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ENTROPY SOLUTIONS FOR ANISOTROPIC UNILATERAL ELLIPTIC PROBLEM WITH NEUMANN BOUNDARY CONDITIONS

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Abstract. We consider the following strongly nonlinear Neumann elliptic problem:

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + H(x,u,\nabla u) + |u|^{p_0-2}u = f(x) + \sum_{i=1}^{N} D^i \phi_i(x,u) & \text{in } \Omega, \\ \sum_{i=1}^{N} (a_i(x,u,\nabla u) - \phi_i(x,u)) \cdot n_i = 0 & \text{on } \partial\Omega, \end{cases}$$

where the Carathéodory functions $a_i(x,s,\xi)$, $H(x,s,\xi)$ and $\phi_i(x,s)$ verify some nonstandard conditions. By applying an approximation method, we prove the existence of entropy solutions for the unilateral problem with L^1-data, and we conclude some regularity results.

Keywords: anisotropic Sobolev space; weak solution; entropy solution; strongly nonlinear problem; Neumann boundary condition; unilateral problem

MSC 2020: 35J60, 35D35

1. Introduction

Let Ω be an open bounded set with Lipschitz boundary $\partial\Omega$. For the class of noncoercive elliptic problems in the classical Sobolev spaces, Boccardo et al. have studied in [11] the degenerate elliptic equation

$$\begin{cases} -\operatorname{div}(A(x,u)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the function f belongs to $L^m(\Omega)$ for $m \geqslant 1$. They have proved the existence of solutions and conclude some regularity results. In [3], Alvino et al. have studied

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the noncoercive elliptic equation

$$
\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}
$$

where $0 \leqslant \theta < 1$; they have proved several existence and regularity results for the solutions. Akdim et al. have shown in [2] the existence of entropy solutions for the unilateral problem associated to the strongly nonlinear and noncoercive elliptic equation of the type

$$
\begin{cases} -\operatorname{div}(b(|u|)|\nabla u|^{p-2}\nabla u) + d(|u|)|\nabla u|^p = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}
$$

where $f(x,s)$ is a Carathéodory function and satisfies only some growth conditions.

Recently, the anisotropic Sobolev spaces has attracted the attention of many scientists and researchers, the impulse for this mainly comes from their important applications in modelling real-world problems in electrorheological, magnetorheological fluids; we refer the reader to [12], [18], [5], [15], [14], [8], [19], [13] for more details.

In [19], Salmani et al. have studied the existence of entropy solutions for a class of unilateral problem associated to the nonlinear anisotropic elliptic equation of the type

$$
\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i} a_i(x,u,\nabla u) = \mu + \sum_{i=1}^{N} \partial_i \phi_i(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}
$$

where $Au = -\sum\limits_{i=1}^{N} \partial_{x_i} a_i(x,u,\nabla u)$ is a Leray-Lions operator with $\phi_i \in C^0(\mathbb{R},\mathbb{R})$ for $i = 1,\dots,N$ and μ belongs to $L^1(\Omega) + W^{-1,\vec{p}'}(\Omega)$. Benaichouche et al. in [6] have proved the existence and the regularity of solutions for the unilateral problem associated to the degenerate anisotropic elliptic equation

$$
\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i\left(\dfrac{a_i(x,\nabla u)}{(1+|u|)^{\theta(p_i-1)}}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}
$$

where $0 \leqslant \theta < 1$ and $f \in L^m(\Omega)$ for $m \geqslant 1$ and $a_i(x,s,\xi)$ is a Carathéodory function that verifies the growth, the monotonicity and the coercivity conditions. Benboubker et al. have studied in [7] the existence of renormalized solutions for the noncoercive strongly nonlinear Neumann problem

$$\text{(1.6)}\qquad \begin{cases} -\displaystyle\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + |u|^{p_0-2}u = f(x,u,\nabla u) & \text{in } \Omega,\\ \displaystyle\sum_{i=1}^{N} a_i(x,u,\nabla u)\eta_i = g(x) & \text{on } \partial\Omega, \end{cases}$$

where the Carathéodory function $f(x,u,\nabla u)$ satisfies a growth condition and g belongs to $L^1(\partial\Omega)$.

The aim of this paper is to prove the existence and regularity of entropy solutions for the unilateral problem associated to the strongly nonlinear and noncoercive Neumann elliptic equation

$$\text{(1.7)}\qquad \begin{cases} Au + H(x,u,\nabla u) + |u|^{p_0-2}u = f(x) + \displaystyle\sum_{i=1}^{N} \operatorname{div}(\phi_i(x,u)) & \text{in } \Omega,\\ \displaystyle\sum_{i=1}^{N} (a_i(x,u,\nabla u) - \phi_i(x,u)) \cdot n_i = 0 & \text{on } \partial\Omega, \end{cases}$$

where A is a noncoercive Leray-Lions operator acting from $W^{1,\vec{p}}(\Omega)$ into its dual $(W^{1,\vec{p}}(\Omega))'$ and the Carathéodory functions $H(x,s,\xi)$ and $\phi_i(x,s)$ verify some growth conditions. We will prove the existence of solutions for the unilateral problem in the cases of L^∞-data and L^1-data.

This paper is structured as follows: Section 2 is devoted to recall some definitions and properties concerning the anisotropic Sobolev spaces. In Section 3, we present the assumptions on the Carathéodory functions $a_i(x,s,\xi)$, $H(x,s,\xi)$ and $\phi_i(x,s)$ for all $i = \{1,\dots,N\}$ under which our problem has at least one entropy solution. In Section 4, we study the existence of weak solutions for the unilateral problem associated to our equation with L^∞-data, then we prove the existence of entropy solutions for the unilateral problem associated to the elliptic equation (1.7) with L^1-data.

2. Preliminaries

Let Ω be an open bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ with smooth boundary $\partial\Omega$. We set $p_1,\dots,p_N$ be N real constant numbers with $1 < p_i < \infty$ for $i = 1,\dots,N$, and we denote

$$\vec{p} = (1, p_1, \dots, p_N) \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for } i = 1,\dots,N.$$

We set

$$\underline{p} = \min\{p_1, p_2, \dots, p_N\} \quad \text{and} \quad p_0 = \max\{p_1, p_2, \dots, p_N\}.$$

We define the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ as

$$W^{1,\vec{p}}(\Omega) = \{u \in W^{1,1}(\Omega) \text{ such that } D^i u \in L^{p_i}(\Omega) \text{ for } i = 1, 2, \dots, N\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}} = \|u\|_{1,1} + \sum_{i=1}^{N} \|D^i u\|_{L^{p_i}(\Omega)}. \tag{2.1}$$

The space $(W^{1,\vec{p}}(\Omega), \|\cdot\|_{1,\vec{p}})$ is a separable and reflexive Banach space (cf. [18]).

Proposition 2.1 (cf. [16], [20]). *Let $u \in W^{1,\vec{p}}(\Omega)$. We have*

(i) *Poincaré Wirtinger inequality: there exists a constant $C_p > 0$ such that*

$$\|u - \operatorname{med}(u)\|_{L^{p_i}(\Omega)} \leqslant C_p \|D^i u\|_{L^{p_i}(\Omega)} \quad \textit{for any } i = 1, \dots, N$$

with

$$\operatorname{med}(u) = \frac{1}{|\Omega|} \int_\Omega |u| \, \mathrm{d}x,$$

(ii) *Sobolev inequality: there exists another constant $C_s > 0$ such that*

$$\|u - \operatorname{med}(u)\|_{L^q(\Omega)} \leqslant \frac{C_s}{N} \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)},$$

where

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} \quad \textit{and} \quad \begin{cases} q = \overline{p}^* = \dfrac{N\overline{p}}{N - \overline{p}} & \textit{if } \overline{p} < N, \\ q \in [1, \infty[& \textit{if } \overline{p} \geqslant N. \end{cases}$$

Lemma 2.1 (cf. [17]). *Let Ω be a bounded open set in $\mathbb{R}^N$ $(N \geqslant 2)$. Then we have the following embeddings:*

- ▷ *if $\underline{p} < N$, then the embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [\underline{p}, \overline{p}^*[$,*
- ▷ *if $\underline{p} = N$, then the embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [\underline{p}, \infty[$,*
- ▷ *if $\underline{p} > N$, then the embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$ is compact.*

The proof of Lemma 2.1 follows from the fact that the embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow W^{1,\underline{p}}(\Omega)$ is continuous, and from the compact embedding theorem in classical Sobolev spaces.

Definition 2.1. Let $k > 0$. We consider the truncation function $T_k(\cdot)\colon \mathbb{R} \mapsto \mathbb{R}$ given by

$$T_k(s) = \begin{cases} s & \text{if } |s| \leqslant k, \\ k\dfrac{s}{|s|} & \text{if } |s| > k, \end{cases}$$

and we define

$$\mathcal{T}^{1,\vec{p}}(\Omega) := \{u\colon \Omega \mapsto \mathbb{R} \text{ measurable, such that } T_k(u) \in W^{1,\vec{p}}(\Omega) \text{ for any } k > 0\}.$$

Proposition 2.2. *Let $u \in \mathcal{T}^{1,\vec{p}}(\Omega)$. For any $i \in \{1, \dots, N\}$ there exists a unique measurable function such that*

$$\text{for all } k > 0 \qquad D^i T_k(u) = v_i.\chi_{\{|u|<k\}} \quad \text{a.e., } x \in \Omega,$$

where χ_A denotes the characteristic function of a measurable set A. The functions v_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if u belongs to $W^{1,1}(\Omega)$, then v_i coincides with the standard distributional derivative of u, that is, $v_i = D^i u$.

The proof of Proposition 2.2 follows the usual techniques developed in [10] for the case of Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [14], [15], [5], [9].

Now, we introduce the set $\mathcal{T}^{1,\vec{p}}_{\text{tr}}(\Omega)$ as a subset of $\mathcal{T}^{1,\vec{p}}(\Omega)$ for which a generalized notion of trace may be defined (see also [4] for the case of constant exponent). More precisely, $\mathcal{T}^{1,\vec{p}}_{\text{tr}}(\Omega)$ is the set of functions u in $\mathcal{T}^{1,\vec{p}}(\Omega)$ such that there exists a sequence $(u_n)_n$ in $W^{1,\vec{p}}(\Omega)$ and a measurable function v on $\partial\Omega$ that verify

(a) $u_n \to u$ a.e., in Ω,
(b) $D^i T_k(u_n) \to D^i T_k(u)$ strongly in $L^1(\Omega)$ for every $k > 0$ and for $i = 1, \dots, N$,
(c) $u_n \to v$ a.e., on $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [4]. Let $u \in W^{1,\vec{p}}(\Omega)$. The trace of u on $\partial\Omega$ will be denoted by $\tau(u)$.

For any $u \in \mathcal{T}^{1,\vec{p}}_{\text{tr}}(\Omega)$, the trace of u on $\partial\Omega$ will be denoted by $\text{tr}(u)$ or u, the operator $\text{tr}(\cdot)$ satisfies the following properties:

(i) if $u \in \mathcal{T}^{1,\vec{p}}_{\text{tr}}(\Omega)$, then $\tau(T_k(u)) = T_k(\text{tr}(u))$ for any $k > 0$,
(ii) if $\varphi \in W^{1,\vec{p}}(\Omega)$, then for any $u \in \mathcal{T}^{1,\vec{p}}_{\text{tr}}(\Omega)$ we have $u - \varphi \in \mathcal{T}^{1,\vec{p}}_{\text{tr}}(\Omega)$ and $\text{tr}(u - \varphi) = \text{tr}(u) - \tau(\varphi)$.

In the case where $u \in W^{1,\vec{p}}(\Omega)$, $\mathrm{tr}(u)$ coincides with $\tau(u)$. Obviously, we have

$$W^{1,\vec{p}}(\Omega) \subset \mathcal{T}^{1,\vec{p}}_{\mathrm{tr}}(\Omega) \subset \mathcal{T}^{1,\vec{p}}(\Omega).$$

Lemma 2.2 (see [1], Lemma 2.1). *Let $g \in L^p(\Omega)$ and let $(g_n)_n$ be a sequence uniformly bounded in $L^p(\Omega)$.*

If $g_n \to g$ almost everywhere in Ω, then $g_n \rightharpoonup g$ weakly in $L^p(\Omega)$.

3. Essential assumptions

Let ψ be a measurable function in Ω with values in $\overline{\mathbb{R}}$ such that $\psi^+ \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ and the closed set

$$K_\psi = \{u \in W^{1,\vec{p}}(\Omega),\ u \geqslant \psi \text{ a.e., in } \Omega\}. \tag{3.1}$$

Note that the set K_ψ has a non empty intersection with $L^\infty(\Omega)$ (since $\psi^+ \in K_\psi \cap L^\infty(\Omega)$).

Let $f \in L^1(\Omega)$. We consider the strongly nonlinear and noncoercive elliptic equation

$$\begin{cases} Au + H(x,u,\nabla u) + |u|^{p_0-2}u = f(x) - \displaystyle\sum_{i=1}^N D^i \phi_i(x,u) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^N (a_i(x,u,\nabla u) - \phi_i(x,u)) \cdot n_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $Au = -\sum_{i=1}^N D^i a_i(x,u,\nabla u)$ is a Leray-Lions operator acting from $W^{1,\vec{p}}(\Omega)$ into its dual, where $a_i(x,s,\xi)\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ are Carathéodory functions for $i = 1,\dots,N$ (measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω), which satisfies the following conditions:

$$(a_i(x,s,\xi) - a_i(x,s,\xi'))(\xi_i - \xi_i') > 0 \quad \text{for } \xi_i \neq \xi_i', \tag{3.3}$$

$$|a_i(x,s,\xi)| \leqslant \beta(K_i(x) + |s|^{p_i-1} + |\xi_i|^{p_i-1}), \tag{3.4}$$

with $\beta > 0$ and the nonnegative function $K_i(\cdot)$ assumed to be in $L^{p_i'}(\Omega)$ for $i = 1,\dots,N$,

$$a_i(x,s,\xi)\xi_i \geqslant b(|s|)|\xi_i|^{p_i} \quad \text{with} \quad \frac{b_0}{(1+|s|)^\lambda} \leqslant b(|s|) \quad \text{for any } s \in \mathbb{R}, \tag{3.5}$$

with $b_0 > 0$ and $0 < \lambda < \min(1, \underline{p} - 1, 1/(p_0 - 1))$.

As a consequence of (3.5) and the continuity of the function $a_i(x, s, \cdot)$ with respect to ξ, we have

$$a_i(x, s, 0) = 0.$$

The lower order term $H(x, s, \xi)\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}^N$ is a Carathéodory function that satisfies only the growth condition

$$|H(x, s, \xi)| \leqslant f_0(x) + \sum_{i=1}^{N} d(|s|)|\xi_i|^{p_i}, \tag{3.6}$$

where $f_0(x)$ is assumed to be a positive measurable function in $L^1(\Omega)$ and $d(|\cdot|)\colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a continuous decreasing function such that $d(|\cdot|)/b(|\cdot|) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

The right-hand side $\phi_i(x, s)\colon \Omega \times \mathbb{R} \mapsto \mathbb{R}$ are Carathéodory function that verifies

$$|\phi_i(x, s)| \leqslant c_i(x)(1 + |s|)^{\gamma_i}, \tag{3.7}$$

with $0 \leqslant \gamma_i < (p_0 - 1)/p_i' - \lambda/p_i$ and the positive measurable functions $c_i(x)$ belonging to $L^{r_i}(\Omega)$ with $r_i \geqslant p_i'(p_0 - 1)/(p_0 - (\lambda + \gamma_i)p_i' + \lambda)$ for $i = 1, \ldots, N$.

4. Main result

Definition 4.1. A measurable function u is an entropy solution for the unilateral problem associated to the strongly nonlinear elliptic equation (4.3) if $u \geqslant \psi$ a.e., in Ω, and $T_k(u) \in K_\psi$ for any $k > 0$, with $|u|^{p_0-1} \in L^1(\Omega)$ and $H(x, u, \nabla u) \in L^1(\Omega)$ such that u verifies the inequality

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) D^i T_k(u - v)\,\mathrm{d}x &+ \int_\Omega H(x, u, \nabla u) T_k(u - v)\,\mathrm{d}x \\ &+ \int_\Omega |u|^{p_0-2} u T_k(u - v)\,\mathrm{d}x \\ &\leqslant \int_\Omega f(x) T_k(u - v)\,\mathrm{d}x + \sum_{i=1}^{N} \int_\Omega \phi_i(x, u) D^i T_k(u - v)\,\mathrm{d}x \end{aligned} \tag{4.1}$$

for any $v \in K_\psi$ and $k > 0$.

Theorem 4.1. *Assume that* (3.3)–(3.5) *and* (3.6)–(3.7) *hold true. Then there exists at least one entropy solution u for the unilateral problem associated to the strongly nonlinear elliptic problem* (3.2).

Now, we are going to recall the following technical lemma which is useful to prove our main results:

Lemma 4.1. *Let $k > 0$, assume that (3.3)–(3.5) hold true, and let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $W^{1,\vec{p}}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,\vec{p}}(\Omega)$, and*

$$\tag{4.2} \begin{aligned} &\int_\Omega (|u_n|^{\underline{p}-2}u_n - |u|^{\underline{p}-2}u)(u_n - u)\,\mathrm{d}x \\ &\quad + \sum_{i=1}^N \int_\Omega (a_i(x, T_k(u_n), \nabla u_n) - a_i(x, T_k(u_n), \nabla u))(D^i u_n - D^i u)\,\mathrm{d}x \to 0 \end{aligned}$$

as $n \to \infty$.

Then $u_n \to u$ strongly in $W^{1,\vec{p}}(\Omega)$ for a subsequence.

Before proving Theorem 4.1, we will need to show the existence of a weak solution for the approximate problem with L^∞-data.

4.1. Existence of weak solutions for an approximate problem with L^∞-data. We consider the strongly nonlinear anisotropic elliptic problem

$$\tag{4.3} \begin{cases} -\displaystyle\sum_{i=1}^N D^i a(x, T_n(u), \nabla u) + H_n(x, u, \nabla u) + |u|^{p_0-2}u & \\ = F(x) - \displaystyle\sum_{i=1}^N D^i \phi_{i,n}(x, u) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^N (a_i(x, u, \nabla u) - \phi_{i,n}(x, u)) \cdot n_i = 0 & \text{on } \partial\Omega, \end{cases}$$

with $F(x) \in L^\infty(\Omega)$, $H_n(x, s, \xi) = T_n(H(x, s, \xi))$ and $\phi_{i,n}(x, u) = T_n(\phi_i(x, u))$ for $i = 1, \ldots, N$.

Definition 4.2. A measurable function u is a weak solution for the unilateral problem associated to the strongly nonlinear elliptic equation (4.3) if $u \in K_\psi$, $|u|^{p_0} \in L^1(\Omega)$, $H_n(x, u, \nabla u) \in L^1(\Omega)$ and u verifies the inequality

$$\tag{4.4} \begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x, T_n(u), \nabla u) D^i(u - v)\,\mathrm{d}x + \int_\Omega |u|^{p_0-2}u(u - v)\,\mathrm{d}x \\ &\quad + \int_\Omega H_n(x, u, \nabla u)(u - v)\,\mathrm{d}x \\ &\leqslant \int_\Omega F(x)(u - v)\,\mathrm{d}x + \sum_{i=1}^N \int_\Omega \phi_{i,n}(x, u) D^i(u - v)\,\mathrm{d}x \end{aligned}$$

for any $v \in K_\psi$.

Theorem 4.2. *Assume that* (3.3)–(3.5) *hold true. Then there exists at least one weak solution u for the unilateral problem associated to the strongly nonlinear elliptic equation* (4.3).

4.1.1. Proof of Theorem 4.2.

Step 1: Approximate problem. We consider the sequence of approximate problem

$$
(4.5)\qquad \begin{cases} A_m u_m + H_n(x, u_m, \nabla u_m) + \dfrac{1}{m}|u_m|^{p_- -2}u_m + |T_m(u_m)|^{p_0-2}T_m(u_m) & \\ = F(x) - \displaystyle\sum_{i=1}^{N} D^i \phi_{i,n}(x, u_m) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} (a_i(x, T_n(u_m), \nabla u_m) - \phi_{i,n}(x, u_m)) \cdot n_i = 0 & \text{on } \partial\Omega, \end{cases}
$$

where A_m is a Leray-Lions operator acting from $W^{1,\vec{p}}(\Omega)$ into its dual, given by

$$
(4.6)\qquad \begin{aligned} \langle A_m u, v\rangle = {} & \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u) D^i v \,\mathrm{d}x + \frac{1}{m}\int_\Omega |u|^{\underline{p}-2} uv \,\mathrm{d}x \\ & + \int_\Omega |T_m(u)|^{p_0-2} T_m(u) v \,\mathrm{d}x \end{aligned}
$$

for any $u, v \in W^{1,\vec{p}}(\Omega)$. We define the operator G_m from $W^{1,\vec{p}}(\Omega)$ into its dual, given by

$$
(4.7)\qquad \langle G_m u, v\rangle = \int_\Omega H_n(x, u, \nabla u) v \,\mathrm{d}x - \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(x, u) D^i v \,\mathrm{d}x
$$

for any $u, v \in W^{1,\vec{p}}(\Omega)$.

Lemma 4.2. *The bounded operator $B_m = A_m + G_m$ acting from $W^{1,\vec{p}}(\Omega)$ into its dual $(W^{1,\vec{p}}(\Omega))'$ is pseudo-monotone and coercive in the following sense: there exists $v_0 \in K_\psi$ such that*

$$
(4.8)\qquad \frac{\langle B_m v, v - v_0\rangle}{\|v\|_{W^{1,\vec{p}}(\Omega)}} \to \infty \quad \textit{as} \quad \|v\|_{W^{1,\vec{p}}(\Omega)} \to \infty \quad \textit{for } v \in K_\psi.
$$

For the proof of Lemma 4.2, see Appendix.

In view of Lemma 4.2, there exists at least one weak solution $u_m \in K_\psi$ for the unilateral problem associated to the elliptic equation (4.5), i.e.,

$$\begin{aligned}(4.9)\quad &\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_m), \nabla u_m) D^i(u_m - v)\,\mathrm{d}x + \int_\Omega H_n(x, u_m, \nabla u_m)(u_m - v)\,\mathrm{d}x \\ &\quad + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2} u_m (u_m - v)\,\mathrm{d}x + \int_\Omega |T(u_m)|^{p_0-2} T_m(u_m)(u_m - v)\,\mathrm{d}x \\ &\quad \leqslant \int_\Omega F(x)(u_m - v)\,\mathrm{d}x + \sum_{i=1}^N \int_\Omega \phi_{i,n}(x, u_m) D^i(u_m - v)\,\mathrm{d}x\end{aligned}$$

for any $v \in K_\psi \cap W^{1,\vec{p}}(\Omega)$.

Step 2: A priori estimates. We set $V_m = u_m - \mu(u_m - \psi^+)$ with μ being a positive constant small enough. Thus, $V_m \in K_\psi \cap W^{1,\vec{p}}(\Omega)$ is an admissible test function for the approximate problem (4.5), and we obtain

$$\begin{aligned}(4.10)\qquad &\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_m), \nabla u_m) D^i(u_m - \psi^+)\,\mathrm{d}x \\ &\quad + \int_\Omega H_n(x, u_m, \nabla u_m)(u_m - \psi^+)\,\mathrm{d}x \\ &\quad + \int_\Omega |T_m(u_m)|^{p_0-2} T_m(u_m)(u_m - \psi^+)\,\mathrm{d}x \\ &\quad + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2} u_m(u_m - \psi^+)\,\mathrm{d}x \\ &\leqslant \int_\Omega F(x)(u_m - \psi^+)\,\mathrm{d}x + \sum_{i=1}^N \int_\Omega \phi_{i,n}(x, u_m) D^i(u_m - \psi^+)\,\mathrm{d}x.\end{aligned}$$

Since $u_m - \psi^+$ has the same sign as u_n and thanks to (3.5), we obtain

$$\begin{aligned}(4.11)\quad &\sum_{i=1}^N \int_\Omega b(|T_n(u_m)|)|D^i u_m|^{p_i}\,\mathrm{d}x + \int_\Omega |T_m(u_m)|^{p_0-1}|u_m - \psi^+|\,\mathrm{d}x \\ &\quad + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}-1}|u_m - \psi^+|\,\mathrm{d}x \\ &\leqslant \int_\Omega |F|\,|u_m - \psi^+|\,\mathrm{d}x + n\sum_{i=1}^N \int_\Omega (|D^i u_m| + |D^i\psi^+|)\,\mathrm{d}x \\ &\quad + \sum_{i=1}^N \int_\Omega |a_i(x, T_n(u_m), \nabla u_m)|\,|D^i\psi^+|\,\mathrm{d}x + n\int_\Omega |u_m - \psi^+|\,\mathrm{d}x.\end{aligned}$$

We have $b_0/(1+n)^\lambda \leqslant b(|T_n(u_m)|) \leqslant b_0$, and using Young's inequality we deduce that

$$(4.12)\quad \sum_{i=1}^N \int_\Omega b(|T_n(u_m)|)|D^i u_m|^{p_i}\,\mathrm{d}x + \int_\Omega |T_m(u_m)|^{p_0-1}|u_m-\psi^+|\,\mathrm{d}x + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}-1}|u_m-\psi^+|\,\mathrm{d}x$$
$$\leqslant C_0 + (\|F\|_{L^\infty(\Omega)}+n)\int_\Omega |u_m-\psi^+|\,\mathrm{d}x + \frac{1}{2}\sum_{i=1}^N \int_\Omega b(|T_n(u_m)|)|D^i u_m|^{p_i}\,\mathrm{d}x + \frac{1}{2}\sum_{i=1}^N \int_\Omega b(|T_n(u_m)|)|D^i \psi^+|^{p_i}\,\mathrm{d}x + \sum_{i=1}^N \int_\Omega |a_i(x,T_n(u_m),\nabla u_m)||D^i\psi^+|\,\mathrm{d}x$$

with C_0 being a positive constant depending on n and not depending on m. For the last term on the right-hand side of (4.12) we have

$$(4.13)\quad \sum_{i=1}^N \int_\Omega |a_i(x,T_n(u_m),\nabla u_m)|\,|D^i\psi^+|\,\mathrm{d}x$$
$$\leqslant \beta\sum_{i=1}^N \int_\Omega (|K_i(x)| + n^{p_i-1} + |\nabla u_m|^{p_i-1})|D^i\psi^+|\,\mathrm{d}x$$
$$\leqslant C_2\sum_{i=1}^N \int_\Omega |K_i(x)|^{p_i'}\,\mathrm{d}x + C_3\sum_{i=1}^N \int_\Omega |D^i\psi^+|^{p_i}\,\mathrm{d}x + C_4\int_\Omega n^{p_0}\,\mathrm{d}x + \frac{1}{4}\sum_{i=1}^N \int_\Omega b(|T_n(u_m)|)|D^i u_m|^{p_i}\,\mathrm{d}x$$
$$\leqslant \frac{1}{4}\sum_{i=1}^N \int_\Omega b(|T_n(u_m)|)|D^i u_m|^{p_i}\,\mathrm{d}x + C_5.$$

By combining (4.12)–(4.13), it follows that

$$(4.14)\quad \frac{b_0}{4(1+n)^\lambda}\sum_{i=1}^N \int_\Omega |D^i u_m|^{p_i}\,\mathrm{d}x + \int_\Omega |T_m(u_m)|^{p_0-1}|u_m-\psi^+|\,\mathrm{d}x + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}-1}|u_m-\psi^+|\,\mathrm{d}x \leqslant C_6.$$

Thus, we conclude that

$$(4.15)\quad \sum_{i=1}^N \int_\Omega |D^i u_m|^{p_i}\,\mathrm{d}x + \int_\Omega |T_m(u_m)|^{p_0-1}|u_m-\psi^+|\,\mathrm{d}x \leqslant C_7.$$

Moreover, we have $u_m \geqslant \psi^+$, and by using the Young's inequality we obtain

$$
\begin{aligned}
(4.16)\qquad & \int_\Omega |T_m(u_m)|^{p_0-1}|u_m-\psi^+|\,\mathrm{d}x \\
&\geqslant \int_\Omega |T_m(u_m)|^{p_0}\,\mathrm{d}x - \int_\Omega |T_m(u_m)|^{p_0-1}|\psi^+|\,\mathrm{d}x \\
&\geqslant \int_\Omega |T_m(u_m)|^{p_0}\,\mathrm{d}x - \frac{1}{p_0'}\int_\Omega |T_m(u_m)|^{p_0}\,\mathrm{d}x - \frac{1}{p_0}\int_\Omega |\psi^+|^{p_0}\,\mathrm{d}x \\
&\geqslant \frac{1}{p_0}\int_\Omega |T_m(u_m)|^{p_0}\,\mathrm{d}x - \frac{1}{p_0}\int_\Omega |\psi^+|^{p_0}\,\mathrm{d}x.
\end{aligned}
$$

Having in mind (4.15), we deduce that

$$
(4.17)\qquad \sum_{i=1}^N \int_\Omega |D^i u_m|^{p_i}\,\mathrm{d}x + \int_\Omega |T_m(u_m)|^{p_0}\,\mathrm{d}x \leqslant C_8,
$$

where C_8 is a positive constant that does not depend on m. Moreover, we obtain

$$
\begin{aligned}
(4.18)\qquad \|T_m(u_m)\|_{1,\vec{p}} &= \|T_m(u_m)\|_{1,1} + \sum_{i=1}^N \|D^i T_m(u_m)\|_{L^{p_i}(\Omega)} \\
&\leqslant \|T_m(u_m)\|_{L^1(\Omega)} + \sum_{i=1}^N \int_\Omega |D^i T_m(u_m)|\,\mathrm{d}x \\
&\quad + \sum_{i=1}^N \Big(\int_\Omega |D^i T_m(u_m)|^{p_i}\,\mathrm{d}x\Big)^{1/p_i} \\
&\leqslant \int_\Omega |T_m(u_m)|^{p_0}\,\mathrm{d}x + 2\sum_{i=1}^N \int_\Omega |D^i u_m|^{p_i}\,\mathrm{d}x \\
&\quad + N + (N+1)\,\mathrm{meas}(\Omega) \\
&\leqslant C_9,
\end{aligned}
$$

where C_9 is a constant that does not depend on m. Thus, the sequence $(u_m)_m$ is uniformly bounded in $W^{1,\vec{p}}(\Omega)$, and we obtain

$$
(4.19)\qquad \begin{cases} u_m \rightharpoonup u & \text{weakly in } W^{1,\vec{p}}(\Omega), \\ u_m \to u & \text{strongly in } L^{\underline{p}}(\Omega) \quad \text{and} \quad \text{a.e., in } \Omega, \\ u_m \rightharpoonup u & \text{weakly in } L^1(\partial\Omega) \quad \text{and} \quad \text{a.e., in } \partial\Omega. \end{cases}
$$

Thus, we obtain

$$
(4.20)\qquad \frac{1}{m}|u_m|^{\underline{p}-2}u_m \to 0 \quad \text{strongly in} \quad L^{\underline{p}'}(\Omega).
$$

Furthermore, using the fact that $u_m \to u$ almost everywhere in Ω and $\phi_{i,n}(x,u_m) \to \phi_{i,n}(x,u)$ almost everywhere in Ω and $|\phi_{i,n}(x,u_m)| \leqslant n$ a.e in Ω, in view of the Lebesgue's dominated convergence theorem, we get

$$\phi_{i,n}(x,u_m) \to \phi_{i,n}(x,u) \quad \text{strongly in } L^{p_i}(\Omega) \quad \text{for } i=1,\dots,N. \tag{4.21}$$

Step 3: The convergence of the gradient almost everywhere. Let η be small enough. By taking $v = u_m - \eta(u_m - u) \in K_\psi$ as a test function for the approximate problem (4.5), we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_m),\nabla u_m)(D^i u_m - D^i u)\,\mathrm{d}x + \int_\Omega H_n(x,u_m,\nabla u_m)(u_m-u)\,\mathrm{d}x\\ + \int_\Omega |T_m(u_m)|^{p_0-2}T_m(u_m)(u_m-u)\,\mathrm{d}x + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2}u_m(u_m-u)\,\mathrm{d}x\\ \leqslant \int_\Omega F(u_m-u)\,\mathrm{d}x + \sum_{i=1}^{N}\int_\Omega \phi_{i,n}(x,u_m)(D^i u_m - D^i u)\,\mathrm{d}x.\end{aligned} \tag{4.22}$$

It follows that

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega (a_i(x,T_n(u_m),\nabla u_m) - a_i(x,T_n(u_m),\nabla u))(D^i u_m - D^i u)\,\mathrm{d}x\\ &\quad + \int_\Omega (|T_m(u_m)|^{p_0-2}T_m(u_m) - |T_m(u)|^{p_0-2}T_m(u))(u_m-u)\,\mathrm{d}x\\ &\leqslant \int_\Omega |F|\,|u_m-u|\,\mathrm{d}x + \int_\Omega |H_n(x,u_m,\nabla u_m)|\,|u_m-u|\,\mathrm{d}x\\ &\quad + \sum_{i=1}^{N}\int_\Omega |\phi_{i,n}(x,u_m)|\,|D^i u_m - D^i u|\,\mathrm{d}x + \int_\Omega |T_m(u)|^{p_0-1}|u_m-u|\,\mathrm{d}x\\ &\quad + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}-1}|u_m-u|\,\mathrm{d}x\\ &\quad + \sum_{i=1}^{N}\int_\Omega |a_i(x,T_n(u_m),\nabla u)|\,|D^i u_m - D^i u|\,\mathrm{d}x\\ &\leqslant \|F\|_{L^\infty(\Omega)}\int_\Omega |u_m-u|\,\mathrm{d}x + n\int_\Omega |u_m-u|\,\mathrm{d}x\\ &\quad + n\sum_{i=1}^{N}\int_\Omega |D^i u_m - D^i u|\,\mathrm{d}x + \int_\Omega |T_m(u)|^{p_0-1}|u_m-u|\,\mathrm{d}x\end{aligned} \tag{4.23}$$

$$+ \frac{1}{m}\int_\Omega |u_m|^{\underline{p}-1}|u_m - u|\,\mathrm{d}x$$
$$+ \sum_{i=1}^N \int_\Omega |a_i(x, T_n(u_m), \nabla u)|\,|D^i u_m - D^i u|\,\mathrm{d}x.$$

For the first and second terms on the right-hand side of (4.23), we have $u_m \to u$ strongly in $L^1(\Omega)$. Then

$$\|F\|_{L^\infty(\Omega)} \int_\Omega |u_m - u|\,\mathrm{d}x + n\int_\Omega |u_m - u|\,\mathrm{d}x \to 0 \quad \text{as } m \to \infty. \tag{4.24}$$

Also, we have $n \in L^{p_i'}(\Omega)$ and $D^i u_m \rightharpoonup D^i u$ weakly in $L^{p_i}(\Omega)$. It follows that

$$\sum_{i=1}^N \int_\Omega n|D^i u_m - D^i u|\,\mathrm{d}x \to 0 \quad \text{as } m \to \infty. \tag{4.25}$$

For the fourth term on the the right-hand side of (4.23), we have $u_m \rightharpoonup u$ weakly in $L^{p_0}(\Omega)$ and since $|T_m(u)|^{p_0-1} \in L^{p_0'}(\Omega)$,

$$\int_\Omega |T_m(u)|^{p_0-1}|u_m - u|\,\mathrm{d}x \to 0 \quad \text{as } m \to \infty. \tag{4.26}$$

Moreover, we have $u_m \rightharpoonup u$ weakly in $L^{\underline{p}}(\Omega)$ and thanks to (4.20) we have

$$\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-1}|u_m - u|\,\mathrm{d}x \to 0 \quad \text{as } m \to \infty. \tag{4.27}$$

For the last term on the right-hand side of (4.23) we have $T_n(u_m) \to T_n(u)$ strongly in $L^{p_i}(\Omega)$, then in view of the Lebesgue dominated convergence theorem, $a_i(x, T_n(u_m), \nabla T_k(u)) \to a_i(x, T_n(u_m), \nabla T_k(u))$ strongly in $L^{p_i}(\Omega)$, and since $D^i T_k(u_m) \rightharpoonup D^i T_k(u)$ weakly in $L^{p_i'}(\Omega)$ for $i = 1, \ldots, N$, we obtain

$$\sum_{i=1}^N \int_\Omega |a_i(x, T_n(u_m), \nabla T_k(u))|\,|D^i u_m - D^i u|\,\mathrm{d}x \to 0 \quad \text{as } m \to \infty. \tag{4.28}$$

By combining (4.23) and (4.24)–(4.28), we conclude that

$$\begin{aligned} 0 \leqslant \lim_{m\to\infty}\Bigg(\sum_{i=1}^N \int_\Omega &(a_i(x, T_n(u_m), \nabla u_m) \\ &- a_i(x, T_n(u_m), \nabla u))(D^i u_m - D^i u)\,\mathrm{d}x \\ &+ \int_\Omega (|T_m(u_m)|^{p_0-2}T_m(u_m) - |T_m(u)|^{p_0-2}T_m(u))(u_m - u)\,\mathrm{d}x\Bigg) = 0. \end{aligned} \tag{4.29}$$

In view of Lemma 4.1, we deduce that

$$\begin{cases} u_m \to u & \text{strongly in } W^{1,\vec{p}}(\Omega), \\ D^i u_m \to D^i u & \text{a.e., in } \Omega. \end{cases} \tag{4.30}$$

Therefore, we have $a_i(x, T_n(u_m), \nabla u_m) \to a_i(x, T_n(u), \nabla u)$ and $H_n(x, u_m, \nabla u_m) \to H_n(x, u, \nabla u)$ almost everywhere in Ω, and in view of (2.2) and Lemma 2.2, we obtain

$$a_i(x, T_n(u_m), \nabla u_m) \rightharpoonup a_i(x, T_n(u), \nabla u) \quad \text{weakly in } L^{p_i'}(\Omega) \text{ for } i = 1, \ldots, N \tag{4.31}$$

and

$$H_n(x, u_m, \nabla u_m) \rightharpoonup H_n(x, u, \nabla u) \quad \text{weakly in } L^{\underline{p}'}(\Omega). \tag{4.32}$$

Step 4: Passage to the limit. By taking $v \in K_\psi$ as a test function for the approximate problem (4.5) we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_m), \nabla u_m)(D^i u_m - D^i v)\,\mathrm{d}x \\ &\quad + \int_\Omega H_n(x, u_m, \nabla u_m)(u_m - v)\,\mathrm{d}x \\ &\quad + \int_\Omega |T(u_m)|^{p_0-2} T_m(u_m)(u_m - v)\,\mathrm{d}x + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2} u_m(u_m - v)\,\mathrm{d}x \\ &\leqslant \int_\Omega F(x)(u_m - v)\,\mathrm{d}x + \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(x, u_m)(D^i u_m - D^i v)\,\mathrm{d}x. \end{aligned} \tag{4.33}$$

Firstly, we have $D^i u_m - D^i v \to D^i u - D^i v$ strongly in $L^{p_i}(\Omega)$ and thanks to (4.21) and (4.31) we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_m), \nabla u_m)(D^i u_m - D^i v)\,\mathrm{d}x \\ &\quad \to \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u)(D^i u - D^i v)\,\mathrm{d}x \quad \text{as } m \to \infty \end{aligned} \tag{4.34}$$

and

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \phi_{i,n}(x, u_m)(D^i u_m - D^i v)\,\mathrm{d}x \\ &\quad \to \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(x, u)(D^i u - D^i v)\,\mathrm{d}x \quad \text{as } m \to \infty. \end{aligned} \tag{4.35}$$

Moreover, in view of (4.19) and Fatou's Lemma, we obtain

$$\int_\Omega |u|^{p_0-2}u(u-v)\,\mathrm{d}x \leqslant \liminf_{m\to\infty}\int_\Omega |T_m(u_m)|^{p_0-2}T_m(u_m)(u_m-v)\,\mathrm{d}x \tag{4.36}$$

and we have $u_m - v \to u - v$ strongly in $L^{\underline{p}}(\Omega)$. Thus, thanks to (4.20) we obtain

$$\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2}u_m(u_m-v)\,\mathrm{d}x \to 0 \quad \text{as } m\to\infty. \tag{4.37}$$

Similarly, we have $F \in L^\infty(\Omega)$, then

$$\int_\Omega F(x)(u_m-v)\,\mathrm{d}x \to \int_\Omega F(x)(u-v)\,\mathrm{d}x \quad \text{as } m\to\infty \tag{4.38}$$

and thanks to (4.32) we get

$$\int_\Omega H_n(x,u_m,\nabla u_m)(u_m-v)\,\mathrm{d}x \to \int_\Omega H_n(x,u,\nabla u)(u-v)\,\mathrm{d}x \quad \text{as } m\to\infty. \tag{4.39}$$

By combining (4.33) and (4.34)–(4.39), we conclude that

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x,T_n(u),\nabla u)(D^iu - D^iv)\,\mathrm{d}x \\ &\quad + \int_\Omega H_n(x,u,\nabla u)(u-v)\,\mathrm{d}x + \int_\Omega |u|^{p_0-2}u(u-v)\,\mathrm{d}x \\ &\leqslant \int_\Omega F(x)(u-v)\,\mathrm{d}x + \sum_{i=1}^N \int_\Omega \phi_{i,n}(x,T_n(u))(D^iu - D^iv)\,\mathrm{d}x, \end{aligned} \tag{4.40}$$

which completes the proof of Theorem 4.2. □

4.2. Existence of entropy solutions for the unilateral problem with L^1-data.

P r o o f of Theorem 4.1. *Step 1: Approximate problem.* We consider the approximate problem

$$\begin{cases} -\displaystyle\sum_{i=1}^N D^ia_i(x,T_n(u_n),\nabla u_n) + H_n(x,u_n,\nabla u_n) + |u_n|^{p_0-2}u_n \\ = f_n(x) - \displaystyle\sum_{i=1}^N D^i\phi_{i,n}(x,T_n(u)) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^N (a_i(x,T_n(u_n),\nabla u_n) - \phi_{i,n}(x,T_n(u)))\cdot n_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.41}$$

with $f_n(x) = T_n(f(x))$, $H_n(x,s,\xi) = T_n(H(x,s,\xi))$ and $\phi_{i,n}(x,s) = T_n(\phi_i(x,s))$. In view of Theorem 4.2, there exists at least one weak solution u_n for the approximate problem (4.41), i.e.,

$$\begin{aligned}
(4.42)\qquad &\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)(D^i u_n - D^i v)\,\mathrm{d}x \\
&\quad + \int_\Omega H_n(x,u_n,\nabla u_n)(u_n - v)\,\mathrm{d}x + \int_\Omega |u_n|^{p_0-2}u_n(u_n - v)\,\mathrm{d}x \\
&\leqslant \int_\Omega f_n(x)(u_n - v)\,\mathrm{d}x + \sum_{i=1}^{N}\int_\Omega \phi_{i,n}(x,T_n(u_n))D^i(u_n - v)\,\mathrm{d}x
\end{aligned}$$

for any $v \in K_\psi$.

Step 2: Weak convergence of truncations. Let $k \geqslant \max(1, \|\psi^+\|_\infty)$, and $0 \leqslant B(s) = 2\int_0^s d(|\tau|)/b(|\tau|)\,\mathrm{d}\tau \leqslant B(\infty) < \infty$ be a finite real number.

Let $\eta > 0$ be small enough. By taking $u_n - \eta T_k(u_n - \psi^+)\mathrm{e}^{B(|u_n|)} \in K_\psi$ as a test function for the approximate problem (4.5), we obtain

$$\begin{aligned}
(4.43)\qquad &\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),D^i u_n)D^i T_k(u_n - \psi^+)\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\
&\quad + 2\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),D^i u_n)D^i u_n T_k(u_n - \psi^+) \\
&\quad \times \frac{d(|u_n|)}{b(|u_n|)}\mathrm{sign}(u_n)\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\
&\quad + \int_\Omega H_n(x,u_n,D^i u_n)T_k(u_n - \psi^+)\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\
&\quad + \int_\Omega |u_n|^{p_0-2}u_n T_k(u_n - \psi^+)\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\
&\leqslant \int_\Omega f_n T_k(u_n - \psi^+)\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\
&\quad + \sum_{i=1}^{N}\int_\Omega \phi_{i,n}(x,T_n(u_n))D^i T_k(u_n - \psi^+)\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\
&\quad + 2\sum_{i=1}^{N}\int_\Omega \phi_{i,n}(x,T_n(u_n))D^i u_n T_k(u_n - \psi^+) \\
&\quad \times \frac{d(|u_n|)}{b(|u_n|)}\mathrm{sign}(u_n)\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x.
\end{aligned}$$

Since $u_n - \psi^+$ has the same sign as u_n and in view of (3.5), (3.6) and (3.7), we deduce that

$$
\begin{aligned}
(4.44)\quad & \sum_{i=1}^{N} \int_{\{|u_n-\psi^+|\leqslant k\}} b(|u_n|)|D^i u_n|^{p_i}\,\mathrm{d}x \\
&- \sum_{i=1}^{N} \int_{\{|u_n-\psi^+|\leqslant k\}} |a_i(x,T_n(u_n),D^i u_n)|\,|D^i\psi^+|\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\
&+ 2\sum_{i=1}^{N} \int_\Omega d(|u_n|)|D^i u_n|^{p_i}|T_k(u_n-\psi^+)|\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\
&+ \int_\Omega |u_n|^{p_0-1}|T_k(u_n-\psi^+)|\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\
\leqslant & \int_\Omega (|f_n|+|f_0|)|T_k(u_n-\psi^+)|\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\
&+ \sum_{i=1}^{N} \int_\Omega d(|u_n|)|D^i u_n)|^{p_i}|T_k(u_n-\psi^+)|\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\
&+ \mathrm{e}^{B(\infty)}\sum_{i=1}^{N} \int_{\{|u_n-\psi^+|\leqslant k\}} c_i(x)(1+|T_n(u_n)|)^{\gamma_i}(|D^i u_n|+|D^i\psi^+|)\,\mathrm{d}x \\
&+ 2\sum_{i=1}^{N} \int_\Omega c_i(x)(1+|T_n(u_n)|)^{\gamma_i}|D^i u_n|\,|T_k(u_n-\psi^+)|\frac{d(|u_n|)}{b(|u_n|)}\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\
\leqslant & k\mathrm{e}^{B(\infty)}(\|f_n\|_{L^1(\Omega)}+\|f_0\|_{L^1(\Omega)}) \\
&+ \sum_{i=1}^{N} \int_\Omega d(|u_n|)|D^i u_n|^{p_i}|T_k(u_n-\psi^+)|\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\
&+ C_0\sum_{i=1}^{N} \int_{\{|u_n-\psi^+|\leqslant k\}} \frac{|c_i(x)|^{p_i'}(1+|T_n(u_n)|)^{\gamma_i p_i'}}{b(|T_n(u_n)|)^{p_i'-1}}\,\mathrm{d}x \\
&+ \frac{1}{2}\sum_{i=1}^{N} \int_{\{|u_n-\psi^+|\leqslant k\}} b(|u_n|)(|D^i u_n)|^{p_i}+|D^i\psi^+|^{p_i})\,\mathrm{d}x \\
&+ C_1\sum_{i=1}^{N} \int_\Omega |c_i(x)|^{p_i'}(1+|T_n(u_n)|)^{\gamma_i p_i'}|T_k(u_n-\psi^+)|\frac{d(|u_n|)}{b(|u_n|)^{p_i'}}\,\mathrm{d}x \\
&+ \frac{1}{2}\sum_{i=1}^{N} \int_\Omega d(|u_n|)|D^i u_n)|^{p_i}|T_k(u_n-\psi^+)|\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x.
\end{aligned}
$$

Concerning the third term on the left-hand side of (4.44), we have

$$
(4.45)\qquad \int_\Omega |u_n|^{p_0-1}|T_k(u_n-\psi^+)\,\mathrm{d}x
$$

$$
\begin{aligned}
&= \int_{\{|u_n-\psi^+|\leqslant k\}} |u_n|^{p_0-2}u_n(u_n-\psi^+)\,\mathrm{d}x \\
&\quad + k\int_{\{|u_n-\psi^+|>k\}} |u_n|^{p_0-1}\,\mathrm{d}x \\
&\geqslant \int_{\{|u_n-\psi^+|\leqslant k\}} |u_n|^{p_0}\,\mathrm{d}x - \int_{\{|u_n-\psi^+|\leqslant k\}} |u_n|^{p_0-1}|\psi^+|\,\mathrm{d}x \\
&\quad + k\int_{\{|u_n-\psi^+|>k\}} |u_n|^{p_0-1}\,\mathrm{d}x \\
&\geqslant \int_{\{|u_n-\psi^+|\leqslant k\}} |u_n|^{p_0}\,\mathrm{d}x - \frac{1}{p_0'}\int_{\{|u_n-\psi^+|\leqslant k\}} |u_n|^{p_0}\,\mathrm{d}x \\
&\quad - \frac{1}{p_0}\int_\Omega |\psi^+|^{p_0}\,\mathrm{d}x + k\int_{\{|u_n-\psi^+|>k\}} |u_n|^{p_0-1}\,\mathrm{d}x \\
&\geqslant \frac{1}{p_0}\int_{\{|u_n-\psi^+|\leqslant k\}} |u_n|^{p_0}\,\mathrm{d}x + k\int_{\{|u_n-\psi^+|>k\}} |u_n|^{p_0-1}\,\mathrm{d}x \\
&\quad - \frac{1}{p_0}\int_\Omega |\psi^+|^{p_0}\,\mathrm{d}x.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(4.46)\qquad &\frac{1}{2}\sum_{i=1}^N \int_{\{|u_n-\psi^+|\leqslant k\}} b(|u_n|)|D^i u_n|^{p_i}\,\mathrm{d}x \\
&\quad + \frac{1}{2}\sum_{i=1}^N \int_\Omega d(|u_n|)|D^i u_n|^{p_i}|T_k(u_n-\psi^+)|\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\
&\quad + \frac{1}{p_0}\int_{\{|u_n-\psi^+|\leqslant k\}} |u_n|^{p_0}\,\mathrm{d}x \\
&\quad + k\int_{\{|u_n-\psi^+|>k\}} |u_n|^{p_0-1}\,\mathrm{d}x - \frac{1}{p_0}\int_{\{|u_n-\psi^+|\leqslant k\}} |\psi^+|^{p_0}\,\mathrm{d}x \\
&\leqslant k\mathrm{e}^{B(\infty)}(\|f_n\|_{L^1(\Omega)} + \|f_0\|_{L^1(\Omega)}) \\
&\quad + C_2\sum_{i=1}^N \int_{\{|u_n-\psi^+|\leqslant k\}} |c_i(x)|^{p_i'}(1+|T_n(u_n)|)^{(\gamma_i+\lambda)p_i'-\lambda}\,\mathrm{d}x \\
&\quad + C_3\left\|\frac{d(|\cdot|)}{b(|\cdot|)}\right\|_{L^\infty(\mathbb{R})} \\
&\quad \times \sum_{i=1}^N \int_\Omega |c_i(x)|^{p_i'}(1+|T_n(u_n)|)^{(\gamma_i+\lambda)p_i'-\lambda}|T_k(u_n-\psi^+)|\,\mathrm{d}x \\
&\quad + \mathrm{e}^{B(\infty)}\sum_{i=1}^N \int_{\{|u_n-\psi^+|\leqslant k\}} |a_i(x,T_n(u_n),D^i u_n)|\,|D^i\psi^+|\,\mathrm{d}x.
\end{aligned}
$$

For the second term of the right-hand side of (4.46), in view of Young's inequality, we obtain after some computations

$$
\begin{aligned}
(4.47)\quad & C_2 \sum_{i=1}^{N} \int_{\{|u_n-\psi^+|\leqslant k\}} |c_i(x)|^{p_i'}(1+|T_n(u_n)|)^{(\gamma_i+\lambda)p_i'-\lambda}\,\mathrm{d}x \\
&\leqslant C_4 \sum_{i=1}^{N} \int_{\{|u_n-\psi^+|\leqslant k\}} \Big(|c_i(x)|^{p_i'} + |c_i(x)|^{\frac{p_i'(p_0-1)}{p_0-1-(\gamma_i+\lambda)p_i'+\lambda}}\Big)|u_n|\,\mathrm{d}x \\
&\quad + \frac{1}{3p_0}\int_{\{|u_n-\psi^+|\leqslant k\}} |u_n|^{p_0}\,\mathrm{d}x \\
&\leqslant C_5 k + \frac{1}{3p_0}\int_{\{|u_n-\psi^+|\leqslant k\}} |u_n|^{p_0}\,\mathrm{d}x.
\end{aligned}
$$

For the third term of the right-hand side of (4.46), using Young's inequality we get

$$
\begin{aligned}
(4.48)\quad & C_3\Big\|\frac{d(|\cdot|)}{b(|\cdot|)}\Big\|_{L^\infty(\mathbb{R})} \sum_{i=1}^{N} \int_{\Omega} |c_i(x)|^{p_i'}(1+|T_n(u_n)|)^{(\gamma_i+\lambda)p_i'-\lambda}|T_k(u_n-\psi^+)|\,\mathrm{d}x \\
&\leqslant C_6 \sum_{i=1}^{N} \int_{\Omega} \Big(|c_i(x)|^{p_i'} + |c_i(x)|^{\frac{p_i'(p_0-1)}{p_0-1-(\gamma_i+\lambda)p_i'+\lambda}}\Big)|T_k(u_n-\psi^+)|\,\mathrm{d}x \\
&\quad + \frac{1}{6p_0}\int_{\Omega} |u_n|^{p_0-1}|T_k(u_n-\psi^+)|\,\mathrm{d}x \\
&\leqslant C_7 k + \frac{1}{6p_0}\int_{\Omega} |u_n|^{p_0-1}|T_k(u_n-\psi^+)|\,\mathrm{d}x.
\end{aligned}
$$

For the last term on the right-hand side of (4.46), in view of assumption (3.4) and by using Young's inequality, we have

$$
\begin{aligned}
(4.49)\quad & \mathrm{e}^{B(\infty)} \int_{\{|u_n-\psi^+|\leqslant k\}} |a_i(x,T_n(u_n),D^i u_n)|\,|D^i\psi^+|\,\mathrm{d}x \\
&\leqslant \mathrm{e}^{B(\infty)}\beta \int_{\{|u_n-\psi^+|\leqslant k\}} (|K_i(x)| + |T_n(u_n)|^{p_i-1} + |D^i u_n|^{p_i-1})|D^i\psi^+|\,\mathrm{d}x \\
&\leqslant C_8 \int_{\{|u_n-\psi^+|\leqslant k\}} |K_i(x)|^{p_i'}\,\mathrm{d}x + \frac{1}{6p_0}\int_{\{|u_n-\psi^+|\leqslant k\}} |T_n(u_n)|^{p_0}\,\mathrm{d}x \\
&\quad + C_9 \int_{\{|u_n-\psi^+|\leqslant k\}} |D^i\psi^+|^{p_i}\,\mathrm{d}x \\
&\quad + \frac{1}{4}\int_{\{|u_n-\psi^+|\leqslant k\}} b(|T_n(u_n)|)|D^i u_n|^{p_i}\,\mathrm{d}x \\
&\quad + C_{10} \int_{\{|u_n-\psi^+|\leqslant k\}} \frac{|D^i\psi^+|^{p_i}}{b(|T_n(u_n)|)^{p_i-1}}\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C_{11} + \frac{1}{6p_0}\int_{\{|u_n-\psi^+|\leqslant k\}} |T_n(u_n)|^{p_0}\,\mathrm{d}x \\
&\quad + \frac{1}{4}\int_{\{|u_n-\psi^+|\leqslant k\}} b(|T_n(u_n)|)|D^i u_n|^{p_i}\,\mathrm{d}x \\
&\quad + C_{10}\int_{\{|u_n-\psi^+|\leqslant k\}} |D^i\psi^+|^{p_i}(1+|u_n|)^{\lambda(p_i-1)}\,\mathrm{d}x \\
&\leqslant C_{11} + \frac{1}{6p_0}\int_{\{|u_n-\psi^+|\leqslant k\}} |T_n(u_n)|^{p_0}\,\mathrm{d}x \\
&\quad + \frac{1}{4}\int_{\{|u_n-\psi^+|\leqslant k\}} b(|u_n|)|D^i u_n|^{p_i}\,\mathrm{d}x \\
&\quad + C_{12}k^{\lambda(p_i-1)}\int_{\{|u_n-\psi^+|\leqslant k\}} |D^i\psi^+|^{p_i}\,\mathrm{d}x \\
&\leqslant C_{13}k^{\lambda(p_i-1)} + \frac{1}{6p_0}\int_{\{|u_n-\psi^+|\leqslant k\}} |T_n(u_n)|^{p_0}\,\mathrm{d}x \\
&\quad + \frac{1}{4}\int_{\{|u_n-\psi^+|\leqslant k\}} b(|u_n|)|D^i u_n|^{p_i}\,\mathrm{d}x.
\end{aligned}
$$

By combining (4.46) and (4.47)–(4.49), we conclude that

$$
\begin{aligned}
(4.50)\qquad &\frac{1}{4}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leqslant k\}} b(|T_n(u_n)|)|D^i u_n|^{p_i}\,\mathrm{d}x \\
&\quad + \frac{1}{2}\sum_{i=1}^{N}\int_{\Omega} d(|u_n|)|D^i u_n)|^{p_i}|T_k(u_n-\psi^+)|\,\mathrm{d}x \\
&\quad + \frac{1}{6p_0}\int_{\{|u_n-\psi^+|\leqslant k\}} |u_n|^{p_0}\,\mathrm{d}x + \frac{k}{3p_0}\int_{\{|u_n-\psi^+|>k\}} |u_n|^{p_0-1}\,\mathrm{d}x \\
&\leqslant C_{14}(k+k^{\lambda(p_i-1)}).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(4.51)\qquad \frac{b_0}{(1+2k)^{\lambda}}\sum_{i=1}^{N}&\int_{\{|u_n-\psi^+|\leqslant k\}} |D^i u_n|^{p_i}\,\mathrm{d}x \\
&\leqslant \sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leqslant k\}} b(|T_n(u_n)|)|D^i u_n|^{p_i}\,\mathrm{d}x \\
&\leqslant C_{15}(k+k^{\lambda(p_i-1)}).
\end{aligned}
$$

Thus, we conclude that

$$
(4.52)\qquad \sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leqslant k\}} |D^i u_n|^{p_i}\,\mathrm{d}x \leqslant C_{16}(k^{1+\lambda}+k^{\lambda p_i}).
$$

Having in mind that $\{|u_n| \leqslant k\} \subset \{|u_n - \psi^+| \leqslant k + \|\psi^+\|_{L^\infty(\Omega)}\}$ and thanks to (4.50) and (4.52) we obtain

$$
\begin{aligned}
(4.53)\qquad & \sum_{i=1}^{N} \int_\Omega |D^i T_k(u_n)|^{p_i}\,\mathrm{d}x + \int_\Omega |T_k(u_n)|^{p_0}\,\mathrm{d}x \\
&= \sum_{i=1}^{N} \int_{\{|u_n|\leqslant k\}} |D^i u_n|^{p_i}\,\mathrm{d}x + \int_\Omega |T_k(u_n)|^{p_0}\,\mathrm{d}x \\
&\leqslant \sum_{i=1}^{N} \int_{\{|u_n-\psi^+|\leqslant k+\|\psi^+\|_{L^\infty(\Omega)}\}} |D^i u_n|^{p_i}\,\mathrm{d}x + \int_{\{|u_n-\psi^+|\leqslant k\}} |u_n|^{p_0}\,\mathrm{d}x \\
&\quad + k\int_{\{|u_n-\psi^+|>k\}} |u_n|^{p_0-1}\,\mathrm{d}x \\
&\leqslant C_{16}(k^{1+\lambda} + k^{\lambda p_i}).
\end{aligned}
$$

Thus, similarly as in (4.18) we conclude that

$$(4.54)\qquad \|T_k(u_n)\|_{1,\vec{p}} \leqslant C_{17}(k^{1+\lambda} + k^{\lambda p_i})$$

with C_{17} being a constant that does not depend on k and n. Thus, the sequence $(T_k(u_n))_{n\in N^*}$ is uniformly bounded in $W^{1,\vec{p}}(\Omega)$, and there exists a subsequence still denoted by $(T_k(u_n))_{n\in N^*}$ and a measurable function $v_k \in W^{1,\vec{p}}(\Omega)$ such that

$$(4.55)\qquad T_k(u_n) \rightharpoonup v_k \quad \text{weakly in } W^{1,\vec{p}}(\Omega).$$

By the compact embedding, we obtain

$$(4.56)\qquad T_k(u_n) \to v_k \quad \text{strongly in } L^1(\Omega) \quad \text{and a.e., in } \Omega.$$

Now, we will prove that $(u_n)_n$ is a Cauchy sequence in Ω.

It is clear that

$$
\begin{aligned}
(4.57)\qquad \operatorname{meas}\{|u_n - u_m| \geqslant \delta\} \leqslant{} & \operatorname{meas}\{|u_n| \geqslant k\} + \operatorname{meas}\{|u_m| \geqslant k\} \\
& + \operatorname{meas}\{|T_k(u_n) - T_k(u_m)| \geqslant \delta\}.
\end{aligned}
$$

On the one hand, thanks to (4.53) we have

$$
\begin{aligned}
(4.58)\qquad k^{p_0}\operatorname{meas}(\{|u_n| > k\}) &= \int_{\{|u_n|>k\}} |T_k(u_n)|^{p_0}\,\mathrm{d}x \\
&\leqslant \int_\Omega |T_k(u_n)|^{p_0}\,\mathrm{d}x \leqslant C_{16}(k^{1+\lambda} + k^{\lambda p_i})
\end{aligned}
$$

with $0 \leqslant \lambda < \min(\underline{p}-1,1)$. It follows that

$$\text{meas}\{|u_n|>k\} \leqslant \frac{C_{16}(k^{1+\lambda}+k^{\lambda p_i})}{k^{p_0}} \to 0 \quad \text{as } k\to\infty. \tag{4.59}$$

Thus, for all $\varepsilon \geqslant 0$, there exists $k_0(\varepsilon)$ such that

$$\text{meas}(\{|u_n|>k\}) \leqslant \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}(\{|u_m|>k\}) \leqslant \frac{\varepsilon}{3} \quad \text{for any } k>k_0(\varepsilon). \tag{4.60}$$

Moreover, thanks to (4.54) we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure on Ω. Then for all $k>0$ and $\delta,\varepsilon>0$ there exists $n_0=n_0(k,\delta,\varepsilon)$ such that

$$\text{meas}(\{|T_k(u_n)-T_k(u_m)| \geqslant \delta\}) \leqslant \frac{\varepsilon}{3} \quad \text{for any } n,m \geqslant n_0(k,\delta,\varepsilon). \tag{4.61}$$

By combining (4.57) and (4.60)–(4.61), we conclude that for any $\varepsilon,\delta>0$ there exists $n_0=n_0(\sigma,\varepsilon)$ such that

$$\text{meas}\{|u_n-u_m|>\delta\} \leqslant \varepsilon \quad \text{for any } n,m \geqslant n_0(\sigma,\varepsilon). \tag{4.62}$$

Therefore, the sequence $(u_n)_n$ is a Cauchy sequence in measure, and there exists a subsequence of $(u_n)_n$, still denoted $(u_n)_n$, that converges to u almost everywhere in Ω. Consequently, we conclude that

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } W^{1,\vec{p}}(\Omega), \\ T_k(u_n) \to T_k(u) & \text{strongly in } L^1(\Omega) \quad \text{and} \quad \text{a.e., in } \Omega, \\ T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } L^1(\partial\Omega) \quad \text{and} \quad \text{a.e., in } \Omega. \end{cases} \tag{4.63}$$

Step 3: Some regularity results Let $h \geqslant k \geqslant \min(1,\|\psi^+\|_\infty)$. We have $v_n = u_n - \eta T_h(u_n-\psi^+)\mathrm{e}^{B(|u_n|)}/h \in K_\psi$ is an admissible test function for η small enough. Thus, by taking v_n as a test function for the approximate problem (4.41), we obtain

$$\begin{aligned} (4.64)\quad & \frac{1}{h}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iT_h(u_n-\psi^+)\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\ &+\int_\Omega H_n(x,u_n,\nabla u_n)\frac{T_h(u_n-\psi^+)}{h}\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\ &+2\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iu_n\frac{|T_h(u_n-\psi^+)|}{h}\frac{d(|u_n|)}{b(|u_n|)}\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\ &+\int_\Omega |u_n|^{p_0-2}u_n\frac{T_h(u_n-\psi^+)}{h}\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \end{aligned}$$

$$\leqslant \int_\Omega f_n(x)\frac{T_h(u_n-\psi^+)}{h}\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x$$
$$+\frac{1}{h}\sum_{i=1}^N\int_\Omega \phi_{i,n}(x,T_n(u_n))D^iT_h(u_n-\psi^+)\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x$$
$$+2\sum_{i=1}^N\int_\Omega \phi_{i,n}(x,T_n(u_n))D^iu_n\frac{|T_h(u_n-\psi^+)|}{h}\frac{d(|u_n|)}{b(|u_n|)}\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x.$$

In view of (3.5), (3.6) and (3.7), and since $T_h(u_n-\psi^+)/h$ has the same sign as u_n, we deduce that

$$\begin{aligned}
(4.65)\quad &\frac{1}{2h}\sum_{i=1}^N\int_{\{|u_n-\psi^+|\leqslant h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,\mathrm{d}x\\
&+\frac{1}{2h}\sum_{i=1}^N\int_{\{|u_n-\psi^+|\leqslant h\}} b(T_n(u_n))|D^iu_n|^{p_i}\,\mathrm{d}x\\
&-\frac{1}{h}\sum_{i=1}^N\int_{\{|u_n-\psi^+|\leqslant h\}} |a_i(x,T_n(u_n),\nabla u_n)|\,|D^i\psi^+|\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x\\
&+\frac{2}{h}\sum_{i=1}^N\int_\Omega d(|u_n|)|D^iu_n|^{p_i}|T_h(u_n-\psi^+)|\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x\\
&+\int_\Omega |u_n|^{p_0-1}\frac{|T_h(u_n-\psi^+)|}{h}\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x\\
\leqslant{}& \mathrm{e}^{B(\infty)}\int_\Omega(|f_n(x)|+|f_0(x)|)\frac{|T_h(u_n-\psi^+)|}{h}\,\mathrm{d}x\\
&+\frac{1}{h}\sum_{i=1}^N\int_\Omega d(|u_n|)|D^iu_n|^{p_i}|T_h(u_n-\psi^+)|\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x\\
&+\frac{\mathrm{e}^{B(\infty)}}{h}\sum_{i=1}^N\int_\Omega c_i(x)(1+|T_n(u_n)|)^{\gamma_i}|D^iT_h(u_n-\psi^+)|\,\mathrm{d}x\\
&+\frac{2}{h}\sum_{i=1}^N\int_\Omega c_i(x)(1+|T_n(u_n)|)^{\gamma_i}D^iu_n|T_h(u_n-\psi^+)|\frac{d(|u_n|)}{b(|u_n|)}\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x\\
\leqslant{}& \mathrm{e}^{B(\infty)}\int_\Omega(|f_n(x)|+|f_0(x)|)\frac{|T_h(u_n-\psi^+)|}{h}\,\mathrm{d}x\\
&+\frac{1}{h}\sum_{i=1}^N\int_\Omega d(|u_n|)|D^iu_n|^{p_i}|T_h(u_n-\psi^+)|\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x\\
&+\frac{C_0}{h}\sum_{i=1}^N\int_\Omega \frac{|c_i(x)|^{p_i'}(1+|T_n(u_n)|)^{\gamma_ip_i'}}{b(|T_n(u_n)|)^{p_i'-1}}\,\mathrm{d}x
\end{aligned}$$

$$+\frac{1}{4h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leqslant h\}} b(T_n(u_n))(|D^i u_n|^{p_i}+|D^i\psi^+|^{p_i})\,\mathrm{d}x$$
$$+\frac{C_1}{h}\sum_{i=1}^{N}\int_\Omega |c_i(x)|^{p_i'}(1+|T_n(u_n)|)^{\gamma_i p_i'}$$
$$\times |T_h(u_n-\psi^+)|\frac{d(|u_n|)}{b(|u_n|)^{p_i'}}\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x$$
$$+\frac{1}{2h}\sum_{i=1}^{N}\int_\Omega d(|u_n|)|D^i u_n|^{p_i}|T_h(u_n-\psi^+)|\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x.$$

It follows that

$$\begin{aligned}(4.66)\qquad &\frac{1}{2h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leqslant h\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n\,\mathrm{d}x\\
&\quad+\frac{1}{4h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leqslant h\}} b(T_n(u_n))|D^i u_n|^{p_i}\,\mathrm{d}x\\
&\quad+\frac{1}{2h}\sum_{i=1}^{N}\int_\Omega d(|u_n|)|D^i u_n|^{p_i}|T_h(u_n-\psi^+)|\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x\\
&\quad+\frac{1}{h}\int_\Omega |u_n|^{p_0-1}|T_h(u_n-\psi^+)|\,\mathrm{d}x\\
&\leqslant \mathrm{e}^{B(\infty)}\int_\Omega (|f(x)|+|f_0(x)|)\frac{|T_h(u_n-\psi^+)|}{h}\,\mathrm{d}x\\
&\quad+\frac{1}{4h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leqslant h\}} b(T_n(u_n))|D^i\psi^+|^{p_i}\,\mathrm{d}x\\
&\quad+\frac{C_2}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leqslant h\}} |c_i(x)|^{p_i'}(1+|T_n(u_n)|)^{(\gamma_i+\lambda)p_i'-\lambda}\,\mathrm{d}x\\
&\quad+\frac{C_3}{h}\left\|\frac{d(|\cdot|)}{b(|\cdot|)}\right\|_{L^\infty(\mathbb{R})}\\
&\quad\times\sum_{i=1}^{N}\int_\Omega |c_i(x)|^{p_i'}(1+|T_n(u_n)|)^{(\gamma_i+\lambda)p_i'-\lambda}|T_h(u_n-\psi^+)|\,\mathrm{d}x\\
&\quad+\frac{\mathrm{e}^{B(\infty)}}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leqslant h\}} |a_i(x,T_n(u_n),\nabla u_n)|\,|D^i\psi^+|\,\mathrm{d}x.\end{aligned}$$

We have $f(x)\in L^1(\Omega)$ and $f_0(x)\in L^1(\Omega)$, and since $\mathrm{meas}\{|u_n|>h\}\to 0$ as h tends to ∞, then $|T_h(u_n-\psi^+)|/h \rightharpoonup 0$ weak-$*$ in $L^\infty(\Omega)$ as h goes to infinity, it follows that

$$\varepsilon_1(h) = \mathrm{e}^{B(\infty)} \int_\Omega (|f(x)| + |f_0(x)|) \frac{|T_h(u_n - \psi^+)|}{h} \, \mathrm{d}x \to 0 \quad \text{as } h \to \infty. \tag{4.67}$$

Also, we have $\psi^+ \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, then

$$\varepsilon_2(h) = \frac{1}{4h} \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leqslant h\}} b(T_n(u_n)) |D^i \psi^+|^{p_i} \, \mathrm{d}x \to 0 \quad \text{as } h \to \infty. \tag{4.68}$$

Concerning the third and fourth terms on the right-hand side of (4.66), in view of Young's inequality we have

$$\begin{aligned} &\frac{C_2}{h} \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leqslant h\}} |c_i(x)|^{p_i'} (1 + |T_n(u_n)|)^{(\gamma_i + \lambda) p_i' - \lambda} \, \mathrm{d}x \\ &\qquad \leqslant \frac{C_4}{h} \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leqslant h\}} \Big(|c_i(x)|^{p_i'} + |c_i(x)|^{\frac{p_i'(p_0 - 1)}{p_0 - (\gamma_i + \lambda) p_i' + \lambda}} \Big) |u_n| \, \mathrm{d}x \\ &\qquad\quad + \frac{1}{3 p_0 h} \int_{\{|u_n - \psi^+| \leqslant h\}} |u_n|^{p_0} \, \mathrm{d}x \\ &\qquad \leqslant \varepsilon_3(h) + \frac{1}{3 p_0 h} \int_{\{|u_n - \psi^+| \leqslant h\}} |u_n|^{p_0} \, \mathrm{d}x. \end{aligned} \tag{4.69}$$

We also have

$$\begin{aligned} &\frac{C_3}{h} \left\| \frac{d(|\cdot|)}{b(|\cdot|)} \right\|_{L^\infty(\mathbb{R})} \sum_{i=1}^N \int_\Omega |c_i(x)|^{p_i'} (1 + |T_n(u_n)|)^{(\gamma_i + \lambda) p_i' - \lambda} |T_h(u_n - \psi^+)| \, \mathrm{d}x \\ &\leqslant \frac{C_5}{h} \sum_{i=1}^N \int_\Omega \Big(|c_i(x)|^{p_i'} + |c_i(x)|^{\frac{p_i'(p_0 - 1)}{p_0 - 1 - (\gamma_i + \lambda) p_i' + \lambda}} \Big) |T_h(u_n - \psi^+)| \, \mathrm{d}x \\ &\quad + \frac{1}{6 p_0 h} \int_\Omega |u_n|^{p_0 - 1} |T_h(u_n - \psi^+)| \, \mathrm{d}x \\ &\leqslant \varepsilon_4(h) + \frac{1}{6 p_0 h} \int_\Omega |u_n|^{p_0 - 1} |T_h(u_n - \psi^+)| \, \mathrm{d}x. \end{aligned} \tag{4.70}$$

For the last term on the right-hand side of (4.66), similarly as in (4.49), we can show that

$$\begin{aligned} &\frac{\mathrm{e}^{B(\infty)}}{h} \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leqslant h\}} |a_i(x, T_n(u_n), \nabla u_n)| \, |D^i \psi^+| \, \mathrm{d}x \\ &\qquad \leqslant \frac{C_6}{h} \int_{\{|u_n - \psi^+| \leqslant h\}} |K_i(x)|^{p_i'} \, \mathrm{d}x + \frac{1}{6 h p_0} \int_{\{|u_n - \psi^+| \leqslant h\}} |T_n(u_n)|^{p_0} \, \mathrm{d}x \end{aligned} \tag{4.71}$$

$$
\begin{aligned}
&+ \frac{1}{8h}\int_{\{|u_n-\psi^+|\leqslant h\}} b(|T_n(u_n)|)|D^i u_n|^{p_i}\,\mathrm{d}x\\
&+ \frac{C_8}{h}\int_{\{|u_n-\psi^+|\leqslant h\}} |D^i\psi^+|^{p_i}\,\mathrm{d}x\\
&+ \frac{C_9}{h}\int_{\{|u_n-\psi^+|\leqslant h\}} \frac{|D^i\psi^+|^{p_i}}{b(|T_n(u_n)|)^{p_i-1}}\,\mathrm{d}x\\
\leqslant{}& \varepsilon_5(h) + \frac{1}{6p_0h}\int_{\{|u_n-\psi^+|\leqslant h\}} |T_n(u_n)|^{p_0}\,\mathrm{d}x\\
&+ \frac{1}{8h}\int_{\{|u_n-\psi^+|\leqslant h\}} b(|T_n(u_n)|)|D^i u_n|^{p_i}\,\mathrm{d}x\\
&+ \frac{C_9}{h}\int_{\{|u_n-\psi^+|\leqslant h\}} |D^i\psi^+|^{p_i}(1+|u_n|)^{\lambda(p_i-1)}\,\mathrm{d}x\\
\leqslant{}& \varepsilon_5(h) + \frac{1}{6p_0h}\int_{\{|u_n-\psi^+|\leqslant h\}} |T_n(u_n)|^{p_0}\,\mathrm{d}x\\
&+ \frac{1}{8h}\int_{\{|u_n-\psi^+|\leqslant h\}} b(|T_n(u_n)|)|D^i u_n|^{p_i}\,\mathrm{d}x\\
&+ \frac{C_9(1+h)^{\lambda(p_i-1)}}{h}\int_{\{|u_n-\psi^+|\leqslant h\}} |D^i\psi^+|^{p_i}\,\mathrm{d}x\\
\leqslant{}& \varepsilon_6(h) + \frac{1}{6p_0h}\int_{\{|u_n-\psi^+|\leqslant h\}} |T_n(u_n)|^{p_0}\,\mathrm{d}x\\
&+ \frac{1}{4h}\int_{\{|u_n-\psi^+|\leqslant h\}} b(|T_n(u_n)|)|D^i u_n|^{p_i}\,\mathrm{d}x.
\end{aligned}
$$

By combining (4.66) and (4.67)–(4.71) we conclude that

$$
\begin{aligned}
(4.72)\qquad &\frac{1}{2h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leqslant h\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n\,\mathrm{d}x\\
&\quad+\frac{1}{8h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leqslant h\}} b(T_n(u_n))|D^i u_n|^{p_i}\,\mathrm{d}x\\
&\quad+\frac{1}{2h}\sum_{i=1}^{N}\int_\Omega d(|u_n|)|D^i u_n|^{p_i}|T_h(u_n-\psi^+)|\,\mathrm{d}x\\
&\quad+\frac{1}{6hp_0}\int_\Omega |u_n|^{p_0-1}|T_h(u_n-\psi^+)|\,\mathrm{d}x\\
&\leqslant \varepsilon_7(h).
\end{aligned}
$$

Having in mind that $\psi^+ \in L^\infty(\Omega)$, we conclude that

$$
(4.73)\qquad \lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\leqslant h\}} a_i(x,u_n,\nabla u_n)D^i u_n\,\mathrm{d}x = 0.
$$

Moreover, we have

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leqslant h\}} b(T_n(u_n))|D^i u_n|^{p_i}\,\mathrm{d}x = 0, \tag{4.74}$$

$$\lim_{h\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{N}\int_{\{|u_n|>h\}} d(|u_n|)|D^i u_n|^{p_i}\,\mathrm{d}x = 0, \tag{4.75}$$

and

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{|u_n|>h\}} |u_n|^{p_0-1}\,\mathrm{d}x = 0. \tag{4.76}$$

Furthermore, we have

$$\begin{aligned}
(4.77)\quad \frac{1}{h}\sum_{i=1}^{N}&\left|\int_{\Omega}\phi_{i,n}(x,T_n(u_n))D^iT_h(u_n)\,\mathrm{d}x\right| \\
&\leqslant \frac{1}{h}\sum_{i=1}^{N}\int_{\Omega} c_i(x)(1+|T_n(u_n)|)^{\gamma_i}|D^iT_h(u_n)|\,\mathrm{d}x \\
&\leqslant \frac{1}{h}\sum_{i=1}^{N}\int_{\Omega}\frac{|c_i(x)|^{p_i'}(1+|T_h(u_n)|)^{\gamma_i p_i'}}{b(|T_h(u_n)|)^{p_i'-1}}\,\mathrm{d}x \\
&\quad + \frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\leqslant h\}} b(T_h(u_n))(|D^iu_n|^{p_i}+|D^i\psi^+|^{p_i})\,\mathrm{d}x \\
&\leqslant \frac{C_2}{h}\sum_{i=1}^{N}\int_{\{|u_n|\leqslant h\}} |c_i(x)|^{p_i'}(1+|T_h(u_n)|)^{(\gamma_i+\lambda)p_i'-\lambda}\,\mathrm{d}x \\
&\quad + \frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\leqslant h\}} b(T_h(u_n))|D^iu_n|^{p_i}\,\mathrm{d}x \\
&\quad + \frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\leqslant h\}} b(T_h(u_n))|D^i\psi^+|^{p_i}\,\mathrm{d}x \\
&\leqslant \varepsilon_8(h) + \frac{1}{h}\int_{\{|u_n|\leqslant h\}}|u_n|^{p_0}\,\mathrm{d}x + \frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\leqslant h\}} b(T_n(u_n))|D^iu_n|^{p_i}\,\mathrm{d}x.
\end{aligned}$$

It follows that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\Omega}\phi_{i,n}(x,T_n(u_n))D^iT_h(u_n)\,\mathrm{d}x = 0. \tag{4.78}$$

Now, we will show that the sequence $(|u_n|^{p_0-2}u_n)_n$ is uniformly equi-integrable.

Let $E \subset \Omega$ be a measurable set. In view of (4.76) we have for any $h > 0$

$$\begin{aligned}(4.79)\qquad \int_E |u_n|^{p_0-1}\,\mathrm{d}x &\leqslant \int_E |T_h(u_n)|^{p_0-1}\,\mathrm{d}x + \int_{E\cap\{|u_n|>h\}} |u_n|^{p_0-1}\,\mathrm{d}x \\ &\leqslant \operatorname{meas}(E)h^{p_0-1} + \varepsilon_9(h).\end{aligned}$$

Thus, for any $\varepsilon > 0$ there exists $\beta(\varepsilon) > 0$ such that

$$(4.80)\qquad \int_E |u_n|^{p_0-1}\,\mathrm{d}x \leqslant \varepsilon \quad \text{for any } E \subset \Omega \quad \text{with} \quad \operatorname{meas}(E) \leqslant \beta(\varepsilon).$$

Then we deduce that the sequence $(|u_n|^{p_0-2}u_n)_n$ is uniformly equi-integrable in $L^1(\Omega)$. Having in mind that $u_n \to u$ a.e., in Ω and by applying Vitali's theorem, we conclude that

$$(4.81)\qquad |u_n|^{p_0-2}u_n \to |u|^{p_0-2}u \quad \text{strongly in } L^1(\Omega).$$

Step 4: Almost everywhere convergence of the gradients. Let $\max\{1, \|\psi^+\|_\infty\} \leqslant k < h$, we set $S_h(s) = 1 - |T_{2h}(s) - T_h(s)|/h$ and $B(s) = 2\int_0^s d(|\tau|)/b(|\tau|)\,\mathrm{d}\tau$.

Let $\varphi(s) = s\exp(\delta^2 s^2/2)$ and $\delta = 3\|d(|\cdot|)/b(|\cdot|)\|_{L^\infty(\mathbb{R})}$. It is clear that

$$\varphi'(s) - \delta|\varphi(s)| \geqslant \frac{1}{2}.$$

By taking $v_n = u_n - \eta\varphi(T_k(u_n) - T_k(u))S_h(u_n)\mathrm{e}^{B(|u_n|)} \in W^{1,\vec{p}}(\Omega)$ with η small enough, v_n is an admissible test function for the approximate problem (4.5), and we obtain

$$\begin{aligned}(4.82)\qquad &\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n)\varphi'(T_k(u_n) - T_k(u))) \\ &\times (D^iT_k(u_n) - D^iT_k(u))S_h(u_n)\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\ &+ 2\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n)D^iu_n\frac{d(|u_n|)}{b(|u_n|)} \\ &\times \operatorname{sign}(u_n)\varphi(T_k(u_n) - T_k(u))S_h(u_n)\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\ &- \frac{1}{h}\sum_{i=1}^N \int_{\{h\leqslant|u_n|\leqslant 2h\}} a_i(x, T_{2h}(u_{2h}), D^iu_{2h}) \\ &\times D^iT_{2h}(u_n)\varphi(T_k(u_n) - T_k(u))|\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x \\ &+ \int_\Omega H_n(x, u_n, \nabla u_n)\varphi(T_k(u_n) - T_k(u))S_h(u_n)\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega |u_n|^{p_0-2}u_n\varphi(T_k(u_n)-T_k(u))S_h(u_n)\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x\\
&\leqslant \int_\Omega f_n(x)\varphi(T_k(u_n)-T_k(u))S_h(u_n)e^{B(|u_n|)}\,\mathrm{d}x\\
&+\sum_{i=1}^N\int_\Omega \phi_i(x,T_n(u_n))\varphi'(T_k(u_n)-T_k(u))\\
&\times(D^iT_k(u_n)-D^iT_k(u))S_h(u_n)\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x\\
&+2\sum_{i=1}^N\int_\Omega \phi_i(x,T_n(u_n))D^iu_n\frac{d(|u_n|)}{b(|u_n|)}\\
&\times\operatorname{sign}(u_n)S_h(u_n)\varphi(T_k(u_n)-T_k(u))\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x\\
&-\frac{\mathrm{e}^{B(\infty)}}{h}\sum_{i=1}^N\int_{\{h\leqslant|u_n|\leqslant 2h\}}\phi_i(x,T_{2h}(u_n))\\
&\times D^iT_{2h}(u_n)\varphi(T_k(u_n)-T_k(u)))\,\mathrm{d}x.
\end{aligned}
$$

Since $T_k(u_n)-T_k(u)$ has the same sign as u_n on the set $\{|u_n|>k\}$ and $S_h(u_n)=1$ on the set $\{|u_n|\leqslant k\}$, by using (3.5) and (3.6)–(3.7) we obtain

$$
\begin{aligned}
(4.83)\quad &\sum_{i=1}^N\int_{\{|u_n|\leqslant k\}} a_i(x,T_n(u_n),\nabla u_n)\varphi'(T_k(u_n)-T_k(u))\\
&\times(D^iT_k(u_n)-D^iT_k(u))\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x\\
&-\sum_{i=1}^N\int_{\{k<|u_n|\leqslant 2h\}} a_i(x,T_n(u_n),\nabla u_n)\varphi'(T_k(u_n)-T_k(u)))\\
&\times D^iT_k(u)S_h(u_n)\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x\\
&-2\mathrm{e}^{B(\infty)}\sum_{i=1}^N\int_{\{|u_n|\leqslant k\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\frac{d(|u_n|)}{b(|u_n|)}\\
&\times|\varphi(T_k(u_n)-T_k(u))|\,\mathrm{d}x\\
&+2\sum_{i=1}^N\int_{\{k<|u_n|\leqslant 2h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\frac{d(|u_n|)}{b(|u_n|)}\\
&\times|\varphi(T_k(u_n)-T_k(u))|\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x\\
&\leqslant \mathrm{e}^{B(\infty)}\int_\Omega(|f(x)|+|f_0(x)|)|\varphi(T_k(u_n)-T_k(u))|\,\mathrm{d}x\\
&+\mathrm{e}^{B(\infty)}\int_\Omega|u_n|^{p_0-1}|\varphi(T_k(u_n)-T_k(u))|\,\mathrm{d}x\\
&+\sum_{i=1}^N\int_\Omega d(|u_n|)|D^iu_n|^{p_i}|\varphi(T_k(u_n)-T_k(u))|\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x
\end{aligned}
$$

$$+\varphi'(2k)\mathrm{e}^{B(\infty)}\sum_{i=1}^{N}\int_{\Omega}c_i(x)(1+|T_n(u_n)|)^{\gamma_i}|D^iT_k(u_n)-D^iT_k(u)|\,\mathrm{d}x$$

$$+2\mathrm{e}^{B(\infty)}\left\|\frac{d(|\cdot|)}{b(|\cdot|)}\right\|_{L^\infty(\Omega)}\sum_{i=1}^{N}\int_{\Omega}c_i(x)(1+|T_n(u_n)|)^{\gamma_i}$$

$$\times|D^iu_n|\,|\varphi(T_k(u_n)-T_k(u))|\,\mathrm{d}x$$

$$+\frac{\mathrm{e}^{B(\infty)}\varphi(2k)}{h}\sum_{i=1}^{N}\int_{\{h\leqslant|u_n|\leqslant 2h\}}\phi_i(x,T_{2h}(u_n))D^iu_n\,\mathrm{d}x$$

$$+\frac{\mathrm{e}^{B(\infty)}\varphi(2k)}{h}\sum_{i=1}^{N}\int_{\{h\leqslant|u_n|\leqslant 2h\}}a_i(x,T_{2h}(u_n),D^iu_n)D^iu_n\,\mathrm{d}x.$$

It follows that

$$\begin{aligned}(4.84)\quad &\sum_{i=1}^{N}\int_{\{|u_n|\leqslant k\}}a_i(x,T_k(u_n),\nabla T_k(u_n))\varphi'(T_k(u_n)-T_k(u)))\\
&\times(D^iT_k(u_n)-D^iT_k(u))\mathrm{e}^{B(|u_n|)}\,\mathrm{d}x\\
&-3\mathrm{e}^{B(\infty)}\sum_{i=1}^{N}\int_{\{|u_n|\leqslant k\}}a_i(x,T_k(u_n),\nabla T_k(u_n))\\
&\times D^iT_k(u_n)\frac{d(|u_n|)}{b(|u_n|)}|\varphi(T_k(u_n)-T_k(u))|\,\mathrm{d}x\\
\leqslant{}&\mathrm{e}^{B(\infty)}\int_{\Omega}(|f(x)|+|f_0(x)|)|\varphi(T_k(u_n)-T_k(u))|\,\mathrm{d}x\\
&+\mathrm{e}^{B(\infty)}\int_{\Omega}|u_n|^{p_0-1}|\varphi(T_k(u_n)-T_k(u))|\,\mathrm{d}x\\
&+\varphi'(2k)\mathrm{e}^{B(\infty)}\sum_{i=1}^{N}\int_{\Omega}c_i(x)(1+|T_{2h}(u_n)|)^{\gamma_i}\\
&\times|D^iT_k(u_n)-D^iT_k(u)|\,\mathrm{d}x\\
&+2\mathrm{e}^{B(\infty)}\left\|\frac{d(|\cdot|)}{b(|\cdot|)}\right\|_{L^\infty(\Omega)}\sum_{i=1}^{N}\int_{\Omega}c_i(x)(1+|T_n(u_n)|)^{\gamma_i}\\
&\times|D^iu_n|\,|\varphi(T_k(u_n)-T_k(u))|\,\mathrm{d}x\\
&+\frac{\mathrm{e}^{B(\infty)}\varphi(2k)}{h}\sum_{i=1}^{N}\int_{\{h\leqslant|u_n|\leqslant 2h\}}|\phi_i(x,T_{2h}(u_n))|\,|D^iu_n|\,\mathrm{d}x\\
&+\frac{\mathrm{e}^{B(\infty)}\varphi(2k)}{h}\sum_{i=1}^{N}\int_{\{h\leqslant|u_n|\leqslant 2h\}}a_i(x,T_{2h}(u_n),D^iu_n)D^iu_n\,\mathrm{d}x\\
&+\varphi'(2k)\mathrm{e}^{B(\infty)}\sum_{i=1}^{N}\int_{\{k<|u_n|\leqslant 2h\}}|a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))|\,|D^iT_k(u)|\,\mathrm{d}x.\end{aligned}$$

For the two first terms on the right-hand side of (4.84) we have $f \in L^1(\Omega)$ and $f_0 \in L^1(\Omega)$, and since $\varphi(T_k(u_n) - T_k(u)) \rightharpoonup 0$ weak-$*$ in $L^\infty(\Omega)$ as n goes to infinity, it follows that

$$\varepsilon_1(n) = \int_\Omega (|f(x)| + |f_0(x)|)|\varphi(T_k(u_n) - T_k(u))| \,\mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{4.85}$$

Similarly, thanks to (4.81) we have $|u_n|^{p_0-1} \to |u_n|^{p_0-1}$ strongly in $L^1(\Omega)$, then

$$\varepsilon_2(n) = \int_\Omega |u_n|^{p_0-1}|\varphi(T_k(u_n) - T_k(u))| \,\mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{4.86}$$

For the third and fourth terms on the right-hand side of (4.84), we have $c_i(x)(1 + |T_{2h}(u_n)|)^{\gamma_i} \to c_i(x)(1 + |T_{2h}(u)|)^{\gamma_i}$ strongly in $L^{p_i'}(\Omega)$, and since $D^i T_k(u_n) \rightharpoonup D^i T_k(u)$ weakly in $L^{p_i}(\Omega)$, it follows that

$$\varepsilon_3(n) = \sum_{i=1}^N \int_\Omega c_i(x)(1 + |T_{2h}(u_n)|)^{\gamma_i} |D^i T_k(u_n) - D^i T_k(u)| \,\mathrm{d}x \to 0 \tag{4.87}$$

as $n \to \infty$. Also, since $|D^i u_n|\varphi(T_k(u_n) - T_k(u))| \rightharpoonup 0$ weakly in $L^{p_i}(\Omega)$, then

$$\varepsilon_4(n) = \sum_{i=1}^N \int_\Omega c_i(x)(1 + |T_n(u_n)|)^{\gamma_i} |D^i u_n|\, |\varphi(T_k(u_n) - T_k(u))| \,\mathrm{d}x \to 0 \tag{4.88}$$

as $n \to \infty$. Moreover, in view of (4.78) and (4.73) we conclude that

$$\varepsilon_5(h) = \frac{1}{h} \sum_{i=1}^N \int_{\{h \leqslant |u_n| \leqslant 2h\}} |\phi_i(x, T_{2h}(u_n))|\, |D^i u_n| \,\mathrm{d}x \to 0 \quad \text{as } h \to \infty, \tag{4.89}$$

and

$$\varepsilon_6(h) = \frac{1}{h} \sum_{i=1}^N \int_{\{h \leqslant |u_n| \leqslant 2h\}} a_i(x, T_{2h}(u_n), D^i u_n) D^i u_n \,\mathrm{d}x \to 0 \tag{4.90}$$

as $h \to \infty$. Concerning the last term on the right-hand side of (4.84), the sequence $(|a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n))|)_n$ is uniformly bounded in $L^{p_i'}(\Omega)$, then there exists a measurable function $\zeta_{2h} \in L^{p_i'}(\Omega)$ such that $|a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n))| \rightharpoonup \zeta_{2h}$ weakly in $L^{p_i'}(\Omega)$ and it follows that

$$\begin{aligned} \varepsilon_7(h) &= \sum_{i=1}^N \int_{\{k < |u_n| \leqslant 2h\}} |a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n))|\, |D^i T_k(u)| \,\mathrm{d}x \\ &\to \sum_{i=1}^N \int_{\{k < |u| \leqslant 2h\}} \zeta_{2h} |D^i T_k(u)| \,\mathrm{d}x = 0 \quad \text{as } h \to \infty. \end{aligned} \tag{4.91}$$

By combining (4.84) and (4.85)–(4.91) we conclude that

$$
\begin{aligned}
(4.92)\qquad & \sum_{i=1}^{N} \int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \\
& \quad \times (D^i T_k(u_n) - D^i T_k(u)) \\
& \quad \times \left(\varphi'(T_k(u_n) - T_k(u)) - \frac{d(|u_n|)}{b(|u_n|)} |\varphi(T_k(u_n) - T_k(u))|\right) \mathrm{e}^{B(|u_n|)} \,\mathrm{d}x \\
& \leqslant \left(\varphi'(2k) + 3\left\|\frac{d(|\cdot|)}{b(|\cdot|)}\right\|_{L^\infty(\Omega)} \varphi(2k)\right) \mathrm{e}^{B(\infty)} \\
& \quad \times \sum_{i=1}^{N} \int_\Omega |a_i(x, T_k(u_n), \nabla T_k(u))|\, |D^i T_k(u_n) - D^i T_k(u)| \,\mathrm{d}x \\
& \quad + 3\mathrm{e}^{B(\infty)} \left\|\frac{d(|\cdot|)}{b(|\cdot|)}\right\|_{L^\infty(\Omega)} \\
& \quad \times \sum_{i=1}^{N} \int_\Omega |a_i(x, T_k(u_n), \nabla T_k(u_n))|\, |D^i T_k(u)|\, |\varphi(T_k(u_n) - T_k(u))| \,\mathrm{d}x \\
& \quad + \varepsilon_8(n, h).
\end{aligned}
$$

For the first term of the right-hand side of (4.92), we have $a_i(x, T_k(u_n), \nabla T_k(u)) \to a_i(x, T_k(u), \nabla T_k(u))$ strongly in $L^{p_i'}(\Omega)$, and since $D^i T_k(u_n) \rightharpoonup D^i T_k(u)$ weakly in $L^{p_i}(\Omega)$, it follows that

$$
(4.93)\qquad \varepsilon_6(n) = \sum_{i=1}^{N} \int_\Omega |a_i(x, T_k(u_n), D^i T_k(u))|\, |D^i T_k(u_n) - D^i T_k(u)| \,\mathrm{d}x \to 0
$$

as $n \to \infty$. Concerning the second term of the right-hand side of (4.92), in view of (3.4) we have $(a_i(x, T_k(u_n), \nabla T_k(u_n)))_n$ is bounded in $L^{p_i'}(\Omega)$, then there exists a measurable function $\vartheta_{k,i}$ in $L^{p_i'}(\Omega)$ such that $a_i(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \vartheta_{k,i}$ in $L^{p_i'}(\Omega)$, and since $|D^i T_k(u)|\, |\varphi(T_k(u_n) - T_k(u))| \to 0$ strongly in $L^{p_i}(\Omega)$, it follows that

$$
\begin{aligned}
(4.94)\qquad \varepsilon_5(n) = & \sum_{i=1}^{N} \int_\Omega |a_i(x, T_k(u_n), D^i T_k(u_n))|\, |D^i T_k(u)| \\
& \times |\varphi(T_k(u_n) - T_k(u))| \,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

By combining (4.92) and (4.93)–(4.94), we conclude that

$$
\begin{aligned}
(4.95)\qquad & \frac{1}{2} \sum_{i=1}^{N} \int_\Omega (a_i(x, T_k(u_n), D^i T_k(u_n)) - a_i(x, T_k(u_n), D^i T_k(u)) \\
& \quad \times (D^i T_k(u_n) - D^i T_k(u)) \,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \sum_{i=1}^{N} \int_\Omega (a_i(x, T_k(u_n), D^i T_k(u_n)) - a_i(x, T_k(u_n), D^i T_k(u))) \\
&\quad \times (D^i T_k(u_n) - D^i T_k(u)) \\
&\quad \times \Bigg(\varphi'(T_k(u_n) - T_k(u)) \\
&\quad - 3 \left\| \frac{d(|\cdot|)}{b(|\cdot|)} \right\|_{L^\infty(\Omega)} |\varphi(T_k(u_n) - T_k(u))| \Bigg) \mathrm{e}^{B(|u_n|)} \,\mathrm{d}x \\
&\leqslant \varepsilon_9(n, h).
\end{aligned}
$$

By letting n tend to infinity, and since $T_k(u_n) \to T_k(u)$ strongly in $L^{p_0}(\Omega)$, we conclude that

$$
\begin{aligned}
(4.96) \qquad & \sum_{i=1}^{N} \int_\Omega (a_i(x, T_k(u_n), D^i T_k(u_n)) - a_i(x, T_k(u_n), D^i T_k(u))) \\
& \quad \times (D^i T_k(u_n) - D^i T_k(u)) \,\mathrm{d}x \\
& \quad + \int_\Omega (|T_k(u_n)|^{p_0-2} T_k(u_n) - |T_k(u)|^{p_0-2} T_k(u)) \\
& \quad \times (T_k(u_n) - T_k(u)) \,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

In view of Lemma 4.1 we conclude that

$$
(4.97) \qquad \begin{cases} T_k(u_n) \to T_k(u) & \text{strongly in } \ W^{1,\vec{p}}(\Omega), \\ D^i T_k(u_n) \to D^i T_k(u) & \text{a.e., in } \Omega. \end{cases}
$$

Therefore, we have $a_i(x, T_n(u_n), \nabla u_n) D^i u_n$ tends to $a_i(x, u, \nabla u) D^i u$ almost everywhere in Ω. Thanks to Fatou's Lemma and (4.73), we conclude that

$$
\begin{aligned}
(4.98) \quad & \lim_{h\to\infty} \frac{1}{h} \sum_{i=1}^{N} \int_{\{|u| \leqslant h\}} a_i(x, u, \nabla u) D^i u \,\mathrm{d}x \\
& \leqslant \lim_{h\to\infty} \liminf_{n\to\infty} \frac{1}{h} \sum_{i=1}^{N} \int_{\{|u_n| \leqslant h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n \,\mathrm{d}x \\
& \leqslant \lim_{h\to\infty} \limsup_{n\to\infty} \frac{1}{h} \sum_{i=1}^{N} \int_{\{|u_n| \leqslant h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n \,\mathrm{d}x = 0.
\end{aligned}
$$

Step 5: The equi-integrability of $(H_n(x, u_n, \nabla u_n))_n$. We will prove that

$$
(4.99) \qquad H_n(x, u_n, \nabla u_n) \to H(x, u, \nabla u) \quad \text{strongly in } L^1(\Omega).
$$

Using Vitali's theorem, it is sufficient to show that the sequence $(H_n(x,u_n,\nabla u_n))_n$ is uniformly equi-integrable. Indeed, thanks to (4.75), we have: for all $\varepsilon \geqslant 0$, there exists $h_0(\varepsilon) \geqslant 0$ such that

$$\sum_{i=1}^{N} \int_{\{|u_n|>h\}} d(|u_n|)|D^i u_n|^{p_i}\,\mathrm{d}x \leqslant \frac{\varepsilon}{3} \quad \text{for any } h \geqslant h_0(\varepsilon). \tag{4.100}$$

On the other hand, thanks to (4.97) we have: for any $\varepsilon, h \geqslant 0$ there exists $\beta(\varepsilon,h) > 0$ such that

$$\sum_{i=1}^{N} \int_{E} d(|u_n|)|D^i T_h(u_n)|^{p_i}\,\mathrm{d}x \leqslant \frac{\varepsilon}{3} \qquad \text{for any } E \subset \Omega \text{ with } \operatorname{meas}(E) \leqslant \beta(\varepsilon,h). \tag{4.101}$$

Thus, we conclude that for any $\varepsilon, h \geqslant 0$ there exists $\beta(\varepsilon,h) > 0$ such that

$$\begin{aligned}\int_E |H_n(x,u_n,\nabla u_n)|\,\mathrm{d}x &\leqslant \int_E |f_0(x)|\,\mathrm{d}x + \sum_{i=1}^{N}\int_E d(|u_n|)|D^i T_h(u_n)|^{p_i}\,\mathrm{d}x \\ &\quad + \sum_{i=1}^{N}\int_{\{|u_n|>h\}} d(|u_n|)|D^i u_n|^{p_i}\,\mathrm{d}x \\ &\leqslant \varepsilon.\end{aligned} \tag{4.102}$$

For any measurable subset $E \subset \Omega$ with $\operatorname{meas}(E) \leqslant \beta(\varepsilon)$, the sequence $(H_n(x,u_n,\nabla u_n))_n$ is uniformly equi-integrable, since $H_n(x,u_n,\nabla u_n) \to H(x,u,\nabla u)$ almost everywhere in Ω, in view of Vitali's theorem, we obtain

$$H_n(x,u_n,\nabla u_n) \to H(x,u,\nabla u) \quad \text{strongly in } L^1(\Omega). \tag{4.103}$$

Step 6: Passage to limit. Now, we will pass to the limit in our approximate problem. Indeed, let $\varphi \in K_\psi \cap L^\infty(\Omega)$ with $M = k + \|\psi^+\|_\infty$.

By taking $v_n = u_n - \eta T_k(u_n - \varphi)$ as a test function for the approximate problem (4.5), we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i T_k(u_n-\varphi)\,\mathrm{d}x \\ &\qquad + \int_\Omega H_n(x,u_n,\nabla u_n)T_k(u_n-\varphi)\,\mathrm{d}x + \int_\Omega |u_n|^{p_0-2}u_n T_k(u_n-\varphi)\,\mathrm{d}x \\ &\leqslant \int_\Omega f_n T_k(u_n-\varphi)\,\mathrm{d}x + \sum_{i=1}^{N}\int_\Omega \phi_i(x,T_n(u_n))D^i T_k(u_n-\varphi)\,\mathrm{d}x.\end{aligned} \tag{4.104}$$

Firstly, since $\{|u_n - \varphi| \leqslant k\} \subseteq \{|u_n| \leqslant M\}$, then the first term on the left-hand side of (4.104) can be written as follow:

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi)\,\mathrm{d}x \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n))(D^i T_M(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\leqslant k\}}\,\mathrm{d}x \\
&= \sum_{i=1}^{N} \int_\Omega (a_i(x, T_M(u_n), \nabla T_M(u_n)) - a_i(x, T_M(u_n), \nabla\varphi)) \\
&\quad \times (D^i T_M(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\leqslant k\}}\,\mathrm{d}x \\
&\quad + \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla\varphi)(D^i T_M(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\leqslant k\}}\,\mathrm{d}x.
\end{aligned}$$

In view of Fatou's lemma, we obtain

$$\begin{aligned}
(4.105)\quad &\liminf_{n\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla u_n) D^i T_k(u_n - \varphi)\,\mathrm{d}x \\
&= \liminf_{n\to\infty} \sum_{i=1}^{N} \int_\Omega (a_i(x, T_M(u_n), \nabla T_M(u_n)) - a_i(x, T_M(u_n), \nabla\varphi)) \\
&\quad \times (D^i T_M(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\leqslant k\}}\,\mathrm{d}x \\
&\quad + \lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla\varphi)(D^i T_M(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\leqslant k\}}\,\mathrm{d}x \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u), \nabla T_M(u))(D^i T_M(u) - D^i\varphi)\chi_{\{|u-\varphi|\leqslant k\}}\,\mathrm{d}x \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) D^i T_k(u - \varphi)\,\mathrm{d}x.
\end{aligned}$$

Secondly, we have $f_n(x) \to f(x)$ strongly in $L^1(\Omega)$ and $T_k(u_n - \varphi) \rightharpoonup T_k(u - \varphi)$ weak-$*$ in $L^\infty(\Omega)$ and in view of (4.81) and (4.103) we conclude that

$$(4.106)\quad \int_\Omega H_n(x, u_n, \nabla u_n) T_k(u_n - \varphi)\,\mathrm{d}x \to \int_\Omega H(x, u, \nabla u) T_k(u - \varphi)\,\mathrm{d}x$$

$$(4.107)\quad \int_\Omega |u_n|^{p_0-2} u_n T_k(u_n - \varphi)\,\mathrm{d}x \to \int_\Omega |u|^{p_0-2} u T_k(u - \varphi)\,\mathrm{d}x$$

and

$$(4.108)\quad \int_\Omega f_n T_k(u_n - \varphi)\,\mathrm{d}x \to \int_\Omega f T_k(u - \varphi)\,\mathrm{d}x.$$

Moreover, we have $\phi_{i,n}(x, T_M(u_n)) \to \phi_i(x, T_M(u))$ strongly in $L^{p_i'}(\Omega)$, and since $D^i T_k(u_n - \varphi) \rightharpoonup D^i T_k(u - \varphi)$ weak-$*$ in $L^{p_i}(\Omega)$ for $i = 1, \ldots, N$, it follows that

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(x, T_M(u_n)) D^i T_k(u_n - \varphi) \,\mathrm{d}x &\\ \to \sum_{i=1}^{N} \int_\Omega \phi_i(x, T_M(u)) D^i T_k(u - \varphi) \,\mathrm{d}x &\\ = \sum_{i=1}^{N} \int_\Omega \phi_i(x, u) D^i T_k(u - \varphi) \,\mathrm{d}x. & \end{aligned} \tag{4.109}$$

Consequently, by combining (4.104) and (4.105)–(4.109), we deduce that

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) D^i T_k(u - \varphi) \,\mathrm{d}x &\\ + \int_\Omega |u|^{p_0 - 2} u T_k(u - \varphi) \,\mathrm{d}x + \int_\Omega H(x, u, \nabla u) T_k(u - \varphi) \,\mathrm{d}x &\\ \leqslant \int_\Omega f(x) T_k(u - \varphi) \,\mathrm{d}x + \sum_{i=1}^{N} \int_\Omega \phi_i(x, u) D^i T_k(u - \varphi) \,\mathrm{d}x, & \end{aligned} \tag{4.110}$$

for all $\varphi \in K_\psi \cap L^\infty(\Omega)$, which completes proof of Theorem 4.1. □

5. Appendix

P r o o f of Lemma 4.2. In view of Hölder's inequality we have for any $u, v \in W^{1,\vec{p}}(\Omega)$

$$\begin{aligned} |\langle B_m u, v\rangle| \leqslant & \sum_{i=1}^{N} \int_\Omega |a_i(x, T_n(u), \nabla u)|\, |D^i v| \,\mathrm{d}x \\ & + \int_\Omega |T_m(u)|^{p_0 - 1} |v| \,\mathrm{d}x + \frac{1}{m} \int_\Omega |u|^{\underline{p} - 1} |v| \,\mathrm{d}x \\ & + \int_\Omega |H_n(x, u, \nabla u)|\, |v| \,\mathrm{d}x + \sum_{i=1}^{N} \int_\Omega |\phi_{i,n}(x, u)|\, |D^i v| \,\mathrm{d}x \\ \leqslant & \sum_{i=1}^{N} \int_\Omega \beta(K_i(x) + n^{p_i - 1} + |D^i u|^{p_i - 1}) |D^i v| \,\mathrm{d}x + m^{p_0 - 1} \int_\Omega |v| \,\mathrm{d}x \\ & + \frac{1}{m} \|u\|_{L^{\underline{p}}(\Omega)}^{\underline{p} - 1} \|v\|_{L^{\underline{p}}(\Omega)} + n \int_\Omega |v| \,\mathrm{d}x + n \sum_{i=1}^{N} \int_\Omega |D^i v| \,\mathrm{d}x \end{aligned} \tag{5.1}$$

$$\leqslant C_0 \sum_{i=1}^{N} \big(\|K_i(x)\|_{L^{p'_i}(\Omega)} + n^{p_i - 1} + \|D^i u\|^{p_i - 1}_{L^{p_i}(\Omega)} \big) \|v\|_{1,\vec{p}}$$
$$+ (m^{p_0 - 1} + 2n)\|v\|_{1,\vec{p}} + \frac{C_1}{m} \|u\|^{\underline{p}-1}_{1,\vec{p}} \|v\|_{1,\vec{p}}$$
$$\leqslant C_2 \big(1 + \|u\|^{p_0 - 1}_{1,\vec{p}}\big) \|v\|_{1,\vec{p}}.$$

Thus, the operator B_m is bounded. For the coercivity we have

$$\begin{aligned}
\text{(5.2)} \qquad \langle B_m v, v\rangle = {} & \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(v), \nabla v) D^i v \,\mathrm{d}x \\
& + \int_\Omega |T_m(v)|^{p_0 - 1} |v| \,\mathrm{d}x + \frac{1}{m} \int_\Omega |v|^{\underline{p}} \,\mathrm{d}x \\
& + \int_\Omega H_n(x, v, \nabla v) v \,\mathrm{d}x - \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(x, v) D^i v \,\mathrm{d}x \\
\geqslant {} & \frac{b_0}{(1+n)^\lambda} \sum_{i=1}^{N} \int_\Omega |D^i v|^{p_i} \,\mathrm{d}x + \frac{1}{m} \int_\Omega |v|^{\underline{p}} \,\mathrm{d}x \\
& - n \int_\Omega |v| \,\mathrm{d}x - n \sum_{i=1}^{N} \int_\Omega |D^i v| \,\mathrm{d}x \\
\geqslant {} & \frac{b_0}{(1+n)^\lambda} \sum_{i=1}^{N} \int_\Omega |D^i v|^{p_i} \,\mathrm{d}x + \frac{C_3}{m} \|v\|^{\underline{p}}_{L^1(\Omega)} - 2nC_4 \|v\|_{1,\vec{p}} \\
\geqslant {} & C_5 \|v\|^{\underline{p}}_{1,\vec{p}} - 2nC_4 \|v\|_{1,\vec{p}}.
\end{aligned}$$

Let $v_0 \in K_\psi$. Similarly as in (5.1), we have

$$\text{(5.3)} \qquad |\langle B_m v, v_0\rangle| \leqslant C_6 \big(1 + \|v\|^{p_0 - 1}_{1,\vec{p}}\big) \|v_0\|_{1,\vec{p}}.$$

Since $\underline{p} - 1 > 0$, we deduce that

$$\begin{aligned}
\frac{|\langle B_m v, v - v_0\rangle|}{\|v\|_{1,\vec{p}}} &= \frac{|\langle B_m v, v\rangle|}{\|v\|_{1,\vec{p}}} - \frac{|\langle B_m v, v_0\rangle|}{\|v\|_{1,\vec{p}}} \\
&\geqslant \frac{C_5}{2} \|v\|^{\underline{p}-1}_{1,\vec{p}} - 2nC_4 - \frac{C_6 \big(1 + \|v\|^{p_0 - 1}_{1,\vec{p}}\big) \|v_0\|_{1,\vec{p}}}{\|v\|_{1,\vec{p}}} \\
&\to \infty \quad \text{as } \|v\|_{1,\vec{p}} \to \infty.
\end{aligned}$$

Therefore, the operator B_m is coercive.

Now, it remains to show that the operator B_m is pseudo-monotone. Let $(u_k)_k$ be sequence in $W^{1,\vec{p}}(\Omega)$ such that

$$
\begin{cases} u_k \rightharpoonup u & \text{weakly in } W^{1,\vec{p}}(\Omega), \\ B_m u_k \rightharpoonup \chi & \text{weakly in } (W^{1,\vec{p}}(\Omega))', \\ \limsup\limits_{k\to\infty} \langle B_m u_k, u_k\rangle \leqslant \langle \chi, u\rangle. \end{cases} \tag{5.4}
$$

We will show that $\chi = B_m u$ and $\langle B_m u_k, u_k\rangle \to \langle \chi, u\rangle$ as $k \to \infty$.

In view of the compact embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$, we have $u_k \to u$ strongly in $L^{\underline{p}}(\Omega)$ and a.e., in Ω for a subsequence denoted again $(u_k)_k$.

In view of (3.4) we have $(a_i(x, T_m(u_k), \nabla u_k))_k$ is bounded in $L^{p_i'}(\Omega)$, then there exists a measurable function $\varphi_i \in L^{p_i'}(\Omega)$ such that

$$
a_i(x, T_m(u_k), \nabla u_k) \rightharpoonup \varphi_i \quad \text{weakly in } L^{p_i'}(\Omega). \tag{5.5}
$$

Also, we have

$$
\frac{1}{m}|u_k|^{\underline{p}-2}u_k \to \frac{1}{m}|u|^{\underline{p}-2}u \quad \text{strongly in } L^{\underline{p}'}(\Omega). \tag{5.6}
$$

Furthermore, in view of the Lebesgue's dominated convergence theorem we obtain

$$
|T_m(u_k)|^{p_0-2}T_m(u_k) \to |T_m(u)|^{p_0-2}T_m(u) \quad \text{strongly in } L^{\underline{p}'}(\Omega) \tag{5.7}
$$

and

$$
\phi_{i,n}(x, u_k) \to \phi_{i,n}(x, u) \quad \text{strongly in } L^{p_i'}(\Omega). \tag{5.8}
$$

Moreover, we have the $(H_n(x, u_k, \nabla u_k))_k$ is a uniformly bounded sequence in $L^{\underline{p}'}(\Omega)$, then there exists a measurable function $\psi_n \in L^{\underline{p}'}(\Omega)$ such that

$$
H_n(x, u_k, \nabla u_k) \rightharpoonup \psi_n \quad \text{weakly in } L^{\underline{p}'}(\Omega). \tag{5.9}
$$

By combining (5.5)–(5.9) we conclude that for any $v \in W^{1,\vec{p}}(\Omega)$

$$
\begin{aligned}
\langle \chi, v\rangle &= \lim_{k\to\infty} \langle B_m u_k, v\rangle \\
&= \lim_{k\to\infty} \Bigg(\sum_{i=1}^{N} \int_\Omega a_i(x, T_m(u_k), \nabla u_k) D^i v \,\mathrm{d}x \\
&\quad + \int_\Omega |T_m(u_k)|^{p_0-2}T_m(u_k) v \,\mathrm{d}x + \frac{1}{m}\int_\Omega |u_k|^{\underline{p}-2}u_k v \,\mathrm{d}x
\end{aligned} \tag{5.10}
$$

$$+\int_\Omega H_n(x,u_k,\nabla u_k)v\,\mathrm{d}x-\sum_{i=1}^N\int_\Omega \phi_{i,n}(x,u_k)D^iv\,\mathrm{d}x\Big)$$
$$=\sum_{i=1}^N\int_\Omega \varphi_i D^iv\,\mathrm{d}x+\int_\Omega |T_m(u)|^{p_0-2}T_m(u)v\,\mathrm{d}x$$
$$+\frac{1}{m}\int_\Omega |u|^{\underline{p}-2}uv\,\mathrm{d}x+\int_\Omega \psi_n v\,\mathrm{d}x-\sum_{i=1}^N\int_\Omega \phi_{i,n}(x,u)D^iv\,\mathrm{d}x.$$

Using (5.4) and (5.10), we conclude that

$$\begin{aligned}\limsup_{k\to\infty}\langle B_m u_k,u_k\rangle &= \limsup_{k\to\infty}\Big(\sum_{i=1}^N\int_\Omega a_i(x,T_m(u_k),\nabla u_k)D^iu_k\,\mathrm{d}x\\&\quad+\int_\Omega |T_m(u_k)|^{p_0-1}\,|u_k|\,\mathrm{d}x+\frac{1}{m}\int_\Omega |u_k|^{\underline{p}}\,\mathrm{d}x\\&\quad+\int_\Omega H_n(x,u_k,\nabla u_k)u_k\,\mathrm{d}x-\sum_{i=1}^N\int_\Omega \phi_{i,n}(x,u_k)D^iu_k\,\mathrm{d}x\Big)\\&\leqslant \sum_{i=1}^N\int_\Omega \varphi_iD^iu\,\mathrm{d}x+\int_\Omega |T_m(u)|^{p_0-1}\,|u|\,\mathrm{d}x\\&\quad+\frac{1}{m}\int_\Omega |u|^{\underline{p}}\,\mathrm{d}x+\int_\Omega \psi_n u\,\mathrm{d}x-\sum_{i=1}^N\int_\Omega \phi_{i,n}(x,u)D^iu\,\mathrm{d}x.\end{aligned}\tag{5.11}$$

In view of (5.6)–(5.9), we have

$$\begin{aligned}&\int_\Omega |T_m(u_k)|^{p_0-1}\,|u_k|\,\mathrm{d}x+\frac{1}{m}\int_\Omega |u_k|^{\underline{p}}\,\mathrm{d}x\\&\qquad\to\int_\Omega |T_m(u)|^{p_0-1}\,|u|\,\mathrm{d}x+\frac{1}{m}\int_\Omega |u|^{\underline{p}}\,\mathrm{d}x\quad\text{as } k\to\infty,\end{aligned}\tag{5.12}$$

and

$$\begin{aligned}&\int_\Omega H_n(x,u_k,\nabla u_k)u_k\,\mathrm{d}x-\sum_{i=1}^N\int_\Omega \phi_{i,n}(x,u_k)D^iu_k\,\mathrm{d}x\\&\qquad\to\int_\Omega \psi_n u\,\mathrm{d}x-\sum_{i=1}^N\int_\Omega \phi_{i,n}(x,u)D^iu\,\mathrm{d}x\quad\text{as } k\to\infty.\end{aligned}\tag{5.13}$$

Thus, having in mind (5.4), (5.10) and (5.11), we get

$$\limsup_{k\to\infty}\sum_{i=1}^N\int_\Omega a_i(x,T_m(u_k),\nabla u_k)D^iu_k\,\mathrm{d}x\leqslant\sum_{i=1}^N\int_\Omega \varphi_iD^iu\,\mathrm{d}x.\tag{5.14}$$

On the other hand, thanks to (3.3) we have

$$\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_m(u_k),\nabla u_k)-a_i(x,T_m(u_k),\nabla u))(D^iu_k-D^iu)\,\mathrm{d}x\geqslant 0, \tag{5.15}$$

then

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_m(u_k),\nabla u_k)D^iu_k\,\mathrm{d}x &\geqslant \sum_{i=1}^{N}\int_{\Omega}a_i(x,T_m(u_k),\nabla u)(D^iu_k-D^iu)\,\mathrm{d}x\\ &\quad+\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_m(u_k),\nabla u_k)D^iu\,\mathrm{d}x.\end{aligned} \tag{5.16}$$

Thanks to (5.9), we get

$$\liminf_{k\to\infty}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_m(u_k),\nabla u_k)D^iu_k\,\mathrm{d}x\geqslant\sum_{i=1}^{N}\int_{\Omega}\varphi_iD^iu\,\mathrm{d}x. \tag{5.17}$$

Combining (5.14) and (5.17), we conclude that

$$\lim_{k\to\infty}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_m(u_k),\nabla u_k)D^iu_k\,\mathrm{d}x=\sum_{i=1}^{N}\int_{\Omega}\varphi_iD^iu\,\mathrm{d}x. \tag{5.18}$$

Thus, in view of (5.12), (5.13) and (5.18), we obtain

$$\lim_{k\to\infty}\langle B_mu_k,u_k\rangle=\langle\chi_m,u\rangle. \tag{5.19}$$

Moreover, thanks to (5.18) we have

$$\lim_{k\to\infty}\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_m(u_k),\nabla u_k)-a_i(x,T_m(u),\nabla u))(D^iu_k-D^iu)\,\mathrm{d}x=0, \tag{5.20}$$

and since $u_k\to u$ strongly in $L^{\underline{p}}(\Omega)$, by applying Lemma 4.1 we deduce that

$$\begin{cases}u_k\to u & \text{strongly in } W^{1,\vec{p}}(\Omega),\\ D^iu_k\to D^iu & \text{a.e., in } \Omega.\end{cases}$$

Then $a_i(x,T_m(u_k),\nabla u_k)\to a_i(x,T_m(u),\nabla u)$ and $H_n(x,u_k,\nabla u_k)\to H_n(x,u,\nabla u)$ almost everywhere in Ω, and thanks to (3.4) we conclude that

$$a_i(x,T_m(u_k),\nabla u_k)\rightharpoonup a_i(x,T_m(u),\nabla u)\quad\text{weakly in } L^{p_i'}(\Omega), \tag{5.21}$$

and

$$H_n(x, u_k, \nabla u_k) \rightharpoonup H_n(x, u, \nabla u) \quad \text{weakly in } L^{\underline{p}'}(\Omega). \tag{5.22}$$

Then, in view of (5.6)–(5.8) and (5.18) we deduce that $B_m u = \chi$. Thus, the proof of Lemma 4.2 is concluded. □

6. Conclusion

In this paper, we have studied the existence of weak solutions for the unilateral problem associated to our Neumann elliptic equation (3.2) with L^∞-data. Furthermore, we have proved the existence of entropy solutions in the case of L^1-data. However, the existence of entropy solutions for our problem without the term $|u|^{p_0-2}u$ remains an open problem.

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