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ON NON-BAIRE RARE SETS IN CATEGORY BASES

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Abstract. We deal with non-Baire rare sets in category bases which forms $\aleph_0$-independent family, where a rare set is a common generalization of both Luzin and Sierpinski set.

Keywords: point-meager Baire base; countable chain condition; rare set; strictly $\aleph_0$-independent family; cofinality of cardinals; Ulam matrix

MSC 2020: 03E10, 03E50, 28A05, 54A05, 54E52

1. Introduction

In 1914, Luzin [4] constructed using continuum hypothesis ($2^{\aleph_0} = \aleph_1$) an uncountable set of reals having countable intersection with every set of first category. A similar construction was given by Mahlo [4] a year before in 1913, but in literature this set is commonly known as the Luzin set probably because Luzin investigated these types of sets more thoroughly and proved a number of its important properties.

The dual of a Luzin set is the Sierpinski set constructed by Sierpinski [4] in 1924 using the same continuum hypothesis. It is an uncountale set of reals having countable intersection with every set of Lebesgue measure zero. From the dual nature of σ-ideal of first category sets and the σ-ideal of Lebesgue null sets and their complementarity in the real line, it follows that the nature of the above two sets are dual of each other. For example, a Luzin set is a set of measure zero (in fact, they are universally null sets) [4], whereas a Sierpinski set is a set of first category; no uncountable subset of a Luzin set can have the Baire property, whereas no uncountable subset of a Sierpinski set is Lebesgue measurable, etc. [4] and they follow from some instrisic properties of the real line that are closely connected with measure and

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category. This paper involves the notion of 'rare set' in a category base which is a common generalization of both a Luzin set and its dual the Sierpinski set.

2. Preliminaries and results

The concept of a category base is a generalization of both measure and topology. Its main objective is to present both measure and Baire category (topology) and also some other aspects of point set classification within a common framework. It was introduced by Morgan II in the seventies of the last century and since then has been developed through a series of papers [1], [7], [6], [10], etc.

Definition 2.1 ([8]). A pair $(X, \mathcal{C})$, where X is a nonempty set and $\mathcal{C}$ is a family of subsets of X, is called a category base if the nonempty members of $\mathcal{C}$ called regions satisfy the following set of axioms:

(1) Every point of X belongs to some region; i.e., $X = \cup\mathcal{C}$.
(2) Let A be a region and $\mathcal{D}$ be any nonempty family of disjont regions having cardinality less than the cardinality of $\mathcal{C}$.
   (i) If $A \cap (\cup\mathcal{D})$ contains a region, then there exists a region $D \in \mathcal{D}$ such that $A \cap D$ contains a region.
   (ii) If $A \cap (\cup\mathcal{D})$ contains no region, then there exists a region $B \subseteq A$ which is disjoint from every region in $\mathcal{D}$.

Definition 2.2 ([8]). In a category base $(X, \mathcal{C})$, a set is called 'singular' if every region contains a subregion which is disjoint from the set itself. A set which can be expressed as countable union of singular sets is called 'meager'. Otherwise, it is called 'abundant'. In particular, it is abundant everywhere (or everywhere in a region) if the set is abundant in every region (or in every subregion of it). A set is 'Baire' (or a Baire set) if every region contains a subregion in which the set or its complement is meager.

Definition 2.3 ([8]). A category base $(X, \mathcal{C})$ is called 'point-meager' if every singleton set in it is meager and a 'Baire base' if every region in it is abundant. It satisfies countable chain condition if every subfamily of disjoint regions in it is countable.

In a category base, a set is called a 'rare set' [8] if its intersection with every meager set is countable. If a category base is point-meager satisfying countable chain condition and the family of all regions has cardinality at most $2^{\aleph_0}$, then under the assumption of continuum hypothesis it can be shown that every abundant set contains a rare set of cardinality $2^{\aleph_0}$ [8]. In a perfect base [8] satisfying countable chain condition, every rare set is non-Baire [8]. In generalizing a result of Sierpinski, Morgan showed (Theorem 15, III, Chapter 2, [8]) that under the assumptions stated above, every abundant set is representable as the union of $2^{\aleph_1}$ non-Baire rare sets such that the intersection of any two different set is countable. In this paper, we prove a result of a different flavour dealing with $\aleph_0$-independent family of non-Baire rare sets in any abundant Baire subset of a point-meager base satisfying countable chain condition.

Definition 2.4. In a category base $(X, \mathcal{C})$, a set F is a full subset of a set E if $F \subseteq E$ and for every abundant Baire set B, $B \cap F$ is abundant whenever $B \cap E$ is so. If E is a Baire set, this is equivalent to stating that $E - F$ cannot contain any abundant Baire set.

The above definition formulated in the pattern of Grzegorek and Labuda provides a common generalization of two analogous concepts of full subsets in measure and category within the common framework of category bases.

As stated above, our result in this paper involves $\aleph_0$-independent family of sets. As a particular case of the general definition given in [9], a family $\{A_i\colon\ i \in I\}$ of sets in X is called $\aleph_0$-independent (or strictly $\aleph_0$-independent) if for each set J having $\operatorname{card}(J) < \aleph_0$ (or $\operatorname{card}(J) \leqslant \aleph_0$) and every function

$$f\colon\ J \mapsto \{0,1\},\ \bigcap\{A_j^{f(j)}\colon\ j \in J\} \neq \emptyset,$$

where $A_j^{f(j)} = A_j$ if $f(j) = 0$ and $A_j^{f(j)} = X - A_j$ if $f(j) = 1$.

The existence of a $\aleph_0$-independent family of subsets of an infinite set E was solved by Tarski [5]. He showed that such a family exists and has cardinality $2^{\operatorname{card}(E)}$. The result has many important applications [2], [9] in solving problems related to measure extension.

Within the framework of category bases, the idea of a $\aleph_0$-independent (or strictly $\aleph_0$-independent) family can be made further strong. We say that:

Definition 2.5. A family $\{A_i\colon\ i \in I\}$ of subsets of an abundant set E is $\aleph_0$-independent (or strictly $\aleph_0$-independent) in E with respect to a category base $(X, \mathcal{C})$ if for each set J with $\operatorname{card}(J) < \aleph_0$ (or $\operatorname{card}(J) \leqslant \aleph_0$) and every function $f \mapsto \{0,1\}$, the set $\bigcap\{A_j^{f(j)}\colon\ j \in J\}$ is a full subset of E.

Let Φ be a set of one-to-one mappings of X onto X which is closed with respect to the composition of mappings and formation of inverses. We say that a family $\mathcal{S}$ of subsets of X is invariant under Φ or simply Φ-invariant [8] if $\phi(\mathcal{S}) = \mathcal{S}$ for all $\phi \in \Phi$, where $\phi(\mathcal{S}) = \{\phi(S)\colon\ S \in \mathcal{S}\}$. It is easy to check that in a category base $(X, \mathcal{C})$, if $\mathcal{C}$ is a Φ-invariant family, then so are the families of meager, abundant and Baire sets [8].

Our theorem in this paper is as follows.

Theorem 2.6. *Let $(X, \mathcal{C})$ be a point-meager, Baire base satisfying countable chain condition (ccc) and the cardinality of the class Φ is at most $2^{\aleph_0}$. Moreover, let $\mathcal{C}$ be Φ-invariant having cardinality at most $2^{\aleph_0}$ and satisfying the condition that for any two regions C and D, there exists $\phi \in \Phi$ such that $C \cap \phi(D)$ contains a region. Then under continuum hypothesis in every abundant Baire set E satisfying $\phi(E) = E$ for all $\phi \in \Phi$, there exists a family of non-Baire rare sets having cardinality $2^{\aleph_1}$ such that for every set A in this family and every $\phi \in \Phi$, $\phi(A)$ is also a rare set and this family is strictly $\aleph_0$-independent and hence $\aleph_0$-independent in E with respect to $(X, \mathcal{C})$.*

A proof of the above theorem depends on the following lemma, which is a particular case of Proposition 2.10 stated in [9].

Lemma 2.7. *If E is an infinite set satisfying the condition $(\operatorname{card} E)^{\aleph_0} = \operatorname{card} E$, then there exists a maximal strictly $\aleph_0$-independent family $\{A_i \colon i \in I\}$ of subsets of E such that $\operatorname{card}(I) = 2^{\operatorname{card}(E)}$.*

In our proof, we also utilize the concept of an Ulam $(\aleph_0, \aleph_1)$-matrix [3], the definition of which is given below.

Definition 2.8. Let E be an infinite set with $\operatorname{card}(E) = \aleph_1$. A double family $(E_{\xi,\zeta})_{\xi<\aleph_0,\zeta<\aleph_1}$ of subsets of E is called an Ulam $(\aleph_0, \aleph_1)$-matrix over E if the following two conditions are satisfied:

(1) $\operatorname{card}(E - \bigcup\{E_{\xi,\zeta} \colon \xi < \aleph_0\}) \leqslant \aleph_0$ for every $\zeta < \aleph_1$.

(2) $E_{\xi,\zeta} \cap E_{\xi,\zeta'} = \emptyset$ for all $\xi < \aleph_0$ and any two distinct ordinals $\zeta < \aleph_1$, $\zeta' < \aleph_1$.

P r o o f of Theorem 2.6. Since $\mathcal{C}$ satisfies countable chain condition having cardinality at most $2^{\aleph_0}$, every meager set in $(X, \mathcal{C})$ is contained in a $\mathcal{K}_{\delta\sigma}$-meager set (Theorem 5, II, Chapter 1, [8]), where $A \in \mathcal{K}$ iff $X - A \in \mathcal{C}$ and moreover as X is abundant, $\operatorname{card}(\mathcal{K}_{\delta\sigma}) = 2^{\aleph_0}$ (Theorem 1, I, Chapter 2, [8]). Let Ω be the smallest ordinal representing $\operatorname{card}(E)$ (we shall use the same notations $\aleph_0$, $\aleph_1$ for the smallest ordinals representing the cardinals $\aleph_0$ and $\aleph_1$) and consider the well orderings

$$\begin{array}{ll} x_1, x_2, \ldots, x_\alpha, \ldots & (\alpha < \Omega), \\ R_1, R_2, \ldots, R_\alpha, \ldots & (\alpha < \aleph_1), \\ \phi_1, \phi_2, \ldots, \phi_\alpha, \ldots & (\alpha < \aleph_1) \end{array}$$

of E, the family of all $\mathcal{K}_{\delta\sigma}$-meager sets and the family Φ of all mappings. These arrangements are justified, because according to the continuum hypothesis, we have $2^{\aleph_0} = \aleph_1$.

We now proceed to construct a subset of E in the following manner by defining a transfinite sequence

$$y_1, y_2, \ldots y_\alpha, \ldots (\alpha < \aleph_1).$$

Select arbitrarily a point x_1 from E and put $y_1 = x_1$. Assume that for any ordinal α ($\alpha < \aleph_1$) we have already selected elements y_1, y_2, …, y_β ($\beta < \alpha$). Let G_α be the group generated by the mappings ϕ_β ($\beta < \alpha$), i.e., G_α consists of all the elements which are of the form $\phi_{\alpha_1}^{n_1} * \phi_{\alpha_2}^{n_2} * \ldots * \phi_{\alpha_k}^{n_k}$, where $k \in \mathbb{N}$ (the set of naturals), n_1, n_2, …, n_k are integers and ordinals α_1, α_2, …, α_k ($< \alpha$).

We set $P_\alpha = \bigcup_{\beta<\alpha} \{\psi^{-1}(R_\beta)\colon\ \psi \in G_\alpha\}$ and $B_\alpha = \{\psi(y_\beta)\colon\ \beta < \alpha, \psi \in G_\alpha\}$. The set B_α is countable and hence meager by hypothesis. Also P_α is meager because $\mathcal{C}$ is a Φ-invariant family. Hence $P_\alpha \cup B_\alpha$ is meager. We choose y_α as the first element from E which lies outside $P_\alpha \cup B_\alpha$.

The sets B_α ($1 < \alpha < \aleph_1$) along with $B_1 = \emptyset$ form an increasing transfinite sequence, i.e., $B_\beta \subseteq B_\alpha$ if $\beta < \alpha < \aleph_1$ and $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$ if α is a limit ordinal. We set $O_\alpha = B_{\alpha+1} - B_\alpha$. Since $y_\alpha = \psi^{(0)} y_\alpha$, so $y_\alpha \in O_\alpha$ and hence, $O_\alpha \neq \emptyset$ for every α. Moreover, the sets O_α are mutually disjoint, i.e., $O_\alpha \cap O_\beta = \emptyset$ if $\alpha \neq \beta$ by construction. We set $A = \bigcup_{\alpha<\aleph_1} O_\alpha$, which is obviously an abundant set contained in E.

Now we consider the Ulam $(\aleph_0, \aleph_1)$-matrix $(\Pi_{\xi,\zeta})_{\xi<\aleph_0,\zeta<\aleph_1}$ on $\aleph_1$ and set $E_{\xi,\zeta} = \bigcup_{\gamma\in\Pi_{\xi,\zeta}} O_\gamma$. Then there exists ξ_0 and a subset Θ of $\aleph_1$ having $\mathrm{card}(\Theta) = \aleph_1$ such that $E_{\xi_0,\zeta}$ is abundant in $(X, \mathcal{C})$ for every $\zeta \in \Theta$ and for any two ζ, ζ' ($\zeta \neq \zeta'$), $E_{\xi_0,\zeta} \cap E_{\xi_0,\zeta'} = \emptyset$. This is so because $(X, \mathcal{C})$ is point-meager, A is abundant and the class of meager sets in $(X, \mathcal{C})$ forms a σ-ideal. By continuum hypothesis, $2^{\aleph_0} = \aleph_1$ and therefore ${\aleph_1}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = \aleph_1$. So by Lemma 2.7, there exists a strictly $\aleph_0$-independent family $\{\Theta_i\colon\ i \in I\}$ of subsets of Θ such that $\mathrm{card}(I) = 2^{\aleph_1}$. Consequently, for every set $J \subseteq I$ having $\mathrm{card}(J) \leqslant \aleph_0$ and every function $f\colon\ J \mapsto \{0, 1\}$, $\bigcap_{j\in J} A_j^{f(j)} \neq \emptyset$, where $A_i = \bigcup_{\zeta\in\Theta_i} E_{\xi_0,\zeta}$.

Evidently, from the construction it follows that each A_i is a rare set. Also for every $\psi \in \Phi$ and every A_i in the family $\{A_i\colon\ i \in I\}$, $\psi(A_i) \triangle A_i$ is countable and therefore $\psi(A_i)$ is a rare set. We claim that no A_i can be a Baire set by showing that every abundant Baire set meets any set T which is the union of some $E_{\xi_0,\zeta}$ nonvacously. If possible, let there be an abundant Baire set B such that $B \cap T = \emptyset$. Let B be abundant everywhere in a region C and T be abundant everywhere in a region D. By hypothesis, there exists $\phi^* \in \Phi$ such that $C \cap \phi^*(D)$ contains a region F (say). Consequently, $F - B$ is meager (Theorem 2, III, Chapter 1, [8]) and $(X, \mathcal{C})$ being a Baire base, $B \cap \phi^*(T)$ is abundant. But this is impossible because B is disjoint from T and $\phi^*(T) \triangle T$ is countable and hence meager. Therefore, $B \cap T \neq \emptyset$. But in the above construction we can always choose $B = A_i$ and $T = E_{\xi_0,\zeta_i}$ such that $A_i \cap E_{\xi_0,\zeta_i} = \emptyset$ even when A_i is a Baire set.

Lastly, the family $\{A_i\colon i \in I\}$ is strictly $\aleph_0$-independent with respect to $(X, \mathcal{C})$. We set $T = \bigcap_{j\in J} A_j^{f(j)}$, where $J \subseteq I$ with $\operatorname{card}(J) \leqslant \aleph_0$ and $f\colon J \mapsto \{0,1\}$ as above. Since every set T can be expressed as $T = \bigcup\{E_{\xi_0,\zeta}\colon \zeta \in \bigcap_{j\in J} \Theta_j^{f(j)}\}$, where $\bigcap_{j\in J} \Theta_j^{f(j)} \neq \emptyset$ and each $E_{\xi_0,\zeta}$ is abundant, so by the same reasoning as given above, T meets every abundant Baire set contained in E. □

R e m a r k 2.9. Let $X = \mathbb{R}$ and Φ represents the group of all translations in $\mathbb{R}$. Then for the category base $(\mathbb{R}, \mathcal{C})$, where $\mathcal{C}$ is the usual topology of $\mathbb{R}$, the hypothesis of Theorem 2.6 is satisfied and in $\mathbb{R}$ we can have a family of Luzin sets without Baire property such that for every set A in this family and every $x \in \mathbb{R}$, $A+x$ is also a Luzin set which obviously follows from the translation invariance of first category sets and the family is strictly $\aleph_0$-independent with respect to the usual topology of $\mathbb{R}$. As it is also true (by regularity of Lebesgue measure) that for every measurable set A of positive measure, $A \cap (A + x)$ is a set of positive measure for every x in an interval around the origin, so for the category base $(\mathbb{R}, \mathcal{C})$, where $\mathcal{C}$ is the family of Lebesgue measurable sets of positive measures, the hypothesis of Theorem 2.6 is satisfied, and so in $\mathbb{R}$ we can have a family of nonmeasurable Sierpinski sets such that for every set A in this family and every $x \in \mathbb{R}$, $A + x$ is also a Sierpinski set, which obviously follows from the translation invariance of measure zero sets and the family is strictly $\aleph_0$-independent with respect to the Lebesgue measure space.

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