s},$$

where  $\epsilon_B$  is as in Section 4.2.4 ;  $K_{i_1+1+i_3} \times K_{i_2}$  is sent to  $f_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3})$  while  $K_s \times J_{i_1} \times \dots \times J_{i_s}$  is sent to  $m_s(f_{i_1} \otimes \dots \otimes f_{i_s})$  by the functor  $C_{-*}^{\text{cell}}$ .

**Step 1** We first explain how to obtain the set-theoretic equality for the boundary

$$\partial J_n = \bigcup_{\substack{i_1+i_2+i_3=n \\ i_2 \geq 2}} J_{i_1+1+i_3} \times K_{i_2} \cup \bigcup_{\substack{i_1+\dots+i_s=n \\ s \geq 2}} K_s \times J_{i_1} \times \dots \times J_{i_s}.$$

The top dimensional strata in the boundary of a  $J_\omega$  are obtained by allowing exactly one of the inequalities

$$\begin{aligned} x_{i_1+1} + \dots + x_{i_1+i_2-1} &\geq \sum_{i_1+1 \leq k < l \leq i_1+i_2} \omega_k \omega_l, \\ x_{i_1} + x_{i_1+i_2} + \dots + x_{i_1+\dots+i_{s-1}} &\leq 2 \sum_{1 \leq t < u \leq s} \Omega_t \Omega_u, \end{aligned}$$

to become an equality. We write  $H_{i_1, i_2, i_3}$  and  $H^{i_1, \dots, i_s}$  for these hyperplanes.

Begin with the  $H_{i_1, i_2, i_3}$  component. Defining two new weights

$$\begin{aligned} \bar{\omega} &:= (\omega_1, \dots, \omega_{i_1}, \omega_{i_1+1} + \dots + \omega_{i_1+i_2}, \omega_{i_1+i_2+1}, \dots, \omega_n), \\ \tilde{\omega} &:= (\omega_{i_1+1}, \dots, \omega_{i_1+i_2}), \end{aligned}$$

the map

$$\begin{aligned} \theta : \mathbb{R}^{i_1+i_3} \times \mathbb{R}^{i_2-1} &\longrightarrow \mathbb{R}^{n-1} \\ (x_1, \dots, x_{i_1+i_3}) \times (y_1, \dots, y_{i_2-1}) &\longmapsto (x_1, \dots, x_{i_1}, y_1, \dots, y_{i_2-1}, x_{i_1+1}, \dots, x_{i_1+i_3}) \end{aligned}$$

induces a bijection between  $J_{\bar{\omega}} \times K_{\tilde{\omega}}$  and the codimension 1 face of  $J_\omega$  corresponding to the intersection with  $H_{i_1, i_2, i_3}$ .

In the case of the  $H^{i_1, \dots, i_s}$  component, we define the weights

$$\begin{aligned} \bar{\omega} &:= (\sqrt{2}\Omega_1, \dots, \sqrt{2}\Omega_s), \\ \tilde{\omega}_t &:= (\omega_{i_1+\dots+i_{t-1}+1}, \dots, \omega_{i_1+\dots+i_{t-1}+i_t}), \quad 1 \leq t \leq s. \end{aligned}$$

This time, the map

$$\theta : \mathbb{R}^{s-1} \times \mathbb{R}^{i_1-1} \times \dots \times \mathbb{R}^{i_s-1} \longrightarrow \mathbb{R}^{n-1}$$

sends an element  $(x_1, \dots, x_{s-1}) \times (y_1^1, \dots, y_{i_1-1}^1) \times \dots \times (y_1^s, \dots, y_{i_s-1}^s)$  to

$$(y_1^1, \dots, y_{i_1-1}^1, x_1, y_1^2, \dots, y_{i_2-1}^2, x_2, y_1^3, \dots, x_{s-1}, y_1^s, \dots, y_{i_s-1}^s).$$

It induces a bijection between  $K_{\bar{\omega}} \times J_{\tilde{\omega}_1} \times \dots \times J_{\tilde{\omega}_s}$  and the codimension 1 face of  $J_\omega$  corresponding to the intersection with  $H^{i_1, \dots, i_s}$ .

**Step 2** We set the orientation on  $\mathbb{R}^{n-1}$ , and hence on  $J_\omega$ , to be such that the vectors

$$f_j^\omega := (0, 0, \dots, 0, -1_j, 0, \dots, 0) ,$$

define a positively oriented basis of  $\mathbb{R}^{n-1}$ . In the coordinates of the basis  $f_j^\omega$ , the half-space  $D_{i_1, i_2, i_3}$  reads as

$$z_{i_1+1} + \dots + z_{i_1+i_2-1} \leq - \sum_{i_1+1 \leq k < l \leq i_1+i_2} \omega_k \omega_l ,$$

and the half-space  $D^{i_1, \dots, i_s}$  as

$$-z_{i_1} - z_{i_1+i_2} - \dots - z_{i_1+\dots+i_{s-1}} \leq 2 \sum_{1 \leq t < u \leq s} \Omega_t \Omega_u$$

In this basis, an outward pointing vector for the boundary  $H_{i_1, i_2, i_3}$  is then

$$\nu := (0, \dots, 0, 1_{i_1+1}, \dots, 1_{i_1+i_2-1}, 0, \dots, 0) ,$$

while an outward pointing vector for the boundary  $H^{i_1, \dots, i_s}$  is

$$\nu := (0, \dots, 0, -1_{i_1}, 0, \dots, 0, -1_{i_1+i_2}, 0, \dots, 0, -1_{i_1+i_2+\dots+i_{s-1}}, 0, \dots, 0) .$$

Now that we have chosen positively oriented bases for the  $J_\omega$ , and chosen outward pointing vectors for each component of their boundaries, we conclude again by computing the image of these bases under the maps  $\theta$ .

In the case of a boundary component  $H_{i_1, i_2, i_3}$ ,

$$j \leq i_1 , \theta(f_j^{\bar{\omega}}) = f_j^\omega \quad j \geq i_1 + 1 , \theta(f_j^{\bar{\omega}}) = f_{j+i_2-1}^\omega \quad \theta(e_j^{\tilde{\omega}}) = -f_{i_1+1}^\omega + f_{i_1+j+1}^\omega .$$

The determinant against the basis  $f_j^\omega$  then has value

$$\det_{f_j^\omega} \left( \nu, \theta(f_j^{\bar{\omega}}), \theta(e_j^{\tilde{\omega}}) \right) = (i_2 - 1)(-1)^{i_1+i_2i_3} .$$

Thus, we recover the  $(-1)^{i_1+i_2i_3} J_{i_1+1+i_3} \times K_{i_2}$  oriented component of the boundary.

Finally, in the case of a boundary component  $H^{i_1, \dots, i_s}$ , we compute that

$$\theta(e_j^{\bar{\omega}}) = -f_{i_1}^\omega + f_{i_1+\dots+i_{j+1}}^\omega \quad \theta(f_j^{\tilde{\omega}_t}) = f_{j+i_1+\dots+i_{t-1}}^\omega .$$

This time,

$$\det_{f_j^\omega} \left( \nu, \theta(e_j^{\bar{\omega}}), \theta(f_j^{\tilde{\omega}_1}), \dots, \theta(f_j^{\tilde{\omega}_s}) \right) = -(s-1)(-1)^{\epsilon_B} .$$

We find again the  $(-1)^{\epsilon_B} K_s \times J_{i_1} \times \dots \times J_{i_s}$  oriented component of the boundary, which concludes the proof of the theorem.  $\square$

## 5. Signs and moduli spaces for $\Omega BAs$ -algebras and $\Omega BAs$ -morphisms

### 5.1 The operad $\Omega BAs$

#### 5.1.1 Definition of the operad $\Omega BAs$

The definition of the operad  $\Omega BAs$  that we now lay out is the one given by Markl and Shnider in [28]. We only expose the material necessary to our construction, and refer to their paper for further details and proofs. In the rest of the section, the notation  $t$  stands for a stable ribbon tree, and the notation  $t_{br}$  denotes a broken stable ribbon tree. Observe that a stable ribbon tree is a broken stable ribbon tree with 0 broken edge. As a result, all constructions performed for broken stable ribbon trees in the upcoming sections will hold in particular for stable ribbon trees.

**Definition 5.1.1** ([28]). Given a broken stable ribbon tree  $t_{br}$ , an *ordering* of  $t_{br}$  is defined to be an ordering of its  $i$  finite internal edges  $e_1, \dots, e_i$ . Two orderings are said to be *equivalent* if one passes from one ordering to the other by an even permutation. An *orientation* of  $t_{br}$  is then defined to be an equivalence class of orderings, and written  $\omega := e_1 \wedge \dots \wedge e_i$ . Each tree  $t_{br}$  has exactly two orientations. Given an orientation  $\omega$  of  $t_{br}$  we will write  $-\omega$  for the second orientation on  $t_{br}$ , called its *opposite orientation*.

**Definition 5.1.2** ([28]). The operad  $\Omega BAs$  is defined as follows. Consider the  $\mathbb{Z}$ -module freely generated by the pairs  $(t_{br}, \omega)$  where  $t_{br}$  is a broken stable ribbon tree and  $\omega$  an orientation of  $t_{br}$ . We define the arity  $n$  space of operations  $\Omega BAs(n)_*$  to be the quotient of this  $\mathbb{Z}$ -module under the relation

$$(t_{br}, -\omega) = -(t_{br}, \omega) .$$

A pair  $(t_{br}, \omega)$  where  $t_{br}$  has  $i$  finite internal edges, is defined to have degree  $-i$ . The partial compositions are then

$$(t_{br}, \omega) \circ_k (t'_{br}, \omega') = (t_{br} \circ_k t'_{br}, \omega \wedge \omega') ,$$

where the tree  $t_{br} \circ_k t'_{br}$  is the broken ribbon tree obtained by grafting  $t'_{br}$  to the  $k$ -th incoming edge of  $t_{br}$ , and the edge resulting from the grafting is broken. The differential  $\partial_{\Omega BAs}$  on  $\Omega BAs(n)_*$  is finally set to send an element  $(t_{br}, e_1 \wedge \dots \wedge e_i)$  to

$$\sum_{j=1}^i (-1)^j ((t_{br}/e_j, e_1 \wedge \dots \wedge \hat{e}_j \wedge \dots \wedge e_i) - ((t_{br})_j, e_1 \wedge \dots \wedge \hat{e}_j \wedge \dots \wedge e_i)) ,$$

where  $t_{br}/e_j$  is the tree obtained from  $t$  by collapsing the edge  $e_j$  and  $(t_{br})_j$  is the tree obtained from  $t_{br}$  by breaking the edge  $e_j$ .

Choosing a distinguished orientation for every stable ribbon tree  $t \in SRT$ , this definition of the operad  $\Omega BAs$  yields the definition as the quasi-free operad

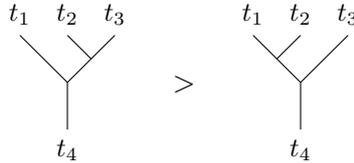
$$\mathcal{F}(\Upsilon, \Psi, \Upsilon, \Upsilon, \dots, SRT_n, \dots) ,$$

given in Section 3.1.4. Albeit Definition 5.1.2 is more tedious at first sight, it allows for easier computations of signs.

**5.1.2 Canonical orientations for the binary ribbon trees ([28])**

For a fixed  $n \geq 2$ , the set of binary ribbon trees  $BRT_n$  can be endowed with a partial order that Tamari introduced in his thesis [40].

**Definition 5.1.3.** The *Tamari order* on  $BRT_n$  is the partial order generated by the covering relations



where  $t_1, t_2, t_3$  and  $t_4$  are binary ribbon trees.

The left-hand side in the above covering relation will be called a *right-leaning configuration*, and the right-hand side a *left-leaning configuration*. Hence given two trees  $t$  and  $t'$  in  $BRT_n$ , the inequality  $t \geq t'$  holds if and only one can pass from  $t$  to  $t'$  by successive transformations of a right-leaning configuration into a left-leaning configuration.

For example in the case of  $BRT_4$ , we obtain the Hasse diagram in Figure 5.1. The Tamari poset is moreover a lattice, hence has a unique maximal element and a unique minimal element, respectively given by the right-leaning and left-leaning combs and denoted  $t_{max}$  and  $t_{min}$ . Given moreover a binary ribbon tree  $t$ , its immediate neighbours are by definition the trees obtained from  $t$  by either transforming exactly one right-leaning configuration of  $t$  into a left-leaning configuration, or transforming exactly one left-leaning configuration of  $t$  into a right-leaning configuration.

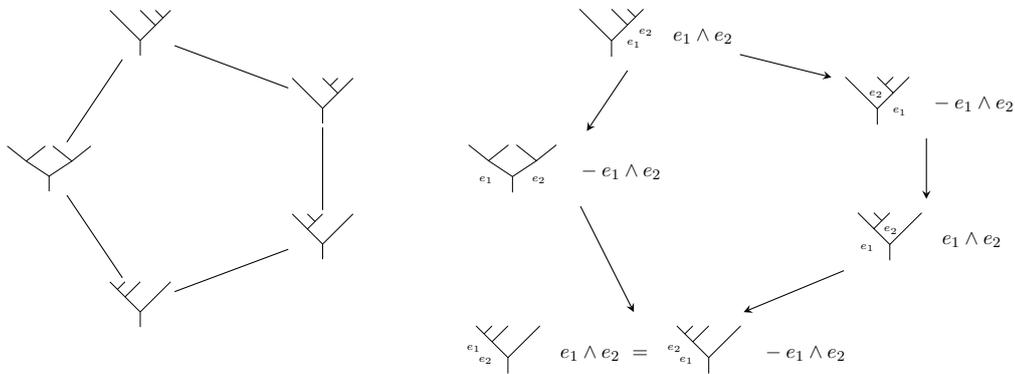


Figure 5.1: On the left, the Hasse diagram of the Tamari poset, where the maximal element is written at the top. On the right, all the canonical orientations for  $BRT_4$  computed going down the Tamari poset.

The canonical orientation on the maximal binary tree is defined as

$$\omega_{can} := e_1 \wedge \dots \wedge e_{n-2} .$$

Using the Tamari order, we can now build inductively canonical orientations on all binary trees.

We start at the maximal binary ribbon tree, and use the following rule on the covering relations

$$\begin{array}{c} t_1 \quad t_2 \quad t_3 \\ \diagdown \quad \diagup \\ \quad \quad e \\ \quad \quad | \\ \quad \quad t_4 \end{array} \quad \omega = \cdots \wedge e \wedge \cdots \quad \longrightarrow \quad \begin{array}{c} t_1 \quad t_2 \quad t_3 \\ \diagdown \quad \diagup \\ e \\ | \\ t_4 \end{array} \quad -\omega = \cdots \wedge (-e) \wedge \cdots ,$$

to define the orientations of its immediate neighbours. We then repeat this rule while going down the Tamari poset until the minimal binary tree is reached.

**Lemma 5.1.4** ([28]). *This process is consistent: it does not depend on the path taken in the Tamari poset from the maximal binary tree to the binary tree whose orientation is being defined.*

**Definition 5.1.5** ([28]). The well-defined orientations obtained under this process are called the *canonical orientations* and written  $\omega_{can}$ .

### 5.1.3 Proof of Proposition 3.1.11

Consider a cell  $\overline{\mathcal{T}}_n(t_{br}) \subset (\overline{\mathcal{T}}_n)_{\Omega BAs}$ , where  $t_{br}$  is a broken stable ribbon tree. An ordering of its finite internal edges  $e_1, \dots, e_i$  induces an isomorphism

$$\overline{\mathcal{T}}_n(t_{br}) \xrightarrow{\sim} [0, +\infty]^i ,$$

where the length  $l_{e_j}$  is seen as the  $j$ -th coordinate in  $[0, +\infty]^i$ . This ordering induces in particular an orientation on  $\mathcal{T}_n(t_{br})$ , by taking the image of the canonical orientation of  $]0, +\infty[^i$  under the isomorphism. We check that two orderings of  $t_{br}$  define the same orientation on  $\mathcal{T}_n(t_{br})$  if and only if they are equivalent: in other words, an orientation of  $t_{br}$  amounts to an orientation of  $\mathcal{T}_n(t_{br})$ .

Consider now the  $\mathbb{Z}$ -module freely generated by the pairs

$$(\overline{\mathcal{T}}_n(t_{br}), \text{choice of orientation } \omega \text{ on the cell } \overline{\mathcal{T}}_n(t_{br})) ,$$

where  $t_{br}$  is a broken stable ribbon tree. The complex  $C_{-*}^{cell}(\overline{\mathcal{T}}_n)$  is exactly defined to be the quotient of this  $\mathbb{Z}$ -module under the relation

$$-(\overline{\mathcal{T}}_n(t_{br}), \omega) = (\overline{\mathcal{T}}_n(t_{br}), -\omega) .$$

The differential of an element  $(\overline{\mathcal{T}}_n(t_{br}), \omega)$  is moreover given by the classical cubical differential on  $[0, +\infty]^i$ . Defining the cell chain complex in this way, the result of Proposition 3.1.11 becomes tautological.

### 5.1.4 Proof of Proposition 3.1.15

We now have all the necessary material to prove Proposition 3.1.15: our goal is to show that the obvious map  $\text{id} : (\overline{\mathcal{T}}_n)_{A_\infty} \rightarrow (\overline{\mathcal{T}}_n)_{\Omega BAs}$  is sent under the functor  $C_{-*}^{cell}$  to the morphism of operads  $A_\infty \rightarrow \Omega BAs$  of [28] acting as

$$m_n \longmapsto \sum_{t \in BRT_n} (t, \omega_{can}) .$$

Beware however that we do not construct a morphism of operads  $K_n \rightarrow (\overline{\mathcal{T}}_n)_{\Omega BAs}$  (see Remark 5.1.6). For this purpose, we will work with the Loday realizations of the associahedra and

use Lemma 3.1.13: we will prove that taking the restriction of the orientation of  $K_n$  chosen in Section 4.3 to the top dimensional cells of its dual subdivision, yields the canonical orientations on these cells under the identification  $(\overline{\mathcal{T}}_n)_{\Omega BAs} \simeq (K_n)_{dual}$ .

We begin by proving this statement for the cell labeled by the right-leaning comb  $t_{max}$ . Consider the orientation on the cell  $\overline{\mathcal{T}}_n(t_{max})$  induced by the canonical ordering  $e_1, \dots, e_{n-2}$  under the isomorphism

$$\overline{\mathcal{T}}_n(t_{max}) \xrightarrow{\sim} [0, +\infty]^{n-2} .$$

The face of  $\overline{\mathcal{T}}_n(t_{max})$  associated to the breaking of the  $i$ -th edge corresponds to the face  $H_{i,n-i,0}$  when seen in the Loday polytope. An outward-pointing vector for the face  $H_{i,n-i,0}$  is moreover

$$\nu_i := (0, \dots, 0, 1_i, \dots, 1_{n-2}) ,$$

where coordinates are taken in the basis  $e_j^\omega$ . The orientation defined by the canonical basis of  $[0, +\infty]^{n-2}$  being exactly the one defined by the ordered list of the outward-pointing vectors to the  $+\infty$  boundary, it is sent to the orientation of the basis  $(\nu_1, \dots, \nu_{n-2})$  in the Loday polytope. We then check that

$$\det_{e_j^\omega}(\nu_j) = 1 .$$

Hence the orientation of  $K_n$  and the one induced by the canonical orientation are the same for the cell  $\overline{\mathcal{T}}_n(t_{max})$ .

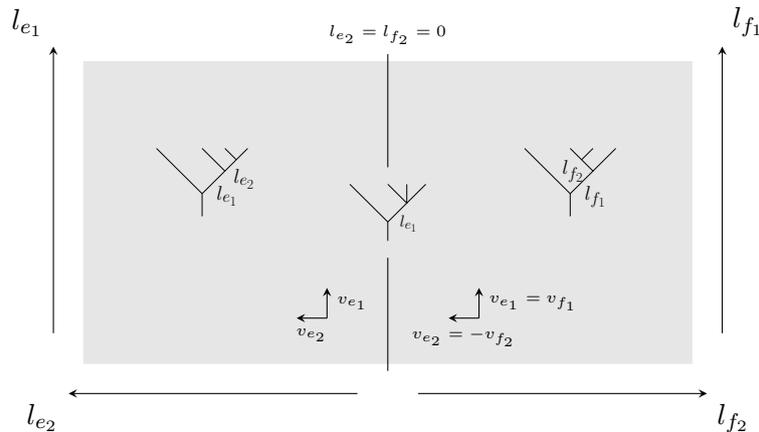


Figure 5.2: Gluing the cells  $\overline{\mathcal{T}}_n(t_{max})$  and  $\overline{\mathcal{T}}_n(t)$  along their common boundary: on this diagram, a vector of the form  $v_e$  is the vector orienting the axis associated to the length  $l_e$

It can easily be seen from Definition 5.1.3 that the cells labeled by the immediate neighbours of the right-leaning comb  $t_{max}$  in the Tamari order are exactly the cells having a codimension 1 stratum in common with this cell. Choose an immediate neighbour  $t$ , and write  $e$  for the edge that has been collapsed to obtain the common codimension 1 stratum. The method to obtain the induced orientation on  $\overline{\mathcal{T}}_n(t)$  follows Figure 5.2. Gluing the cells  $\overline{\mathcal{T}}_n(t_{max})$  and  $\overline{\mathcal{T}}_n(t)$  along their common boundary, we obtain a new copy of  $[0, +\infty]^{n-2}$  which can be divided into two halves  $t_{max}$  and  $t$ . We then orient the total space  $[0, +\infty]^{n-2}$  as the  $t_{max}$  half. Reading the induced orientation on the  $t$  half, it is the one obtained from the  $t_{max}$  half by reversing the axis associated to the edge  $e$ . By construction, this orientation is exactly the one obtained by restricting the global orientation on  $K_n$  to an orientation on  $\mathcal{T}_n(t)$ . Finally, going down the Tamari order, we can read the induced orientation on the top dimensional cells one immediate neighbour after

another. And the rule to do this step-by-step process is exactly the one given in Section 5.1.2 on the covering relations. Hence, by construction, the global orientation on  $K_n$  restricts to the canonical orientations on binary trees, which concludes the proof of Proposition 3.1.15.

REMARK 5.1.6. The operad  $(\overline{\mathcal{T}}_n)_{\Omega BAs}$  is in fact naturally isomorphic to the  $W$ -construction  $WAss$  of the standard associative operad  $Ass$ , as explained in [6]. It is unclear to the author whether explicit morphisms of topological operads  $K \rightarrow WAss$  or  $WAss \rightarrow K$  were already constructed in the literature or not - where  $K$  denotes any topological operad isomorphic to the Loday associahedra operad. We should however mention in this regard that in [5, Theorem 1.4.10] Barber constructs an explicit isomorphism of topological operads  $WK \xrightarrow{\sim} K$ .

**5.2 The moduli spaces  $\mathcal{CT}_n(t_{br,c})$**  The goal of this section is two-fold: complete the definition of the moduli spaces  $\mathcal{CT}_n(t_c)$  introduced in Definition 3.2.15 and compute the signs appearing in the codimension 1 strata of their compactification in order to complete the definition of the differential on the operadic bimodule  $M_{\Omega BAs}$  (Definition 3.2.15) in Lemma 5.3.3 of Section 5.3.1.

**5.2.1 Definition of the moduli spaces  $\mathcal{CT}_n(t_{br,c})$**

We will write  $t_{br,c}$  for a broken 2-colored stable ribbon tree (Definition 3.2.11) and  $t_c$  for an (unbroken) 2-colored stable ribbon tree. We will moreover call the unique stable 2-colored tree of arity 1  $\vdash$  the *trivial 2-colored tree*.

**Definition 5.2.1.** We define the *underlying broken stable ribbon tree*  $t_{br}$  of a broken 2-colored stable ribbon tree  $t_{br,c}$  to be the broken stable ribbon tree obtained by first deleting all the  $\vdash$  in  $t_{br,c}$ , and then forgetting all the remaining gauges of  $t_{br,c}$ . We then refer to a gauge in  $t_{br,c}$  which is associated to a non-trivial gauged tree as a *non-trivial gauge* of  $t_{br,c}$ .

We refer to Figure 5.3 for an instance of association  $t_{br,c} \mapsto t_{br}$ . We now define the moduli spaces  $\mathcal{CT}_n(t_{br,c})$  in Definition 5.2.5 following a step-by-step approach in Steps 5.2.2 to 5.2.4.

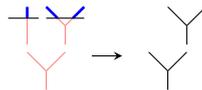


Figure 5.3: An instance of association  $t_{br,c} \mapsto t_{br}$ , where  $t_{br,c}$  has one trivial gauge and one non-trivial gauge

**Step 5.2.2.** Consider a 2-colored stable ribbon tree  $t_c$  whose gauge does not intersect any of its vertices. Locally at any vertex directly adjacent to the gauge, the intersection between the gauge and the edges of  $t_c$  corresponds to one of the following two cases



Write  $r$  for the root, the unique vertex adjacent to the outgoing edge. For a vertex  $v$ , we denote  $d(r, v)$  the distance separating it from the root: the sum of the lengths of the edges appearing in the unique non self-crossing path going from  $r$  to  $v$ . Associating lengths  $l_e > 0$  to all edges of  $t$ , we then associate the following inequalities to the two above cases

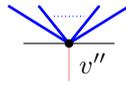
$$-\lambda > d(r, v) \qquad -\lambda < d(r, v') .$$

Note that this set of inequalities amounts to seeing the gauge as going towards  $-\infty$  when going up, and towards  $+\infty$  as going down. The moduli space  $\mathcal{CT}_n(t_c)$  is then defined as

$$\mathcal{CT}_n(t_c) := \{(\lambda, \{l_e\}_{e \in E(t)}) , \lambda \in \mathbb{R}, l_e > 0, -\lambda > d(r, v), -\lambda < d(r, v')\} ,$$

where the set of inequalities on  $\lambda$  is prescribed by the 2-colored tree  $t_c$ .

**Step 5.2.3.** Consider now a 2-colored stable ribbon tree  $t_c$  whose gauge may intersect some of its vertices. To the two local pictures of Step 5.2.2, one has to add the case



to which we associate the equality

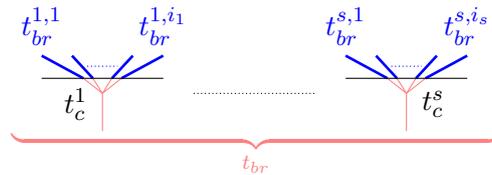
$$-\lambda = d(r, v'').$$

The moduli space  $\mathcal{CT}_n(t_c)$  is this time defined as

$$\mathcal{CT}_n(t_c) := \{(\lambda, \{l_e\}_{e \in E(t)}) , \lambda \in \mathbb{R}, l_e > 0, -\lambda > d(r, v), -\lambda < d(r, v'), -\lambda = d(r, v'')\} ,$$

where the set of equalities and inequalities on  $\lambda$  is prescribed by the 2-colored tree  $t_c$ .

**Step 5.2.4.** Finally, consider a 2-colored broken stable ribbon tree  $t_{br,c}$ , whose gauges may intersect some of its vertices. We order the non-trivial unbroken 2-colored ribbon trees appearing in  $t_{br,c}$  from left to right, as



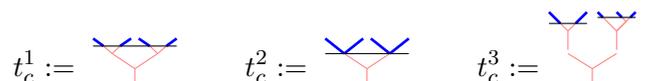
where  $t_{br}^{1,1}, \dots, t_{br}^{1,i_1}, \dots, t_{br}^{s,1}, \dots, t_{br}^{s,i_s}$  and  $t_{br}$  are broken stable ribbon trees, and the non-trivial unbroken 2-colored ribbon trees are represented in the picture as 2-colored corollae  $t_c^1, \dots, t_c^s$  for the sake of readability. We write moreover  $r_1, \dots, r_s$  and  $\lambda_1, \dots, \lambda_s$  for their respective roots and gauges.

**Definition 5.2.5.** Given a 2-colored broken stable ribbon tree  $t_{br,c}$ , we define the moduli space

$$\mathcal{CT}_n(t_{br,c}) := \left\{ \begin{array}{l} (\lambda_1, \dots, \lambda_s, \{l_e\}_{e \in E(t_{br,c})}) , \lambda_i \in \mathbb{R}, l_e > 0, \\ -\lambda_i > d(r_i, v), -\lambda_i < d(r_i, v'), -\lambda_i = d(r_i, v'') \end{array} \right\} ,$$

where the set of equalities and inequalities on  $\lambda_i$  is prescribed by the unbroken 2-colored tree  $t_c^i$  as in Steps 5.2.2 to 5.2.4.

**EXAMPLE 5.2.6.** We consider the unbroken 2-colored trees  $t_c^1$  and  $t_c^2$  and the broken 2-colored tree  $t_c^3$  defined as follows



Applying Definition 5.2.5 we have that

$$\begin{aligned} \mathcal{CT}_4(t_c^1) &= \{(\lambda, \{l_1, l_2\}), -\lambda > l_1, l_2 \text{ and } l_1, l_2 > 0\} , \\ \mathcal{CT}_4(t_c^2) &= \{(\lambda, \{l_1, l_2\}), -\lambda = l_1 = l_2 > 0\} \subset \mathcal{CT}_4(t_c^1) , \\ \mathcal{CT}_4(t_c^3) &= \{(\lambda_1, \lambda_2, l), \lambda_1 < 0, -\lambda_2 > l > 0\} . \end{aligned}$$

### 5.2.2 Orienting the moduli spaces $\mathcal{CT}_n(t_{br,c})$

**Definition 5.2.7.** We define an *ordering/orientation* on a broken 2-colored stable ribbon tree  $t_{br,c}$ , to be an ordering/orientation on the broken ribbon tree  $t_{br}$  (Definition 5.1.1).

We will denote orderings on trees with the symbol  $\Omega$  and orientations on trees with the symbol  $\omega$ . We now explain how to orient the moduli spaces  $\mathcal{CT}_n(t_{br,c})$  in Definition 5.2.11, following the step-by-step approach adopted in the previous section in Steps 5.2.8 to 5.2.10.

**Step 5.2.8.** Begin with a 2-colored stable ribbon tree  $t_c$  whose gauge does not intersect any of its vertices. An ordering  $\Omega$  on  $t_c$  identifies  $\mathcal{CT}_n(t_c)$  with a polyhedral cone

$$\mathcal{CT}_n(t_c) \subset ] - \infty, +\infty[\times]0, +\infty[^{e(t)},$$

defined by the inequalities  $-\lambda > d(r, v)$  and  $-\lambda < d(r, v')$ . This polyhedral cone has dimension  $e(t) + 1$ , and we choose to orient it as an open subset of  $] - \infty, +\infty[\times]0, +\infty[^{e(t)}$  endowed with its canonical orientation.

**Step 5.2.9.** Consider now a 2-colored stable ribbon tree  $t_c$  whose gauge may intersect some of its vertices. This time, an ordering  $\Omega$  on  $t_c$  identifies  $\mathcal{CT}_n(t_c)$  with a polyhedral cone

$$\mathcal{CT}_n(t_c) \subset ] - \infty, +\infty[\times]0, +\infty[^{e(t)},$$

defined by the inequalities  $-\lambda > d(r, v)$  and  $-\lambda < d(r, v')$ , to which we add the equalities  $-\lambda = d(r, v'')$ . If there are exactly  $j$  gauge-vertex intersections in the gauged tree  $t_c$ , this polyhedral cone has codimension  $j$  in  $] - \infty, +\infty[\times]0, +\infty[^{e(t)}$  (it is given by  $j$  equalities  $-\lambda = d(r, v'')$ ), hence has dimension  $e(t) + 1 - j$ .

Order now the  $j$  intersections from left to right



and consider the tree  $t'_c$  obtained by replacing these intersections by



One can see  $t_c$  as lying in the boundary of  $t'_c$ , by allowing the inequalities  $-\lambda > d(r, v_k)$  to become equalities  $-\lambda = d(r, v_k)$  for  $k = 1, \dots, j$ . This determines in particular  $j$  vectors  $\nu_k$  corresponding to the outward-pointing vectors to the boundary of the half-space  $-\lambda \geq d(r, v_k)$ . We finally choose to coorient (and hence orient)  $\mathcal{CT}_n(t_c)$  inside  $] - \infty, +\infty[\times]0, +\infty[^{e(t)}$  with the vectors  $(\nu_1, \dots, \nu_j)$ .

**Step 5.2.10.** Lastly, consider a 2-colored broken stable ribbon tree  $t_{br,c}$ , whose gauges may intersect some of its vertices. Suppose there are exactly  $s$  non-trivial unbroken 2-colored trees  $t_c^1, \dots, t_c^s$  appearing in  $t_{br,c}$ , which are ordered from left to right as previously. Suppose also that in each tree  $t_c^i$ , there are  $j_i$  gauge-vertex intersections. An ordering  $\Omega$  on  $t_{br,c}$  identifies  $\mathcal{CT}_n(t_{br,c})$  with a polyhedral cone

$$\mathcal{CT}_n(t_{br,c}) \subset ] - \infty, +\infty[^s \times ]0, +\infty[^{e(t_{br})},$$

defined by the set of equalities and inequalities on the  $\lambda_i$ , and where the factor  $] - \infty, +\infty[^s$  corresponds to  $(\lambda_1, \dots, \lambda_s)$ . This polyhedral cone has dimension  $e(t_{br}) + s - \sum_{i=1}^s j_i$ . Now, as in

Step 5.2.9, order all gauge-vertex intersections from left to right in every tree  $t_c^i$ , and construct a new tree  $t'_{br,c}$ . Seeing  $\mathcal{CT}_n(t_{br,c})$  as lying in the boundary of  $\mathcal{CT}_n(t'_{br,c})$ , this determines again a collection of outward-pointing vectors  $\nu_{i,1}, \dots, \nu_{i,j_i}$  for  $i = 1, \dots, s$ . We then coorient  $\mathcal{CT}_n(t_{br,c})$  inside  $] - \infty, +\infty[^s \times ]0, +\infty[^{e(t_{br})}$  with the vectors  $(\nu_{1,1}, \dots, \nu_{1,j_1}, \dots, \nu_{s,1}, \dots, \nu_{s,j_s})$ .

**Definition 5.2.11.** Given a 2-colored broken stable ribbon tree  $t_{br,c}$  together with an ordering  $\Omega$ , we define  $\mathcal{CT}_n(t_{br,c}, \Omega)$  to be the moduli space  $\mathcal{CT}_n(t_{br,c})$  endowed with the orientation described in Steps 5.2.8 to 5.2.10.

For two equivalent orderings  $\Omega_1$  and  $\Omega_2$  on  $t_{br,c}$ , the oriented space  $\mathcal{CT}_n(t_{br,c}, \Omega_1)$  and the oriented space  $\mathcal{CT}_n(t_{br,c}, \Omega_2)$  are then naturally isomorphic as oriented spaces.

EXAMPLE 5.2.12. We keep the notations  $t_c^1$  and  $t_c^2$  of Example 5.2.6 and order the edges of  $t_c^1$  and  $t_c^2$  from left to right. We then have that the moduli space  $\mathcal{CT}_4(t_c^1)$  is oriented as an open subset of  $] - \infty, +\infty[ \times ]0, +\infty[^2$  and that the moduli space  $\mathcal{CT}_4(t_c^2) \subset \mathcal{CT}_4(t_c^1)$  is cooriented in  $\mathcal{CT}_4(t_c^1) \subset ] - \infty, +\infty[ \times ]0, +\infty[^2$  by the vectors  $(1, 1, 0)$  and  $(1, 0, 1)$ .

REMARK 5.2.13. We point out that for a fixed broken stable ribbon tree type  $t_{br}$  together with an ordering, all 2-colored trees  $t_{br,c}$  whose underlying ribbon tree is  $t_{br}$  determine a partition of  $] - \infty, +\infty[^s \times ]0, +\infty[^{e(t_{br})}$  in polyhedral cones. This is illustrated in Figure 5.4.

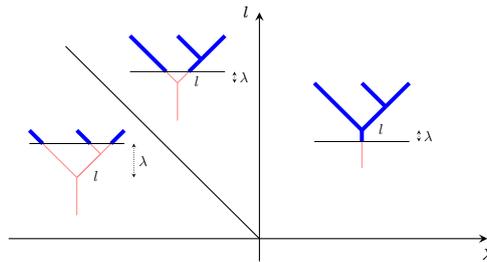


Figure 5.4

### 5.2.3 Codimension 1 strata of the compactification $\overline{\mathcal{CT}}_n(t_c)$

For a 2-colored stable ribbon tree  $t_c$ , the compactified moduli space  $\overline{\mathcal{CT}}_n(t_c)$  has codimension 1 strata given by the four components introduced in Definition 3.2.13: (int-collapse), (gauge-vertex), (above-break) and (below-break). Choose an ordering  $\Omega$  for  $t_c$ . We will now compute the signs appearing in the boundary of the compactification of the oriented moduli space  $\mathcal{CT}_n(t_c, \Omega)$  in Sections 5.2.4 to 5.2.7.

#### 5.2.4 The (int-collapse) boundary component

Consider a 2-colored stable ribbon tree  $t_c$ . The (int-collapse) boundary corresponds to the collapsing of an internal edge that does not intersect the gauge of the tree  $t$ . Choosing an ordering  $\Omega = e_1, \dots, e_i$ , suppose that it is the  $p$ -th edge of  $t$  which collapses. Write moreover  $(t/e_p)_c$  for the resulting 2-colored tree, and  $\Omega_p := e_1, \dots, \widehat{e_p}, \dots, e_i$  for the induced ordering on the edges of  $t/e_p$ .

**Proposition 5.2.14** ((int-collapse) sign). *For a 2-colored stable ribbon tree  $t_c$  whose gauge intersects  $j$  of its vertices, the boundary component  $\mathcal{CT}_n((t/e_p)_c, \Omega_p)$  corresponding to the collapsing of the  $p$ -th edge of  $t$  bears a  $(-1)^{p+1+j}$  sign in the boundary of  $\overline{\mathcal{CT}}_n(t_c, \Omega)$ .*

*Proof. Case 1.* We begin by considering the case of a 2-colored tree  $t_c$  whose gauge does not intersect any of its vertices. Suppose first that the collapsing edge is located above the gauge. A neighborhood of the boundary can then be parametrized as

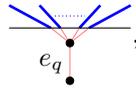
$$\begin{aligned} \Phi : ] - 1, 0] \times \mathcal{CT}_n((t/e_p)_c, \Omega_p) &\longrightarrow \overline{\mathcal{CT}}_n(t_c, \Omega) \\ (\delta, \lambda, l_1, \dots, \widehat{l_p}, \dots, l_i) &\longmapsto (\lambda, l_1, \dots, l_p := -\delta, \dots, l_i) . \end{aligned}$$

This map has sign  $(-1)^{p+1}$ , and the component  $\mathcal{CT}_n((t/e_p)_c, \Omega_p)$  consequently bears a  $(-1)^{p+1}$  sign in the boundary of  $\overline{\mathcal{CT}}_n(t_c, \Omega)$ .

*Case 2.* Suppose next that the collapsing edge is located below the gauge. We define a parametrization of a neighborhood of the boundary

$$] - 1, 0] \times \mathcal{CT}_n((t/e_p)_c, \Omega_p) \longrightarrow \overline{\mathcal{CT}}_n(t_c, \Omega)$$

as follows:  $\lambda$  is sent to  $\lambda + \delta$ ; if the edge  $e_q$  is located directly below a gauge-edge intersection



then we send  $l_q$  to  $l_q - \delta$ ; for all the other edges  $e_q$  of  $(t/e_p)$ , we send  $l_q$  to  $l_q$ ; finally, we set  $l_p := -\delta$ . We check again that this map has sign  $(-1)^{p+1}$ . Hence, in general, for a 2-colored tree  $t_c$  whose gauge does not intersect any of its vertices, the component  $\mathcal{CT}_n((t/e_p)_c, \Omega_p)$  bears a  $(-1)^{p+1}$  sign in the boundary of  $\overline{\mathcal{CT}}_n(t_c, \Omega)$ .

*Case 3.* Move on to the case of a 2-colored stable ribbon tree  $t_c$  whose gauge may intersect some of its vertices. Order the  $j$  gauge-vertex intersections from left to right as depicted in Section 5.2.2. We are going to distinguish three cases, but will eventually end up with the same sign in each case. Suppose to begin with that the collapsing edge  $e_p$  is located above the gauge, and is not adjacent to a gauge-vertex intersection. Then, denoting  $(t/e_p)'_c$  the tree obtained via the same process as  $t'_c$ , we check that parametrization introduced in (Case 1)

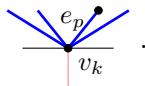
$$\Phi : ] - 1, 0] \times \mathcal{CT}_n((t/e_p)'_c, \Omega_p) \longrightarrow \overline{\mathcal{CT}}_n(t'_c, \Omega) ,$$

restricts to a parametrization of a neighborhood of the boundary

$$\phi : ] - 1, 0] \times \mathcal{CT}_n((t/e_p)_c, \Omega_p) \longrightarrow \overline{\mathcal{CT}}_n(t_c, \Omega) .$$

We also check that  $\Phi$  sends the outward-pointing vectors  $\nu_k^{(t/e_p)}$  associated to the gauge-vertex intersections in  $(t/e_p)_c$ , to the outward-pointing vectors  $\nu_k^t$  associated to the gauge-vertex intersections in  $t_c$ . Computing the sign of  $\phi$  amounts to computing the sign of  $\Phi$  and then exchanging the direction  $\delta$  with the outward-pointing vectors  $\nu_1^t, \dots, \nu_j^t$ . The total sign is hence  $(-1)^{p+1+j}$ .

*Case 4.* Suppose, as second case, that the collapsing edge  $e_p$  is located above the gauge, and directly adjacent to a gauge-vertex intersection.



We cannot use the trees  $(t/e_p)'_c$  and  $t'_c$  as in the last paragraph, as the gauge would then cut the edge  $e_p$  in the 2-colored tree  $t'_c$ . A small change is required. We form the tree  $t''_c$  as the tree  $t'_c$ , but instead of moving the gauge up at the vertex  $v_k$ , we move it down. The tree  $(t/e_p)''_c$  is

defined similarly. Applying the same argument as previously, we compute again a  $(-1)^{p+1+j}$  sign for the boundary.

*Case 5.* Finally, suppose that the collapsing edge  $e_p$  is located below the gauge. It may this time be directly adjacent to a gauge-vertex intersection. Introducing again the trees  $(t/e_p)'_c$  and  $t'_c$ , and using this time the parametrization of (Case 2), we find a  $(-1)^{p+1+j}$  sign for the boundary. Note that there is a small adjustment to make in the proof for the outward-pointing vectors. Indeed, the outward-pointing vector  $\nu_k^{(t/e_p)}$  gets again sent to the outward-pointing vector  $\nu_k^t$ , except if the edge  $e_p$  is located in the non-self crossing path going from the vertex  $v_k$  intersected by the gauge to the root. For such an intersection, the vector  $\nu_k^{(t/e_p)}$  is sent to  $\nu_k^t - e_p$  by the map  $\Phi$ , where  $e_p$  is the positive direction for the length  $l_p$ . Though the vector  $\nu_k^t - e_p$  is not equal to  $\nu_k^t$ , it is still outward-pointing to the half-space  $-\lambda \geq d(r, v_k)$ . As a result,  $\Phi(\nu_1^{(t/e_p)}), \dots, \Phi(\nu_j^{(t/e_p)})$  defines indeed the same coorientation of  $\mathcal{CT}_n(t_c, \Omega)$  as  $\nu_1^t, \dots, \nu_j^t$ .  $\square$

### 5.2.5 The (gauge-vertex) boundary component

Consider a 2-colored stable ribbon tree  $t_c$  whose gauge may intersect some of its vertices. We order the gauge-vertex intersections from left to right as depicted in Section 5.2.2. The (gauge-vertex) boundary corresponds to the gauge crossing exactly one additional vertex of  $t$ . We suppose that this intersection takes place between the  $k$ -th and  $k + 1$ -th intersections of  $t_c$ . We write moreover  $t_c^0$  for the resulting 2-colored tree, and introduce again the tree  $t'_c$  of Section 5.2.2.

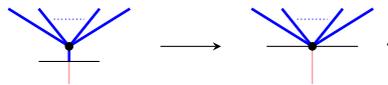
**Proposition 5.2.15** ((gauge-vertex) sign). *Suppose the crossing results from a move*



Then the boundary component  $\mathcal{CT}_n(t_c^0, \Omega)$  has sign  $(-1)^{j+k}$  in the boundary of  $\overline{\mathcal{CT}}_n(t_c, \Omega)$ .

*Proof.* Indeed the orientation induced on  $\mathcal{CT}_n(t_c^0, \Omega)$  in the boundary of  $\overline{\mathcal{CT}}_n(t_c, \Omega)$ , is defined by the coorientation  $(\nu_1, \dots, \nu_k, \widehat{\nu}, \nu_{k+1}, \dots, \nu_j, \nu)$  inside  $\mathcal{CT}_n(t'_c, \Omega)$ . The orientation defined by  $\Omega$  on  $\mathcal{CT}_n(t_c^0, \Omega)$ , is the one defined by the coorientation  $(\nu_1, \dots, \nu_k, \nu, \nu_{k+1}, \dots, \nu_j)$  inside  $\mathcal{CT}_n(t'_c, \Omega)$ . Hence, these two orientations differ by a  $(-1)^{j+k}$  sign.  $\square$

**Proposition 5.2.16** ((gauge-vertex) sign). *Suppose the crossing results from a move*



Then the boundary component  $\mathcal{CT}_n(t_c^0, \Omega)$  has sign  $(-1)^{j+k+1}$  in the boundary of  $\overline{\mathcal{CT}}_n(t_c, \Omega)$ .

*Proof.* Again the orientation induced on  $\mathcal{CT}_n(t_c^0, \Omega)$  in the boundary of  $\overline{\mathcal{CT}}_n(t_c, \Omega)$ , is defined by the coorientation  $(\nu_1, \dots, \nu_k, \widehat{\nu}, \nu_{k+1}, \dots, \nu_j, -\nu)$  inside  $\mathcal{CT}_n(t'_c, \Omega)$ . The orientation defined by  $\Omega$  on  $\mathcal{CT}_n(t_c^0, \Omega)$ , is the one defined by the coorientation  $(\nu_1, \dots, \nu_k, \nu, \nu_{k+1}, \dots, \nu_j)$  inside  $\mathcal{CT}_n(t'_c, \Omega)$ . Hence, these two orientations differ by a  $(-1)^{j+k+1}$  sign.  $\square$

### 5.2.6 The (above-break) boundary component

The (above-break) boundary corresponds either to the breaking of an internal edge of  $t$ , that is located above the gauge or intersects the gauge, or, when the gauge is below the root, to the outgoing edge breaking between the gauge and the root. Choosing an ordering  $\Omega = e_1, \dots, e_i$ , suppose that it is the  $p$ -th edge of  $t$  which breaks and write moreover  $(t_p)_c$  for the resulting broken 2-colored tree.

**Proposition 5.2.17** ((above-break) sign). *For a 2-colored stable ribbon tree  $t_c$  whose gauge intersects  $j$  vertices, the boundary component  $\mathcal{CT}_n((t_p)_c, \Omega_p)$  corresponding to the breaking of the  $p$ -th edge of  $t$  bears a  $(-1)^{p+j}$  sign in the boundary of  $\overline{\mathcal{CT}}_n(t_c, \Omega)$ , where we set  $e_0$  for the outgoing edge of  $t$ .*

*Proof. Case 1.* We begin by considering the case of a 2-colored tree  $t_c$  whose gauge does not intersect any of its vertices. Suppose first that the breaking edge does not intersect the gauge. A neighborhood of the boundary can then be parametrized as

$$\begin{aligned} ]0, +\infty] \times \mathcal{CT}_n((t_p)_c, \Omega_p) &\longrightarrow \overline{\mathcal{CT}}_n(t_c, \Omega) \\ (\delta, \lambda, l_1, \dots, \widehat{l}_p, \dots, l_i) &\longmapsto (\lambda, l_1, \dots, l_p := \delta, \dots, l_i) . \end{aligned}$$

This map has sign  $(-1)^p$ . In the case when the breaking edge does intersect the gauge, a neighborhood of the boundary can be parametrized as

$$\begin{aligned} ]0, +\infty] \times \mathcal{CT}_n((t_p)_c, \Omega_p) &\longrightarrow \overline{\mathcal{CT}}_n(t_c, \Omega) \\ (\delta, \lambda, l_1, \dots, \widehat{l}_p, \dots, l_i) &\longmapsto (\lambda, l_1, \dots, l_p := \delta - \lambda, \dots, l_i) , \end{aligned}$$

where we set this time  $l_p := \delta - \lambda$  in order for the inequality  $-\lambda < d(r, v')$  to hold in this case. This parametrization again has sign  $(-1)^p$ .

*Case 2.* The case of a 2-colored tree  $t_c$  whose gauge may intersect some of its vertices is treated as in Section 5.2.4. We check again that the parametrization maps  $\Phi$  introduced in the previous paragraph, restrict to parametrizations of a neighborhood of the boundary

$$]0, +\infty] \times \mathcal{CT}_n((t_p)_c, \Omega_p) \longrightarrow \overline{\mathcal{CT}}_n(t_c, \Omega) ,$$

and that  $\Phi$  sends moreover the coorientation of  $\mathcal{CT}_n((t_p)_c, \Omega_p)$  to the coorientation of  $\mathcal{CT}_n(t_c, \Omega)$ . These coorientations introduce as previously an additional  $(-1)^j$  sign.

*Case 3.* Finally, suppose that the gauge of  $t_c$  intersects its outgoing edge and compute the sign of the (above-break) boundary component corresponding to the gauge going towards  $+\infty$ . A parametrization of a neighborhood of the boundary is simply given by

$$\begin{aligned} ]0, +\infty] \times \mathcal{CT}_n((t_0)_c, \Omega_p) &\longrightarrow \overline{\mathcal{CT}}_n(t_c, \Omega) \\ (\delta, l_1, \dots, l_i) &\longmapsto (\lambda := \delta, l_1, \dots, l_i) . \end{aligned}$$

This map has sign 1. □

### 5.2.7 The (below-break) boundary component

The (below-break) boundary finally corresponds to the breaking of edges of  $t$  that are located below the gauge or intersect it, such that there is exactly one edge breaking in each non-self crossing path from an incoming edge to the root. Write  $(t_{br})_c$  for the resulting broken 2-colored tree. Consider now an ordering  $\Omega = e_1, \dots, e_i$  of  $t_c$ . We order again from left to right the  $s$  non-trivial unbroken 2-colored trees  $t_c^1, \dots, t_c^s$  of  $(t_{br})_c$ , and denote moreover  $e_{j_1}, \dots, e_{j_s}$  the internal edges of  $t$  whose breaking produce the trees  $t_c^1, \dots, t_c^s$ . Beware that we do not necessarily have that  $j_1 < \dots < j_s$ . We denote  $\varepsilon(j_1, \dots, j_s; \Omega)$  the sign obtained after modifying  $\Omega$  by moving  $e_{j_k}$  to the  $k$ -th spot in  $\Omega$ , and write  $\Omega_0$  for the newly obtained ordering on  $t_c$ . Twisting the orientation on  $\mathcal{CT}_n(t_c, \Omega)$  by  $(-1)^{\varepsilon(j_1, \dots, j_s; \Omega)}$  amounts to identifying it with  $\mathcal{CT}_n(t_c, \Omega_0)$ .

**Proposition 5.2.18** ((below-break) sign). *For a 2-colored stable ribbon tree  $t_c$  whose gauge intersects  $j$  vertices, the boundary component  $\mathcal{CT}_n((t_{br})_c, \Omega_{br})$  corresponding to the breaking of the internal edges  $e_{j_1}, \dots, e_{j_s}$  of  $t$  bears a  $(-1)^{\varepsilon(j_1, \dots, j_s; \Omega) + 1 + j}$  sign in the boundary of  $\overline{\mathcal{CT}}_n(t_c, \Omega)$ .*

*Proof.* We begin by assuming that  $j_1 = 1, \dots, j_s = s$ , and will explain how to deal with the general case at the end of the proof. We set to this extent  $\Omega_{br} := e_{s+1}, \dots, e_i$ . We moreover introduce two more pieces of notation. We will denote  $\mathcal{E}_\infty$  the set of incoming edges of  $t$  which are crossed by the gauge and correspond to the trivial 2-colored trees in  $(t_{br})_c$ . In other words, the set of edges which are breaking in the (below-break) boundary component associated to  $(t_{br})_c$  is  $\mathcal{E}_\infty \cup \{e_{j_1}, \dots, e_{j_s}\}$ . For an edge  $e$ , internal or external, we will moreover write  $w_e$  for the vertex adjacent to  $e$  which is closest to the root  $r$  of  $t$ , and set  $w_u := w_{e_u}$  for  $u = 1, \dots, s$ .

*Case 1.* Start by considering the case of a 2-colored tree  $t_c$  whose gauge does not intersect any of its vertices. Suppose first that among the breaking internal edges, none of them intersects the gauge. We define a parametrization of a neighbourhood of the boundary

$$]0, +\infty] \times \mathcal{CT}_n((t_{br})_c, \Omega_{br}) \longrightarrow \overline{\mathcal{CT}}_n(t_c, \Omega)$$

by sending  $(\delta, \lambda_1, \dots, \lambda_s, l_{s+1}, \dots, l_i)$  to the element of  $\mathcal{CT}_n(t_c, \Omega)$  whose entries are defined as

$$\begin{aligned} \lambda &:= -\delta + \sum_{u=1}^s (\lambda_u - d(r, w_u)) - \sum_{e \in \mathcal{E}_\infty} d(r, w_e) , \\ l_v &:= \delta + \sum_{\substack{u=1, \dots, s \\ u \neq v}} (-\lambda_u + d(r, w_u)) + \sum_{e \in \mathcal{E}_\infty} d(r, w_e) && \text{for } v = e_1, \dots, e_s , \\ l_k &:= l_k && \text{for } k = s + 1, \dots, i . \end{aligned}$$

We compute that this map has sign  $-1$ .

*Case 2.* Suppose now that among the breaking internal edges of  $t_c$ , some of them may intersect the gauge. We denote  $\mathcal{N}_\cap \subset \{1, \dots, s\}$  for the set of indices corresponding to the breaking internal edges which intersect the gauge, and  $\mathcal{N}_\emptyset \subset \{1, \dots, s\}$  for the set of indices corresponding to the breaking of internal edges which do not intersect the gauge. We define this time a parametrization of a neighbourhood of the boundary

$$]0, +\infty] \times \mathcal{CT}_n((t_{br})_c, \Omega_{br}) \longrightarrow \overline{\mathcal{CT}}_n(t_c, \Omega)$$

by sending  $(\delta, \lambda_1, \dots, \lambda_s, l_{s+1}, \dots, l_i)$  to the element of  $\mathcal{CT}_n(t_c, \Omega)$  whose entries are set to be

$$\begin{aligned} \lambda &:= -\delta + \sum_{u \in \mathcal{N}_\emptyset} (\lambda_u - d(r, w_u)) - \sum_{u \in \mathcal{N}_\cap} d(r, w_u) - \sum_{e \in \mathcal{E}_\infty} d(r, w_e) , \\ l_v &:= \delta + \sum_{\substack{u \in \mathcal{N}_\emptyset \\ u \neq v}} (-\lambda_u + d(r, w_u)) + \sum_{u \in \mathcal{N}_\cap} d(r, w_u) + \sum_{e \in \mathcal{E}_\infty} d(r, w_e) && \text{for } v \in \mathcal{N}_\emptyset , \\ l_v &:= \delta + \lambda_v + \sum_{u \in \mathcal{N}_\emptyset} (-\lambda_u + d(r, w_u)) + \sum_{\substack{u \in \mathcal{N}_\cap \\ u \neq v}} d(r, w_u) + \sum_{e \in \mathcal{E}_\infty} d(r, w_e) && \text{for } v \in \mathcal{N}_\cap , \\ l_k &:= l_k && \text{for } k = s + 1, \dots, i . \end{aligned}$$

We compute that this map has again sign  $-1$ .

*Case 3.* Consider now the case of a 2-colored tree  $t_c$  whose gauge intersects  $j$  of its vertices. We check as in the previous proofs that the parametrization maps introduced in the previous paragraphs, restrict to parametrizations of a neighborhood of the boundary

$$]0, +\infty] \times \mathcal{CT}_n((t_{br})_c, \Omega_{br}) \longrightarrow \overline{\mathcal{CT}}_n(t_c, \Omega) ,$$

and that these maps send moreover the coorientation of  $\mathcal{CT}_n((t_{br})_c, \Omega_{br})$  to the coorientation of  $\mathcal{CT}_n(t_c, \Omega)$ . These coorientations introduce an additional  $(-1)^j$  sign.

*General case.* We have thus computed the sign of the (below-break) boundary when  $j_1 = 1, \dots, j_s = s$ . Now, consider the general case where we do not necessarily have that  $j_1 = 1, \dots, j_s = s$ . We can apply the previous constructions and find the desired sign for the associated (below-break) component.  $\square$

### 5.3 The operadic bimodule $M_{\Omega BAs}$

#### 5.3.1 Proof of Proposition 3.2.20

We use the formalism of orientations on 2-colored trees in this proof, so that our description of the operadic bimodule  $M_{\Omega BAs}$  be compatible with the definition of [28] for the operad  $\Omega BAs$ . As before,  $t_{br,c}$  will stand for a broken 2-colored stable ribbon tree, while  $t_c$  will denote an unbroken 2-colored stable ribbon tree. We also respectively write  $t_{br}$  and  $t$  for their underlying stable ribbon trees.

**Lemma 5.3.1.** *Consider the  $\mathbb{Z}$ -module freely generated by the pairs  $(t_{br,c}, \omega)$ . The arity  $n$  space of operations  $M_{\Omega BAs}(n)_*$  is the quotient of this  $\mathbb{Z}$ -module under the relation*

$$(t_{br,c}, -\omega) = -(t_{br,c}, \omega) .$$

An element  $(t_{br,c}, \omega)$  where  $t_{br,c}$  has  $e(t_{br})$  finite internal edges and  $g$  non-trivial gauges which intersect  $j$  vertices of  $t_{br}$  has degree  $j - (e(t_{br}) + g)$ . The operad  $\Omega BAs$  then acts on  $M_{\Omega BAs}$  as follows

$$\begin{aligned} (t_{br,c}, \omega) \circ_i (t'_{br}, \omega') &= (t_{br,c} \circ_i t'_{br}, \omega \wedge \omega') , \\ \mu((t_{br}, \omega), (t_{br,c}^1, \omega_1), \dots, (t_{br,c}^s, \omega_s)) &= (-1)^\dagger (\mu(t_{br}, t_{br,c}^1, \dots, t_{br,c}^s), \omega \wedge \omega_1 \wedge \dots \wedge \omega_s) , \end{aligned}$$

where the tree  $t_{br,c} \circ_i t'_{br}$  is the 2-colored broken ribbon tree obtained by grafting  $t'_{br}$  to the  $i$ -th incoming edge of  $t_{br,c}$  and  $\mu(t_{br}, t_{br,c}^1, \dots, t_{br,c}^s)$  is the 2-colored broken ribbon tree defined by grafting each  $t_{br,c}^j$  to the  $j$ -th incoming edge of  $t_{br}$ . Writing  $g_i$  for the number of non-trivial gauges and  $j_i$  for the number of gauge-vertex intersections of  $t_{br,c}^i$ ,  $i = 1, \dots, s$ , and setting  $t_{br}^0 := t_{br}$  and  $g_0 = j_0 = 0$ ,

$$\dagger := \sum_{i=1}^s g_i \sum_{l=0}^{i-1} e(t_{br}^l) + \sum_{i=1}^s j_i \sum_{l=0}^{i-1} (e(t_{br}^l) + g_l - j_l) .$$

*Proof.* The description of  $M_{\Omega BAs}(n)_*$  as a graded  $\mathbb{Z}$ -module stems from the same arguments used in the proof of Proposition 3.1.11 in Section 5.1.3. It remains to check that the signs for the action-composition maps are indeed the ones determined by the compactified moduli spaces  $(\overline{\mathcal{CT}}_n)_{\Omega BAs}$ . The computation for  $\circ_i$  is straightforward. Consider now the map

$$\begin{aligned} \mu : \mathcal{T}(t_{br}, \Omega) \times \mathcal{CT}(t_{br,c}^1, \Omega_1) \times \dots \times \mathcal{CT}(t_{br,c}^s, \Omega_s) &\longrightarrow \mathcal{CT}(\mu(t_{br}, t_{br,c}^1, \dots, t_{br,c}^s), \Omega \cdot \Omega_1 \cdot \dots \cdot \Omega_s) \\ (L_\Omega, (\Lambda_1, L_{\Omega_1}), \dots, (\Lambda_s, L_{\Omega_s})) &\longmapsto (\Lambda_1, \dots, \Lambda_s, L_\Omega, L_{\Omega_1}, \dots, L_{\Omega_s}) , \end{aligned}$$

where  $L_{\Omega_i}$  stands for the list of lengths of  $t_{br}^i$  according to the ordering  $\Omega_i$ ,  $\Omega \cdot \Omega_1 \cdots \Omega_s$  is the concatenation of the orderings  $\Omega, \Omega_1, \dots, \Omega_s$  and  $\Lambda_i := (\lambda_{i,1}, \dots, \lambda_{i,g_i})$  stands for the list of non-trivial gauges of  $t_{br,c}^i$ . We compute that, in the absence of gauge vertex intersections, this map has sign

$$(-1)^{\sum_{i=1}^s g_i \sum_{l=0}^{i-1} e(t_{br}^l)} .$$

Assuming that there are some gauge-vertex intersections, the combinatorics of coorientations introduce an additional sign

$$(-1)^{\sum_{i=1}^s j_i \sum_{l=0}^{i-1} (e(t_{br}^l) + g_l - j_l)} .$$

In total, we recover the sign  $(-1)^\dagger$ , which concludes the proof. □

Choosing a distinguished orientation for every 2-colored stable ribbon tree  $t_c \in SCRT$ , this definition of the operadic bimodule  $M_{\Omega BAs}$  amounts to defining it as the free operadic bimodule in graded modules

$$\mathcal{F}^{\Omega BAs, \Omega BAs}(\uparrow, \downarrow, \downarrow, \downarrow, \dots, SCRT_n, \dots) .$$

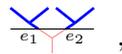
REMARK 5.3.2. We point out that a second formula for  $\dagger$  is

$$\dagger = \sum_{i=1}^s g_i \left( |t_{br}| + \sum_{l=1}^{i-1} |t_{br}^l| \right) + \sum_{i=1}^s j_i \left( |t_{br}| + \sum_{l=1}^{i-1} |t_{br,c}^l| \right) .$$

**Lemma 5.3.3.** *The differential of a 2-colored stable ribbon tree  $(t_c, \omega)$  is the signed sum of all codimension 1 contributions*

$\partial(t_c, \omega) = \sum \pm(\text{int} - \text{collapse}) + \sum \pm(\text{gauge} - \text{vertex}) + \sum \pm(\text{above} - \text{break}) + \sum \pm(\text{below} - \text{break})$  ,  
 where our choice of notation for the terms of the sums is as in Definition 3.2.13 and where the signs are as computed in Propositions 5.2.14 to 5.2.18.

EXAMPLE 5.3.4. We compute for instance that after choosing the orientation  $e_1 \wedge e_2$  on



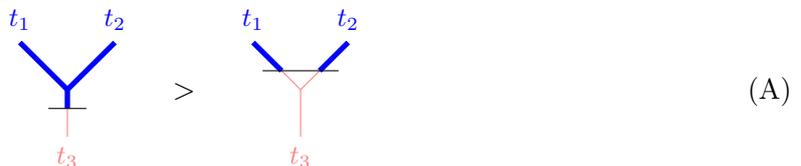
the signs in Example 3.2.21 are

$$\begin{aligned} \partial\left(\frac{\downarrow \downarrow}{\uparrow}, e_1 \wedge e_2\right) = & \left(\frac{\downarrow \downarrow}{\uparrow}, e_1 \wedge e_2\right) - \left(\frac{\downarrow \downarrow}{\uparrow}, e_1 \wedge e_2\right) - \left(\frac{\downarrow \downarrow}{\uparrow}, e_1 \wedge e_2\right) \\ & + \left(\frac{\downarrow \downarrow}{\uparrow}, e_1\right) - \left(\frac{\downarrow \downarrow}{\uparrow}, e_2\right) - \left(\frac{\downarrow \downarrow}{\uparrow}, \emptyset\right) . \end{aligned}$$

**5.3.2 Canonical orientations for 2-colored binary ribbon trees**

For a fixed  $n \geq 2$ , the set of 2-colored binary ribbon trees  $CBRT_n$  can be endowed with a partial order, inspired by the Tamari order on  $BRT_n$ . It is introduced in [20].

**Definition 5.3.5** ([20]). The *Tamari order on  $CBRT_n$*  is the partial order generated by the covering relations



where  $t_1, t_2$  and  $t_3$  are binary ribbon trees,

$$\begin{array}{c} t_c^1 \quad t_c^2 \quad t_c^3 \\ \diagdown \quad \diagup \\ \text{---} \\ t \end{array} > \begin{array}{c} t_c^1 \quad t_c^2 \quad t_c^3 \\ \diagdown \quad \diagup \\ \text{---} \\ t \end{array} \tag{B.1}$$

where  $t_c^1, t_c^2, t_c^3$  are 2-colored binary ribbon trees and  $t$  is a binary ribbon tree, and

$$\begin{array}{c} t_1 \quad t_2 \quad t_3 \\ \diagdown \quad \diagup \\ \text{---} \\ t_c \end{array} > \begin{array}{c} t_1 \quad t_2 \quad t_3 \\ \diagdown \quad \diagup \\ \text{---} \\ t_c \end{array} \tag{B.2}$$

where  $t_1, t_2, t_3$  are binary ribbon trees and  $t_c$  is a 2-colored binary ribbon tree.

For example in the case of  $CBRT_4$ , we obtain the Hasse diagram in Figure 5.5. This Tamari-like poset has a unique maximal element and a unique minimal element, respectively given by the right-leaning comb whose gauge intersects the outgoing edge, and the left-leaning comb whose gauge intersects all incoming edges.

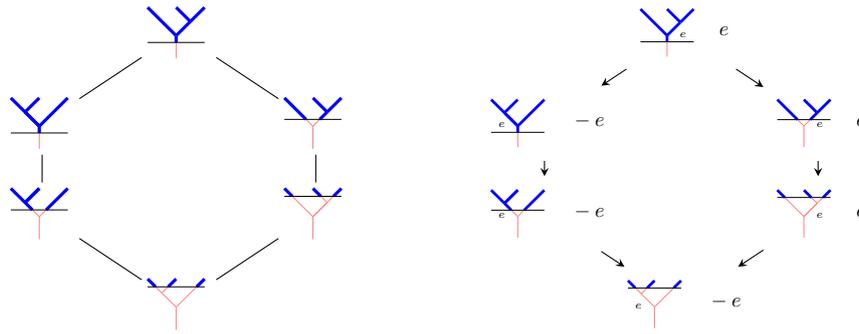


Figure 5.5: On the left, the Hasse diagram of the poset  $CBRT_3$ , where the maximal element is written at the top. On the right, all the canonical orientations for  $CBRT_3$  computed going down the poset.

The canonical orientation on the maximal 2-colored binary tree is defined as

$$\begin{array}{c} \diagdown \quad \diagup \\ \text{---} \\ e_1 \end{array} \dots \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \\ e_{n-2} \end{array} \quad \omega_{can} := e_1 \wedge \dots \wedge e_{n-2} .$$

Using this Tamari-like order, we can now build inductively canonical orientations on all 2-colored binary trees. We start at the maximal 2-colored binary tree, and transport the orientation  $\omega_{can}$  to its immediate neighbours as follows: the immediate neighbours of  $t_c^{max}$  obtained under the covering relation (A) are endowed with the orientation  $\omega_{can}$ , while the ones obtained under the covering relations (B) are endowed with the orientation  $-\omega_{can}$ . We then repeat this operation while going down the poset until the minimal 2-colored binary tree is reached.

**Lemma 5.3.6.** *This process is consistent, it does not depend on the path taken in the poset from  $t_c^{max}$  to the 2-colored binary tree whose orientation is being defined.*

*Proof.* An adaptation of the proof of Lemma 5.1.4 in [28] shows that it is enough to prove that the diagrams described by  $K_4$  and  $J_3$  commute in order to conclude. This is proven in Figures 5.1 and 5.5.  $\square$

**Definition 5.3.7.** These well-defined orientations will again be called the *canonical orientations* and written  $\omega_{can}$ .

It is in fact straightforward to check that they coincide with the canonical orientations on the underlying binary trees.

REMARK 5.3.8. Lemmas 5.1.4 and 5.3.6 are MacLane’s coherence type lemmas. A heuristic explanation for Lemma 5.1.4 can be given as follows. A path between two trees  $t$  and  $t'$  in the Tamari poset corresponds to a path in the 1-skeleton of  $K_n$ . The faces of the 2-skeleton of  $K_n$  consist moreover of the products

$$K_2 \times \cdots \times K_2 \times K_3 \times K_2 \times \cdots \times K_2 \times K_3 \times K_2 \times \cdots \times K_2 ,$$

$$K_2 \times \cdots \times K_2 \times K_4 \times K_2 \times \cdots \times K_2 .$$

The first type of face corresponds to a square diagram that tautologically commutes, while the second type of face corresponds to the  $K_4$  diagram. Given now two paths from  $t$  to  $t'$ , they delineate a family of faces in the 2-skeleton of  $K_n$ . Translating this into algebra, as all faces translate into commuting diagrams, the two paths produce the same orientation. See also [10].

### 5.3.3 Proof of Proposition 3.2.25

We can now show that the map  $\text{id} : (\overline{\mathcal{CT}}_n)_{A_\infty} \rightarrow (\overline{\mathcal{CT}}_n)_{\Omega BAs}$  is sent under the functor  $C_{-*}^{cell}$  to a morphism of operadic bimodules  $M_\infty \rightarrow M_{\Omega BAs}$  acting as

$$f_n \longmapsto \sum_{t_c \in CBRT_n} (t_c, \omega_{can}) .$$

Beware that we do not construct a morphism of operadic bimodules  $J_n \rightarrow (\overline{\mathcal{CT}}_n)_{\Omega BAs}$ . We will work with the Forcey–Loday realizations of the multiplihedra  $J_n$  and use Lemma 3.2.23, to prove that taking the restriction of the orientation of  $J_n$  chosen in Section 4.4 to the top dimensional cells of its dual subdivision yields the canonical orientations on these cells in the  $\overline{\mathcal{CT}}_n$  viewpoint. We follow in this regard the exact same line of proof as in Section 5.1.4.

This statement is at first shown for the maximal 2-colored binary tree  $t_c^{max}$ , the right-leaning comb whose gauge crosses the outgoing edge. The orientation on the cell  $\overline{\mathcal{CT}}_n(t_c^{max})$  induced by the canonical ordering  $e_1, \dots, e_{n-2}$  defines an isomorphism

$$\overline{\mathcal{CT}}_n(t_c^{max}) \xrightarrow{\sim} [0, +\infty] \times [0, +\infty]^{n-2} ,$$

where the factor  $[0, +\infty]$  corresponds to the gauge  $\lambda$ , and the factor  $[0, +\infty]^{n-2}$  to the lengths of the inner edges. The face of  $\overline{\mathcal{CT}}_n(t_c^{max})$  associated to the gauge going to  $+\infty$  corresponds to the face  $H_{0,n,0}$  when seen in the Forcey–Loday polytope, while the face associated to the breaking of the  $i$ -th edge corresponds to the face  $H_{i,n-i,0}$ . An outward-pointing vector for the face  $H_{i,n-i,0}$  is moreover

$$\nu_i := (0, \dots, 0, 1_{i+1}, \dots, 1_{n-1}) ,$$

where coordinates are taken in the basis  $f_j^\omega$ . The orientation defined by the canonical basis of  $[0, +\infty] \times [0, +\infty]^{n-2}$  is exactly the one defined by the ordered list of the outward-pointing

vectors to the  $+\infty$  boundary. This orientation is thus sent to the orientation defined by the basis  $(\nu_0, \dots, \nu_{n-2})$  in the Forcey–Loday polytope. It remains to check that

$$\det_{f_j^\omega}(\nu_j) = 1 .$$

As a result, the orientation induced by  $J_n$  and the one defined by the canonical orientation coincide for the cell  $\overline{\mathcal{CT}}_n(t_c^{max})$ .

The rest of the proof is a mere adaptation of the proof of Section 5.1.4. The cells labeled by the 2-colored binary trees which are immediate neighbours of the maximal 2-colored binary tree, are exactly the ones having a codimension 1 stratum in common with  $\overline{\mathcal{CT}}_n(t_c^{max})$ . Choosing one such tree  $t_c$ , and gluing the cells  $\overline{\mathcal{CT}}_n(t_c)$  and  $\overline{\mathcal{CT}}_n(t_c^{max})$  along their common boundary, one can read the induced orientation on  $\overline{\mathcal{CT}}_n(t_c)$ . In the case when the immediate neighbour  $t_c$  is obtained under the covering relation (A), the cells  $\overline{\mathcal{CT}}_n(t_c)$  and  $\overline{\mathcal{CT}}_n(t_c^{max})$  are in fact both oriented as subspaces of  $] - \infty, +\infty[\times]0, +\infty[^{n-2}$ . In the case when the immediate neighbour  $t_c$  is obtained under the covering relations (B), we send the reader back to Section 5.1.4 for explanations on why a  $-1$  twist of the orientation has to be introduced. In each case, the induced orientation is exactly the canonical orientation on  $\overline{\mathcal{CT}}_n(t_c)$ . This argument can now be repeated going down the poset, and the induced orientation will always coincide with the canonical orientation on the cell, which concludes the proof of the theorem.

## PART II: GEOMETRY

### 6. $A_\infty$ and $\Omega BAs$ -algebra structures on the Morse cochains

Let  $M$  be an oriented closed Riemannian manifold endowed with a Morse function  $f$  together with a Morse-Smale metric. Following [17], the Morse cochains  $C^*(f)$  form a deformation retract of the singular cochains on  $M$ . The cup product naturally endows the singular cochains  $C_{sing}^*(M)$  with a dg algebra structure. Theorem 1.6.1 then ensures that it can be transferred to an  $A_\infty$ -algebra structure on the Morse cochains  $C^*(f)$ . The following question then naturally arises. The differential on the Morse cochains is defined by a count of negative gradient trajectories connecting critical points of  $f$ . Is it possible to define higher multiplications  $m_n$  on  $C^*(f)$  by counting the points of 0-dimensional moduli spaces, such that they fit into a structure of  $A_\infty$ -algebra ?

We have seen in the previous part that the polytopes encoding the operad  $A_\infty$  are the associahedra and that they can be realized as the compactified moduli spaces of stable metric ribbon trees. A natural candidate would thus be an interpretation of metric ribbon trees in Morse theory. A naive approach would be to define trees whose internal edges correspond to finite Morse trajectories and whose external edges correspond to semi-infinite Morse trajectories, as in Figure 6.1. These moduli spaces are however not well defined, as two trajectories coming from two distinct critical points cannot intersect. A second problem is that moduli spaces of trajectories issued from the same critical point do not intersect transversely. Abouzaid bypasses this problem in [3] by perturbing the equation around each vertex, so that a transverse intersection can be achieved. This is illustrated in Figure 6.1.

Trees obtained in this way will be called *perturbed Morse gradient trees*. Let  $t$  be a stable ribbon tree type and  $y, x_1, \dots, x_n$  a collection of critical points of the Morse function  $f$ . We prove in this section that for a generic choice of perturbation data  $\mathbb{X}_t$  on the moduli space  $\mathcal{T}_n(t)$ ,

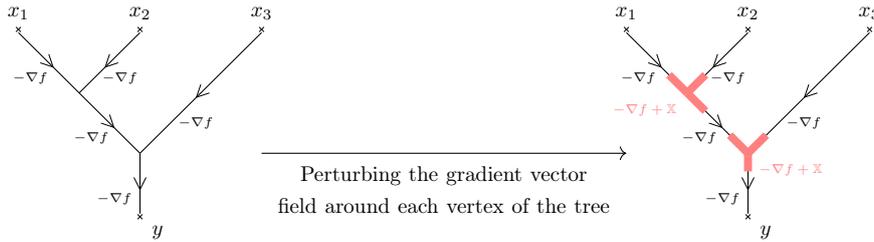


Figure 6.1

the moduli space of perturbed Morse gradient trees modeled on  $t$  and connecting  $x_1, \dots, x_n$  to  $y$ , denoted  $\mathcal{T}_t(y; x_1, \dots, x_n)$ , is an orientable manifold (Proposition 6.3.3). Under some additional generic assumptions on the choices of perturbation data  $\mathbb{X}_t$ , these moduli spaces are compact in the 0-dimensional case, and can be compactified to compact manifolds with boundary in the 1-dimensional case (Theorems 6.4.5 and 6.4.6). We are finally able to define operations on the Morse cochains  $C^*(f)$  through counts of perturbed Morse gradient trees: these operations define an  $\Omega BAs$ -algebra structure on  $C^*(f)$  (Theorem 6.5.1). Our constructions are carried out using the viewpoint of [3, Sections 2 and 7] on perturbed Morse gradient trees that we recall in Section 6.2, and borrowing some terminology and notations used in [35]. Technical details are moreover postponed to Sections 8 and 9.

**6.1 Conventions** We will study Morse theory of the Morse function  $f : M \rightarrow \mathbb{R}$  using its negative gradient vector field  $-\nabla f$ . Denote  $d$  the dimension of the manifold  $M$  and  $\phi^s$  the flow of  $-\nabla f$ . For a critical point  $x$  define its unstable and stable manifolds

$$W^U(x) := \{z \in M, \lim_{s \rightarrow -\infty} \phi^s(z) = x\}$$

$$W^S(x) := \{z \in M, \lim_{s \rightarrow +\infty} \phi^s(z) = x\} .$$

Their dimensions are such that  $\dim(W^U(x)) + \dim(W^S(x)) = d$ . We then define the *degree of a critical point*  $x$  to be  $|x| := \dim(W^S(x))$ . This degree is often referred to as the *coindex of  $x$*  in the litterature.

We will moreover work with Morse cochains. For two critical point  $x \neq y$ , define

$$\mathcal{T}(y; x) := W^S(y) \cap W^U(x) / \mathbb{R}$$

to be the moduli space of negative gradient trajectories connecting  $x$  to  $y$ . Denote moreover  $\mathcal{T}(x; x) = \emptyset$ . Under the Morse-Smale assumption on  $f$  and the Riemannian metric on  $M$ , for  $x \neq y$  the moduli space  $\mathcal{T}(y; x)$  has dimension  $\dim(\mathcal{T}(y; x)) = |y| - |x| - 1$ . The Morse differential  $\partial_{Morse} : C^*(f) \rightarrow C^*(f)$  is then defined to count descending negative gradient trajectories

$$\partial_{Morse}(x) := \sum_{|y|=|x|+1} \#\mathcal{T}(y; x) \cdot y .$$

We refer to Section 9.2 for additional details on the moduli spaces introduced in this section.

**6.2 Perturbed Morse gradient trees**

**Definition 6.2.1** ([3]). Let  $T := (t, \{l_e\}_{e \in E(t)})$  be a metric tree, where  $\{l_e\}_{e \in E(t)}$  are the lengths of its internal edges. A *choice of perturbation data* on  $T$  consists of the following data:

(i) a vector field

$$[0, l_e] \times M \xrightarrow{\mathbb{X}_e} TM ,$$

that vanishes on  $[1, l_e - 1]$ , associated to each internal edge  $e$  of  $t$  ;

(ii) a vector field

$$[0, +\infty[ \times M \xrightarrow{\mathbb{X}_{e_0}} TM ,$$

that vanishes away from  $[0, 1]$ , associated to the outgoing edge  $e_0$  of  $t$  ;

(iii) a vector field

$$]-\infty, 0] \times M \xrightarrow{\mathbb{X}_{e_i}} TM ,$$

that vanishes away from  $[-1, 0]$ , associated to each incoming edge  $e_i$  ( $1 \leq i \leq n$ ) of  $t$ .

Note that when  $l_e \leq 2$ , the vanishing condition on  $[1, l_e - 1]$  is empty, that is we do not require any specific vanishing property for  $\mathbb{X}_e$ . For brevity's sake we will write  $D_e$  for all segments  $[0, l_e]$  as well as for all semi-infinite segments  $]-\infty, 0]$  and  $[0, +\infty[$  in the rest of the paper.

**Definition 6.2.2** ([3]). A perturbed Morse gradient tree  $T^{Morse}$  associated to  $(T, \mathbb{X})$  is the data for each edge  $e$  of  $t$  of a smooth map  $\gamma_e : D_e \rightarrow M$  such that  $\gamma_e$  is a trajectory of the perturbed negative gradient  $-\nabla f + \mathbb{X}_e$ , i.e.

$$\dot{\gamma}_e(s) = -\nabla f(\gamma_e(s)) + \mathbb{X}_e(s, \gamma_e(s)) ,$$

and such that the endpoints of these trajectories coincide as prescribed by the edges of the tree  $T$ .

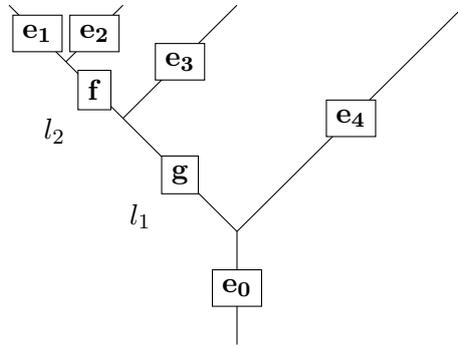


Figure 6.2: Choosing perturbation data  $\mathbb{X}$  for this metric tree, we have that  $\phi_{1,\mathbb{X}} = \phi_{g,\mathbb{X}}^{l_1} \circ \phi_{f,\mathbb{X}}^{l_2} \circ \phi_{e_1,\mathbb{X}}^1$ ,  $\phi_{2,\mathbb{X}} = \phi_{g,\mathbb{X}}^{l_1} \circ \phi_{f,\mathbb{X}}^{l_2} \circ \phi_{e_2,\mathbb{X}}^1$ ,  $\phi_{3,\mathbb{X}} = \phi_{g,\mathbb{X}}^{l_1} \circ \phi_{e_3,\mathbb{X}}^1$  and  $\phi_{4,\mathbb{X}} = \phi_{e_4,\mathbb{X}}^1$

A perturbed Morse gradient tree  $T^{Morse}$  associated to  $(T, \mathbb{X})$  is determined by the data of the time -1 points on its incoming edges plus the time 1 point on its outgoing edge. Indeed, for each edge  $e$  of  $t$ , we write  $\phi_{e,\mathbb{X}}$  for the flow of  $-\nabla f + \mathbb{X}_e$ . We moreover define for every incoming edge  $e_i$  ( $1 \leq i \leq n$ ) of  $T$ , the diffeomorphism  $\phi_{i,\mathbb{X}}$  to be the composition of all flows obtained by following the time -1 point of the metric tree on  $e_i$  along the only non-self crossing path connecting it to the root. We also set  $\phi_{0,\mathbb{X}}$  for the flow of  $\phi_{e_0,\mathbb{X}}$  at time -1, where  $e_0$  is the outgoing edge of  $t$ . This is depicted on Figure 6.2. Setting

$$\Phi_{T,\mathbb{X}} : M \times \cdots \times M \xrightarrow{\phi_{0,\mathbb{X}} \times \cdots \times \phi_{n,\mathbb{X}}} M \times \cdots \times M ,$$

and  $\Delta$  for the thin diagonal of  $M \times \cdots \times M$ , it is then clear that:

**Proposition 6.2.3** ([3]). *There is a one-to-one correspondence*

$$\left\{ \begin{array}{l} \text{perturbed Morse gradient trees} \\ \text{associated to } (T, \mathbb{X}) \end{array} \right\} \longleftrightarrow (\Phi_{T, \mathbb{X}})^{-1}(\Delta) .$$

The vector fields on the external edges are equal to  $-\nabla f$  away from a length 1 segment, hence the trajectories associated to these edges all converge to critical points of the function  $f$ . For critical points  $y$  and  $x_1, \dots, x_n$ , the map  $\Phi_{T, \mathbb{X}}$  can be restricted to

$$W^S(y) \times W^U(x_1) \times \dots \times W^U(x_n) ,$$

such that the inverse image of the diagonal yields all perturbed Morse gradient trees associated to  $(T, \mathbb{X})$  connecting  $x_1, \dots, x_n$  to  $y$ .

**6.3 Moduli spaces of perturbed Morse gradient trees** Let  $t$  be a stable ribbon tree and  $\Omega$  be an ordering on  $t$  (Definition 5.1.1). Recall that  $E(t)$  stands for the set of internal edges of  $t$ , and  $\bar{E}(t)$  for the set of all its edges. We previously saw that a choice of perturbation data on a metric ribbon tree  $T := (t, (l_e)_{e \in E(t)})$  is the data of maps  $\mathbb{X}_{T, f} : D_f \times M \rightarrow TM$ , for every edge  $f \in \bar{E}(t)$  of  $t$ . Define the cone  $C_f \subset \mathcal{T}_n(t, \Omega) \times \mathbb{R} \simeq ]0, +\infty[^{e(t)} \times \mathbb{R}$  to be

- (i)  $\{((l_e)_{e \in E(t)}, s) \text{ such that } 0 \leq s \leq l_f\}$  if  $f$  is an internal edge ;
- (ii)  $\{((l_e)_{e \in E(t)}, s) \text{ such that } s \leq 0\}$  if  $f$  is an incoming edge ;
- (iii)  $\{((l_e)_{e \in E(t)}, s) \text{ such that } s \geq 0\}$  if  $f$  is the outgoing edge.

Then a choice of perturbation data for every metric ribbon tree in  $\mathcal{T}_n(t)$  yields a map

$$\mathbb{X}_{t, f} : C_f \times M \rightarrow TM ,$$

for every edge  $f$  of  $t$ .

**Definition 6.3.1.** A choice of perturbation data  $\mathbb{X}_t$  is said to be *smooth* if all the maps  $\mathbb{X}_{t, f} : C_f \times M \rightarrow TM$  extend to smooth maps  $]0, +\infty[^{e(t)} \times \mathbb{R} \times M \rightarrow TM$ .

**Definition 6.3.2.** Let  $\mathbb{X}_t$  be a smooth choice of perturbation data on  $\mathcal{T}_n(t)$ . For critical points  $y$  and  $x_1, \dots, x_n$ , we define the moduli space

$$\mathcal{T}_t^{\mathbb{X}_t}(y; x_1, \dots, x_n) := \left\{ \begin{array}{l} \text{perturbed Morse gradient trees associated to } (T, \mathbb{X}_T) \\ \text{and connecting } x_1, \dots, x_n \text{ to } y, \text{ for } T \in \mathcal{T}_n(t) \end{array} \right\} .$$

Introduce now the map

$$\phi_{\mathbb{X}_t} : \mathcal{T}_n(t) \times W^S(y) \times W^U(x_1) \times \dots \times W^U(x_n) \rightarrow M^{\times n+1} ,$$

whose restriction to every  $T \in \mathcal{T}_n(t)$  is as defined at the end of Section 6.2:

**Proposition 6.3.3.** (i) *The moduli space  $\mathcal{T}_t^{\mathbb{X}_t}(y; x_1, \dots, x_n)$  can be rewritten as*

$$\mathcal{T}_t^{\mathbb{X}_t}(y; x_1, \dots, x_n) = \phi_{\mathbb{X}_t}^{-1}(\Delta) ,$$

where  $\Delta$  is the thin diagonal of  $M^{\times n+1}$ .

(ii) *Given a choice of perturbation data  $\mathbb{X}_t$  making  $\phi_{\mathbb{X}_t}$  transverse to the diagonal  $\Delta$ , the moduli space  $\mathcal{T}_t^{\mathbb{X}_t}(y; x_1, \dots, x_n)$  is an orientable manifold of dimension*

$$\dim(\mathcal{T}_t(y; x_1, \dots, x_n)) = e(t) + |y| - \sum_{i=1}^n |x_i| .$$

(iii) Choices of perturbation data  $\mathbb{X}_t$  such that  $\phi_{\mathbb{X}_t}$  is transverse to  $\Delta$  exist.

*Proof.* Item (i) is straightforward and Item (ii) stems from the fact that if  $\phi_{\mathbb{X}_t}$  is transverse to  $\Delta$ , the moduli spaces  $\mathcal{T}_t^{\mathbb{X}_t}(y; x_1, \dots, x_n)$  are manifolds of codimension

$$\text{codim}(\mathcal{T}_t(y; x_1, \dots, x_n)) = \text{codim}_{M^{\times n+1}}(\Delta) = nd,$$

where  $d := \dim(M)$ . Note that we have chosen to grade the Morse cochains using the coindex in order for this convenient dimension formula to hold. We refer to Section 8.2 for details on Item (iii).  $\square$

## 6.4 Compactifications

### 6.4.1 Compactification of the 1-dimensional manifolds $\mathcal{T}_t^{\mathbb{X}_t}(y; x_1, \dots, x_n)$

We now would like to compactify the 1-dimensional moduli spaces  $\mathcal{T}_t^{\mathbb{X}_t}(y; x_1, \dots, x_n)$  to 1-dimensional manifolds with boundary. They are defined as the inverse image in  $\mathcal{T}_n(t) \times W^S(y) \times W^U(x_1) \times \dots \times W^U(x_n)$  of the diagonal  $\Delta$  under  $\phi_{\mathbb{X}_t}$ . The boundary components in the compactification should hence come from those of  $\mathcal{T}_n(t)$ , of the unstable manifolds  $W^U(x_i)$ , and of the stable manifold  $W^S(y)$ . In other words, they should respectively come from internal edges of the perturbed Morse gradient tree collapsing, or breaking at a critical point (boundary of  $\mathcal{T}_n(t)$ ), its semi-infinite incoming edges breaking at a critical point (boundary of  $W^U(x_i)$ ) and its semi-infinite outgoing edge breaking at a critical point (boundary of  $W^S(y)$ ). We illustrate some of these phenomena in Figure 6.3.

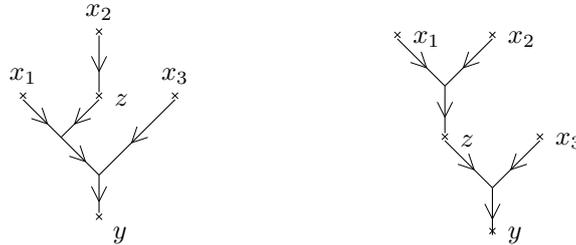


Figure 6.3: Two examples of perturbed Morse gradient trees breaking at a critical point

**Definition 6.4.1.** Given a smooth perturbation data  $\mathbb{X}_t$  for all  $t \in SRT_i$ ,  $2 \leq i \leq n$ , we denote  $\mathbb{X}_n := (\mathbb{X}_t)_{t \in SRT_n}$  and call it a *choice of perturbation data on the moduli space  $\mathcal{T}_n$* .

Following the previous discussion, we would like the boundary of the compactification of the moduli space  $\mathcal{T}_t^{\mathbb{X}_t}(y; x_1, \dots, x_n)$  to be given by the following spaces:

- (i) corresponding to an internal edge collapsing (int-collapse):

$$\mathcal{T}_{t'}^{\mathbb{X}_{t'}}(y; x_1, \dots, x_n)$$

where  $t' \in SRT_n$  are all the trees obtained by collapsing exactly one internal edge of  $t$ ;

- (ii) corresponding to an internal edge breaking (int-break):

$$\mathcal{T}_{t_1}^{\mathbb{X}_{t_1}}(y; x_1, \dots, x_{i_1}, z, x_{i_1+i_2+1}, \dots, x_n) \times \mathcal{T}_{t_2}^{\mathbb{X}_{t_2}}(z; x_{i_1+1}, \dots, x_{i_1+i_2}),$$

where  $t_2$  is seen to lie above the  $(i_1 + 1)$ -th incoming edge of  $t_1$ ;

(iii) corresponding to an external edge breaking (Morse):

$$\mathcal{T}(y; z) \times \mathcal{T}_t^{\mathbb{X}_t}(z; x_1, \dots, x_n) \quad \text{and} \quad \mathcal{T}_t^{\mathbb{X}_t}(y; x_1, \dots, z, \dots, x_n) \times \mathcal{T}(z; x_i) .$$

While the (Morse) boundary simply comes from the fact that external edges are Morse trajectories away from a length 1 segment, the analysis for the (int-collapse) and (int-break) boundaries requires some additional conditions on the perturbation data.

### 6.4.2 Smooth choices of perturbation data

We begin by tackling the conditions coming with the (int-collapse) boundary. Let  $t$  be a stable ribbon tree type and consider a choice of perturbation data on  $\mathcal{T}_n(t)$ : it is a choice of perturbation data  $\mathbb{X}_T$  for every  $T \in \mathcal{T}_n(t) \simeq ]0, +\infty[^{e(t)}$ . Denote  $coll(t) \subset SRT_n$  the set of all trees obtained by collapsing internal edges of  $t$ . A choice of perturbation data  $(\mathbb{X}_{t'})_{t' \in coll(t)}$  then corresponds to a choice of perturbation data  $\mathbb{X}_T$  for every  $T \in [0, +\infty[^{e(t)}$ . Following Section 6.3, such a choice of perturbation data is equivalent to a map

$$\tilde{\mathbb{X}}_{t,f} : \tilde{C}_f \times M \longrightarrow TM ,$$

for every edge  $f$  of  $t$ , where  $\tilde{C}_f \subset [0, +\infty[^{e(t)} \times \mathbb{R} \subset \mathbb{R}^{e(t)} \times \mathbb{R}$  is defined in a similar fashion to  $C_f$ .

**Definition 6.4.2.** A choice of perturbation data  $(\mathbb{X}_{t'})_{t' \in coll(t)}$  is said to be *smooth* if all maps  $\tilde{\mathbb{X}}_{t,f}$  extend to smooth maps  $\mathbb{R}^{e(t)} \times \mathbb{R} \times M \rightarrow TM$ . A choice of perturbation data  $\mathbb{X}_n$  is said to be *smooth* if for every  $t \in SRT_n$ , the choice of perturbation data  $(\mathbb{X}_{t'})_{t' \in coll(t)}$  is smooth.

### 6.4.3 Gluing-compatible choices of perturbation data

We now tackle the conditions coming with the (int-break) boundary. We work again with a fixed stable ribbon tree type  $t$ . Consider a choice of perturbation data  $\mathbb{X}_t = (\mathbb{X}_{t,e})_{e \in \bar{E}(t)}$  on  $\mathcal{T}_n(t)$ . We have to specify what happens on the  $\mathbb{X}_{t,e}$  when the length of an internal edge  $f$  of  $t$ , denoted  $l_f$ , goes towards  $+\infty$ . Write  $t_1$  and  $t_2$  for the trees obtained by breaking  $t$  at the edge  $f$ .

(i) For  $e \in \bar{E}(t)$  and  $e \neq f$ , assuming for instance that  $e \in t_1$ , we require that

$$\lim_{l_f \rightarrow +\infty} \mathbb{X}_{t,e} = \mathbb{X}_{t_1,e} .$$

(ii) For  $f = e$ ,  $\mathbb{X}_{t,f}$  yields two parts when  $l_f \rightarrow +\infty$ : the part corresponding to the infinite edge in  $t_1$  and the part corresponding to the infinite edge in  $t_2$ . We then require that they coincide respectively with  $\mathbb{X}_{t_1,f}$  and  $\mathbb{X}_{t_2,f}$ .

We now illustrate each of these two cases with an example. Begin with an example of the first case, where  $e \neq f$ . This is represented on Figure 6.4. We only represent the perturbation  $\mathbb{X}_{t,f_3}$  on this figure for clarity's sake. The perturbation datum  $\mathbb{X}_{t,f_3}^\infty$  could a priori depend on  $l_{f_1}$ : the requirement  $\mathbb{X}_{t,f_3}^\infty = \mathbb{X}_{t_1,f_3}$  says in particular that it is independent of  $l_{f_1}$ . Similarly, we illustrate the second case, where  $e = f$ , on Figure 6.5. A priori,  $\mathbb{X}_{t,f_2}^+$  and  $\mathbb{X}_{t,f_2}^-$  can depend on both  $l_{f_1}$  and  $l_{f_3}$ : the requirement  $\mathbb{X}_{t,f_2}^+ = \mathbb{X}_{t_2,f_2}$  says in particular that  $\mathbb{X}_{t,f_2}^+$  is independent of  $l_{f_3}$ , and the same goes for  $\mathbb{X}_{t,f_2}^- = \mathbb{X}_{t_1,f_2}$  with respect to  $l_{f_1}$ .

**Definition 6.4.3.** A choice of perturbation data  $(\mathbb{X}_i)_{2 \leq i \leq n}$  is said to be *gluing-compatible* if it satisfies the conditions of Items (i) and (ii) for lengths of edges going toward  $+\infty$ .

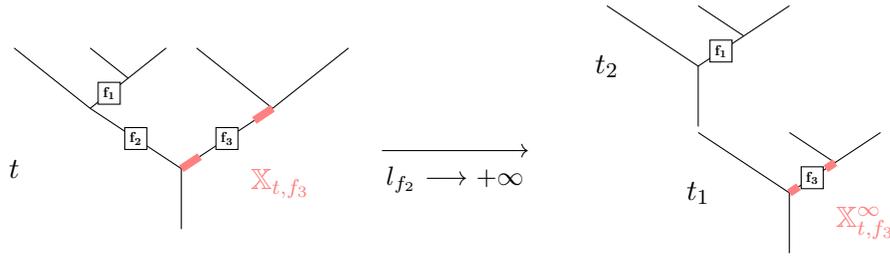


Figure 6.4

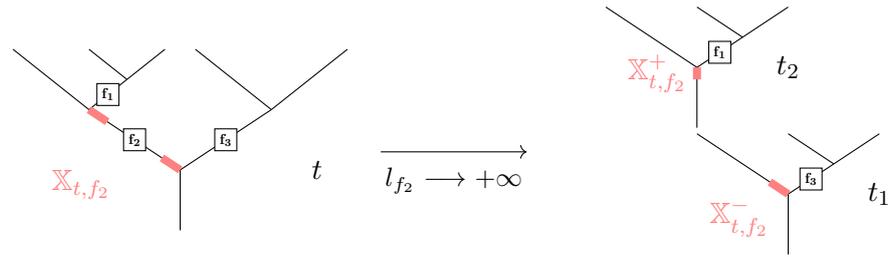


Figure 6.5

As explained in the two previous examples, a gluing-compatible choice of perturbation data has in particular the property that when lengths of edges go towards  $+\infty$ , the perturbation datum on each edge only depends on the lengths of the tree that the edge belongs to. If we only required that perturbation data on each edge have limits when lengths of edges go towards  $+\infty$  without further assumptions, the (int-break) boundary component of Section 6.4.1 would in fact appear as a fiber product over the codimension 1 boundary of the compactified moduli space  $\overline{\mathcal{T}}_n(t)$ , since these limits might depend on the lengths of all edges of the broken tree obtained from  $t$ .

### 6.4.4 Admissible choices of perturbation data

**Definition 6.4.4.** A choice of perturbation data  $(\mathbb{X}_n)_{n \geq 2}$  being smooth and gluing-compatible, and such that all maps  $\phi_{\mathbb{X}_t}$  are transverse to  $\Delta$  is said to be *admissible*.

**Theorem 6.4.5.** *Admissible choices of perturbation data on the moduli spaces  $\mathcal{T}_n$  exist.*

*Proof.* See Section 8.2. □

**Theorem 6.4.6.** *Let  $(\mathbb{X}_n)_{n \geq 2}$  be an admissible choice of perturbation data. The 0-dimensional moduli spaces  $\mathcal{T}_t^{\mathbb{X}}(y; x_1, \dots, x_n)$  are compact. The 1-dimensional moduli spaces  $\mathcal{T}_t^{\mathbb{X}}(y; x_1, \dots, x_n)$  can be compactified to 1-dimensional manifolds with boundary  $\overline{\mathcal{T}}_t^{\mathbb{X}}(y; x_1, \dots, x_n)$ , whose boundary is described in Section 6.4.1.*

*Proof.* This theorem results from the techniques of [35, Chapter 6]. □

Consider in fact an internal edge  $f \in E(t)$  and write  $t_1$  and  $t_2$  for the trees obtained by breaking  $t$  at the edge  $f$ , where  $t_2$  is seen to lie above  $t_1$ . Given critical points  $y, z, x_1, \dots, x_n$  suppose moreover that the moduli spaces  $\mathcal{T}_{t_1}(y; x_1, \dots, x_{i_1}, z, x_{i_1+i_2+1}, \dots, x_n)$  and  $\mathcal{T}_{t_2}(z; x_{i_1+1}, \dots, x_{i_1+i_2})$  are 0-dimensional. Let  $T_1^{Morse}$  and  $T_2^{Morse}$  be two perturbed Morse gradient trees which belong

respectively to the former and the latter moduli spaces. Theorem 6.4.6 implies in particular that there exists  $R > 0$  and an embedding

$$\#_{T_1^{Morse}, T_2^{Morse}} : [R, +\infty] \longrightarrow \overline{\mathcal{T}}_t(y; x_1, \dots, x_n)$$

parametrizing a neighborhood of the boundary  $\{T_1^{Morse}\} \times \{T_2^{Morse}\} \subset \partial \overline{\mathcal{T}}_t^{Morse}$ , i.e. sending  $+\infty$  to  $(T_1^{Morse}, T_2^{Morse}) \in \partial \overline{\mathcal{T}}_t^{Morse}$ . Such a map is called a *gluing map* for  $T_1^{Morse}$  and  $T_2^{Morse}$ . We will construct explicit gluing maps in Section 9.4.3.

**6.5  $\Omega BAs$ -algebra structure on the Morse cochains** We now have all the necessary material to define an  $\Omega BAs$ -algebra structure on the Morse cochains  $C^*(f)$ .

**Theorem 6.5.1.** *Let  $\mathbb{X} := (\mathbb{X}_n)_{n \geq 2}$  be an admissible choice of perturbation data. Defining for every  $n$  and  $t \in SRT_n$  the operations  $m_t$  as*

$$m_t : C^*(f) \otimes \dots \otimes C^*(f) \longrightarrow C^*(f)$$

$$x_1 \otimes \dots \otimes x_n \longmapsto \sum_{|y| = \sum_{i=1}^n |x_i| - e(t)} \# \mathcal{T}_t^{\mathbb{X}}(y; x_1, \dots, x_n) \cdot y,$$

they endow the Morse cochains  $C^*(f)$  with an  $\Omega BAs$ -algebra structure.

*Proof.* The proof of this theorem is the subject of Section 9.4. Putting it shortly, counting the boundary points of the 1-dimensional orientable compactified moduli spaces  $\overline{\mathcal{T}}_t^{\mathbb{X}}(y; x_1, \dots, x_n)$  whose boundary is described in Section 6.4.1 yields the  $\Omega BAs$ -equations

$$[\partial_{Morse}, m_t] = \sum_{t' \in coll(t)} \pm m_{t'} + \sum_{t_1 \#_i t_2 = t} \pm m_{t_1} \circ_i m_{t_2}. \quad \square$$

In fact, the collection of operations  $\{m_t\}$  does not exactly define an  $\Omega BAs$ -algebra structure: one of the two differentials  $\partial_{Morse}$  appearing in the bracket  $[\partial_{Morse}, \cdot]$  has to be twisted by a specific sign for the  $\Omega BAs$ -equations to hold. We will speak about a *twisted  $\Omega BAs$ -algebra structure* (Definition 9.3.1). In the case when  $M$  is odd-dimensional, this twisted  $\Omega BAs$ -algebra is exactly an  $\Omega BAs$ -algebra.

**REMARK 6.5.2.** If we want to recover an  $A_\infty$ -algebra structure on the Morse cochains, it suffices to apply the morphism of operads  $A_\infty \rightarrow \Omega BAs$  of Proposition 3.1.15. In [3], Abouzaid constructs a geometric  $A_\infty$ -morphism  $C_{sing}^*(M) \rightarrow C^*(f)$ , where the Morse cochains are endowed with the  $A_\infty$ -algebra structure induced by the  $\Omega BAs$ -algebra structure of Theorem 6.5.1. This  $A_\infty$ -morphism is in fact a quasi-isomorphism. This implies in particular that the Morse cochains  $C^*(f)$  endowed with the  $A_\infty$ -algebra structure of Theorem 6.5.1 are quasi-isomorphic as an  $A_\infty$ -algebra to the Morse cochains endowed with the  $A_\infty$ -algebra structure induced by Theorem 1.6.1. Abouzaid’s construction of the  $A_\infty$ -morphism  $C_{sing}^*(M) \rightarrow C^*(f)$  could be adapted to our present framework, and lifted to an  $\Omega BAs$ -morphism. We will however not give more details on that matter.

**REMARK 6.5.3.** We point out that Morse theory is the natural viewpoint that connects symplectic topology to differential topology, as pseudo-holomorphic curves tend to degenerate in the low-energy regime to Morse gradient flow trees: if symplectic topology is to be thought of as a quantization of differential topology, then pseudo-holomorphic curve theory is the quantization of Morse theory (see [14] and [11]). While the  $\Omega BAs$ -algebra structure on the Morse cochains

stems from Lemma 3.1.13, the  $A_\infty$ -category structure on the Fukaya category of a symplectic manifold then stems from the fact that the compactified moduli spaces of stable disks with  $n + 1$  marked boundary points  $\overline{\mathcal{D}}_{n,1}$  are naturally isomorphic to the associahedra  $K_n$  endowed with their  $A_\infty$ -cell decomposition (see [37] for instance).

### 7. $A_\infty$ and $\Omega BAs$ -morphisms between the Morse cochains

Let  $M$  be an oriented closed Riemannian manifold endowed with a Morse function  $f$  together with a Morse-Smale metric. We have proven in Section 6 that upon choosing admissible perturbation data on the moduli spaces of stable metric ribbon trees  $\mathcal{T}_n(t)$ , we can define moduli spaces of perturbed Morse gradient trees, whose count will define the operations  $m_t$ ,  $t \in SRT$ , of an  $\Omega BAs$ -algebra structure on the Morse cochains  $C^*(f)$ .

Consider now another Morse function  $g$  on  $M$ . Apply again Theorem 1.6.1 to the Morse cochains  $C^*(f)$  and  $C^*(g)$ , which are deformation retracts of the singular cochains on  $M$ . Endowing them with their induced  $A_\infty$ -algebra structures, this yields a diagram

$$(C^*(f), m_n^{ind}) \xrightarrow{\sim} (C_{sing}^*(M), \cup) \xrightarrow{\sim} (C^*(g), m_n^{ind}),$$

where each arrow is an  $A_\infty$ -quasi-isomorphism, hence an  $A_\infty$ -quasi-isomorphism  $(C^*(f), m_n^{ind}) \rightarrow (C^*(g), m_n^{ind})$ . Let moreover  $\mathbb{X}^g$  be an admissible perturbation data for  $g$ . The previous quasi-isomorphism motivates the following question: endowing  $C^*(f)$  and  $C^*(g)$  with the  $\Omega BAs$ -algebra structures of Theorem 6.5.1, can we construct an  $\Omega BAs$ -morphism

$$(C^*(f), m_t^{\mathbb{X}^f}) \longrightarrow (C^*(g), m_t^{\mathbb{X}^g})$$

by counting perturbed Morse trees ?

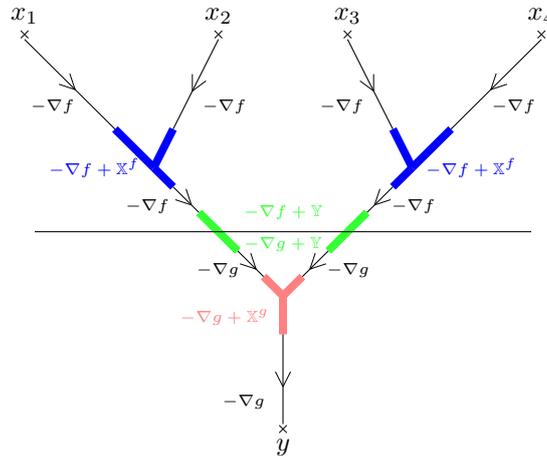


Figure 7.1: An example of a perturbed 2-colored Morse gradient tree, where the  $x_i$  are critical points of  $f$  and  $y$  is a critical point of  $g$

While stable metric ribbon trees encode  $\Omega BAs$ -algebra structures, we have seen that 2-colored stable metric ribbon trees encode  $\Omega BAs$ -morphisms. The answer to the previous question is then positive, and the morphism will be constructed using moduli spaces of 2-colored perturbed Morse gradient trees. As in Section 6, 2-colored Morse gradient trees will be defined by perturbing Morse gradient equations around each vertex of the tree, where the Morse gradient is  $-\nabla f$

above the gauge and  $-\nabla g$  below the gauge. This is illustrated in Figure 7.1. The figure is incorrect, because we will not choose the perturbation to be equal to  $\mathbb{X}^f$  above the gauge and to  $\mathbb{X}^g$  below, but conveys the correct intuition on the construction we unfold in this section.

The structure of this section follows the same lines as Section 6, and the only difficulty will consist in adapting properly our arguments to the combinatorics of 2-colored ribbon trees. Under a generic choice of perturbation data on the moduli spaces  $\mathcal{CT}_n$ , the moduli spaces of 2-colored perturbed Morse gradient trees connecting  $x_1, \dots, x_n \in \text{Crit}(g)$  to  $y \in \text{Crit}(g)$ , that we denote  $\mathcal{CT}_{t_c}(y; x_1, \dots, x_n)$ , are orientable manifolds. They are moreover compact when 0-dimensional and can be compactified to compact manifolds with boundary when 1-dimensional (Theorems 7.3.3 and 7.3.4). Counting 2-colored Morse gradient trees then defines an  $\Omega BAs$ -morphism from  $C^*(f)$  to  $C^*(g)$ , called a *continuation morphism* (Theorem 7.4.1). We prove in Theorem 7.4.5 that continuation morphisms are in fact quasi-isomorphisms. We finally discuss in Section 7.5 the two problems that naturally arise from our construction of continuation morphisms and that will be the respective starting points to the part II and part III to this series of articles.

**7.1 Perturbed 2-colored Morse gradient trees** A 2-colored metric ribbon tree  $T_c$  will be written  $(t_c, \{L_{f_c}\}_{f_c \in E(t_c)})$  from the viewpoint of Definition 3.2.1 and  $(t_c, \lambda, \{l_e\}_{e \in E(t)})$  from the viewpoint of Definition 3.2.2, where  $t$  denotes the underlying stable ribbon tree of  $t_c$ .

**Definition 7.1.1.** A *choice of perturbation data*  $\mathbb{Y}$  on a 2-colored metric ribbon tree  $T_c$  is defined to be a choice of perturbation data on the metric ribbon tree  $(t_c, \{L_{f_c}\})$  in the sense of Definition 6.2.1.

**Definition 7.1.2.** A *2-colored perturbed Morse gradient tree*  $T_c^{Morse}$  associated to a pair 2-colored metric ribbon tree and perturbation data  $(T_c, \mathbb{Y})$  is the data

- (i) for each edge  $f_c$  of  $t_c$  which is above the gauge of a smooth map

$$D_{f_c} \xrightarrow{\gamma_{f_c}} M ,$$

such that  $\gamma_{f_c}$  is a trajectory of the perturbed negative gradient  $-\nabla f + \mathbb{Y}_{f_c}$ ,

- (ii) for each edge  $f_c$  of  $t_c$  which is below the gauge of a smooth map

$$D_{f_c} \xrightarrow{\gamma_{f_c}} M ,$$

such that  $\gamma_{f_c}$  is a trajectory of the perturbed negative gradient  $-\nabla g + \mathbb{Y}_{f_c}$ ,

and such that the endpoints of these trajectories coincide as prescribed by the edges of the tree  $t_c$ .

REMARK 7.1.3. We point out that the above definitions still work for  $\dagger$ . A choice of perturbation data for  $\dagger$  is the data of vector fields

$$[0, +\infty[ \times M \xrightarrow{\mathbb{Y}_+} TM \quad ] - \infty, 0] \times M \xrightarrow{\mathbb{Y}_-} TM ,$$

which vanish away from a length 1 segment, and a 2-colored perturbed Morse gradient tree associated to  $(\dagger, \mathbb{Y})$  is then simply the data of two smooth maps

$$] - \infty, 0] \xrightarrow{\gamma_-} M \quad [0, +\infty[ \xrightarrow{\gamma_+} M ,$$

such that  $\gamma_-$  is a trajectory of  $-\nabla f + \mathbb{Y}_-$  and  $\gamma_+$  is a trajectory of  $-\nabla g + \mathbb{Y}_+$ .

A 2-colored perturbed Morse gradient tree can be equivalently defined by following the flows of  $-\nabla f + \mathbb{Y}$  and  $-\nabla g + \mathbb{Y}$  along the metric ribbon tree  $(t_c, L_{f_c})$ , as it is determined by the data of the time -1 points on its incoming edges plus the time 1 point on its outgoing edge. Introduce again the map

$$\Phi_{T_c, \mathbb{Y}} : M \times \cdots \times M \xrightarrow{\phi_{0, \mathbb{Y}} \times \cdots \times \phi_{n, \mathbb{Y}}} M \times \cdots \times M ,$$

defined as in Section 6.2, and set  $\Delta$  for the diagonal of  $M^{\times n+1}$

**Proposition 7.1.4.** *There is a one-to-one correspondence*

$$\left\{ \begin{array}{l} \text{2-colored perturbed Morse gradient trees} \\ \text{associated to } (T_c, \mathbb{Y}) \end{array} \right\} \longleftrightarrow (\Phi_{T_c, \mathbb{Y}})^{-1}(\Delta) .$$

The vector fields on the incoming edges are equal to  $-\nabla f$  away from a length 1 segment, hence the trajectories associated to these edges all converge to critical points of the function  $f$ , while the vector field on the outgoing edge is equal to  $-\nabla g$  away from a length 1 segment, hence the trajectory associated to these edge converges to a critical point of the function  $g$ . For critical points  $y$  of the function  $g$  and  $x_1, \dots, x_n$  of the function  $f$ , the map  $\Phi_{T_c, \mathbb{Y}}$  can be restricted to

$$W_g^S(y) \times W_f^U(x_1) \times \cdots \times W_f^U(x_n) ,$$

such that the inverse image of the diagonal yields all 2-colored perturbed Morse gradient trees associated to  $(T_c, \mathbb{Y})$  connecting  $x_1, \dots, x_n$  to  $y$ .

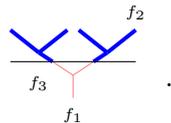
**7.2 Moduli spaces of 2-colored perturbed Morse gradient trees** Let  $t_c$  be a 2-colored stable ribbon tree of arity  $n$  together with an ordering  $\Omega$  on  $t_c$ . We write  $(*)_{t_c}$  for the set of inequalities and equalities on  $(l_e)_{e \in E(t)}$  and  $\lambda$ , which define the polyhedral cone  $\mathcal{CT}_n(t_c, \Omega) \subset \mathbb{R}^{e(t)+1}$  in Definition 5.2.5. Define for all  $f_c \in \overline{E}(t_c)$ , the cone  $C_{f_c} \subset \mathcal{CT}_n(t_c, \Omega) \times \mathbb{R} \subset \mathbb{R}^{e(t)+1} \times \mathbb{R}$  to be

- (i)  $\{(\lambda, (l_e)_{e \in E(T)}, s)$  such that  $(*)_{t_c}$ ,  $0 \leq s \leq L_{f_c}(\lambda, \{l_e\}_{e \in E(T)})\}$  if  $f_c$  is an internal edge ;
- (ii)  $\{(\lambda, (l_e)_{e \in E(T)}, s)$  such that  $(*)_{t_c}$ ,  $s \leq 0\}$  if  $f_c$  is an incoming edge ;
- (iii)  $\{(\lambda, (l_e)_{e \in E(T)}, s)$  such that  $(*)_{t_c}$ ,  $s \geq 0\}$  if  $f_c$  is the outgoing edge.

EXAMPLE 7.2.1. We consider the following 2-colored tree  $t_c$  with ordering  $e_1 < e_2$  on the underlying ribbon tree  $t$



We then denote  $f_1, f_2$  and  $f_3$  the following three edges of the 2-colored tree  $t_c$



The cones  $C_{f_i} \subset \mathbb{R}^4$  are then equal to

$$\begin{aligned} C_{f_1} &= \{(\lambda, l_1, l_2, s), 0 < -\lambda < l_1, l_2 \text{ and } s \geq 0\} , \\ C_{f_2} &= \{(\lambda, l_1, l_2, s), 0 < -\lambda < l_1, l_2 \text{ and } s \leq 0\} , \\ C_{f_3} &= \{(\lambda, l_1, l_2, s), 0 < -\lambda < l_1, l_2 \text{ and } 0 \leq s \leq -\lambda\} . \end{aligned}$$

Then a choice of perturbation data for every 2-colored metric ribbon tree in  $\mathcal{CT}_n(t_c)$ , yields maps  $\mathbb{Y}_{t_c, f_c} : C_{f_c} \times M \rightarrow TM$  for every edge  $f_c$  of  $t_c$ .

**Definition 7.2.2.** A choice of perturbation data  $\mathbb{Y}_{t_c}$  is said to be *smooth* if all the maps  $\mathbb{Y}_{t_c, f_c}$  extend to smooth maps  $\mathbb{R}^{e(t)+1} \times \mathbb{R} \times M \rightarrow TM$ .

**Definition 7.2.3.** Let  $\mathbb{Y}_{t_c}$  be a smooth choice of perturbation data on the moduli space  $\mathcal{CT}_n(t_c)$ . Given  $y \in \text{Crit}(g)$  and  $x_1, \dots, x_n \in \text{Crit}(f)$ , we define the moduli spaces

$$\mathcal{CT}_{t_c}^{\mathbb{Y}_{t_c}}(y; x_1, \dots, x_n) := \left\{ \begin{array}{l} \text{2-colored perturbed Morse gradient trees associated to } (T_c, \mathbb{Y}_{T_c}) \\ \text{and connecting } x_1, \dots, x_n \text{ to } y \text{ for } T_c \in \mathcal{CT}_n(t_c) \end{array} \right\}.$$

Using the smooth map

$$\phi_{\mathbb{Y}_{t_c}} : \mathcal{CT}_n(t_c) \times W^S(y) \times W^U(x_1) \times \dots \times W^U(x_n) \rightarrow M^{\times n+1},$$

this moduli space can be rewritten as

$$\mathcal{CT}_{t_c}^{\mathbb{Y}_{t_c}}(y; x_1, \dots, x_n) = \phi_{\mathbb{Y}_{t_c}}^{-1}(\Delta).$$

**Proposition 7.2.4.** (i) Given a choice of perturbation data  $\mathbb{Y}_{t_c}$  making  $\phi_{\mathbb{Y}_{t_c}}$  transverse to the diagonal  $\Delta \subset M^{\times n+1}$ , the moduli spaces  $\mathcal{CT}_{t_c}^{\mathbb{Y}_{t_c}}(y; x_1, \dots, x_n)$  are orientable manifolds of dimension

$$\dim(\mathcal{CT}_{t_c}(y; x_1, \dots, x_n)) = +|y| - \sum_{i=1}^n |x_i| - |t_c|.$$

(ii) Choices of perturbation data  $\mathbb{Y}_{t_c}$  such that  $\phi_{\mathbb{Y}_{t_c}}$  is transverse to the diagonal  $\Delta$  exist.

*Proof.* The proof of this proposition is identical to the proof of Proposition 6.3.3. □

### 7.3 Compactifications

#### 7.3.1 Compactification of the 1-dimensional manifolds $\mathcal{CT}_{t_c}^{\mathbb{Y}_{t_c}}(y; x_1, \dots, x_n)$

We would like to compactify the 1-dimensional moduli spaces  $\mathcal{CT}_{t_c}^{\mathbb{Y}_{t_c}}(y; x_1, \dots, x_n)$  to 1-dimensional manifolds with boundary. Their boundary components are going to be given by those coming from the compactification of the moduli space  $\mathcal{CT}_n(t_c)$ , and the compactifications of the unstable manifolds  $W^U(x_i)$  and of the stable manifold  $W^S(y)$ .

Choose admissible perturbation data  $\mathbb{X}^f$  and  $\mathbb{X}^g$  for the functions  $f$  and  $g$ . Choose moreover smooth perturbation data  $\mathbb{Y}_{t_c}$  for all  $t_c \in SCRT_i$ ,  $1 \leq i \leq n$ . We will again denote  $\mathbb{Y}_n := (\mathbb{Y}_{t_c})_{t_c \in SCRT_n}$ , and call it a choice of perturbation data on  $\mathcal{CT}_n$ . Fixing a 2-colored stable ribbon tree  $t_c \in SCRT_n$  we would like to compactify the 1-dimensional moduli space  $\mathcal{CT}_{t_c}^{\mathbb{Y}_{t_c}}(y; x_1, \dots, x_n)$  using the perturbation data  $\mathbb{X}^f$ ,  $\mathbb{X}^g$  and  $(\mathbb{Y}_i)_{1 \leq i \leq n}$ , such that its boundary would be given by the following spaces:

(i) an external edge breaks at a critical point (Morse):

$$\mathcal{T}(y; z) \times \mathcal{CT}_{t_c}^{\mathbb{Y}_{t_c}}(z; x_1, \dots, x_n) \text{ and } \mathcal{CT}_{t_c}^{\mathbb{Y}_{t_c}}(y; x_1, \dots, z, \dots, x_n) \times \mathcal{T}(z; x_i);$$

(ii) an internal edge of the tree  $t$  collapses (int-collapse):

$$\mathcal{CT}_{t'_c}^{\mathbb{Y}_{t'_c}}(y; x_1, \dots, x_n)$$

where  $t'_c \in SCRT_n$  are all the 2-colored trees obtained by collapsing exactly one internal edge, which does not cross the gauge ;

- (iii) the gauge moves to cross exactly one additional vertex of the underlying stable ribbon tree (gauge-vertex):

$$\mathcal{CT}_{t'_c}^{\mathbb{Y}_{t'_c}}(y; x_1, \dots, x_n)$$

where  $t'_c \in SCRT_n$  are all the 2-colored trees obtained by moving the gauge to cross exactly one additional vertex of  $t$  ;

- (iv) an internal edge located above the gauge or intersecting it breaks or, when the gauge is below the root, the outgoing edge breaks between the gauge and the root (above-break):

$$\mathcal{CT}_{t_c^1}^{\mathbb{Y}_{t_c^1}}(y; x_1, \dots, x_{i_1}, z, x_{i_1+i_2+1}, \dots, x_n) \times \mathcal{T}_{t^2}^{\mathbb{X}_{t^2}^f}(z; x_{i_1+1}, \dots, x_{i_1+i_2}) ;$$

- (v) edges (internal or incoming) that are possibly intersecting the gauge, break below it, such that there is exactly one edge breaking in each non-self crossing path from an incoming edge to the root (below-break):

$$\mathcal{T}_{t^0}^{\mathbb{X}_{t^0}^g}(y; y_1, \dots, y_s) \times \mathcal{CT}_{t_c^1}^{\mathbb{Y}_{t_c^1}}(y_1; x_1, \dots) \times \dots \times \mathcal{CT}_{t_c^s}^{\mathbb{Y}_{t_c^s}}(y_s; \dots, x_n) .$$

### 7.3.2 Smooth and gluing-compatible choices of perturbation data

The (Morse) boundaries are again a simple consequence of the fact that external edges are Morse trajectories away from a length 1 segment. Perturbation data that behave well with respect to the (int-collapse) and (gauge-vertex) boundaries are defined using simple adjustments of the discussion in Section 6.4.2, i.e. by asking that all maps  $\tilde{\mathbb{Y}}_{t_c, f_c}$  extend to smooth maps  $\mathbb{R}^{e(t)+1} \times \mathbb{R} \times M \rightarrow TM$ . Hence, it only remains to specify the required behaviours under the breaking of edges.

We begin with the (above-break) boundary. Writing  $t_c$  for the 2-colored ribbon tree associated to  $t_c$ , it corresponds to the breaking of an internal edge  $f_c$  of  $t_c$  situated above the set of colored vertices. Denote  $t_c^1$  and  $t^2$  the trees obtained by breaking  $t_c$  at the edge  $f_c$ , where  $t^2$  is seen to lie above  $t_c^1$ . We have to specify, for each edge  $e_c \in \overline{E}(t_c)$ , what happens to the perturbation  $\mathbb{Y}_{t_c, e}$  at the limit.

- (i) For  $e_c \in \overline{E}(t^2)$  and  $\neq f_c$ , we require that

$$\lim \mathbb{Y}_{t_c, e_c} = \mathbb{X}_{t^2, e_c}^f .$$

- (ii) For  $e_c \in \overline{E}(t_c^1)$  and  $\neq f_c$ , we require that

$$\lim \mathbb{Y}_{t_c, e_c} = \mathbb{Y}_{t_c^1, e_c} .$$

- (iii) For  $f_c = e_c$ ,  $\mathbb{Y}_{t_c, f_c}$  yields two parts at the limit: the part corresponding to the outgoing edge of  $t^1$  and the part corresponding to the incoming edge of  $t_c^1$ . We then require that they coincide respectively with the perturbation  $\mathbb{X}_{t^2, e_c}^f$  and  $\mathbb{Y}_{t_c^1, e_c}$ .

Leaving the notations aside, an example of each case is illustrated in Figure 7.2.

We conclude with the (below-break) boundary. Denote  $t_c^1, \dots, t_c^s$  and  $t^0$  the trees obtained by the chosen breaking of  $t_c$  below the gauge, where  $t_c^1, \dots, t_c^s$  are seen to lie above  $t^0$ .

- (i) For  $e_c \in \overline{E}(t_c^i)$  and not among the breaking edges, we require that

$$\lim \mathbb{Y}_{t_c, e_c} = \mathbb{Y}_{t_c^i, e_c} .$$

- (ii) For  $e_c \in \overline{E}(t^0)$  and not among the breaking edges, we require that

$$\lim \mathbb{Y}_{t_c, e_c} = \mathbb{X}_{t^0, e_c}^g .$$

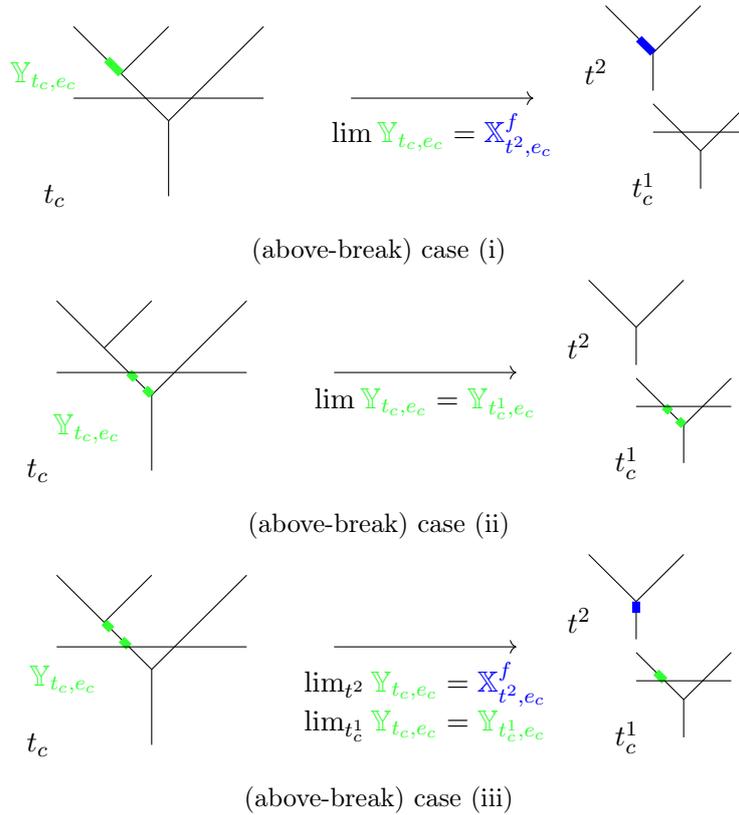


Figure 7.2

(iii) For  $f_c$  among the breaking edges,  $\mathbb{Y}_{t_c, f_c}$  yields two parts at the limit: the part corresponding to the outgoing edge of a  $t_c^j$  and the part corresponding to the incoming edge of  $t^0$ . We then require that they coincide respectively with the perturbation  $\mathbb{Y}_{t_c^j}$  and  $\mathbb{X}_{t^0}^g$ .

This is again illustrated on Figure 7.3.

- Definition 7.3.1.** (i) A choice of perturbation data  $\mathbb{Y}$  on the moduli spaces  $\mathcal{CT}_n$  is said to be *smooth* if it is compatible with the (int-collapse) and (gauge-vertex) boundaries.
- (ii) A smooth choice of perturbation data is said to be *gluing-compatible w.r.t.  $\mathbb{X}^f$  and  $\mathbb{X}^g$*  if it satisfies the (above-break) and (below-break) conditions described in this section.

As we explained in Section 6.4.3, a gluing-compatible choice of perturbation data has in particular the property that when lengths of edges go towards  $+\infty$ , the perturbation datum on each edge only depends on the lengths of the tree/the 2-colored tree that the edge belongs to. This ensures that the (above-break) and (below-break) boundary components of Section 7.3.1 actually appear as standard products and not as fiber products over the codimension 1 boundary of the compactified moduli spaces  $\overline{\mathcal{CT}}_n(t_c)$ .

### 7.3.3 Admissible choices of perturbation data

**Definition 7.3.2.** Smooth and consistent choices of perturbation data  $(\mathbb{Y}_n)_{n \geq 1}$  such that all maps  $\phi_{\mathbb{Y}_{t_c}}$  are transverse to the diagonal  $\Delta$  are called *admissible w.r.t.  $\mathbb{X}^f$  and  $\mathbb{X}^g$*  or simply *admissible*.

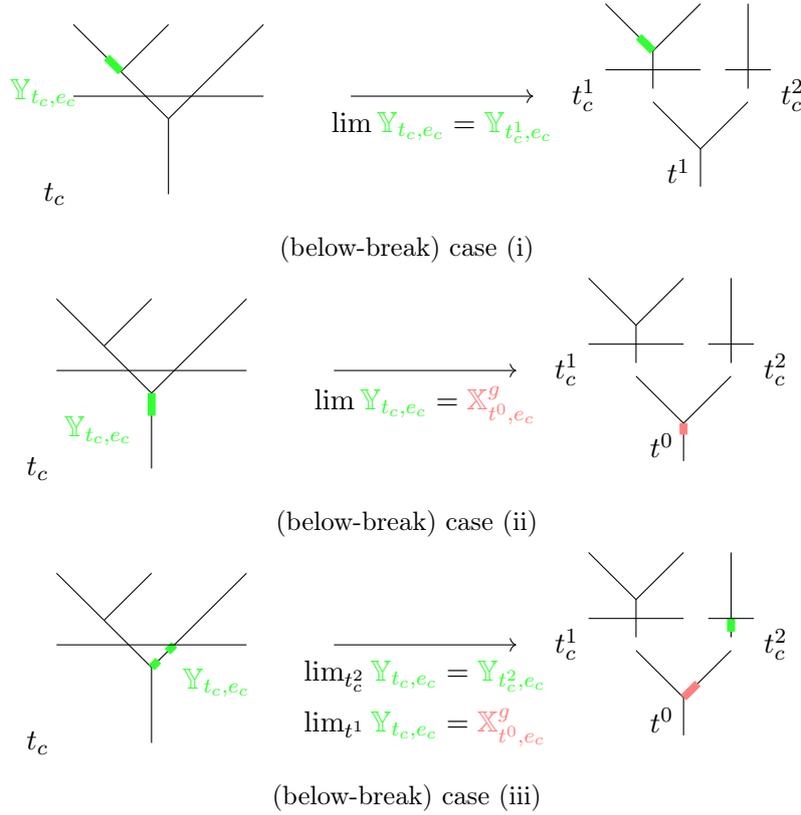


Figure 7.3

**Theorem 7.3.3.** *Given admissible choices of perturbation data  $\mathbb{X}^f$  and  $\mathbb{X}^g$  on the moduli spaces  $\mathcal{T}_n$ , choices of perturbation data on the moduli spaces  $\mathcal{CT}_n$  that are admissible w.r.t.  $\mathbb{X}^f$  and  $\mathbb{X}^g$  exist.*

*Proof.* The proof is identical to the proof of Theorem 6.4.5. □

**Theorem 7.3.4.** *Let  $(\mathbb{Y}_n)_{n \geq 1}$  be an admissible choice of perturbation data on the moduli spaces  $\mathcal{CT}_n$ . The 0-dimensional moduli spaces  $\mathcal{CT}_{t_c}^{\mathbb{Y}}(y; x_1, \dots, x_n)$  are compact. The 1-dimensional moduli spaces  $\mathcal{CT}_{t_c}^{\mathbb{Y}}(y; x_1, \dots, x_n)$  can be compactified to 1-dimensional manifolds with boundary, whose boundary is described in Section 7.3.1.*

*Proof.* This theorem is again a consequence of the techniques of [35, Chapter 6]. □

We point out that Theorem 7.3.4 implies in particular the existence of gluing maps

$$\begin{aligned} \#_{T_c^{1,Morse}, T_c^{2,Morse}}^{\text{above-break}} &: [R, +\infty] \longrightarrow \overline{\mathcal{CT}}_{t_c}(y; x_1, \dots, x_n) \\ \#_{T_c^{0,Morse}, T_c^{1,Morse}, \dots, T_c^{s,Morse}}^{\text{below-break}} &: [R, +\infty] \longrightarrow \overline{\mathcal{CT}}_{t_c}(y; x_1, \dots, x_n) \end{aligned}$$

where our choice of notation is as in Section 6.4.4. We will construct explicit gluing maps in Section 9.5.4.

**7.4 Continuation morphisms** Let  $\mathbb{X}^f$  and  $\mathbb{X}^g$  be admissible choices of perturbation data for the Morse functions  $f$  and  $g$ . Denote  $(C^*(f), m_t^{\mathbb{X}^f})$  and  $(C^*(g), m_t^{\mathbb{X}^g})$  the  $\Omega BAs$ -algebras constructed in Theorem 6.5.1.

**Theorem 7.4.1.** *Let  $(\mathbb{Y}_n)_{n \geq 1}$  be a choice of perturbation on the moduli spaces  $\mathcal{CT}_n$  that is admissible w.r.t.  $\mathbb{X}^f$  and  $\mathbb{X}^g$ . Defining for every  $n$  and  $t_c \in \text{SCRT}_n$  the operations  $\mu_{t_c}$  as*

$$\begin{aligned} \mu_{t_c}^{\mathbb{Y}} : C^*(f) \otimes \cdots \otimes C^*(f) &\longrightarrow C^*(g) \\ x_1 \otimes \cdots \otimes x_n &\longmapsto \sum_{|y| = \sum_{i=1}^n |x_i| + |t_c|} \#\mathcal{CT}_{t_c}^{\mathbb{Y}}(y; x_1, \dots, x_n) \cdot y . \end{aligned}$$

they fit into an  $\Omega\text{BAS}$ -morphism  $\mu^{\mathbb{Y}} : (C^*(f), m_t^{\mathbb{X}^f}) \rightarrow (C^*(g), m_t^{\mathbb{X}^g})$ .

*Proof.* The proof is similar to the proof of Theorem 6.5.1, and postponed to Section 9.5. In fact, we will prove that the collection of operations  $\{\mu_{t_c}\}$  does not exactly define an  $\Omega\text{BAS}$ -morphism but rather a *twisted  $\Omega\text{BAS}$ -morphism*. In the case when  $M$  is odd-dimensional, this twisted  $\Omega\text{BAS}$ -morphism is exactly an  $\Omega\text{BAS}$ -morphism between two  $\Omega\text{BAS}$ -algebras.  $\square$

REMARK 7.4.2. If we want to go back to the algebraic framework of  $A_\infty$ -algebras, an  $A_\infty$ -morphism between the induced  $A_\infty$ -algebra structures on the Morse cochains is simply obtained under the morphism of operadic bimodules  $M_\infty \rightarrow M_{\Omega\text{BAS}}$  of Proposition 3.2.25.

REMARK 7.4.3. In symplectic topology,  $A_\infty$ -functors between Fukaya categories are constructed by counting pseudo-holomorphic quilted disks with marked boundary points and Lagrangian boundary and seam conditions. This is the subject of the work of [31] and [30].

**Definition 7.4.4.**  $\Omega\text{BAS}$ -morphisms  $\mu^{\mathbb{Y}}$  associated to admissible choices of perturbation data  $\mathbb{Y}$  will be called *continuation morphisms*.

**Theorem 7.4.5.** *Continuations morphisms are quasi-isomorphisms.*

*Proof.* We want to prove that the arity 1 component  $\mu_{\dagger}^{\mathbb{Y}} : C^*(f) \rightarrow C^*(g)$  is a map which induces an isomorphism in cohomology. In this regard, consider three perturbation data on  $\mathcal{CT}_1 := \{\dagger\}$ ,  $\mathbb{Y}_{\dagger}^{fg}$ ,  $\mathbb{Y}_{\dagger}^{gf}$  and  $\mathbb{Y}_{\dagger}^{ff}$ , defining chain maps

$$\mu_{\dagger}^{\mathbb{Y}^{ij}} : C^*(i) \longrightarrow C^*(j) .$$

We will introduce in Section 9.6 moduli spaces of perturbed trees  $\mathcal{H}(y; x)$  and prove that their count defines a homotopy  $h : C^*(f) \rightarrow C^*(f)$  between the chain maps  $\mu_{\dagger}^{\mathbb{Y}^{gf}} \circ \mu_{\dagger}^{\mathbb{Y}^{fg}}$  and  $\mu_{\dagger}^{\mathbb{Y}^{ff}}$  in Lemma 9.6.2. Specializing to the case where  $\mathbb{Y}_{\dagger}^{ff}$  is null, the equality  $\mu_{\dagger}^{\mathbb{Y}^{ff}} = \text{id}$  then concludes the proof.  $\square$

REMARK 7.4.6. In Remark 6.5.2, we explained that given any Morse function  $f$  together with an admissible choice of perturbation data  $\mathbb{X}^f$ , the Morse cochains  $C^*(f)$  and the singular cochains  $C_{\text{sing}}^*(M)$  are quasi-isomorphic as  $\Omega\text{BAS}$ -algebras. In particular, given another Morse function  $g$  together with an admissible choice of perturbation data  $\mathbb{X}^g$ , the Morse cochains  $C^*(f)$  and  $C^*(g)$  are quasi-isomorphic as  $\Omega\text{BAS}$ -algebras. Continuation morphisms realize such quasi-isomorphisms explicitly.

**7.5 Towards higher algebra** Two questions naturally arise from our construction of continuation morphisms. They will respectively be the starting point to the part II and part III to this series of articles.

**Problem 1.** Given two Morse functions  $f, g$ , choices of perturbation data  $\mathbb{X}^f$  and  $\mathbb{X}^g$ , and choices of perturbation data  $\mathbb{Y}$  and  $\mathbb{Y}'$ , is  $\mu^{\mathbb{Y}}$  always  $A_\infty$ -homotopic (resp.  $\Omega BAs$ -homotopic) to  $\mu^{\mathbb{Y}'}$  ? I.e., when can the following diagram be filled in the  $A_\infty$  (resp.  $\Omega BAs$ ) world

$$\begin{array}{ccc}
 & \mu^{\mathbb{Y}} & \\
 & \curvearrowright & \\
 C^*(f) & & C^*(g) \quad ? \\
 & \curvearrowleft & \\
 & \mu^{\mathbb{Y}'} & \\
 & \Downarrow & \\
 & & 
 \end{array}$$

In which sense, with which notion of homotopy can it be filled ? And in general, which notion of higher operadic algebra naturally encodes this type of problem ?

This problem is solved in [32] by introducing the notions of  $n$ - $A_\infty$ -morphisms and  $n$ - $\Omega BAs$ -morphisms. In this article, we will show that the simplicial set consisting of higher morphisms defined by a count of perturbed Morse gradient trees is a Kan complex which is contractible, giving a higher categorical meaning to the fact that continuation morphisms in Morse theory are well-defined up to homotopy at chain level.

**Problem 2.** Given three Morse functions  $f_0, f_1, f_2$ , choices of perturbation data  $\mathbb{X}^i$ , and choices of perturbation data  $\mathbb{Y}^{ij}$  defining continuation morphisms

$$\begin{aligned}
 \mu^{\mathbb{Y}^{01}} &: (C^*(f_0), m_t^{\mathbb{X}^0}) \longrightarrow (C^*(f_1), m_t^{\mathbb{X}^1}) , \\
 \mu^{\mathbb{Y}^{12}} &: (C^*(f_1), m_t^{\mathbb{X}^1}) \longrightarrow (C^*(f_2), m_t^{\mathbb{X}^2}) , \\
 \mu^{\mathbb{Y}^{02}} &: (C^*(f_0), m_t^{\mathbb{X}^0}) \longrightarrow (C^*(f_2), m_t^{\mathbb{X}^2}) ,
 \end{aligned}$$

can we construct an  $A_\infty$ -homotopy (or an  $\Omega BAs$ -homotopy) such that  $\mu^{\mathbb{Y}^{12}} \circ \mu^{\mathbb{Y}^{01}} \simeq \mu^{\mathbb{Y}^{02}}$  through this homotopy ? That is can the following diagram be filled in the  $A_\infty$  (resp.  $\Omega BAs$ ) world

$$\begin{array}{ccc}
 C^*(f_0) & \xrightarrow{\mu^{\mathbb{Y}^{01}}} & C^*(f_1) \\
 & \searrow \mu^{\mathbb{Y}^{02}} & \swarrow \mu^{\mathbb{Y}^{12}} \\
 & & C^*(f_2)
 \end{array}$$

Which higher operadic algebra naturally arises from this basic question ?

We point out that the proof of Theorem 7.4.5 solves the arity 1 step of this problem. It will be addressed in an upcoming paper, in which it will appear that the combinatorics of  $n$ - $A_\infty$ -morphisms and of multicolored trees provide a natural framework to solve this question. This question is moreover closely related to the work of Mau, Wehrheim and Woodward on pseudo-holomorphic quilted disks ([30]) and of Bottman on witch curves and the 2-associahedra ([7, 8]).

### 8. Transversality

The goal of this section is to prove Theorem 6.4.5. In this regard, we recall at first the parametric transversality lemma and then build an admissible choice of perturbation data  $(\mathbb{X}_n)_{n \geq 2}$  on the moduli spaces  $\mathcal{T}_n$ , proceeding by induction on the number of internal edges  $e(t)$  of a stable ribbon tree  $t$ . It moreover appears in our construction that all arguments adapt nicely to the framework of 2-colored trees and admissible choices of perturbation data  $(\mathbb{Y}_n)_{n \geq 1}$  on the moduli spaces  $\mathcal{CT}_n$ .

**8.1 Parametric transversality lemma** We begin by recalling Smale’s generalization of the classical Sard theorem. See [38] or [34] for a complete proof:

**Theorem 8.1.1** (Sard-Smale theorem). *Let  $X$  and  $Y$  be separable Banach manifolds. Suppose that  $f : X \rightarrow Y$  is a Fredholm map of class  $C^l$  with  $l \geq \max(1, \text{ind}(f) + 1)$ . Then the set  $Y_{\text{reg}}(f)$  of regular values of  $f$  is residual in  $Y$  in the sense of Baire.*

Theorem 8.1.1 implies in particular the following corollary in transversality theory, that will constitute the cornerstone of our proof of Theorem 6.4.5:

**Corollary 8.1.2** (Parametric transversality lemma). *Let  $\mathfrak{X}$  be a Banach space,  $M$  and  $N$  two finite-dimensional manifolds and  $S \subset N$  a submanifold of  $N$ . Suppose that  $f : \mathfrak{X} \times M \rightarrow N$  is a map of class  $C^l$  with  $l \geq \max(1, \dim(M) + \dim(S) - \dim(N) + 1)$  and that it is transverse to  $S$ . Then the set*

$$\mathfrak{X}_{\pitchfork S} := \{\mathbb{X} \in \mathfrak{X} \text{ such that } f_{\mathbb{X}} \pitchfork S\}$$

*is residual in  $\mathfrak{X}$  in the sense of Baire.*

*Proof.* The map  $f$  being transverse to  $S$ , the inverse image  $f^{-1}(S)$  is a Banach submanifold of  $\mathfrak{X} \times M$ . Consider the standard projection  $p_{\mathfrak{X}} : \mathfrak{X} \times M \rightarrow \mathfrak{X}$  and denote  $\pi := p_{\mathfrak{X}}|_{f^{-1}(S)}$ . Following [4, Lemma 19.2], this map is Fredholm and has index  $\dim(M) + \dim(S) - \dim(N)$ . Moreover, drawing from an argument in [34, Section 3.2], there is an equality  $\mathfrak{X}_{\text{reg}}(\pi) = \mathfrak{X}_{\pitchfork S}$ . One can then conclude by applying Theorem 8.1.1 to the map  $\pi$ .  $\square$

**8.2 Proof of Theorem 6.4.5**

**8.2.1 The case  $e(t) = 0$**

If  $e(t) = 0$ , the tree  $t$  is a corolla. Fix an integer  $l$  such that

$$l \geq \max \left( 1, |y| - \sum_{i=1}^n |x_i| + 1 \right) .$$

We define  $C^l$ -choices of perturbation data in a similar fashion to smooth choices of perturbation data. A  $C^l$ -choice of perturbation data  $\mathbb{X}_t$  on  $\mathcal{T}_n(t)$  then simply corresponds to a  $C^l$ -choice of perturbation datum on each external edge of  $t$ . Define the parametrization space

$$\mathfrak{X}_t^l := \{C^l\text{-perturbation data } \mathbb{X}_t \text{ on the moduli space } \mathcal{T}_n(t)\} .$$

This parametrization space is a Banach space. The linear combination of choices of perturbation data is simply defined as the linear combination of each perturbation datum  $\mathbb{X}_{t,e}$  with  $e$  an external edge of  $t$ . The vector space  $\mathfrak{X}_t^l$  is moreover Banach as each perturbation datum  $\mathbb{X}_{t,e}$  vanishes away from a length 1 segment in  $D_e$ .

Given critical points  $y$  and  $x_1, \dots, x_n$ , introduce the  $C^l$ -map

$$\phi_t : \mathfrak{X}_t^l \times \mathcal{T}_n(t) \times W^S(y) \times W^U(x_1) \times \dots \times W^U(x_n) \longrightarrow M^{\times n+1} ,$$

such that for every  $\mathbb{X}_t \in \mathfrak{X}_t^l$ ,  $\phi_t(\mathbb{X}_t, \cdot) = \phi_{\mathbb{X}_t}$ . Note that we should in fact write  $\phi_t^{y, x_1, \dots, x_n}$  as the domain of  $\phi_t$  depends on  $y, x_1, \dots, x_n$ . The map  $\phi_t$  is then a submersion. This is proven in [3, Lemma 7.3] and Abouzaid explains it informally in the following terms: "[this lemma] is the

infinitesimal version of the fact that perturbing the gradient flow equation on a bounded subset of an edge integrates to an essentially arbitrary diffeomorphism".

In particular the map  $\phi_t$  is transverse to the diagonal  $\Delta \subset M^{\times n+1}$ . Applying Corollary 8.1.2, there exists a residual set  $\mathfrak{Y}_t^{l;y,x_1,\dots,x_n} \subset \mathfrak{X}_t^l$  such that for every choice of perturbation data  $\mathbb{X}_t \in \mathfrak{Y}_t^{l;y,x_1,\dots,x_n}$  the map  $\phi_{\mathbb{X}_t}$  is transverse to the diagonal  $\Delta \subset M^{\times n+1}$ . Considering the intersection

$$\mathfrak{Y}_t^l := \bigcap_{y,x_1,\dots,x_n} \mathfrak{Y}_t^{y,x_1,\dots,x_n} \subset \mathfrak{X}_t$$

which is again residual, any  $\mathbb{X}_t \in \mathfrak{Y}_t^l$  yields a  $C^l$ -choice of perturbation data on  $\mathcal{T}_n(t)$  such that all the maps  $\phi_{\mathbb{X}_t}$  are transverse to the diagonal  $\Delta \subset M^{\times n+1}$ . It remains to prove this statement in the smooth case.

### 8.2.2 Achieving smoothness à la Taubes

Using an argument drawn from [34, Section 3.2] and attributed to Taubes, we now prove that the set

$$\mathfrak{Y}_t := \left\{ \begin{array}{l} \text{smooth choices of perturbation data } \mathbb{X}_t \text{ on } \mathcal{T}_n(t) \text{ such that} \\ \text{all the maps } \phi_{\mathbb{X}_t} \text{ are transverse to the diagonal } \Delta \subset M^{\times n+1} \end{array} \right\}$$

is residual in the Fréchet space

$$\mathfrak{X}_t := \{ \text{smooth choices of perturbation data } \mathbb{X}_t \text{ on } \mathcal{T}_n(t) \} .$$

Choose an exhaustion by compact sets  $L_0 \subset L_1 \subset L_2 \subset \dots$  of the space  $\mathcal{T}_n(t) \times W^S(y) \times W^U(x_1) \times \dots \times W^U(x_n)$ . Define

$$\mathfrak{Y}_{t,L_m} := \left\{ \begin{array}{l} \text{smooth choices of perturbation data } \mathbb{X}_t \text{ on } \mathcal{T}_n(t) \text{ such that} \\ \text{all maps } \phi_{\mathbb{X}_t} \text{ are transverse on } L_m \text{ to the diagonal of } M^{\times n+1} \end{array} \right\}$$

and note that

$$\mathfrak{Y}_t = \bigcap_{m=0}^{+\infty} \mathfrak{Y}_{t,L_m} .$$

We will prove that each  $\mathfrak{Y}_{t,L_m} \subset \mathfrak{Y}_t$  is open and dense in  $\mathfrak{X}_t$  to conclude that  $\mathfrak{Y}_t$  is indeed residual.

Fix  $m \geq 0$ . To prove that the set  $\mathfrak{Y}_{t,L_m}$  is open in  $\mathfrak{X}_t$  it suffices to prove that for every  $l$ , the set  $\mathfrak{Y}_{t,L_m}^l$  is open in  $\mathfrak{X}_t^l$ , where  $\mathfrak{Y}_{t,L_m}^l$  is defined by replacing "smooth" by " $C^l$ " in the definition of  $\mathfrak{Y}_{t,L_m}$ . This last result is a simple consequence of the fact that "being transverse on a compact subset" is an open property: if the map  $\phi_{\mathbb{X}_t^0}$  is transverse on  $L_m$  to the diagonal  $\Delta \subset M^{\times n+1}$  then for  $\mathbb{X}_t \in \mathfrak{X}_t^l$  sufficiently close to  $\mathbb{X}_t^0$  the map  $\phi_{\mathbb{X}_t}$  is again transverse on  $L_m$  to the diagonal on  $L_m$ .

Let now  $\mathbb{X}_t \in \mathfrak{X}_t$ . As  $\mathbb{X}_t \in \mathfrak{X}_t^l$  and the set  $\mathfrak{Y}_t^l$  is dense in  $\mathfrak{X}_t^l$ , there exists a sequence  $\mathbb{X}_t^l \in \mathfrak{Y}_t^l$  such that for all  $l$

$$\|\mathbb{X}_t - \mathbb{X}_t^l\|_{C^l} \leq 2^{-l} .$$

Note that  $\mathbb{X}_t^l \in \mathfrak{Y}_{t,L_m}^l$ . Now since the set  $\mathfrak{Y}_{t,L_m}^l$  is open in  $\mathfrak{X}_t^l$  for the  $C^l$ -topology, there exists  $\varepsilon_l > 0$  such that for all  $\mathbb{X}_t^l \in \mathfrak{X}_t^l$  if

$$\|\mathbb{X}_t^l - \mathbb{X}_t^l\|_{C^l} \leq \min(2^{-l}, \varepsilon_l) ,$$

then  $\mathbb{X}_t^l \in \mathfrak{Y}_{t,L_m}^l$ . Choosing  $\mathbb{X}_t^l$  to be smooth, this yields a sequence of smooth choices of perturbation data lying in  $\mathfrak{Y}_{t,L_m}$  and converging to  $\mathbb{X}_t$ , which concludes the proof.

### 8.2.3 Induction step and conclusion

Let  $k \geq 0$  and suppose that we have constructed an admissible choice of perturbation data  $(\mathbb{X}_t^0)_{e(t) \leq k}$ . This notation should not be confused with the notation  $(\mathbb{X}_i)_{i \leq k}$ : the former corresponds to a choice of perturbation data on the strata  $\mathcal{T}(t)$  of dimension  $\leq k$  while the latter corresponds to a choice of perturbation data on the moduli spaces  $\mathcal{T}_i$  with  $i \leq k$ . Let  $t$  be a stable ribbon tree with  $e(t) = k + 1$  and choose

$$l \geq \max \left( 1, e(t) + |y| - \sum_{i=1}^n |x_i| + 1 \right) .$$

We want to construct a choice of perturbation data  $\mathbb{X}_t$  on  $\mathcal{T}_n(t)$  which is smooth, gluing-compatible and such that each map  $\phi_{\mathbb{X}_t}$  is transverse to the diagonal  $\Delta \subset M^{\times n+1}$ .

Under a choice of identification  $\overline{\mathcal{T}}_n(t) \simeq [0, +\infty]^{e(t)}$ , define  $\underline{\mathcal{T}}_n(t) \subset \overline{\mathcal{T}}_n(t)$  as the inverse image of  $[0, +\infty]^{e(t)}$ . Introduce the parametrization space

$$\mathfrak{X}_t^l := \left\{ \begin{array}{l} C^l\text{-perturbation data } \mathbb{X}_t \text{ on } \underline{\mathcal{T}}_n(t) \text{ such that} \\ \mathbb{X}_t|_{\mathcal{T}(t')} = \mathbb{X}_{t'}^0 \text{ for all } t' \in \text{coll}(t) \text{ and such that} \\ \lim_{l_e \rightarrow +\infty} \mathbb{X}_t = \mathbb{X}_{t_1}^0 \#_e \mathbb{X}_{t_2}^0 \text{ for all } e \in E(t) \end{array} \right\} ,$$

where  $t_1 \#_e t_2 = t$ , and  $\lim_{l_e \rightarrow +\infty} \mathbb{X}_t = \mathbb{X}_{t_1}^0 \#_e \mathbb{X}_{t_2}^0$  denotes the gluing-compatibility condition described in Section 6.4.3. Following [35] this parametrization space is an affine space which is Banach. One can indeed show that the  $l_e \rightarrow +\infty$  conditions imply that each  $\mathbb{X}_t \in \mathfrak{X}_t^l$  is bounded in the  $C^l$ -norm, and that the  $C^l$ -norm is thus well defined on  $\mathfrak{X}_t^l$  although  $\underline{\mathcal{T}}_n(t)$  is not compact.

Consider the  $C^l$ -map

$$\phi_t : \mathfrak{X}_t^l \times \mathcal{T}_n(t) \times W^S(y) \times W^U(x_1) \times \cdots \times W^U(x_n) \longrightarrow M^{\times n+1} .$$

Using the same argument as in Section 8.2.1, the map  $\phi_t$  is again transverse to the diagonal  $\Delta \subset M^{\times n+1}$ . Applying Corollary 8.1.2 and proceeding as in the case  $e(t) = 0$ , there exists a residual set  $\mathfrak{Y}_t^l \subset \mathfrak{X}_t^l$  such that for every choice of perturbation data  $\mathbb{X}_t \in \mathfrak{Y}_t^l$  the map  $\phi_{\mathbb{X}_t}$  is transverse to the diagonal  $\Delta \subset M^{\times n+1}$ . Using the previous argument à la Taubes, we can moreover prove the same statement in the smooth context. By definition of the parametrization spaces  $\mathfrak{X}_t$  this construction yields indeed an admissible choice of perturbation data  $(\mathbb{X}_t)_{e(t) \leq k+1}$ , which concludes the proof of Theorem 6.4.5 by induction.

## 9. Signs, orientations and gluing

We now complete and conclude the proofs of Theorems 6.5.1, 7.4.1 and 7.4.5, by making explicit all orientation conventions on the moduli spaces of Morse gradient trees and computing the signs involved therein. We use to this extent the ad hoc formalism of signed short exact sequences of vector bundles. A particular attention will be paid to the behaviour of orientations under gluing in our proofs.

## 9.1 Orientations and short exact sequences

### 9.1.1 Signed short exact sequences of vector spaces

Consider a short exact sequence of vector spaces

$$0 \longrightarrow V_2 \longrightarrow W \longrightarrow V_1 \longrightarrow 0 .$$

It induces a direct sum decomposition  $W = V_1 \oplus V_2$ . Suppose that the vector spaces  $W$ ,  $V_1$  and  $V_2$  are oriented. We denote  $(-1)^\varepsilon$  the sign obtained by comparing the orientation on  $W$  to the one induced by the direct sum  $V_1 \oplus V_2$ . We will then say that the short exact sequence has sign  $(-1)^\varepsilon$ . In particular, when  $(-1)^\varepsilon = 1$ , we will say that the short exact sequence is *positive*.

Now, consider two short exact sequences

$$0 \longrightarrow V_2 \longrightarrow W \longrightarrow V_1 \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow V_2' \longrightarrow W' \longrightarrow V_1' \longrightarrow 0 ,$$

of respective signs  $(-1)^\varepsilon$  and  $(-1)^{\varepsilon'}$ . Then the short exact sequence obtained by summing them

$$0 \longrightarrow V_2 \oplus V_2' \longrightarrow W \oplus W' \longrightarrow V_1 \oplus V_1' \longrightarrow 0 ,$$

has sign  $(-1)^{\varepsilon+\varepsilon'+\dim(V_1')\dim(V_2)}$ . Indeed, the direct sum decomposition writes as

$$W \oplus W' = (-1)^\varepsilon(V_1 \oplus V_2) \oplus (-1)^{\varepsilon'}(V_1' \oplus V_2') \simeq (-1)^{\varepsilon+\varepsilon'+\dim(V_1')\dim(V_2)}V_1 \oplus V_1' \oplus V_2 \oplus V_2' .$$

### 9.1.2 Orientation and transversality

Given two manifolds  $M, N$ , a codimension  $k$  submanifold  $S \subset N$  and a smooth map

$$\phi : M \longrightarrow N$$

which is transverse to  $S$ , the inverse image  $\phi^{-1}(S)$  is a codimension  $k$  submanifold of  $M$ . Moreover, choosing a complementary  $\nu_S$  to  $TS$ , the transversality assumption yields the following short exact sequence of vector bundles

$$0 \longrightarrow T\phi^{-1}(S) \longrightarrow TM|_{\phi^{-1}(S)} \xrightarrow{d\phi} \nu_S \longrightarrow 0 .$$

Suppose now that  $M$  is oriented and that  $S$  is cooriented (Definition 4.1.2). The submanifold  $\phi^{-1}(S)$  is then oriented by requiring that the previous short exact sequence be positive.

**Definition 9.1.1.** This choice of orientation on  $\phi^{-1}(S)$  will be called the *natural orientation* on  $\phi^{-1}(S)$ .

In the particular case of two submanifolds  $S$  and  $R$  of  $M$  which intersect transversely, we will use the inclusion map  $S \hookrightarrow M$ , which is transverse to  $R \subset M$ , to define the intersection  $S \cap R$ . The orientation will then be defined using the positive short exact sequence

$$0 \longrightarrow T(S \cap R) \longrightarrow TS|_{S \cap R} \longrightarrow \nu_R \longrightarrow 0 ,$$

or equivalently with the direct sum decomposition

$$TS = \nu_R \oplus T(S \cap R) .$$

The intersection  $R \cap S$  (in contrast to  $S \cap R$ ) is oriented by interchanging  $S$  and  $R$  in the above discussion. The two orientations on the intersection differ then by a  $(-1)^{\text{codim}(S)\text{codim}(R)}$  sign.

## 9.2 Standard moduli spaces in Morse theory and their orientations

### 9.2.1 Orienting the unstable and stable manifolds

Recall that for a critical point  $x$  of a Morse function  $f$ , its unstable and stable manifolds are respectively defined as

$$W^U(x) := \{z \in M, \lim_{s \rightarrow -\infty} \phi^s(z) = x\}$$

$$W^S(x) := \{z \in M, \lim_{s \rightarrow +\infty} \phi^s(z) = x\},$$

where we denote  $\phi^s$  the flow of  $-\nabla f$ , and its degree is defined as  $|x| := \dim(W^S(x))$ .

The unstable and stable manifolds are respectively diffeomorphic to a  $(d - |x|)$ -dimensional ball and a  $|x|$ -dimensional ball. They are hence orientable. They intersect moreover transversely in a unique point, which is  $x$ . Assume now that the manifold  $M$  is orientable and oriented. We choose for the rest of this section an arbitrary orientation on  $W^U(x)$ , and endow  $W^S(x)$  with the unique orientation such that the concatenation of orientations  $or_{W^U(x)} \wedge or_{W^S(x)}$  at  $x$  coincides with the orientation  $or_M$ .

### 9.2.2 Orienting the moduli spaces $\mathcal{T}(y; x)$

For two critical points  $x \neq y$ , the moduli spaces of negative gradient trajectories  $\mathcal{T}(y; x)$  can be defined in two ways. The first point of view hinges on the fact that  $\mathbb{R}$  acts on  $W^S(y) \cap W^U(x)$ , by defining  $s \cdot p = \phi^s(p)$  for  $s \in \mathbb{R}$  and  $p \in W^S(y) \cap W^U(x)$ . The moduli space  $\mathcal{T}(y; x)$  is then defined as the quotient associated to this action, i.e. by defining  $\mathcal{T}(y; x) := W^S(y) \cap W^U(x) / \mathbb{R}$ . The second point of view is to consider the transverse intersection with the level set of a regular value  $a$  such that  $f(x) > f(a) > f(y)$ ,

$$\mathcal{T}(y; x) := W^S(y) \cap W^U(x) \cap f^{-1}(a).$$

Using this description, and coorienting the level set  $f^{-1}(a)$  with  $-\nabla f$ , the spaces  $\mathcal{T}(y; x)$  can easily be oriented with the formalism of Section 9.1.2 on transverse intersections:

$$TW^S(y) \simeq TW^S(x) \oplus T(W^S(y) \cap W^U(x)) \simeq TW^S(x) \oplus -\nabla f \oplus T\mathcal{T}(y; x).$$

Note that the space  $W^S(y) \cap W^U(x)$  consists in a union of negative gradient trajectories  $\gamma : \mathbb{R} \rightarrow M$ . We will therefore use the notation  $\dot{\gamma}$  for  $-\nabla f$ , which will become handy in Section 9.2.3.

We point out that the moduli spaces  $\mathcal{T}(y; x)$  are constructed in a different way than the moduli spaces  $\mathcal{T}_t(y; x_1, \dots, x_n)$ : they cannot naturally be viewed as an arity 1 case of the moduli spaces of gradient trees. This observation will be of importance in our upcoming discussion on signs for the  $\Omega BAs$ -algebra structure on the Morse cochains.

Finally, the moduli spaces  $\mathcal{T}(y; x)$  are manifolds of dimension

$$\dim(\mathcal{T}(y; x)) = |y| - |x| - 1,$$

which can be compactified to manifolds with corners  $\overline{\mathcal{T}}(y; x)$ , by allowing convergence towards broken negative gradient trajectories. See for instance [42]. In the case when they are 1-dimensional, their boundary is given by the signed union

$$\partial \overline{\mathcal{T}}(y; x) = \bigcup_{z \in \text{Crit}(f)} -\mathcal{T}(y; z) \times \mathcal{T}(z; x).$$

We moreover recall from Section 6.1 that we work under the convention  $\mathcal{T}(x; x) = \emptyset$ .

### 9.2.3 Compactifications of the unstable and stable manifolds

Using the moduli spaces  $\mathcal{T}(y; x)$ , we can now compactify the manifolds  $W^S(y)$  and  $W^U(x)$  to compact manifolds with corners  $\overline{W^S}(y)$  and  $\overline{W^U}(x)$ , as in [17]. With the choices of orientations made in Section 9.2.2, the top dimensional strata in their boundary are given by

$$\begin{aligned}\partial\overline{W^S}(y) &= \bigcup_{z \in \text{Crit}(f)} (-1)^{|z|+1} W^S(z) \times \mathcal{T}(y; z), \\ \partial\overline{W^U}(x) &= \bigcup_{z \in \text{Crit}(f)} (-1)^{(d-|z|)(|x|+1)} W^U(z) \times \mathcal{T}(z; x),\end{aligned}$$

where  $d$  is the dimension of the ambient manifold  $M$ .

The pictures in the neighborhood of the critical point  $z$  are represented in Figure 9.1. For instance, in the case of  $\partial\overline{W^S}(y)$ , an element of  $W^S(y)$  is seen as lying on a negative semi-infinite trajectory converging to  $y$ , and an outward-pointing vector to the boundary is given by  $-\dot{\gamma}$ . We hence have that

$$-\dot{\gamma} \oplus TW^S(z) \oplus T\mathcal{T}(y; z) = (-1)^{|z|} TW^S(z) \oplus -\dot{\gamma} \oplus T\mathcal{T}(y; z) = (-1)^{|z|+1} TW^S(y).$$

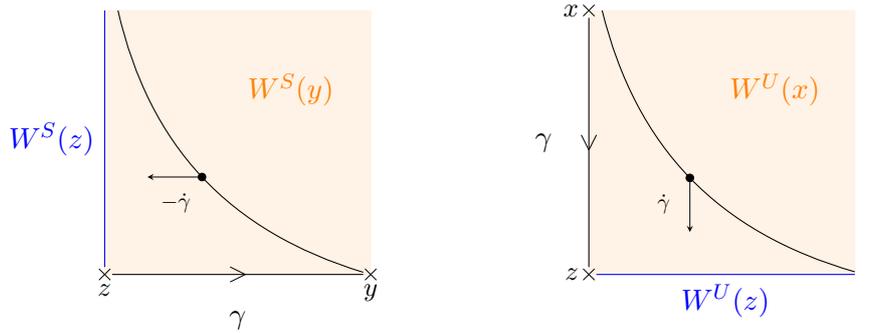


Figure 9.1

### 9.2.4 Euclidean neighborhood of a critical point

Following [42], we will assume in the rest of this part that the pair consisting of the Morse function and the metric on the manifold  $M$  is Euclidean. Denote  $B_\delta^k := \{x \in \mathbb{R}^k, |x| < \delta\}$ . Such a pair is said to be *Euclidean* if it is Morse-Smale and is such that for each critical point  $z \in \text{Crit}(f)$  there exists a local chart  $\phi : B_\delta^{d-|z|} \times B_\delta^{|z|} \xrightarrow{\sim} U_z \subset M$ , such that  $\phi(0) = z$  and such that the function  $f$  and the metric  $g$  read as

$$\begin{aligned}f(x_1, \dots, x_{n-|z|}, y_1, \dots, y_{|z|}) &= f(p) - \frac{1}{2}(x_1^2 + \dots + x_{n-|z|}^2) + \frac{1}{2}(y_1^2 + \dots + y_{|z|}^2) \\ g &= \sum_{i=1}^{n-|z|} dx_i \otimes dx_i + \sum_{i=1}^{|z|} dy_i \otimes dy_i\end{aligned}$$

in the chart  $\phi$ . In this chart, we then have that

$$\begin{aligned}W^U(z) &:= \{y_1 = \dots = y_{|z|} = 0\} \\ W^S(z) &:= \{x_1 = \dots = x_{n-|z|} = 0\},\end{aligned}$$

and  $M = W^U(z) \times W^S(z)$ . Hence any point of  $U_z$  can be uniquely written as a sum  $x + y$  where  $x \in W^U(z)$  and  $y \in W^S(z)$ . Choosing now  $s \in \mathbb{R}$  such that the the image of  $x + y$  under the Morse flow map  $\phi^s$  still lies in  $U_z$ , we have that

$$\phi^s(x + y) = e^s x + e^{-s} y .$$

These observations will prove crucial in the proof of Proposition 9.4.2 in Section 9.4.3.

### 9.3 Preliminaries for the proofs of Theorems 6.5.1, 7.4.1 and 7.4.5

#### 9.3.1 Counting the points on the boundary of an oriented 1-dimensional manifold

Consider an oriented 1-dimensional manifold with boundary. Then its boundary  $\partial M$  is oriented. Assume it can be written set-theoretically as a disjoint union

$$\partial M = \bigsqcup_i N_i .$$

Suppose now that each  $N_i$  comes with its own orientation, and write  $(-1)^{\dagger_i}$  for the sign obtained by comparing this orientation to the boundary orientation. As oriented manifolds, the union writes as

$$\partial M = \bigsqcup_i (-1)^{\dagger_i} N_i .$$

The  $N_i$  being 0-dimensional, they can be seen as collections of points each coming with a + or – sign. Noticing that an orientable 1-dimensional manifold with boundary is either a segment or a circle, and writing  $\#N_i$  for the signed count of points of  $N_i$ , the previous equality finally implies that

$$\sum (-1)^{\dagger_i} \#N_i = 0 .$$

This basic observation is key to constructing most algebraic structures arising Morse theory and in symplectic topology.

For instance, for a critical point  $x$ , counting the boundary points of the 1-dimensional manifolds  $\overline{\mathcal{T}}(y; x)$  implies that

$$\partial^{Morse} \circ \partial^{Morse}(x) = \sum_{\substack{y \in \text{Crit}(f) \\ |y|=|x|+2}} \sum_{\substack{z \in \text{Crit}(f) \\ |z|=|x|+1}} \# \mathcal{T}(y; z) \# \mathcal{T}(z; x) \cdot y = 0 .$$

The equations for  $\Omega BAs$ -algebras and  $\Omega BAs$ -morphisms will be proven using this method.

#### 9.3.2 Reformulating the $\Omega BAs$ -equations

We fix for each  $t \in SRT_n$  an orientation  $\omega_t$ . Given a  $t \in SRT_n$  the orientation  $\omega_t$  defines an orientation of the moduli space  $\mathcal{T}_n(t)$ , and we write moreover  $m_t$  for the operations  $(t, \omega)$ . The  $\Omega BAs$ -equations for an  $\Omega BAs$ -algebra then read as

$$[\partial, m_t] = \sum_{t' \in \text{coll}(t)} (-1)^{\dagger_{\Omega BAs}} m_{t'} + \sum_{t_1 \#_i t_2 = t} (-1)^{\dagger_{\Omega BAs}} m_{t_1} \circ_i m_{t_2} ,$$

where the notations for trees are as defined previously. The signs  $(-1)^{\dagger_{\Omega BAs}}$  are obtained as in Section 5.1.3, by computing the signs of  $\mathcal{T}_n(t')$  and  $\mathcal{T}_{i_1+1+i_3}(t_1) \times_i \mathcal{T}_{i_2}(t_2)$  in the boundary of  $\mathcal{T}_n(t)$ . We will not need to compute their explicit value, and will hence keep this useful notation  $(-1)^{\Omega BAs}$  to refer to them.

### 9.3.3 Twisted $A_\infty$ -algebras and twisted $\Omega BAs$ -algebras

It is clear using the method of Section 9.3.1 that the operations  $m_t$  of Theorem 6.5.1 will endow the Morse cochains  $C^*(f)$  with a structure of  $\Omega BAs$ -algebra over  $\mathbb{Z}/2$ . Working over integers will prove more difficult and we introduce first to this extent the notion of twisted  $A_\infty$ -algebras and twisted  $\Omega BAs$ -algebras.

**Definition 9.3.1.** (i) A *twisted  $A_\infty$ -algebra* is a dg module  $A$  endowed with two different differentials  $\partial_1$  and  $\partial_2$ , and a sequence of degree  $2 - n$  operations  $m_n : A^{\otimes n} \rightarrow A$  such that

$$[\partial, m_n] = - \sum_{\substack{i_1+i_2+i_3=n \\ 2 \leq i_2 \leq n-1}} (-1)^{i_1+i_2i_3} m_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3}),$$

where  $[\partial, \cdot]$  denotes the bracket for the maps  $(A^{\otimes n}, \partial_1) \rightarrow (A, \partial_2)$ .

(ii) A *twisted  $\Omega BAs$ -algebra* is defined similarly.

We refer to them as *twisted*, as these algebras will occur in the upcoming lines by setting  $\partial_2 := (-1)^\sigma \partial_1$ , that is by simply twisting the differential  $\partial_1$  by a specific sign.

We now make explicit the formulae obtained by evaluating the  $\Omega BAs$ -equations on  $A^{\otimes n}$ , as we will need them in our proof of Theorem 6.5.1:

$$\begin{aligned} & -\partial_2 m_t(a_1, \dots, a_n) + (-1)^{|t| + \sum_{j=1}^{i-1} |a_j|} m_t(a_1, \dots, a_{i-1}, \partial_1 a_i, a_{i+1}, \dots, a_n) \\ & + \sum_{t_1 \# t_2 = t} (-1)^{\dagger_{\Omega BAs} + |t_2|} \sum_{j=1}^{i_1} |a_j| m_{t_1}(a_1, \dots, a_{i_1}, m_{t_2}(a_{i_1+1}, \dots, a_{i_1+i_2}), a_{i_1+i_2+1}, \dots, a_n) \\ & + \sum_{t' \in \text{coll}(t)} (-1)^{\dagger_{\Omega BAs}} m_{t'}(a_1, \dots, a_n) \\ & = 0. \end{aligned}$$

REMARK 9.3.2. We point out that these two definitions cannot be phrased in terms of operads, as  $\text{Hom}((A, \partial_1), (A, \partial_2))$  is an  $(\text{End}_{(A, \partial_1)}, \text{End}_{(A, \partial_2)})$ -operadic bimodule but is NOT an operad: the composition maps on  $\text{Hom}((A, \partial_1), (A, \partial_2))$  are associative, but they fail to be compatible with the differential  $[\partial, \cdot]$ . As a result, a twisted  $A_\infty$ -algebra cannot be described as a morphism of operads from  $A_\infty$  to  $\text{Hom}((A, \partial_1), (A, \partial_2))$ . However, a twisted  $\Omega BAs$ -algebra structure always transfers to a twisted  $A_\infty$ -algebra structure. Indeed, while the functorial proof of Proposition 3.1.15 does not work anymore, we point out that it still contains the proof that a sequence of operations  $m_t$  defining a twisted  $\Omega BAs$ -algebra structure on  $A$  can always be arranged in a sequence of operations  $m_n$  defining a twisted  $A_\infty$ -algebra structure on  $A$ .

### 9.3.4 The maps $\psi_{e_i, \mathbb{X}_t}$

Consider again a stable ribbon tree  $t$  and order its external edges clockwise, starting with  $e_0$  at the outgoing edge. Given a choice of perturbation data  $\mathbb{X}_t$ , we illustrate in Figure 9.2 a mean to visualize the map

$$\phi_{\mathbb{X}_t} : \mathcal{T}_n(t) \times W^S(y) \times W^U(x_1) \times \dots \times W^U(x_n) \longrightarrow M^{\times n+1}$$

defined in Section 6.3. We introduce a family of maps defined in a similar fashion. Consider  $e_i$  an incoming edge of  $t$ . Define the map

$$\psi_{e_i, \mathbb{X}_t} : \mathcal{T}_n(t) \times W^S(y) \times W^U(x_1) \times \dots \times \widehat{W^U(x_i)} \times \dots \times W^U(x_n) \longrightarrow M^{\times n}$$

to be the map which, for a fixed metric tree  $T$ , takes a point of a  $W^U(x_j)$  for  $j \neq i$  to the point in  $M$  obtained by following the only non-self crossing path from the time  $-1$  point on  $e_j$  to the time  $-1$  point on  $e_i$  in  $T$  through the perturbed gradient flow maps associated to  $\mathbb{X}_T$ , and which takes a point of  $W^S(y)$  to the point in  $M$  obtained by following the only non-self crossing path from the time  $1$  point on  $e_0$  to the time  $-1$  point on  $e_i$  in  $T$  through the perturbed gradient flow maps associated to  $\mathbb{X}_T$ . The map  $\psi_{e_0, \mathbb{X}_t}$  is defined similarly for the outgoing edge  $e_0$ . These two definitions are illustrated on two examples in Figure 9.2.

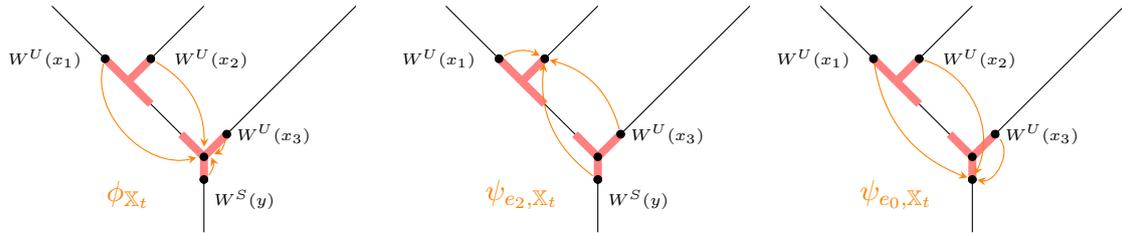


Figure 9.2: Representations of a map  $\phi_{\mathbb{X}_t}$ , a map  $\psi_{e_2, \mathbb{X}_t}$  and a map  $\psi_{e_0, \mathbb{X}_t}$

### 9.4 The twisted $\Omega BAs$ -algebra structure on the Morse cochains

#### 9.4.1 Proof of Theorem 6.5.1

**Definition 9.4.1.** (i) The space  $\tilde{\mathcal{T}}_t^{\mathbb{X}}(y; x_1, \dots, x_n)$  is defined to be the oriented manifold  $\mathcal{T}_t^{\mathbb{X}}(y; x_1, \dots, x_n)$  whose natural orientation (see Definition 9.1.1) has been twisted by a sign of parity

$$\sigma(t; y; x_1, \dots, x_n) := dn(1 + |y| + |t|) + |t||y| + d \sum_{i=1}^n |x_i|(n - i) .$$

(ii) Similarly, we define  $\tilde{\mathcal{T}}(y; x)$  to be the oriented manifold  $\mathcal{T}(y; x)$  whose natural orientation has been twisted by a sign of parity

$$\sigma(y; x) := 1 .$$

The operations  $m_t$  and the differential on  $C^*(f)$  are then defined as

$$m_t(x_1, \dots, x_n) = \sum_{|y| = \sum_{i=1}^n |x_i| + |t|} \# \tilde{\mathcal{T}}_t^{\mathbb{X}}(y; x_1, \dots, x_n) \cdot y ,$$

$$\partial_{Morse}(x) = \sum_{|y| = |x| + 1} \# \tilde{\mathcal{T}}(y; x) \cdot y .$$

**Proposition 9.4.2.** *If  $\tilde{\mathcal{T}}_t(y; x_1, \dots, x_n)$  is 1-dimensional, the boundary of its compactification decomposes as the disjoint union of the following components*

- (i)  $(-1)^{|y| + \dagger_{\Omega BAs} + |t_2| \sum_{i=1}^{i_1} |x_i|} \tilde{\mathcal{T}}_{t_1}(y; x_1, \dots, x_{i_1}, z, x_{i_1+i_2+1}, \dots, x_n) \times \tilde{\mathcal{T}}_{t_2}(z; x_{i_1+1}, \dots, x_{i_1+i_2}) ;$
- (ii)  $(-1)^{|y| + \dagger_{\Omega BAs}} \tilde{\mathcal{T}}_{t'}(y; x_1, \dots, x_n)$  for  $t' \in coll(t) ;$
- (iii)  $(-1)^{|y| + \dagger_{Koszul} + (d+1)|x_i|} \tilde{\mathcal{T}}_t(y; x_1, \dots, z, \dots, x_n) \times \tilde{\mathcal{T}}(z; x_i)$  where  $\dagger_{Koszul} = |t| + \sum_{j=1}^{i-1} |x_j| ;$
- (iv)  $(-1)^{|y| + 1} \tilde{\mathcal{T}}(y; z) \times \tilde{\mathcal{T}}_t(z; x_1, \dots, x_n).$

*Proof.* See Sections 9.4.2 to 9.4.4. □

Theorem 6.5.1 is then a simple corollary to Proposition 9.4.2, that is proven by applying the method of Section 9.3.1:

**Theorem 6.5.1.** *The operations  $m_t$  endow  $(C^*(f), \partial_{Morse}^{Tw}, \partial_{Morse})$  with a twisted  $\Omega$ BAAs-algebra structure, where*

$$(\partial_{Morse}^{Tw})^k := (-1)^{(d+1)k} \partial_{Morse}^k .$$

It appears in particular from the definition of  $\partial_{Morse}^{Tw}$  that when  $M$  is odd-dimensional, this  $\Omega$ BAAs-algebra structure is untwisted, i.e.  $\partial_{Morse}^{Tw} = \partial_{Morse}$ .

REMARK 9.4.3. We point out that the twisted structure arise from the two uncompatible orientation conventions on an intersection  $R \cap S$  and  $S \cap R$  as explained in Section 9.1.2. Indeed, we decided to orient  $\mathcal{T}(y; x)$  inside the intersection  $W^S(y) \cap W^U(x)$ . The signs then compute nicely for the boundary component  $\tilde{\mathcal{T}}(y; z) \times \tilde{\mathcal{T}}_t(z; x_1, \dots, x_n)$ , and the twist in  $\partial_{Morse}^{Tw}$  arises in the boundary component  $\tilde{\mathcal{T}}_t(y; x_1, \dots, z, \dots, x_n) \times \tilde{\mathcal{T}}(z; x_i)$ . Orienting  $\mathcal{T}(y; x)$  inside the intersection  $W^U(x) \cap W^S(y)$  would make these two boundary components switch roles. In that case, redefining the twist on the orientation of the moduli space  $\mathcal{T}(y; x)$  as given by the parity of  $\sigma(y; x) := 1 + |x|$ , we check that the operations  $m_t$  would define a twisted  $\Omega$ BAAs-algebra structure on  $(C^*(f), \partial_{Morse}, \partial_{Morse}^{Tw})$ .

#### 9.4.2 Proof of Proposition 9.4.2: Item (i)

We resort to the formalism of short exact sequences of vector bundles (Section 9.1) to handle orientations in this section. For the sake of readability, we will write  $N$  rather than  $TN$  for the tangent bundle of a manifold  $N$  in the upcoming computations.

The moduli space  $\mathcal{T}_t(y; x_1, \dots, x_n)$  is defined as the inverse image of the diagonal  $\Delta \subset M^{\times n+1}$  under the map

$$\phi_{\mathbb{X}_t} : \mathcal{T}_n(t) \times W^S(y) \times W^U(x_1) \times \dots \times W^U(x_n) \longrightarrow M^{\times n+1} ,$$

where the factors of  $M^{\times n+1}$  are labeled in the order  $M_y \times M_{x_1} \times \dots \times M_{x_n}$ . Orienting the domain and codomain of  $\phi_{\mathbb{X}_t}$  by taking the product orientations, and orienting  $\Delta$  as  $M$ , defines the natural orientation on  $\mathcal{T}_t(y; x_1, \dots, x_n)$  as in Section 9.1.2. Choose  $M^{\times n}$  labeled by  $x_1, \dots, x_n$  as complementary to  $\Delta$ . Then the orientation induced on  $M^{\times n}$  by the orientations on  $M^{\times n+1}$  and on  $\Delta$ , differs by a  $(-1)^{d^2 n}$  sign from the product orientation of  $M^{\times n}$ . In the language of short exact sequences,  $\mathcal{T}_t(y; x_1, \dots, x_n)$  is oriented by the short exact sequence

$$0 \longrightarrow \mathcal{T}_t(y; x_1, \dots, x_n) \longrightarrow \mathcal{T}_n(t) \times W^S(y) \times \prod_{i=1}^n W^U(x_i) \longrightarrow M^{\times n} \longrightarrow 0 ,$$

which has a sign of parity

$$dn . \tag{A}$$

In the case of  $\mathcal{T}_{t_1}^{Morse} := \mathcal{T}_{t_1}(y; x_1, \dots, x_{i_1}, z, x_{i_1+i_2+1}, \dots, x_n)$ , we choose  $M^{\times i_1+1+i_3}$  labeled by  $y, x_1, \dots, x_{i_1}, x_{i_1+i_2+1}, \dots, x_n$  as complementary to  $\Delta$ . The orientation induced on  $M^{\times i_1+1+i_3}$ , by the orientations on  $M^{\times i_1+2+i_3}$  and on  $\Delta$ , differs by a  $(-1)^{d^2 i_3}$  sign from the product orientation of  $M^{\times i_1+1+i_3}$ . Hence the short exact sequence

$$0 \longrightarrow \mathcal{T}_{t_1}^{Morse} \longrightarrow \mathcal{T}_{i_1+1+i_3}(t_1) \times W^S(y) \times \prod_{i=1}^{i_1} W^U(x_i) \times W^U(z) \times \prod_{i=i_1+i_2+1}^n W^U(x_i) \longrightarrow M^{\times i_1+1+i_3} \longrightarrow 0 ,$$

has a sign of parity

$$di_3 . \tag{B}$$

In the case of  $\mathcal{T}_{t_2}^{Morse} := \mathcal{T}_{t_2}(z; x_{i_1+1}, \dots, x_{i_1+i_2})$ , we choose  $M^{\times i_2}$  labeled by  $x_{i_1+1}, \dots, x_{i_1+i_2}$  as complementary to  $\Delta$ . The orientation induced on  $M^{\times i_2}$  differs this time by a  $(-1)^{d^2 i_2}$  sign from the product orientation. The short exact sequence

$$0 \longrightarrow \mathcal{T}_{t_2}^{Morse} \longrightarrow \mathcal{T}_{i_2}(t_2) \times W^S(z) \times \prod_{i=i_1+1}^{i_1+i_2} W^U(x_i) \longrightarrow M^{\times i_2} \longrightarrow 0 ,$$

has now a sign given by the parity of

$$di_2 . \tag{C}$$

Following the convention on the sum of signed short exact sequences in Section 9.1.1, taking the product

$$0 \longrightarrow \mathcal{T}_{t_1}^{Morse} \times \mathcal{T}_{t_2}^{Morse} \longrightarrow \mathcal{T}_{i_1+1+i_3}(t_1) \times W^S(y) \times \prod_{i=1}^{i_1} W^U(x_i) \times W^U(z) \times \prod_{i=i_1+i_2+1}^n W^U(x_i) \times \mathcal{T}_{i_2}(t_2) \times W^S(z) \times \prod_{i=i_1+1}^{i_1+i_2} W^U(x_i) \longrightarrow M^{\times i_1+1+i_3} \times M^{\times i_2} \longrightarrow 0$$

doesn't introduce a sign, as  $\mathcal{T}_{t_1}^{Morse}$  and  $\mathcal{T}_{t_2}^{Morse}$  are 0-dimensional.

In the previous short exact sequence,  $M^{\times i_1+1+i_3} \times M^{\times i_2}$  is labeled by

$$y, x_1, \dots, x_{i_1}, x_{i_1+i_2+1}, \dots, x_n, x_{i_1+1}, \dots, x_{i_1+i_2} .$$

We rearrange this labeling into

$$y, x_1, \dots, x_n ,$$

which induces a sign given by the parity of

$$di_2 i_3 . \tag{D}$$

We also rearrange the expression

$$\mathcal{T}_{i_1+1+i_3}(t_1) \times W^S(y) \times \prod_{i=1}^{i_1} W^U(x_i) \times W^U(z) \times \prod_{i=i_1+i_2+1}^n W^U(x_i) \times \mathcal{T}_{i_2}(t_2) \times W^S(z) \times \prod_{i=i_1+1}^{i_1+i_2} W^U(x_i) ,$$

into

$$W^U(z) \times W^S(z) \times \mathcal{T}_{i_1+1+i_3}(t_1) \times \mathcal{T}_{i_2}(t_2) \times W^S(y) \times \prod_{i=1}^n W^U(x_i) .$$

The parity of the produced sign is that of

$$\begin{aligned} & |z| \left( |t_2| + \sum_{i=i_1+i_2+1}^n (d - |x_i|) \right) + m \left( |t_1| + |y| + \sum_{i=1}^{i_1} (d - |x_i|) \right) \\ & + |t_2| \left( |y| + \sum_{i=1}^{i_1} (d - |x_i|) + \sum_{i=i_1+i_2+1}^n (d - |x_i|) \right) + \left( \sum_{i=i_1+1}^{i_1+i_2} (d - |x_i|) \right) \left( \sum_{i=i_1+i_2+1}^n (d - |x_i|) \right) . \end{aligned} \tag{E}$$

Introduce now the factor  $[L, +\infty[$ , corresponding to the length  $l_e$  increasing towards  $+\infty$ , where  $e$  is the edge of  $t$  whose breaking produces  $t_1$  and  $t_2$  and where  $L$  is chosen to be large enough as in Section 9.4.3. Following the convention of Section 9.1.2, the short exact sequence

$$0 \longrightarrow [L, +\infty[ \times \mathcal{T}_{t_1}^{Morse} \times \mathcal{T}_{t_2}^{Morse} \longrightarrow [L, +\infty[ \times W^U(z) \times W^S(z) \times \mathcal{T}(t_1) \times \mathcal{T}(t_2) \times \times W^S(y) \times \prod_{i=1}^n W^U(x_i) \longrightarrow M^{\times n+1} \longrightarrow 0 ,$$

induces a sign change whose parity is given by

$$d(n+1) . \tag{F}$$

Define the map

$$\psi : M \times \mathcal{T}_n(t) \times W^S(y) \times \prod_{i=1}^n W^U(x_i) \longrightarrow M \times M^{\times n+1} ,$$

which is defined on the factors  $\mathcal{T}_n(t) \times W^S(y) \times \prod_{i=1}^n W^U(x_i)$  as  $\phi$  and is defined on  $M \times \mathcal{T}_n(t)$  by seeing  $M$  as the point lying in the middle of the edge  $e$  in  $t$ . This map is depicted on Figure 9.3. The inverse image of the diagonal of  $M \times M^{\times n+1}$  is exactly  $\mathcal{T}_t(y; x_1, \dots, x_n)$ . Fix now a sufficiently great  $L > 0$ . We prove in Section 9.4.3 that after orienting  $[L, +\infty[ \times \mathcal{T}_{t_1}^{Morse} \times \mathcal{T}_{t_2}^{Morse}$  with the previous short exact sequence, the orientation obtained on  $\mathcal{T}_t^{Morse}$  by gluing is exactly the one given by the short exact sequence

$$0 \longrightarrow \mathcal{T}_t^{Morse} \longrightarrow [L, +\infty[ \times M \times \mathcal{T}(t_1) \times \mathcal{T}(t_2) \times W^S(y) \times \prod_{i=1}^n W^U(x_i) \xrightarrow{d\psi} M^{\times n+1} \longrightarrow 0$$

where our convention on orientations for the unstable and stable manifolds of  $z$  implies that  $W^U(z) \times W^S(z)$  yields indeed the orientation of  $M$ , and  $M^{\times n+1}$  is labeled by  $y, x_1, \dots, x_n$ .

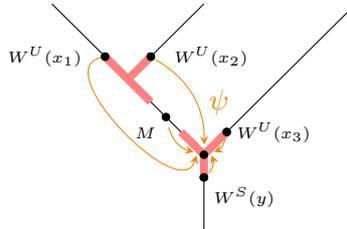


Figure 9.3: Representation of the map  $\psi$

Transform the coorientation labeled by  $y, x_1, \dots, x_n$  into the coorientation labeled by  $M, x_1, \dots, x_n$  and rearrange the factors  $[L, +\infty[ \times M \times \mathcal{T}(t_1) \times \mathcal{T}(t_2) \times \dots$  into  $M \times [L, +\infty[ \times \mathcal{T}(t_1) \times \mathcal{T}(t_2) \times \dots$ . This produces a sign change of parity

$$d + d \equiv 0 . \tag{G}$$

We can moreover now delete the two  $M$  factors associated to the label  $M$  in  $M^{\times n+1}$  and in  $M \times [L, +\infty[ \times \mathcal{T}(t_1) \times \mathcal{T}(t_2) \times \dots$  in order to obtain the short exact sequence

$$0 \longrightarrow \mathcal{T}_t(y; x_1, \dots, x_n) \longrightarrow [L, +\infty[ \times \mathcal{T}(t_1) \times \mathcal{T}(t_2) \times W^S(y) \times \prod_{i=1}^n W^U(x_i) \longrightarrow M^{\times n} \longrightarrow 0 ,$$

where  $M^{\times n} = M_{x_1} \times \dots \times M_{x_n}$ .

Transforming finally  $[L, +\infty[\times \mathcal{T}(t_1) \times \mathcal{T}(t_2)$  into  $\mathcal{T}_n(t)$  gives a sign of parity

$$\dagger_{\Omega BAs} . \tag{H}$$

In closing, the short exact sequence

$$0 \longrightarrow \mathcal{T}_t(y; x_1, \dots, x_n) \longrightarrow \mathcal{T}_n(t) \times W^S(y) \times \prod_{i=1}^n W^U(x_i) \longrightarrow M^{\times n} \longrightarrow 0 ,$$

has sign given by the parity of  $A$  when  $\mathcal{T}_t^{Morse}$  is endowed with its natural orientation. It has sign given by the parity of  $B + C + D + E + F + G + H$  when  $\mathcal{T}_t^{Morse}$  is endowed with the orientation induced by  $[L, +\infty[\times \mathcal{T}_{t_1}^{Morse} \times \mathcal{T}_{t_2}^{Morse}$ , where the first factor is the length  $l_e$  obtained after gluing (see Section 9.4.3) and determines the outward-pointing direction  $\nu_e$  to the boundary component  $\mathcal{T}_{t_1}^{Morse} \times \mathcal{T}_{t_2}^{Morse}$ .

We thus obtain that with our choice of orientation on the moduli spaces  $\mathcal{T}_t^{Morse}$ , the sign of  $\mathcal{T}_{t_1}(y; x_1, \dots, x_{i_1}, z, x_{i_1+i_2+1}, \dots, x_n) \times \mathcal{T}_{t_2}(z; x_{i_1+1}, \dots, x_{i_1+i_2})$  in the boundary of the moduli space of dimension 1  $\mathcal{T}_t(y; x_1, \dots, x_n)$  is given by the parity of

$$\begin{aligned} (*) \quad & A + B + C + D + E + F + G + H \\ & = |z|t_2 + d|y| + d|t_1| + (n + 1)d + \sum_{i=1}^{i_1} d|x_i| + |t_2||y| + d i_1 |t_2| + d i_2 \sum_{i=i_1+i_2+1}^n |x_i| + \dagger_{\Omega BAs} + |t_2| \sum_{i=1}^{i_1} |x_i| . \end{aligned}$$

Hence the sign of  $\widetilde{\mathcal{T}}_{t_1}(y; x_1, \dots, x_{i_1}, z, x_{i_1+i_2+1}, \dots, x_n) \times \widetilde{\mathcal{T}}_{t_2}(z; x_{i_1+1}, \dots, x_{i_1+i_2})$  in the boundary of the 1-dimensional moduli space  $\widetilde{\mathcal{T}}_t(y; x_1, \dots, x_n)$  is given by the parity of

$$\begin{aligned} & \sigma(t; y; x_1, \dots, x_n) + \sigma(t_1; y; x_1, \dots, x_{i_1}, z, x_{i_1+i_2+1}, \dots, x_n) + \sigma(t_2; z; x_{i_1+1}, \dots, x_{i_1+i_2}) + (*) \\ & = |y| + \dagger_{\Omega BAs} + |t_2| \sum_{i=1}^{i_1} |x_i| . \end{aligned}$$

This concludes the proof of Item (i) in Proposition 9.4.2.

### 9.4.3 Proof of Proposition 9.4.2: gluing and orientations

We prove in this subsection that after orienting  $[L, +\infty[\times \mathcal{T}_{t_1}^{Morse} \times \mathcal{T}_{t_2}^{Morse}$  with the short exact sequence

$$0 \longrightarrow [L, +\infty[\times \mathcal{T}_{t_1}^{Morse} \times \mathcal{T}_{t_2}^{Morse} \longrightarrow [L, +\infty[\times W^U(z) \times W^S(z) \times \mathcal{T}(t_1) \times \mathcal{T}(t_2) \times W^S(y) \times \prod_{i=1}^n W^U(x_i) \longrightarrow M^{\times n+1} \longrightarrow 0 ,$$

the orientation induced on  $\mathcal{T}_t^{Morse}$  by gluing is the one given by the short exact sequence

$$0 \longrightarrow \mathcal{T}_t^{Morse} \longrightarrow [L, +\infty[\times M \times \mathcal{T}(t_1) \times \mathcal{T}(t_2) \times W^S(y) \times \prod_{i=1}^n W^U(x_i) \xrightarrow{d_\psi} M^{\times n+1} \longrightarrow 0 .$$

The proof boils down to the following lemma.

**Lemma 9.4.4.** *Let  $M$  and  $N$  be manifolds and  $S \subset N$  a submanifold of  $N$ . Suppose that  $M$ ,  $N$  and  $S$  are orientable and oriented. Let  $f : [0, 1] \times M \rightarrow N$  be a smooth map such that  $f_1 := f(1, \cdot) : M \rightarrow N$  is transverse to  $S$ . Let  $x \in f_1^{-1}(S)$ . Then there exist an open subset  $V$  of  $M$  containing  $x$  and  $0 \leq t_1 < 1$  such that*

- (i) *The map  $f|_{[t_1, 1] \times V} : [t_1, 1] \times V \rightarrow N$  is transverse to  $S$ . In particular the inverse image  $f|_{[t_1, 1] \times V}^{-1}(S)$  is then a submanifold of  $[t_1, 1] \times V$ .*

(ii) There exists an orientation-preserving embedding

$$f|_{[t_1, 1] \times V}^{-1}(S) \longrightarrow [t_1, 1] \times f_1^{-1}(S)$$

equal to the identity on  $f_1|_V^{-1}(S)$  and preserving the  $t$  coordinate, where we orient  $[t_1, 1] \times f_1^{-1}(S)$  with the short exact sequence

$$0 \longrightarrow [t_1, 1] \times f_1^{-1}(S) \longrightarrow [0, 1] \times M \longrightarrow \nu_S \longrightarrow 0$$

and we orient  $f|_{[t_1, 1] \times V}^{-1}(S)$  with the short exact sequence

$$0 \longrightarrow f|_{[t_1, 1] \times V}^{-1}(S) \longrightarrow [0, 1] \times M \longrightarrow \nu_S \longrightarrow 0 .$$

*Proof.* Choose an adapted chart for  $S$  around  $f_1(x)$ , i.e. a chart  $\phi : U' \subset N \rightarrow \mathbb{R}^n$  such that

$$\phi(U' \cap S) = \{(y_1, \dots, y_{n-s}, x_1, \dots, x_s) \in \mathbb{R}^n, y_1 = \dots = y_{n-s} = 0\} ,$$

where  $n$  and  $s$  respectively denote the dimensions of  $N$  and  $S$ . Using the local normal form theorem for submersions, there exists a local chart  $\psi : U \subset M \rightarrow \mathbb{R}^m$  around  $x$  such that the map  $f_1$  reads as

$$(y_1, \dots, y_{n-s}, x_1, \dots, x_{m+s-n}) \longmapsto (y_1, \dots, y_{n-s}, F_1(\vec{y}, \vec{x}), \dots, F_s(\vec{y}, \vec{x}))$$

in the local charts  $\psi$  and  $\phi$ , where the  $F_i$  are smooth maps and  $\vec{y} := y_1, \dots, y_{n-s}$ ,  $\vec{x} := x_1, \dots, x_{m+s-n}$  and  $m := \dim(M)$ . In these local charts,

$$U \cap f_1^{-1}(U' \cap S) = \{(y_1, \dots, y_{n-s}, x_1, \dots, x_{m+s-n}) \in \mathbb{R}^m, y_1 = \dots = y_{n-s} = 0\} .$$

The property "being transverse to  $S$ " being open, there exists a neighborhood  $W$  of  $x$  in  $M$  and  $t_0 \in [0, 1[$  such that the map  $f|_{[t_0, 1] \times W} : [t_0, 1] \times W \rightarrow N$  is transverse to  $S$ . Suppose  $W \subset U$  and consider now the projection  $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^{m+s-n}$  given by

$$(y_1, \dots, y_{n-s}, x_1, \dots, x_{m+s-n}) \longmapsto (x_1, \dots, x_{m+s-n})$$

and define the smooth map

$$\iota := \text{id}_t \times \pi : f|_{[t_0, 1] \times W}^{-1}(S) \longrightarrow [0, 1] \times f_1^{-1}(S)$$

in the local charts  $\phi$  and  $\psi$ . The differential of this map is invertible at  $(1, x)$ . The inverse function theorem then ensures that there exists  $t_1 \in [t_0, 1[$  and a neighborhood  $V \subset W$  of  $x$  such that the map

$$\iota : f|_{[t_1, 1] \times V}^{-1}(S) \longrightarrow [0, 1] \times f_1^{-1}(S)$$

is a diffeomorphism on its image.

Orient now  $[0, 1] \times f_1^{-1}(S)$  and  $f|_{[t_1, 1] \times V}^{-1}(S)$  with the previous short exact sequences. It remains to show that the map  $\iota$  is orientation-preserving. The proof of this result can be reduced to a proof in linear algebra, i.e. by considering a smooth family of linear maps  $f : [0, 1] \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  such that  $f_1$  reads as

$$(y_1, \dots, y_{n-s}, x_1, \dots, x_{m+s-n}) \longmapsto (y_1, \dots, y_{n-s}, F_1(\vec{y}, \vec{x}), \dots, F_s(\vec{y}, \vec{x})) ,$$

and the linear subspace  $S = \{0\} \times \mathbb{R}^s \subset \mathbb{R}^n$ . Then there exists  $t_0 \in [0, 1]$  such that  $f|_{[t_0, 1] \times \mathbb{R}^m}$  is transverse to  $S$ , and we can consider the smooth map

$$\iota := \text{id}_t \times \pi : f|_{[t_0, 1] \times \mathbb{R}^m}^{-1}(S) \longrightarrow [0, 1] \times f_1^{-1}(S)$$

which is a diffeomorphism on its image. Basic computations finally show that the map  $\iota$  is indeed orientation-preserving.  $\square$

We now go back to our initial problem. Let  $T_1^{Morse} \in \mathcal{T}_1^{Morse}$  and  $T_2^{Morse} \in \mathcal{T}_2^{Morse}$ , where we refer to Section 9.4.2 for notations. Consider a local Euclidean chart  $\phi_z : U_z \rightarrow \mathbb{R}^d$  for the critical point  $z$  as in Section 9.2.4. Introduce the map  $ev : [0, +\infty] \times U_z \rightarrow U_z \times U_z$  reading as

$$(\delta, x + y) \mapsto (e^{-2\delta}x + y, x + e^{-2\delta}y)$$

in the chart  $\phi_z$ . The pair  $ev(\delta, x + y)$  corresponds to the two endpoints of the unique finite Morse trajectory parametrized by  $[-\delta, \delta]$  and meeting  $e^{-\delta}x + e^{-\delta}y$  at time 0.

Consider the trajectory  $\gamma_{e,1} : ]-\infty, 0] \rightarrow M$  and the trajectory  $\gamma_{e,2} : [0, +\infty[ \rightarrow M$ , respectively associated to the incoming edge of  $T_1^{Morse}$  and to the outgoing edge of  $T_2^{Morse}$  which result from the breaking of the edge  $e$  in  $t$ . Choose  $L$  large enough such that  $\gamma_{e,1}(-L)$  and  $\gamma_{e,2}(L)$  belong to  $U_z$ . Introduce the map  $f := ev \times (\phi^{-(L-1)})^{\times i_1+1+i_3} \circ \psi_{e, \mathbb{X}_{t_1}} \times (\phi^{L-1})^{\times i_2} \circ \psi_{e, \mathbb{X}_{t_2}}$  acting as

$$\begin{aligned} [0, +\infty] \times U_z \times \mathcal{T}_{i_1+1+i_3}(t_1) \times W^S(y) \times \prod_{i=1}^{i_1} W^U(x_i) \times \prod_{i=i_1+i_2+1}^n W^U(x_i) \times \mathcal{T}_{i_2}(t_2) \times \prod_{i=i_1+1}^{i_1+i_2} W^U(x_i) \\ \longrightarrow M^{\times 2} \times M^{\times i_1+1+i_3} \times M^{\times i_2}, \end{aligned}$$

where  $\phi^{L-1}$  stands for the time  $L - 1$  Morse flow and the maps  $\psi_{e, \mathbb{X}_{t_2}}$  and  $\psi_{e, \mathbb{X}_{t_1}}$  have been introduced in Section 9.2.4. This map is depicted in Figure 9.4.

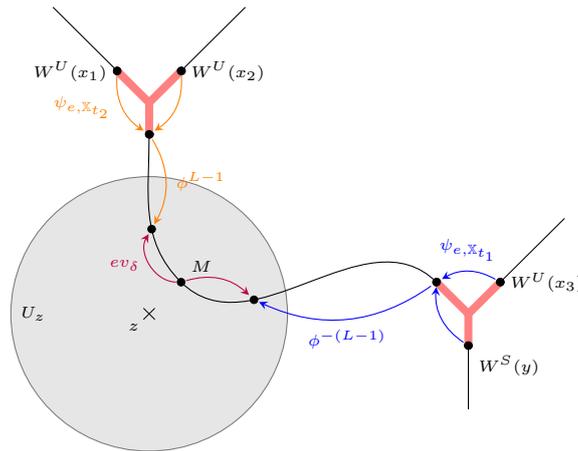


Figure 9.4: Representation of the map  $f$ . Beware that the label  $M$  corresponds to the point  $e^{-\delta}x + e^{-\delta}y$  and not to the point  $x + y$ .

Define the  $2d$ -dimensional submanifold  $\Lambda \subset M^{\times 2} \times M^{\times i_1+1+i_3} \times M^{\times i_2}$  to be

$$\Lambda := \left\{ \begin{array}{l} (m_z^1, m_z^2, m_y, m_1, \dots, m_{i_1}, m_{i_1+1+i_2}, \dots, m_n, m_{i_1+1}, \dots, m_{i_1+i_2}) \\ \text{such that } m_z^1 = m_{i_1+1} = \dots = m_{i_1+i_2} \quad \text{and} \\ m_z^2 = m_y = m_1 = \dots = m_{i_1} = m_{i_1+1+i_2} = \dots = m_n \end{array} \right\}.$$

The pair  $(T_1^{Morse}, T_2^{Morse})$  then belongs to the inverse image  $f_{+\infty}^{-1}(\Lambda)$ . By assumption on the choice of perturbation data  $(\mathbb{X}_n)_{n \geq 2}$ , the map  $f_{+\infty}$  is moreover transverse to  $\Lambda$ . Applying Lemma 9.4.4 to the map  $f$  at the point  $(T_1^{Morse}, T_2^{Morse})$ , there exists  $R > 0$  and an embedding

$$\#_{T_1^{Morse}, T_2^{Morse}} : [R, +\infty] \longrightarrow \overline{\mathcal{T}}_t(y; x_1, \dots, x_n).$$

Note that the parameter  $\delta$  corresponds to an edge of length  $2L + 2\delta$  in the resulting glued tree. Upon reordering the factors of the domain of  $f$ , it is finally easy to check that this lemma also implies the result on orientations stated at the beginning of this subsection.

#### 9.4.4 Proof of Proposition 9.4.2: Items (ii) to (iv)

Repeating the beginning of Section 9.4.2, for the moduli spaces  $\mathcal{T}_{t'}(y; x_1, \dots, x_n)$ , where  $t' \in \text{coll}(t)$ , and  $\mathcal{T}_t(y; x_1, \dots, x_n)$ , we choose  $M^{\times n}$  labeled by  $x_1, \dots, x_n$  as complementary to the diagonal  $\Delta \subset M^{\times n+1}$ . The parity of the total sign change coming from these coorientation choices is

$$dn + dn = 0 . \quad (\text{A})$$

Introduce the factor  $]0, L]$ , corresponding to the length  $l_e$  going towards 0, where  $e$  is the edge of  $t$  whose collapsing produces  $t'$ . Applying again Lemma 9.4.4 and following convention of Section 9.1.1, the short exact sequence

$$0 \longrightarrow \mathcal{T}_t(y; x_1, \dots, x_n) = ]0, L] \times \mathcal{T}_{t'}(y; x_1, \dots, x_n) \longrightarrow ]0, L] \times \mathcal{T}_n(t') \times W^S(y) \times \prod_{i=1}^n W^U(x_i) \longrightarrow M^{\times n} \longrightarrow 0 ,$$

introduces a sign change whose parity is given by

$$dn . \quad (\text{B})$$

Transforming finally  $]0, L] \times \mathcal{T}_n(t')$  into  $\mathcal{T}_n(t)$  gives a sign of parity

$$\dagger_{\Omega BAs} . \quad (\text{C})$$

Adding these contributions, we obtain that the sign of  $\mathcal{T}_{t'}(y; x_1, \dots, x_n)$  in the boundary of the 1-dimensional moduli space  $\mathcal{T}_t(y; x_1, \dots, x_n)$  is given by the parity of

$$A + B + C = dn + \dagger_{\Omega BAs} . \quad (*)$$

The sign of  $\widetilde{\mathcal{T}}_{t'}(y; x_1, \dots, x_n)$  in the boundary of the 1-dimensional moduli space  $\widetilde{\mathcal{T}}_t(y; x_1, \dots, x_n)$  is hence given by the parity of

$$\sigma(t; y; x_1, \dots, x_n) + \sigma(t'; y; x_1, \dots, x_n) + (*) = |y| + \dagger_{\Omega BAs} .$$

Finally, the signs for the (Morse) boundary can be computed following the exact same lines of the two previous proofs. This concludes the proof of Items (ii) to (iv) in Proposition 9.4.2.

### 9.5 The twisted $\Omega BAs$ -morphism between the Morse cochains

#### 9.5.1 Reformulating the $\Omega BAs$ -equations

For the rest of this section, we endow each  $t_c \in \mathcal{S}C\mathcal{R}T_n$  with the orientation chosen for the underlying ribbon tree  $t$  of  $t_c$  in Section 9.3.2. These choices of orientations endow each moduli space  $\mathcal{C}\mathcal{T}_n(t_c)$  with an orientation. We write moreover  $\mu_{t_c}$  for the operations  $(t_c, \omega)$  of  $M_{\Omega BAs}$ . The  $\Omega BAs$ -equations for an  $\Omega BAs$ -morphism then read as

$$\begin{aligned} [\partial, \mu_{t_c}] &= \sum_{t'_c \in \text{coll}(t_c)} (-1)^{\dagger_{\Omega BAs}} \mu_{t'_c} + \sum_{t'_c \in g\text{-vert}(t_c)} (-1)^{\dagger_{\Omega BAs}} \mu_{t'_c} + \sum_{t_c^1 \#_i t_c^2 = t_c} (-1)^{\dagger_{\Omega BAs}} \mu_{t_c^1} \circ_i \mu_{t_c^2} \\ &+ \sum_{t^0 \#_i (t_c^1, \dots, t_c^s) = t_c} (-1)^{\dagger_{\Omega BAs}} m_{t^0} \circ (\mu_{t_c^1} \otimes \dots \otimes \mu_{t_c^s}) , \end{aligned}$$

where the notations for trees are transparent. The signs  $(-1)^{\dagger_{\Omega BAs}}$  are obtained as in Section 9.3.2.

### 9.5.2 Twisted $A_\infty$ -morphisms and twisted $\Omega BAs$ -morphisms

It is again clear using the method of Section 9.3.1 that the operations  $\mu_{t_c}$  of Theorem 7.4.1 define an  $\Omega BAs$ -morphism over  $\mathbb{Z}/2$ . Working over integers will require the following notions:

**Definition 9.5.1.** (i) Let  $(A, \partial_1, \partial_2, m_n)$  and  $(B, \partial_1, \partial_2, m_n)$  be two twisted  $A_\infty$ -algebras. A *twisted  $A_\infty$ -morphism* from  $A$  to  $B$  is defined to be a sequence of degree  $1 - n$  operations  $f_n : A^{\otimes n} \rightarrow B$  such that

$$[\partial, f_n] = \sum_{\substack{i_1+i_2+i_3=n \\ i_2 \geq 2}} (-1)^{i_1+i_2 i_3} f_{i_1+1+i_3}(\text{id}^{\otimes i_1} \otimes m_{i_2} \otimes \text{id}^{\otimes i_3}) - \sum_{\substack{i_1+\dots+i_s=n \\ s \geq 2}} (-1)^{\epsilon_B} m_s(f_{i_1} \otimes \dots \otimes f_{i_s})$$

where  $[\partial, \cdot]$  denotes the bracket for the maps  $(A^{\otimes n}, \partial_1) \rightarrow (B, \partial_2)$ .

(ii) A *twisted  $\Omega BAs$ -morphism* between twisted  $\Omega BAs$ -algebras is defined similarly.

The formulae obtained by evaluating the  $\Omega BAs$ -equations on  $A^{\otimes n}$  then read as

$$\begin{aligned} & - \partial_2 \mu_{t_c}(a_1, \dots, a_n) + (-1)^{|t_c| + \sum_{j=1}^{i-1} |a_j|} \mu_{t_c}(a_1, \dots, a_{i-1}, \partial_1 a_i, a_{i+1}, \dots, a_n) \\ & + \sum_{t_c^\dagger \# t^2 = t} (-1)^{\dagger_{\Omega BAs} + |t^2| \sum_{j=1}^{i-1} |a_j|} \mu_{t_c^\dagger}(a_1, \dots, a_{i_1}, m_{t^2}(a_{i_1+1}, \dots, a_{i_1+i_2}), a_{i_1+i_2+1}, \dots, a_n) \\ & + \sum_{t^1 \# (t_c^1, \dots, t_c^s) = t_c} (-1)^{\dagger_{\Omega BAs} + \dagger_{Koszul}} m_{t^0}(\mu_{t_c^1}(a_1, \dots, a_{i_1}), \dots, \mu_{t_c^s}(a_{i_1+\dots+i_{s-1}+1}, \dots, a_n)) \\ & + \sum_{t'_c \in \text{coll}(t_c)} (-1)^{\dagger_{\Omega BAs}} \mu_{t'_c}(a_1, \dots, a_n) + \sum_{t'_c \in g\text{-vert}(t_c)} (-1)^{\dagger_{\Omega BAs}} \mu_{t'_c}(a_1, \dots, a_n) \\ & = 0, \end{aligned}$$

where

$$\dagger_{Koszul} = \sum_{r=1}^s |t_c^r| \left( \sum_{t=1}^{r-1} \sum_{j=1}^{i_t} |a_{i_1+\dots+a_{i_{t-1}}+j}| \right).$$

**REMARK 9.5.2.** Again Definition 9.5.1 cannot be phrased using an operadic viewpoint. However, a twisted  $\Omega BAs$ -morphism between twisted  $\Omega BAs$ -algebras always descends to a twisted  $A_\infty$ -morphism between twisted  $A_\infty$ -algebras, for the same reason as in Remark 9.3.2.

### 9.5.3 Proof of Theorem 7.4.1

**Definition 9.5.3.** We define  $\widetilde{\mathcal{CT}}_{t_c}^{\mathbb{Y}}(y; x_1, \dots, x_n)$  to be the oriented manifold  $\mathcal{CT}_{t_c}^{\mathbb{Y}}(y; x_1, \dots, x_n)$  whose natural orientation has been twisted by a sign of parity

$$\sigma(t_c; y; x_1, \dots, x_n) := dn(1 + |y| + |t_c|) + |t_c||y| + d \sum_{i=1}^n |x_i|(n - i).$$

The moduli spaces  $\widetilde{\mathcal{T}}(y; x)$  and  $\widetilde{\mathcal{T}}_t(y; x_1, \dots, x_n)$  are moreover defined as in Definition 9.4.1. The operations  $\mu_{t_c} : C^*(f)^{\otimes n} \rightarrow C^*(g)$  are then defined as

$$\mu_{t_c}(x_1, \dots, x_n) = \sum_{|y| = \sum_{i=1}^n |x_i| + |t_c|} \# \widetilde{\mathcal{CT}}_{t_c}^{\mathbb{Y}}(y; x_1, \dots, x_n) \cdot y.$$

**Proposition 9.5.4.** *If  $\widetilde{\mathcal{CT}}_{t_c}^{\mathbb{Y}}(y; x_1, \dots, x_n)$  is 1-dimensional, its boundary decomposes as the disjoint union of the following components*

- (i)  $(-1)^{|y|+\dagger_{\Omega BAs}+|t^2|\sum_{i=1}^{i_1}|x_i|}\widetilde{\mathcal{CT}}_{t_c^1}(y; x_1, \dots, x_{i_1}, z, x_{i_1+i_2+1}, \dots, x_n) \times \widetilde{\mathcal{T}}_{t^2}(z; x_{i_1+1}, \dots, x_{i_1+i_2})$  ;
- (ii)  $(-1)^{|y|+\dagger_{\Omega BAs}+\dagger_{Koszul}}\widetilde{\mathcal{T}}_{t^1}(y; y_1, \dots, y_s) \times \widetilde{\mathcal{CT}}_{t_c^1}(y_1; x_1, \dots) \times \cdots \times \widetilde{\mathcal{CT}}_{t_c^s}(y_s; \dots, x_n)$  ;
- (iii)  $(-1)^{|y|+\dagger_{\Omega BAs}}\widetilde{\mathcal{CT}}_{t_c'}(y; x_1, \dots, x_n)$  for  $t' \in \text{coll}(t)$  ;
- (iv)  $(-1)^{|y|+\dagger_{\Omega BAs}}\widetilde{\mathcal{CT}}_{t_c'}(y; x_1, \dots, x_n)$  for  $t' \in g - \text{vert}(t)$  ;
- (v)  $(-1)^{|y|+\dagger_{Koszul}+(m+1)|x_i|}\widetilde{\mathcal{CT}}_{t_c}(y; x_1, \dots, z, \dots, x_n) \times \widetilde{\mathcal{T}}(z; x_i)$  where  $\dagger_{Koszul} = |t_c| + \sum_{j=1}^{i-1} |x_j|$  ;
- (vi)  $(-1)^{|y|+1}\widetilde{\mathcal{T}}(y; z) \times \widetilde{\mathcal{CT}}_{t_c}(z; x_1, \dots, x_n)$ .

*Proof.* The proof relies on the same computations of signed short exact sequences as in the proof of Proposition 9.4.2. For the sake of concision, we choose not to include them.  $\square$

Theorem 7.4.1 is then again a simple corollary to Proposition 9.5.4, which is proven by applying the method of Section 9.3.1:

**Theorem 7.4.1.** *The operations  $\mu_{t_c}$  define a twisted  $\Omega BAs$ -morphism between the Morse cochains  $(C^*(f), \partial_{Morse}^{Tw}, \partial_{Morse})$  and  $(C^*(g), \partial_{Morse}^{Tw}, \partial_{Morse})$ .*

When  $M$  is odd-dimensional, this twisted  $\Omega BAs$ -morphism is a standard  $\Omega BAs$ -morphism between  $\Omega BAs$ -algebras, as  $\partial_{Morse}^{Tw} = \partial_{Morse}$ .

#### 9.5.4 Gluing

We finally construct explicit gluing maps for the (above-break) and (below-break) boundary components using Lemma 9.4.4. Gluing maps for the (above-break) boundary components

$$\#_{T_c^{1,Morse}, T_c^{2,Morse}}^{\text{above-break}} : [R, +\infty] \longrightarrow \overline{\mathcal{CT}}_{t_c}(y; x_1, \dots, x_n)$$

are built exactly as in Section 9.4.3.

In the (below-break) case, consider critical points  $y, y_1, \dots, y_s \in \text{Crit}(g)$  and  $x_1, \dots, x_n \in \text{Crit}(f)$  such that the moduli spaces  $\mathcal{T}_{t^0}(y; y_1, \dots, y_s)$  and  $\mathcal{CT}_{t_c^r}(y_r; x_{i_1+\dots+i_{r-1}+1}, \dots, x_{i_1+\dots+i_r})$  both have dimension 0. Let  $T^{0,Morse} \in \mathcal{T}_{t^0}^{Morse}$  and  $T_c^{r,Morse} \in \mathcal{CT}_{t_c^r}^{Morse}$ . Fix moreover an Euclidean neighborhood  $U_{z_r}$  of each critical point  $z_r$  and choose  $L$  large enough such that for  $r = 1, \dots, s$ ,  $\gamma_{e_r, T^{0,Morse}}(-L)$  and  $\gamma_{e_0, T_c^{r,Morse}}(L)$  belong to  $U_{z_r}$ . Define finally the map  $\sigma_{e_0, \mathbb{X}_{t^0}} : M \rightarrow M^{\times s}$  in a similar fashion to the maps  $\psi_{e_i, \mathbb{X}_i}$ , as depicted for instance in Figure 9.5.

The gluing map for the perturbed Morse trees  $T^{0,Morse}$  and  $T_c^{r,Morse}$

$$\#_{T^{0,Morse}, T_c^{1,Morse}, \dots, T_c^{s,Morse}}^{\text{below-break}} : [R, +\infty] \longrightarrow \overline{\mathcal{CT}}_{t_c}(y; x_1, \dots, x_n)$$

can then be defined by applying Lemma 9.4.4 to the map

$$[0, +\infty] \times \prod_{r=1}^s U_{z_r} \times \mathcal{T}_s(t^0) \times W^S(y) \times \prod_{r=1}^s \left( \mathcal{CT}_{t_c^r}(t_c^r) \times \prod_{i=i_1+\dots+i_{r-1}+1}^{i_1+\dots+i_r} W^U(x_i) \right) \longrightarrow M^{\times 2s} \times M^{\times s} \times \prod_{r=1}^s M^{\times i_r} .$$

defined as follows:

- (i) the factor  $\mathcal{T}_s(t^0) \times W^S(y)$  is sent to  $M^{\times s}$  under the map  $(\phi^{-(L-1)})^{\times s} \circ \sigma_{e_0, t^0}$  ;
- (ii) the factor  $\mathcal{CT}_{t_c^r}(t_c^r) \times \prod W^U(x_i)$  is sent to  $M^{\times i_r}$  under the map  $(\phi^{(L-1)})^{\times i_r} \circ \sigma_{e_0, t_c^r}$  ;
- (iii) the factor  $[0, +\infty] \times \prod_{r=1}^s U_{z_r}$  is sent to  $M^{\times 2s}$  under the map  $ev_{l_\delta^1}^{U_{z_1}} \times \cdots \times ev_{l_\delta^s}^{U_{z_s}}$  where  $\delta$  denotes the parameter in  $[0, +\infty]$  and the lengths  $l_\delta^r$  are defined as in the proof of Proposition 5.2.18 in order for them to define a 2-colored metric ribbon tree.

In particular, we have explicit formulae depending on  $\delta$  for the resulting edges in the glued tree.

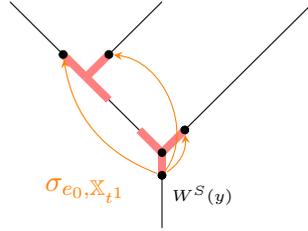


Figure 9.5: Representation of the map  $\sigma_{e_0, X_{t1}}$ .

### 9.6 Proof of Theorem 7.4.5

#### 9.6.1 The moduli spaces $\mathcal{H}(y; x)$

Consider three perturbation data on  $\mathcal{CT}_1 := \{+\}$ ,  $\mathbb{Y}_+^{fg}$ ,  $\mathbb{Y}_+^{gf}$  and  $\mathbb{Y}_+^{ff}$ . For the sake of readability, we will write  $\mathbb{Y}^{ij} := \mathbb{Y}_+^{ij}$  in the rest of this section. They define chain maps

$$\mu^{\mathbb{Y}^{ij}} : (C^*(i), \partial_{Morse}^{Tw}) \longrightarrow (C^*(j), \partial_{Morse}) .$$

Note also that the choices of perturbation data  $\mathbb{X}^f$  and  $\mathbb{X}^g$  are not necessary for this construction.

Begin by considering the moduli space of metric trees  $\mathcal{H}$ , represented in two equivalent ways in Figure 9.6. Adapting the discussions of Section 6.2, we infer without difficulty the notion of *smooth choice of perturbation data on  $\mathcal{H}$* . Given such a choice of perturbation data  $\mathbb{W}$ , we then say that it is consistent with the  $\mathbb{Y}^{ij}$  if it is such that, when  $l \rightarrow 0$ ,  $\lim(\mathbb{W}) = \mathbb{Y}^{ff}$ , and when  $l \rightarrow +\infty$ , the limit  $\lim(\mathbb{W})$  on the above part of the broken tree is  $\mathbb{Y}^{fg}$  and the limit  $\lim(\mathbb{W})$  on the bottom part of the broken tree is  $\mathbb{Y}^{gf}$ .



Figure 9.6: The moduli space  $\mathcal{H}$

**Definition 9.6.1.** For  $x$  and  $y$  two critical points of the Morse function  $f$ , define  $\mathcal{H}^{\mathbb{W}}(y; x)$  to be the moduli space of perturbed Morse gradient trees modeled on  $\frac{+}{+}$ , and such that the two external edges correspond to perturbed Morse trajectories for  $f$ , and the internal edge corresponds to a perturbed Morse trajectory for  $g$ .

We then check that a generic choice of perturbation data  $\mathbb{W}$  makes them into orientable manifolds of dimension

$$\dim(\mathcal{H}^{\mathbb{W}}(y; x)) = |y| - |x| + 1 .$$

The 1-dimensional moduli spaces  $\mathcal{H}(y; x)$  can be compactified into compact manifolds with boundary  $\overline{\mathcal{H}}(y; x)$ , whose boundary is given by the three following phenomena:

- (i) an external edge breaks at a critical point of  $f$  ;
- (ii) the length of the internal edge tends towards 0: this yields the moduli spaces

$$\mathcal{CT}^{\mathbb{Y}^{ff}}(y; x) ;$$

(iii) the internal edge breaks at a critical point of  $g$ : this yields the moduli spaces

$$\bigcup_{z \in \text{Crit}(g)} \mathcal{CT}^{\mathbb{Y}^{gf}}(y; z) \times \mathcal{CT}^{\mathbb{Y}^{fg}}(z; x) .$$

### 9.6.2 Proof of Theorem 7.4.5

Introduce now the degree -1 map  $h : C^*(f) \rightarrow C^*(f)$  defined as

$$h(x) := \sum_{|y|=|x|-1} \#\mathcal{H}^{\mathbb{W}}(y; x) \cdot y .$$

**Lemma 9.6.2.** *The map  $h$  defines an homotopy between  $(-1)^d \mu^{\mathbb{Y}^{gf}} \circ \mu^{\mathbb{Y}^{fg}}$  and  $\mu^{\mathbb{Y}^{ff}}$  i.e. is such that*

$$(-1)^d \mu^{\mathbb{Y}^{gf}} \circ \mu^{\mathbb{Y}^{fg}} - \mu^{\mathbb{Y}^{ff}} = \partial_{\text{Morse}} h + h \partial_{\text{Morse}}^{\text{Tw}} .$$

*Proof.* We define the moduli space  $\mathcal{H}(y; x)$  as before, by introducing the map

$$\phi_{\mathbb{W}} : \mathcal{H} \times W^S(y) \times W^U(x) \longrightarrow M \times M ,$$

and setting  $\mathcal{H}(y; x) := \phi^{-1}(\Delta)$  where  $\Delta$  is the diagonal of  $M \times M$ . We recall moreover that  $\sigma(+; y; x) = d(1 + |y|)$ ,  $\sigma(y; x) = 1$  and that

$$\mu^{\mathbb{Y}^{ij}}(x) = \sum_{|y|=|x|} \#\widetilde{\mathcal{CT}}_+^{\mathbb{Y}^{ij}}(y; x) \cdot y \quad \partial_{\text{Morse}}(x) = \sum_{|y|=|x|+1} \#\widetilde{\mathcal{T}}(y; x) \cdot y .$$

We then set

$$\sigma(\frac{\dagger}{\dagger}; y; x) = (d+1)|y| ,$$

and write  $\widetilde{\mathcal{H}}(y; x)$  for the moduli space  $\mathcal{H}(y; x)$  endowed with the orientation obtained by twisting its natural orientation by a sign of parity  $\sigma(\frac{\dagger}{\dagger}; y; x)$ . We can now define the map  $h : C^*(f) \rightarrow C^*(f)$  by

$$h(x) := \sum_{|y|=|x|-1} \#\widetilde{\mathcal{H}}(y; x) \cdot y .$$

If  $\widetilde{\mathcal{H}}(y; x)$  is 1-dimensional, its boundary decomposes as the disjoint union of the following four types of components

$$\begin{array}{ll} (-1)^{|y|+d} \widetilde{\mathcal{CT}}^{\mathbb{Y}^{gf}}(y; z) \times \widetilde{\mathcal{CT}}^{\mathbb{Y}^{fg}}(z; x) & (-1)^{|y|+1} \widetilde{\mathcal{CT}}^{\mathbb{Y}^{ff}}(y; x) \\ (-1)^{|y|+1} \widetilde{\mathcal{T}}(y; z) \times \widetilde{\mathcal{H}}(z; x) & (-1)^{|y|+1+(d+1)|x|} \widetilde{\mathcal{H}}(y; z) \times \widetilde{\mathcal{T}}(z; x) . \end{array}$$

Counting the boundary points of these 1-dimensional moduli spaces implies that

$$(-1)^d \mu^{\mathbb{Y}^{gf}} \circ \mu^{\mathbb{Y}^{fg}} - \mu^{\mathbb{Y}^{ff}} = \partial_{\text{Morse}} h + h \partial_{\text{Morse}}^{\text{Tw}} . \quad \square$$

Theorem 7.4.5 is then a simple corollary to the relation of Lemma 9.6.2, as it descends in cohomology to the relation

$$(-1)^d [\mu^{\mathbb{Y}^{gf}}] \circ [\mu^{\mathbb{Y}^{fg}}] = [\mu^{\mathbb{Y}^{ff}}] .$$

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