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ON SOME PROPERTIES OF TOPOLOGICAL MV-COALGEBRAS

CYRILLE NGANTEU, MAURICE KIANPI, HILAIRE MBIAKOP

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Abstract. We investigate some properties of topological MV-coalgebras, where MV-coalgebras are coalgebras of the functor which assigns every BL-algebra to its MV-center. We show that the limit of the inverse system arising from a family of Boolean deductive systems is isomorphic to its completion, and characterize Hausdorff topological MV-coalgebras. Moreover, we show that the category of topological MV-coalgebras is strong-monotopological over the category of MV-coalgebras. Finally, we establish a coalgebraic link between BL-algebras and DRI-monoids and deduce the (co)completeness of a category of coalgebras over DRI-monoids.

Keywords: BL-algebra; topological MV-coalgebra; topological category

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1. INTRODUCTION

During the last decades, the interest in the theory of coalgebras for a functor have grown rapidly since the work of Aczel [1], which introduced this notion in order to generalize transition systems. In [28] Rutten developed the theory of Universal Coalgebras, which is dual to Universal Algebra according to category theory. Coalgebras arise as suitable objects for describing well known structures, such as Kripke Frames, labeled transition systems and a large variety of automata. Coalgebras are studied mainly in three directions: in computer science, where many important notions such as bisimulations, coinduction or observational equivalence can be defined coalgebraically (see, e.g., [9] or [22]), in coalgebraic logic, since many variations of modal logics possess coalgebraic semantics (see [20], [21], [29]), and as a development of category theory (see [2], [3], [15], [17]).

BL-algebras are the Lindenbaum-Tarski algebras of the Basic Fuzzy Logic, invented by Hájek [10]. Important examples of BL-algebras include MV-algebras and

Gödel-algebras. Filters, which coincide in these structures with deductive systems, play an important role in their study and therefore have been extensively investigated (see [11], [18], [32], [34]).

The combination of topology and algebra has been investigated for a large variety of residuated algebraic structures such as MTL-algebras [8], FL_{ew} -algebras [23], MV-algebras [14], and BL-algebras [5], [7] and [35]. It appears that these topological algebras (algebras endowed with a topology with respect to which all operations are continuous) have very good properties, mainly when the topology is induced by a family of filters or ideals. The aim of this paper is to provide a coalgebraic approach of these constructions and results, showing how the structure morphism of a coalgebra influences the (categorical) topologicity of an algebraic structure, in the case of BL-algebras.

We proceed as follows. In Section 2, we recall some definitions needed to understand our results, and in Section 3 we establish some results on (Boolean) deductive systems, which will play a crucial role in the next section. In Section 4, we investigate the properties of topological MV-coalgebras and characterize Hausdorff topological MV-coalgebras. This characterization permits us to show that the MV-center of a BL-algebra is a closed subset with respect to our topology. We also compute the limit of the inverse system introduced by the authors in [24] and show that it is isomorphic to its completion. In Section 5, we investigate the topologicity in the sense of [4] and [16] of topological MV-coalgebras, which leads to the (co)completeness of the category of topological MV-coalgebras. Moreover, we show that the category of topological MV-coalgebras is isomorphic to a category of coalgebras over topological DRI-monoids and deduce the (co)completeness of the latter category.

2. PRELIMINARIES

We recall some definitions and basic results that can be found in [10] and [34].

An algebraic structure $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a *commutative residuated lattice* if it satisfies the following conditions:

- (BL1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;
- (BL2) $(L, *, 1)$ is a commutative monoid;
- (BL3) $*$ is a left adjoint of \rightarrow , that is, $x * z \leq y$ if and only if $z \leq x \rightarrow y$.

A *BL-algebra* is a commutative residuated lattice which satisfies the following:

- (BL4) $x \wedge y = x * (x \rightarrow y)$;
- (BL5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

A BL-algebra L is called a *Gödel-algebra* if $x^2 = x * x = x$ for every $x \in L$. L is called an *MV-algebra* if $\overline{\overline{x}} = x$ for all $x \in L$, where $\overline{x} = x \rightarrow 0$. The

MV-center of a BL-algebra L is the largest MV-algebra contained in L . It is given by $MV(L) = \{\bar{x}; x \in L\}$.

The following holds in any BL-algebra L :

Lemma 2.1 ([31], [34], [32]). *For all $x, y, z \in L$,*

- (1) $x \leq y$ if and only if $x \rightarrow y = 1$;
- (2) $x * y \leq x \wedge y$;
- (3) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;
- (4) if $x \leq y$ then $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$;
- (5) $x \leq y \rightarrow (x * y)$; $x * (x \rightarrow y) \leq y$;
- (6) $x * \bar{x} = 0$;
- (7) $(x * y) \rightarrow z = x \rightarrow (y \rightarrow z)$;
- (8) $1 \rightarrow x = x$; $x \rightarrow 1 = 1$; $x \rightarrow x = 1$; $x \leq y \rightarrow x$; $x \leq \bar{\bar{x}}$; $\bar{\bar{x}} = \bar{x}$;
- (9) $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$;
- (10) $x \rightarrow \bar{\bar{y}} = \bar{y} \rightarrow \bar{x}$.

A *filter* of L is a nonempty subset F of L such that for all $x, y \in L$,

- (F1) $x, y \in F$ implies $x * y \in F$;
- (F2) $x \in F$ and $x \leq y$ imply $y \in F$.

A subset D of a BL-algebra L is called a *deductive system* if

- (DS1) $1 \in D$;
- (DS2) $x \in D$ and $x \rightarrow y \in D$ imply $y \in D$.

Deductive systems have been widely studied in BL-algebras namely to characterize fragments of Basic fuzzy logic (see, e.g., [33] and [34]); it is obvious that for a nonempty subset F of L , F is a deductive system if and only if it is a filter.

Let L_1 and L_2 be two BL-algebras, a map $f: L_1 \rightarrow L_2$ is called a *homomorphism of BL-algebras (BL-morphism)* if $f(0) = 0$ and $f(x \alpha y) = f(x) \alpha f(y)$ for all $\alpha \in \{*, \rightarrow\}$. We obviously have $f(1) = 1$ for any BL-homomorphism f and it is shown in [13] that for any BL-morphism f , $f(x \alpha y) = f(x) \alpha f(y)$ with $\alpha \in \{\vee, \wedge\}$ and if $x \leq y$, then $f(x) \leq f(y)$. The *kernel* of a BL-morphism $f: L_1 \rightarrow L_2$ is the set $\text{Ker}(f) := \{x \in L_1: f(x) = 1\}$. Clearly, f is injective iff $\text{Ker}(f) = \{1\}$.

BL-algebras and BL-morphisms form a category. We will denote it by \mathcal{BL} . The one-element BL-algebra $\{0 = 1\}$ is called the *degenerate BL-algebra* [31], Remark 8 and we will denote it by \mathbf{G}_1 . The two-element nondegenerate BL-algebra $\{0, 1\}$ is called the *trivial BL-algebra* and we will denote it by \mathbf{G}_2 . These two algebras are examples of BL-algebras which are both Gödel-algebras and MV-algebras.

Definition 2.2. Let \mathcal{C} be a category.

- (1) A *coalgebra* for an endofunctor $F: \mathcal{C} \rightarrow \mathcal{C}$ is a pair (A, α) , where A is an object of \mathcal{C} and $\alpha: A \rightarrow F(A)$ is a \mathcal{C} -morphism.
- (2) A *homomorphism* between two coalgebras (A, α) and (B, β) for F is a \mathcal{C} -morphism $f: A \rightarrow B$ such that $\beta \circ f = F(f) \circ \alpha$.
- (3) Coalgebras for F and their homomorphisms form a category denoted by $\mathit{Coalg}(F)$.

Let $\mathbf{i}: \mathbb{M}\mathbb{V} \hookrightarrow \mathcal{B}\mathcal{L}$ be the inclusion functor and $\mathbb{M}\mathbb{V}$ the functor which assigns to each BL-algebra its MV-center and which acts on BL-morphisms as identity. The composite $\mathbf{i} \circ \mathbb{M}\mathbb{V}$ is a covariant endofunctor on the category of BL-algebras, called the *MV-functor*. Therefore $\mathbb{M}\mathbb{V}$ -coalgebras are pairs (L, α) , where L is a BL-algebra and $\alpha: L \rightarrow \mathbb{M}\mathbb{V}(L)$ is a BL-morphism. The category $\mathit{Coalg}(\mathbb{M}\mathbb{V})$ has been introduced by the authors in [24]. The morphisms have been characterized and the (co)completeness of the category have been proved. Other Coalgebras over BL-algebras have been defined (see, e.g., [25]) but in this paper, we focus on $\mathbb{M}\mathbb{V}$ -coalgebras.

Definition 2.3. Let (X, τ) be a topological space and (Y, τ') a topological subspace of X . A continuous map $r: X \rightarrow Y$ is called a *retraction* if $r \circ i = id_Y$, where $i: Y \hookrightarrow X$ is the standard inclusion.

Theorem 2.4. Let $r: (X, \tau) \rightarrow (Y, \tau')$ be a retraction. If (X, τ) is separated, then Y is a closed subset of X .

3. SOME FACTS ABOUT DEDUCTIVE SYSTEMS IN BL-ALGEBRAS

For any deductive system F of a BL-algebra $L = (L, \wedge, \vee, *, \rightarrow, 0, 1)$, we can define a relation θ_F on L as follows: for all $x, y \in L$,

$$(x\theta_F y) \iff ((x \rightarrow y) \wedge (y \rightarrow x) \in F).$$

It is well known that θ_F is a congruence on L (see, e.g., [10]) and since the class of BL-algebras is a variety, the quotient structure L/θ_F is also a BL-algebra for which for all $x, y \in L$, $[x \alpha y] := [x] \alpha [y]$, where $\alpha \in \{\wedge, \vee, *, \rightarrow\}$, and $[x] := [x]_{\theta_F}$. A congruence θ on L is said *induced* by F if $[1]_{\theta} = F$. θ_F is clearly induced by F .

Definition 3.1 ([34], [32]). A deductive system D of a BL-algebra L is called *Boolean* if for all $x \in L$, $x \vee \bar{x} \in D$.

Example 3.2. Consider the Product algebra (see in the introduction section of [32]), defined on the unit interval by: $x * y = x \times y$, and

$$x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{y}{x} & \text{otherwise.} \end{cases}$$

For any $x \in [0; 1]$, $\bar{x} = 1$ if $x = 0$ and $\bar{x} = 0$ if $x \neq 0$. So $x \vee \bar{x} \neq 0$. Thus $]0; 1]$ is a Boolean deductive system of the product algebra.

These deductive systems and their generalizations have been widely studied by many authors (see, e.g., [32], [33] and [34]) and the following has been proved:

Proposition 3.3. *Let L be a BL-algebra and D a deductive system of L . The following are equivalent:*

- (i) D is a Boolean deductive system;
- (ii) L/D is a Boolean algebra;
- (iii) If $\bar{x} \rightarrow x \in D$, then $x \in D$.

We now establish the following result, which will be useful in the next section:

Lemma 3.4. *Let D be a deductive system of a BL-algebra L . For all $x, y \in L$,*

$$x \rightarrow y \in D \text{ implies } \bar{y} \rightarrow \bar{x} \in D.$$

The converse holds when D is Boolean.

Proof. Let $x, y \in L$ such that $x \rightarrow y \in D$. We have by Lemma 2.1 (3),

$$\bar{y} \rightarrow \bar{x} = (y \rightarrow 0) \rightarrow (x \rightarrow 0) = x \rightarrow \bar{\bar{y}}.$$

Since $y \leq \bar{\bar{y}}$, we obtain by Lemma 2.1 (4) that $x \rightarrow y \leq \bar{y} \rightarrow \bar{x}$. Since D is a deductive system, we have $\bar{y} \rightarrow \bar{x} \in D$.

For the converse, suppose that D is Boolean and $\bar{y} \rightarrow \bar{x} \in D$. Then $y \vee \bar{y} \in D$. So by Lemma 2.1 (9),

$$y \vee \bar{y} \leq (\bar{y} \rightarrow y) \rightarrow y.$$

So $(\bar{y} \rightarrow y) \rightarrow y \in D$ and since $\bar{\bar{y}} \leq \bar{y} \rightarrow y$, we obtain $(\bar{y} \rightarrow y) \rightarrow y \leq \bar{\bar{y}} \rightarrow y$. Thus, $\bar{\bar{y}} \rightarrow y \in D$ implying $[y]_D = [\bar{\bar{y}}]_D$. Therefore,

$$\begin{aligned} [x \rightarrow y]_D &= [x]_D \rightarrow [y]_D = [x]_D \rightarrow [\bar{\bar{y}}]_D \\ &= [\bar{y}]_D \rightarrow [\bar{x}]_D \text{ by Lemma 2.1 (10)} = [\bar{y} \rightarrow \bar{x}]_D. \end{aligned}$$

It follows that $(\bar{y} \rightarrow \bar{x}) \rightarrow (x \rightarrow y) \in D$, which implies by hypothesis that $x \rightarrow y \in D$. □

Remark 3.5. Let D be a deductive system of L and $L \xrightarrow{f} L'$ be a BL-morphism. Then for all $x \in L$, $f([x]_D) \subseteq [f(x)]_{\langle f(D) \rangle}$.

Proof. Let $y \in f([x]_D)$. Then there exists $z \in L$ such that

$$y = f(z) \text{ and } (z \rightarrow x) \wedge (x \rightarrow z) \in D.$$

We have

$$\begin{aligned} (y \rightarrow f(x)) \wedge (f(x) \rightarrow y) &= f(z \rightarrow x) \wedge f(x \rightarrow z) \\ &= f((z \rightarrow x) \wedge (x \rightarrow z)) \in f(D) \subseteq \langle f(D) \rangle. \end{aligned}$$

Hence $y \in [f(x)]_{\langle f(D) \rangle}$. □

4. TOPOLOGICAL MV-COALGEBRAS

In this section, we provide MV-coalgebras with a topology based on deductive systems as it has been done by many researchers for universal algebras. We show that in the coalgebraic framework the results remain true if we add some compatibility hypothesis.

Definition 4.1. Let (L, α) be an MV-coalgebra.

- (i) Let τ be a topology on L . $((L, \alpha), \tau)$ is called a *topological MV-coalgebra* if (L, τ) is a topological BL-algebra and α is continuous, i.e., for any $x \in L$ and any subset V of L containing $\alpha(x)$, there exists an open set U containing x such that $\alpha(U) \subseteq V$. If (L, τ) is Hausssdorf, then $((L, \alpha), \tau)$ is called a *Hausssdorf topological MV-coalgebra*.
- (ii) Let D be a deductive system of L . D is said to be α -stable if $\alpha(D) \subseteq D$.

Remark 4.2. Let $((L, \alpha), \tau)$ be a topological MV-coalgebra. Then $(MV(L), \tau'')$ is a topological space, where the open subsets of $MV(L)$ with the topology induced by a family of α -stable deductive systems $(D_i)_{i \in I}$ of $MV(L)$ are in the form $U = [a]_{D_i} \cap MV(L)$, with $a \in L$.

For any MV-coalgebra (L, α) , the class of α -stable deductive systems of L is not empty since it contains $\{1\}$.

A poset (I, \leq) is said to be *upward directed* provided that for any $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$. Let I be an upward directed set and $D = \{D_i, i \in I\}$ be a family of deductive systems of a BL-algebra L . Then D is called a *system of deductive systems* or simply a *system* of L if $i \leq j$ implies $D_j \subseteq D_i$,

for any $i, j \in I$. An *inverse system* in a category \mathcal{C} is a family $(B_i, \varphi_{i,j})_{i,j \in I}$ of objects indexed by an upward directed set I , with a family of morphisms $\varphi_{i,j}: B_i \rightarrow B_j$ for $i \leq j$, satisfying the following conditions:

- (1) $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$ for any $k \leq j \leq i$,
- (2) $\varphi_{i,i} = id_{B_i}$ for any $i \in I$.

Definition 4.3. Let (L, τ) be a topological space. The topology τ is called a *linear topology* on L if there exists a base B for τ such that any element of B containing 1 is a deductive system of L .

Theorem 4.4 ([24]). *Let (L, α) be an $\mathbb{M}\mathbb{V}$ -coalgebra and $D = \{D_i, i \in I\}$ be an α -stable system of L (i.e., each D_i is an α -stable deductive system of $L, i \in I$). Then*

- (i) *the set $B = \{[x]_{D_i}, x \in L, i \in I\}$ is a base for a topology on L and τ_B and the topology induced by B is linear;*
- (ii) *$((L, \alpha), \tau_B)$ is a topological $\mathbb{M}\mathbb{V}$ -coalgebra.*

In the sequel $((L, \alpha), \tau)$ denotes a topological $\mathbb{M}\mathbb{V}$ -coalgebra, where τ is the topology induced by a system $\{D_i, i \in I\}$ of α -stable deductive systems.

Proposition 4.5. *Let τ' be the quotient topology on L/D , where D is an α -stable deductive system of L such that $D \subseteq \bigcap_{i \in I} D_i$. Define $L/D \xrightarrow{\beta} MV(L/D)$ by $\beta([x]_D) = [\alpha(x)]_D$. Then*

- (i) *the canonical surjection $L \xrightarrow{\pi} L/D$ is an open homomorphism between (L, α) and $(L/D, \beta)$;*
- (ii) *$((L/D, \beta), \tau')$ is a topological $\mathbb{M}\mathbb{V}$ -coalgebra.*

Proof. (i) It has been shown in [35], Theorem 3.16 that π is open. On the other hand, it is obviously checked that β is a BL-morphism. We just have to prove that $MV(\pi) \circ \alpha = \beta \circ \pi$. Let $x \in L$. Then

$$\beta \circ \pi(x) = \beta([x]_D) = [\alpha(x)]_D = MV(\pi) \circ \alpha(x)(*).$$

Thus π is a homomorphism of $\mathbb{M}\mathbb{V}$ -coalgebras.

(ii) It is clear that $MV(\pi)$ and α are continuous and therefore, so is $MV(\pi) \circ \alpha$. Let $[x]_D \in L/D$, and $V \subseteq L/D$ such that $\beta([x]_D) \in V$. Then $\beta \circ \pi(x) \in V$, and by (*) we obtain $MV(\pi) \circ \alpha(x) \in V$. Since $MV(\pi) \circ \alpha$ is continuous, there exists an open subset U of L such that $x \in U$ and $MV(\pi) \circ \alpha(U) \subseteq V$, i.e., $\beta \circ \pi(U) \subseteq V$. Since π is open, $\pi(U)$ is an open set. Thus, β is continuous and therefore, $((L/D, \beta), \tau')$ is a topological $\mathbb{M}\mathbb{V}$ -coalgebra. \square

In what follows, we give a characterization of Hausdorff topological \mathbb{MV} -coalgebras. The following lemma will be useful for that purpose.

Lemma 4.6. *Let X be a nonempty subset of $MV(L)$. Then $\overline{X} = \bigcap_{i \in I} (X / \langle \alpha(D_i) \rangle)$, where \overline{X} is the topological closure of X and $X / \langle \alpha(D_i) \rangle = \bigcup_{x \in X} ([x]_{\langle \alpha(D_i) \rangle})$.*

Proof. Let $x \in L$.

$$\begin{aligned}
 x \in \overline{X} &\Leftrightarrow (x \in [a]_{\langle \alpha(D_i) \rangle} \cap MV(L) \text{ implies } [a]_{\langle \alpha(D_i) \rangle} \cap MV(L) \cap X \neq \emptyset \\
 &\hspace{15em} \text{for any } [a]_{\langle \alpha(D_i) \rangle} \in \tau' \\
 &\Leftrightarrow [x]_{\langle \alpha(D_i) \rangle} \cap X \neq \emptyset \quad \text{for any } i \in I \\
 &\Leftrightarrow x \in X / \langle \alpha(D_i) \rangle \quad \text{for any } i \in I \\
 &\Leftrightarrow x \in \bigcap_{i \in I} (X / \langle \alpha(D_i) \rangle).
 \end{aligned}$$

Therefore $\overline{X} = \bigcap_{i \in I} (X / \langle \alpha(D_i) \rangle)$. □

Theorem 4.7. *Let $((L, \alpha), \tau)$ be a topological \mathbb{MV} -coalgebra. $((L, \alpha), \tau)$ is Hausdorff iff $\bigcap_{i \in I} \langle \alpha(D_i) \rangle = \{1\}$.*

Proof. Suppose that $((L, \alpha), \tau)$ is a Hausdorff topological \mathbb{MV} -coalgebra. Then $\{1\}$ is closed. Then by Lemma 4.6, $\{1\} = \overline{\{1\}} = \bigcap_{i \in I} \langle \alpha(D_i) \rangle$.

Conversely, suppose that $\bigcap_{i \in I} \langle \alpha(D_i) \rangle = \{1\}$. Let x and y be distinct elements of L . Then we cannot have both $x \rightarrow y = 1$ and $y \rightarrow x = 1$. Suppose $x \rightarrow y \neq 1$. Then by the hypothesis, there exists $k \in I$ such that $x \rightarrow y \notin \langle \alpha(D_k) \rangle$. Hence $x \notin [y]_{\langle \alpha(D_k) \rangle}$. So $[x]_{\langle \alpha(D_k) \rangle} \cap [y]_{\langle \alpha(D_k) \rangle} = \emptyset$. □

Proposition 4.8. *Let $((L, \alpha), \tau)$ be a topological \mathbb{MV} -coalgebra.*

The function $r: (L, \tau) \rightarrow (MV(L), \tau'')$ such that $r(x) = \overline{x}$, is a retraction.

Proof. By Lemma 2.1 (8), we have $r \circ i = id_{MV(L)}$, where $i: MV(L) \rightarrow L$ is the standard inclusion. It remains to show the continuity of r . For this purpose, we will show that the inverse image of any open set in $(MV(L), \tau'')$ is an open set in (L, τ) . Let $[y]_D \cap MV(L) \in \tau''$, where D is a deductive system and $x \in L$ such that $r(x) \in [y]_D \cap MV(L)$. Then $\overline{x} \in [y]_D$. $[x]_D$ is an open set which contains x . Moreover, let $z \in [x]_D$. Then $(x \rightarrow z) \wedge (z \rightarrow x) \in D$. It follows from Lemma 3.4 that $(\overline{x} \rightarrow \overline{z}) \wedge (\overline{z} \rightarrow \overline{x}) \in D$, so $\overline{z} \in [\overline{x}]_D = [y]_D$. Thus $z \in r^{-1}([y]_D \cap MV(L))$ and then r is continuous. □

The characterization of Hausdorff topological $\mathbb{M}\mathbb{V}$ -coalgebras given in Theorem 4.7 and the above proposition leads to the following:

Corollary 4.9. *Let $((L, \alpha), \tau)$ be a topological $\mathbb{M}\mathbb{V}$ -coalgebra. If $\bigcap_{i \in I} \langle \alpha(D_i) \rangle = \{1\}$, then $MV(L)$ is a closed subset of L .*

Theorem 4.10. *Let (L, α) be an $\mathbb{M}\mathbb{V}$ -coalgebra, $D = \{D_i, i \in I\}$ be an α -stable system of Boolean deductive systems of L . For each $i \in I$, define $\alpha_i: L/D_i \rightarrow \alpha(L/D_i)$ by*

$$\alpha_i([x]_{D_i}) = [\alpha(x)]_{D_i}.$$

Consider the family of maps $(\varphi_{ij})_{i \leq j \in I}$ defined by $\varphi_{ij}: (L/D_j, \alpha_j) \rightarrow (L/D_i, \alpha_i)$ such that for all $x \in L$,

$$\varphi_{ij}([x]_{D_j}) = [x]_{D_i}.$$

Then

- (i) $((L/D_i, \alpha_i)_{i \in I}; (\varphi_{ij})_{i \leq j \in I})$ is an inverse system in $\mathcal{BL}_{\mathbb{M}\mathbb{V}}$,
- (ii) $\varprojlim ((L/D_i, \alpha_i)_{i \in I}; (\varphi_{ij})_{i \leq j \in I}) = ((X, \xi), \phi_j)$, where

$$X = \left\{ ([x]_{D_i})_{i \in I} \in \prod_{i \in I} L/D_i; x_i \in [x_j]_{D_j}, \triangleright < i \right\},$$

$$\xi: X \rightarrow MV(X)$$

$$([x_i]_{D_i})_{i \in I} \mapsto ([\alpha(x_i)]_{D_i})_{i \in I}$$

and

$$\phi_j: X \rightarrow L/D_j$$

$$([x_i]_{D_i})_{i \in I} \mapsto [x_j]_{D_j}.$$

Proof. (i) Similar as in [24].

(ii) It is easily checked that X is a BL-subalgebra of the product $\prod_{i \in I} L/D_i$. Let us show that ξ is a BL-morphism. Let $([x_i]_{D_i})_{i \in I}$ and $([y_i]_{D_i})_{i \in I}$ in X such that $[x_i]_{D_i} = [y_i]_{D_i}$ for all $i \in I$. We have

$$\begin{aligned} \xi(([x_i]_{D_i})_{i \in I}) &= ([\alpha(x_i)]_{D_i})_{i \in I} = (\alpha_i([x_i]_{D_i}))_{i \in I} \\ &= (\alpha_i([y_i]_{D_i}))_{i \in I} = ([\alpha(y_i)]_{D_i})_{i \in I} = \xi(([y_i]_{D_i})_{i \in I}). \end{aligned}$$

Moreover, for all $i \in I$, there exists $z_i \in L$ such that $\alpha(x_i) = \overline{z_i}$. So

$$\xi(([x_i]_{D_i})_{i \in I}) = ([\overline{z_i}]_{D_i})_{i \in I} = \overline{([z_i]_{D_i})_{i \in I}}.$$

Let $j < i$, then $x_i \in [x_j]_{D_j}$ by the definition of X . Hence,

$$(x_i \rightarrow x_j) \wedge (x_j \rightarrow x_i) \in D_j.$$

Since α is a BL-morphism, it follows from the stability of the deductive systems that

$$(\alpha(x_i) \rightarrow \alpha(x_j)) \wedge (\alpha(x_j) \rightarrow \alpha(x_i)) \in D_j.$$

Thus,

$$(\overline{z_i} \rightarrow \overline{z_j}) \wedge (\overline{z_j} \rightarrow \overline{z_i}) \in D_j.$$

This leads by Lemma 3.4 to

$$(z_i \rightarrow z_j) \wedge (z_j \rightarrow z_i) \in D_j.$$

Hence $z_i \in [z_j]_{D_j}$, implying that $([z_i]_{D_i})_{i \in I} \in X$. This means that $\xi((x_i)_{i \in I}) \in MV(X)$ and so, ξ is well defined. Since α is a BL-morphism, ξ preserves the BL-algebras operations and is therefore a BL-morphism.

It is easily checked that ϕ_j is a homomorphism of \mathbb{MV} -coalgebras for all $j \in I$. Let $i, j \in I$ such that $j < i$. For any $(x_i)_{i \in I} \in X$ we have

$$\begin{aligned} \varphi_{ij} \circ \phi_i((x_i)_{i \in I}) &= \varphi_{ij}([\alpha(x_i)]_{D_i}) = [x_i]_{D_j} \\ &= [x_j]_{D_j} \text{ by definition of } X = \phi_j((x_i)_{i \in I}). \end{aligned}$$

Thus, $\varphi_{ij} \circ \phi_i = \phi_j$.

It remains to show the universal property. Let (X', ξ') be a \mathbb{MV} -coalgebra, and $(X' \xrightarrow{\lambda_i} L/D_i)_{i \in I}$ a family of homomorphisms of \mathbb{MV} -coalgebras such that $\varphi_{ij} \circ \lambda_i = \lambda_j$, for all $i, j \in I$ such that $j < i$. Define

$$\begin{aligned} \lambda: X' &\rightarrow X \\ x &\mapsto (\lambda_i(x))_{i \in I}. \end{aligned}$$

For any $x \in X'$, $\lambda_i(x) = [y_i]_{D_i}$ with $y_i \in L$ for all $i \in I$. For any $j \in I$ such that $j < i$, we have $\varphi_{ij} \circ \lambda_i(x) = \lambda_j(x)$. So $\varphi_{ij}([y_i]_{D_i}) = [y_j]_{D_j}$, which implies that $[y_i]_{D_j} = [y_j]_{D_j}$ and then $y_i \in [y_j]_{D_j}$, i.e., $(\lambda_i(x))_{i \in I} \in X$. This proves that λ is well defined. Since λ_i is a BL-morphism, then so is λ . We obviously have $\phi_i \circ \lambda = \lambda_i$ for all $i \in I$.

$$\begin{array}{ccccc}
& & \lambda & & \\
& & \curvearrowright & & \\
X' & \xrightarrow{\lambda_i} & L/D_i & \xleftarrow{\phi_i} & X \\
\downarrow \xi' & & \downarrow \alpha_i & & \downarrow \xi \\
MV(X') & \xrightarrow{MV(\lambda_i)} & MV(L/D_i) & \xleftarrow{MV(\phi_i)} & MV(X) \\
& & \curvearrowleft & & \\
& & MV(\lambda) & &
\end{array}$$

Let $x \in X'$. We have:

$$\begin{aligned}
MV(\lambda) \circ \xi'(x) &= \lambda \circ \xi'(x) = (\lambda_i(\xi'(x)))_{i \in I} \\
&= (\alpha_i \circ \lambda_i(x))_{i \in I} \text{ (by commutativity of the first square)} \\
&= (\alpha_i \circ \phi_i \circ \lambda(x))_{i \in I} \\
&= (MV(\phi_i) \circ \xi \circ \lambda(x))_{i \in I} \text{ (by commutativity of the second square)} \\
&= (\phi_i \circ \xi \circ \lambda(x))_{i \in I} = \xi \circ \lambda(x).
\end{aligned}$$

Thus, λ is an MV -homomorphism.

Let $\lambda': (X', \xi') \rightarrow (X, \xi)$ be another MV -homomorphism such that $\phi_i \circ \lambda' = \lambda_i$ for all $i \in I$. Then $\phi_i \circ \lambda'(x) = \phi_i \circ \lambda(x)$ for all $x \in X'$. Suppose that $\lambda(x) = ([y_k]_{D_k})_{k \in I}$ and $\lambda'(x) = ([y'_k]_{D_k})_{k \in I}$. Then

$$\phi_i([y_k]_{D_k})_{k \in I} = \phi_i([y'_k]_{D_k})_{k \in I}.$$

So $[y_i]_{D_i} = [y'_i]_{D_i}$ for all $i \in I$. Therefore, $\lambda = \lambda'$. □

Let C be the set of all Cauchy sequences on (L, τ) and C_1 the set of all sequences which converge to 1. It has been shown in [35], Theorem 4.7 and Proposition 4.8 that C is a BL-algebra and C_1 is a deductive system of C . Thus, the quotient C/C_1 is a BL-algebra called the *completion* of L with respect to the topology induced by the system of deductive systems. In what follows we equip the completion with a structure morphism and show that the coalgebra obtained is isomorphic to the inverse limit of the system defined above.

Theorem 4.11. *Let (L, α) be an MV -coalgebra. With the notation of Theorem 4.10, (X, ξ) is isomorphic to $(C/C_1, \gamma)$, where $\gamma([(x_i)_i]_{C_1}) = [(\alpha(x_i))_i]_{C_1}$.*

Proof. Since α is continuous, $(\alpha(x_i))_i$ is a Cauchy sequence. Now let $(x_i)_i$ and $(y_i)_i$ be two Cauchy sequences such that $[(x_i)_i]_{C_1} = [(y_i)_i]_{C_1}$. Then $((x_i)_i \rightarrow (y_i)_i) \wedge ((y_i)_i \rightarrow (x_i)_i) \in C_1$. By continuity of α and the fact that α is a BL-morphism, we have $((\alpha(x_i))_i \rightarrow (\alpha(y_i))_i) \wedge ((\alpha(y_i))_i \rightarrow (\alpha(x_i))_i) \in C_1$. Thus γ is well defined. With similar arguments, one proves that γ is a BL-morphism and thus, $(C/C_1, \gamma)$ is an $\mathbb{M}\mathbb{V}$ -coalgebra.

Consider $f: X \rightarrow C/C_1$ defined by $f([(x_i)_{D_i}]_i) = [(x_i)_i]_{C_1}$. Then f is an isomorphism of BL-algebras (see [35], proof of Theorem 4.11). Moreover, $MV(f) \circ \xi([(x_i)_{D_i}]_{i \in I}) = f([(\alpha(x_i)]_{D_i})_{i \in I}) = [(\alpha(x_i))_i]_{C_1} = \gamma \circ f([(x_i)_{D_i}]_{i \in I})$. So f is a homomorphism of coalgebras. Therefore, (X, ξ) and $(C/C_1, \gamma)$ are isomorphic. \square

5. CATEGORICAL PROPERTIES OF TOPOLOGICAL $\mathbb{M}\mathbb{V}$ -COALGEBRAS

Definition 5.1. A concrete category (\mathcal{C}, U) over an (E, \mathbf{M}) -category \mathcal{X} is said to be **\mathbf{M} -topological** provided that every structured source in \mathbf{M} has a unique initial lift. If for example $\mathbf{M} = \text{Mono-Sources}$, the term *monotopological* is used.

The following property of \mathbf{M} -topological categories can be found in [13], Corollary 5.2, and Corollary 6.4:

Lemma 5.2. *Let \mathcal{C} be a \mathbf{M} -topological category over \mathcal{X} . If \mathcal{X} is (co)complete, then so is \mathcal{C} .*

The following result can be found in [16] and will be useful in the sequel.

Lemma 5.3. *If \mathcal{C} is an $(\text{Epi}, \text{StrongMono})$ -category and F preserves strong mono-sources, then $\mathcal{C}_{\mathbb{F}}$ is an $(\text{Epi}, \text{StrongMono})$ -category with $(\text{Epi}, \text{StrongMono})$ -factorizations created by the forgetful functor $U_{\mathbb{F}}$.*

It is well known (see, e.g., [6]) that for any topological functor $\mathbb{V}: \mathcal{C} \rightarrow \mathcal{SET}$, and any forgetful functor $\mathbb{U}: \mathcal{X} \rightarrow \mathcal{SET}$, where \mathcal{X} is a category of universal algebra, the category $\mathcal{C}\mathcal{X}$ with objects (A, X) , $(A \in \text{ob}(\mathcal{C}), X \in \text{ob}(\mathcal{X}))$ such that $\mathbb{V}A = \mathbb{U}X$ is topological over \mathcal{X} . This leads to the following result:

Lemma 5.4. *Let TopBL be the category of topological BL-algebras with continuous BL-morphisms. Then TopBL is topological over \mathcal{BL} .*

The question rising up from the above observation is: Does the result hold if \mathcal{X} is a category of coalgebras over universal algebras? The following theorem is a partial answer in the setting of $\mathbb{M}\mathbb{V}$ -coalgebras.

Theorem 5.5. *The category $\text{TopBL}_{\mathbb{M}\mathbb{V}}$ of topological $\mathbb{M}\mathbb{V}$ -coalgebras and continuous $\mathbb{M}\mathbb{V}$ -homomorphisms is strong mono-topological over $\mathcal{BL}_{\mathbb{M}\mathbb{V}}$.*

Proof. Since \mathcal{BL} is an $(Epi, StrongMono)$ -category and $\mathbb{M}\mathbb{V}$ is limit-preserving, it follows from Lemma 5.3 that $\mathcal{BL}_{\mathbb{M}\mathbb{V}}$ is an $(Epi, StrongMono)$ -category.

Let $\mathbb{T}: \mathcal{Top}\mathcal{BL}_{\mathbb{M}\mathbb{V}} \rightarrow \mathcal{BL}_{\mathbb{M}\mathbb{V}}$ and $\mathbb{U}: \mathcal{Top}\mathcal{BL} \rightarrow \mathcal{BL}$ be the standard forgetful functors. We have to show that \mathbb{T} is strong monotopological.

Let $S = ((L, \alpha) \xrightarrow{f_i} \mathbb{T}((L_i, \alpha_i), \tau_i))_I$ be a \mathbb{T} -structured strong monosource in $\mathcal{BL}_{\mathbb{M}\mathbb{V}}$. Then $(L \xrightarrow{f_i} U(L_i, \tau_i))_I$ is an \mathbb{U} -structured source in \mathcal{BL} . So, by Lemma 5.4 it admits a unique \mathbb{U} -initial lift $(L, \alpha) \xrightarrow{\bar{f}_i} (L_i, \tau_i)_I$. Let us show that $S' = ((L, \alpha), \tau) \xrightarrow{\bar{f}_i} ((L_i, \alpha_i), \tau_i)_I$ is a \mathbb{T} -initial lift of S . It is clear that $S' = TS$.

Let $x \in MV(L)$, and V be a subset of $MV(L)$ such that $\alpha(x) \in V$. Since the diagram

$$\begin{array}{ccc} L & \xrightarrow{f_i} & L_i \\ \alpha \downarrow & & \downarrow \alpha_i \\ MV(L) & \xrightarrow{MV(f_i)} & MV(L_i) \end{array}$$

commutes for all $i \in I$, and by the fact that α_i and f_i are continuous, we obtain that $MV(f_i) \circ \alpha$ is a continuous function and $MV(f_i) \circ \alpha(x) \in f_i(V)$ for all $i \in I$. Hence, there exists an open subset U of L containing x such that

$$MV(f_i) \circ \alpha(U) \subseteq MV(f_i)(V) \quad \text{for all } i \in I,$$

i.e., $f_i \circ \alpha(U) \subseteq f_i(V)$ for all $i \in I$. Since S is a strong monosource, we obtain $\alpha(U) \subseteq V$. Thus, α is continuous and therefore, $((L, \alpha), \tau)$ is a topological $\mathbb{M}\mathbb{V}$ -coalgebra.

Let us show now that S' is \mathbb{T} -initial. Let $S'' = ((L', \alpha'), \tau') \xrightarrow{g_i} ((L_i, \alpha_i), \tau_i)_I$. Let $(L', \alpha') \xrightarrow{h} (L, \alpha)$ be an $\mathbb{M}\mathbb{V}$ -homomorphism such that $TS'' = TS' \circ h$. Then h is a BL-morphism from $\mathbb{U}(L', \tau')$ to $\mathbb{U}(L, \tau)$ such that

$$\mathbb{U}((L', \tau') \xrightarrow{g_i} (L_i, \tau_i))_I = \mathbb{U}((L, \tau) \xrightarrow{\bar{f}_i} (L_i, \tau_i))_I \circ h.$$

Since $((L, \tau) \xrightarrow{\bar{f}_i} (L_i, \tau_i))_I$ is initial, there exists $\bar{h}: (L', \tau') \xrightarrow{\bar{h}} (L, \tau)$ such that

$$((L', \tau') \xrightarrow{g_i} (L_i, \tau_i))_I = ((L, \tau) \xrightarrow{\bar{f}_i} (L_i, \tau_i))_I \circ \bar{h}$$

and $h = U\bar{h}$. \bar{h} is an $\mathbb{M}\mathbb{V}$ -homomorphism since so is h ; moreover, \bar{h} is continuous by construction. Thus, \bar{h} is a $\mathcal{Top}\mathcal{BL}_{\mathbb{M}\mathbb{V}}$ -morphism. It is easily checked that $S' = S \circ \bar{h}$ and $T\bar{h} = h$. Therefore, S' is a T -initial source. The unicity of S' follows from the faithfulness of \mathbb{T} . \square

Since $\mathcal{BL}_{\mathbb{M}\mathbb{V}}$ is complete and cocomplete, the above theorem combined with Lemma 5.2 leads to the following result:

Corollary 5.6. *$\mathcal{Top}\mathcal{BL}_{\mathbb{M}\mathbb{V}}$ is complete and cocomplete.*

We now investigate the link between topological $\mathbb{M}\mathbb{V}$ -coalgebras and topological DRI-monoids. Dually residuated lattices ordered monoids (DRI-monoids for short) have been introduced by Swamy [30] as a generalization of some algebraic structures such as l-groups, Brouwerian and Boolean algebras. These algebraic structures have been widely studied in the literature (see, e.g., [19], [26] or [27]), and a strong link with BL-algebras has been found. We first recall some definitions.

Definition 5.7. A DRI-monoid is an algebra $(A, +, -, \sqcap, \sqcup, 0)$ of signature $(2,2,2,2,0)$ which satisfies the following for any $x, y, z \in A$:

- (i) $(A, +, 0)$ is an Abelian monoid;
- (ii) (A, \sqcap, \sqcup) is a lattice;
- (iii) $(A, +, \sqcap, \sqcup, 0)$ is an l-monoid;
- (iv) if \sqsubseteq denotes the order on A , induced by the lattice (A, \sqcap, \sqcup) , then $x - y$ is the smallest $z \in A$ such that $x \sqsubseteq y + z$;
- (v) $((x - y) \sqcup 0) + y \sqsubseteq x \sqcup y$.

Definition 5.8.

- (i) A DRI-monoid is said to be *bounded* if the underlined lattice is bounded.
- (ii) A DRI-monoid is called *representable* if it is isomorphic to a subdirect product of linearly ordered DRI-monoids (i.e., DRI-chains).

A DRI-monoid $(A, +, -, \sqcap, \sqcup, 0)$ is representable if it satisfies the identity

$$(x - y) \sqcap (y - x) \sqsubseteq 0.$$

In [26], Rachůnek proves that BL-algebras are in a one to one correspondence with bounded representable DRI-monoids. If we denote by $r\mathcal{M}$ the category of bounded representable DRI-monoids, then using Theorem 1 and Theorem 2 in [26], it is easily checked that the following correspondences are functorial and mutually inverses:

$$\begin{aligned} \mathbb{F}: \mathcal{BL} &\rightarrow r\mathcal{M} \\ (L, \wedge, \vee, *, \rightarrow, 0, 1) &\mapsto (L, +, -, \sqcap, \sqcup, \tilde{0}, \tilde{1}), \\ \mathbb{G}: r\mathcal{M} &\rightarrow \mathcal{BL} \\ (A, +, -, \sqcap, \sqcup, \tilde{0}, \tilde{1}) &\mapsto (A, \wedge, \vee, *, \rightarrow, 0, 1), \end{aligned}$$

where \mathbb{F} and \mathbb{G} act on morphisms as identity and for all $x, y \in L$,

$$x \rightarrow y = y - x, \quad x * y = x + y, \quad x \wedge y = x \sqcup y, \quad x \vee y = x \sqcap y, \quad 0 = \tilde{1} \text{ and } 1 = \tilde{0}.$$

Therefore, we have the following:

Lemma 5.9. \mathcal{BL} and $r\mathcal{M}$ are isomorphic categories.

Let $(A, +, -, \sqcap, \sqcup, \tilde{0}, \tilde{1})$ be representable bounded DRI-monoid. A subset of A is called an *ideal* if

- (i) $x, y \in I$ implies $x + y \in I$;
- (ii) $y \in I$ and $x \sqsubseteq y$ imply $x \in I$.

We straightforwardly obtain the following results:

Lemma 5.10. Let L be a BL-algebra and A be a representable bounded DRI-monoid. The following properties hold for any subset I of A and any subset F of L .

- (i) I is an ideal of A iff it is a filter of $\mathbb{G}(A)$;
- (ii) F is a filter of L iff it is an ideal of $\mathbb{F}(L)$.

Let $\mathbb{H} = \mathbb{F} \circ \text{MV} \circ \mathbb{G}$. Then \mathbb{H} is a covariant endofunctor on the category $r\mathcal{M}$. Since \mathbb{F} and \mathbb{G} are inverses, the diagrams

$$\begin{array}{ccc} \mathcal{BL} & \begin{array}{c} \xrightarrow{\mathbb{F}} \\ \xleftarrow{\mathbb{G}} \end{array} & r\mathcal{M} \\ \text{MV} \downarrow & & \downarrow \mathbb{H} \\ \mathcal{BL} & \begin{array}{c} \xrightarrow{\mathbb{F}} \\ \xleftarrow{\mathbb{G}} \end{array} & r\mathcal{M} \end{array}$$

commute, i.e., $\mathbb{H} \circ \mathbb{F} = \mathbb{F} \circ \text{MV}$ and $\text{MV} \circ \mathbb{G} = \mathbb{G} \circ \mathbb{H}$. Now consider the correspondences

$$\begin{aligned} \mathbb{F}_* &: \text{TopCoalg}(\text{MV}) \rightarrow \text{TopCoalg}(\mathbb{H}) \\ (L, L \xrightarrow{\alpha} \text{MV}(L), \tau) &\mapsto (\mathbb{F}(L), \mathbb{F}(L) \xrightarrow{\mathbb{F}(\alpha)} \mathbb{H} \circ \mathbb{F}(L), \tau) \end{aligned}$$

and

$$\begin{aligned} \mathbb{G}_* &: \text{TopCoalg}(\mathbb{H}) \rightarrow \text{TopCoalg}(\text{MV}) \\ (A, A \xrightarrow{\beta} \mathbb{H}(A), \tau) &\mapsto (\mathbb{G}(L), \mathbb{G}(L) \xrightarrow{\mathbb{G}(\alpha)} \text{MV} \circ \mathbb{G}(L), \tau), \end{aligned}$$

where $TopCoalg(\mathbb{H})$ is the category of topological \mathbb{H} -coalgebras in which the topology is defined as the linear topology for \mathbb{MV} -coalgebras, but with ideals. Lemma 5.10 ensures that \mathbb{F}_* and \mathbb{G}_* are well defined on objects.

Let f be a morphism in $TopCoalg(\mathbb{MV})$. Since $\mathbb{H} \circ \mathbb{F} = \mathbb{F} \circ \mathbb{MV}$, $\mathbb{F}_*(f) = f$ is a morphism in $TopCoalg(\mathbb{H})$. So \mathbb{F}_* is well defined on morphisms. It is easily verified that \mathbb{G}_* is well defined too. Moreover, for all $((L, \alpha), \tau) \in TopCoalg(\mathbb{H})$:

$$\begin{aligned} \mathbb{F}_* \circ \mathbb{G}_*((L, \alpha), \tau) &= \mathbb{F}_*(\mathbb{G}(L) \xrightarrow{\mathbb{G}(\alpha)} \mathbb{MV} \circ \mathbb{G}(L), \tau) \\ &= (\mathbb{F} \circ \mathbb{G}(L), \mathbb{F} \circ \mathbb{G}(L) \xrightarrow{\mathbb{F} \circ \mathbb{G}(\alpha)} \mathbb{F} \circ \mathbb{MV}(L), \tau) \\ &= ((L, \alpha), \tau). \end{aligned}$$

By similar computations, one proves that we also have $\mathbb{G}_* \circ \mathbb{F}_* = Id_{TopCoalg(\mathbb{MV})}$. Hence, we obtain the following result:

Theorem 5.11. *TopCoalg(MV) and TopCoalg(H) are isomorphic.*

The above theorem combined with Corollary 5.6 lead to the following:

Corollary 5.12. *TopCoalg(H) is complete and cocomplete.*

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