Mathematica Bohemica

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Mathematica Bohemica, Vol. 151 (2026), No. 2, 249–272

Persistent URL: `http://dml.cz/dmlcz/153623`

151 (2026) MATHEMATICA BOHEMICA No. 2, 249–272

ON MODULE CLASSES OF GENERALIZED SEMIPERFECT MODULES

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Received May 17, 2024. Published online March 10, 2025.
Communicated by Simion Breaz

Abstract. We introduce Rad-cc-supplemented module which generalizes the general concept of co-coatomically-supplemented modules; a module W is Rad-cc-supplemented if each co-coatomic submodule of W has a Rad-supplement in W. In Section 2, we present various properties of these modules. In Section 3, we examine the characterization of modules over commutative domains. In Section 4, we explore the concept of $\oplus$-Rad-cc-supplemented modules, which generalizes a generalized notion of $\oplus$-co-coatomically-supplemented modules in R. Alizade, S. Güngör (2018). A module W is $\oplus$-Rad-cc-supplemented if each co-coatomic $A \leqslant W$ is of a Rad-supplement which is a direct summand of W. In the concluding section of this paper, we investigate into its characteristics by introducing Rad-cc-semiperfect modules.

Keywords: co-coatomic submodules; Rad-cc-supplemented modules; totally Rad-cc-supplemented modules

MSC 2020: 16D10, 16D99

1. Introduction

All rings considered in this paper are associative with an identity element. Every module is considered a unitary left module. Let S be a ring and W be a module meeting these requirements. The notation $A \leqslant W$ indicates that A is a submodule of W. The notation $A \leqslant_{\oplus} W$ means that A is a direct summand of W. $A \leqslant W$ is called as *small* in W provided that $A + B \neq W$ for any proper submodule B of W, denoted by $A \ll W$, and we denote by $\mathrm{Rad}(W)$ the sum of all small submodules of W. In contrast to this concept, a submodule A of W is called *essential* in W, symbolically denoted by $A \trianglelefteq W$, if its intersection with nonzero submodules of W is nonzero. A submodule A of W is called a *cofinite submodule* whenever $\frac{W}{A}$ is finitely generated. W is called *coatomic*, if each proper submodule of W is included

DOI: 10.21136/MB.2025.0053-24

in a maximal submodule in W, see [24]. Furthermore, in [2] co-coatomic submodules are defined as a generalization of cofinite submodules as follows: $A \leqslant W$ is called *co-coatomic*, if the factor module $\frac{W}{A}$ is coatomic. A supplement submodule T of A in W is a minimal element of the set $\{B \leqslant W;\ W = A + B\}$ which is equivalent to $W = A+T$ and $A \cap T \ll T$. A module W is called as (*cofinitely*) *supplemented* when each (cofinite) submodule of W is supplemented ([1], [23]). $A \leqslant W$ is referred to as a *Rad-supplement* of B in W provided that $A + B = W$ and $A \cap B \subseteq \mathrm{Rad}(A)$. The module W is called *amply Rad-supplemented* if, for any $A, B \leqslant W$ with $W = A+B$, B includes a Rad-supplement of A in W. W is called *Rad-supplemented* if each $B \leqslant W$ has a Rad-supplement in W [8]. Additionally, as stated in [18], a module W is considered *cofinitely Rad-supplemented* when each cofinite submodule of W has a Rad-supplement within the module W.

In [2], a broader category of supplemented modules is introduced, referred to as co-coatomically supplemented modules. Specifically, a module W is classified as *co-coatomically supplemented* if each co-coatomic submodule of W has a supplement within W. Construction on this concept in [2], $\oplus$-co-coatomically supplemented modules are presented. A module W is designated as $\oplus$*-co-coatomically supplemented* if each co-coatomic submodule of W has a supplement that is a direct summand of W [3]. In both studies, the properties of these modules are examined in detail.

For convenience, we denote the classes of Rad-cc-semiperfect modules, (amply) Rad-cc-supplemented modules, $\oplus$-Rad-cc-supplemented modules and totally Rad-cc-supplemented modules briefly as $\mathcal{SP}_{\text{Rad-cc}}$, $(\mathcal{AS}_{\text{Rad-cc}})$, $\mathcal{S}_{\text{Rad-cc}}$, $\mathcal{S}_{\oplus\text{-Rad-cc}}$, and $\mathcal{TS}_{\text{Rad-cc}}$ in our work.

Inspired by the definitions given above, in Section 2, we introduce the class $\mathcal{S}_{\text{Rad-cc}}$ of modules W where each co-coatomic submodule has a Rad-supplement in W, if each co-coatomic submodule of W has a Rad-supplement in W. Each cofinite submodule of a module is co-coatomic, as finitely generated modules are also coatomic. So every module which belongs to the class $\mathcal{S}_{\text{Rad(cc)}}$ is cofinitely Rad-supplemented. A module W is called as belonging to the class of $\mathcal{TS}_{\text{Rad-cc}}$ if every submodule of W is in $\mathcal{S}_{\text{Rad-cc}}$. We present the main results related with these concepts. We have shown that if the W module is semilocal with $\mathrm{Rad}(W) \ll W$, the concept of the Rad-supplemented modules coincides with the concept we have defined. Furthermore it has been shown that each w-local module belongs to the class $\mathcal{S}_{\text{Rad-cc}}$, any factor module on finite sum of these modules also belongs to the class $\mathcal{S}_{\text{Rad-cc}}$. It is proven that the necessary and sufficient condition for every module over a left V-ring to belong to the class $\mathcal{S}_{\text{Rad-cc}}$ is that the module has to be semisimple. We also give an example of a module that belongs to the class $\mathcal{S}_{\text{Rad-cc}}$ but is not Rad-supplemented. It is obtained that if the factor module is formed by taking a radical submodule of $W \in \mathcal{S}_{\text{Rad-cc}}$, then the module belongs also to the class $\mathcal{S}_{\text{Rad-cc}}$. In the last part of this section, we build rings which belong to the class $\mathcal{S}_{\text{Rad-cc}}$ as modules over themselves. A characterization of this ring is obtained using the concepts from the literature. It is proved that each module, which belongs to the class $\mathcal{S}_{\text{Rad-cc}}$ defined over a semilocal ring, is generalized semiperfect if and only if the ring is left perfect. Furthermore, it is obtained that the concepts of modules belonging to the class $\mathcal{S}_{\text{Rad-cc}}$ and of supplemented modules coincide for every coatomic module defined over a commutative Noetherian semilocal ring.

In Section 3, we study modules which belong to the class $\mathcal{S}_{\text{Rad-cc}}$ over commutative domains. In Section 4, modules belonging to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$ are investigated and some properties of these modules defined are obtained. It is proved that each w-local module belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$ and, as a result, each direct summand of the w-local module belongs to the class $\mathcal{S}_{\text{Rad-cc}}$. Also, equivalent conditions are given for an indecomposable module W to belong to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$. In addition, it is shown that the factor module formed by taking the fully invariant submodule of a module W which belongs to the class $\mathcal{S}_{\text{Rad-cc}}$ also belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$. In addition the module concept introduced in this section coincides with semisimple module over a left V-ring. Moreover, some properties of this module concept defined over Dedekind domains are also proved. In Section 5, modules that belong to the class $\mathcal{SP}_{\text{Rad-cc}}$ are defined with the help of a generalized projective cover. Again, equivalent conditions are given for a module to be co-coatomically semiperfect. Based on previous studies, similar features in the literature have been studied for this module concept.

2. Module class $\mathcal{S}_{\text{Rad-cc}}$

Definition 2.1. If for each co-coatomic submodule $A \leqslant W$, there exists $T \leqslant B$, such that T is a Rad-supplement of A, the module W is referred as belonging to the class $\mathcal{S}_{\text{Rad-cc}}$. Furthermore, W is referred as belonging to the class $\mathcal{AS}_{\text{Rad-cc}}$ if, for any $B \leqslant W$ such that $W = A + B$ for a co-coatomic submodule $A \leqslant W$, there exists $T \leqslant B$, such that T is a Rad-supplement of A.

Proposition 2.1. *Let W be a semilocal module satisfying the condition*

$$\text{Rad}(W) \ll W.$$

Then $W \in \mathcal{S}_{\text{Rad-cc}}$ if and only if W is Rad-supplemented.

P r o o f. ($\Leftarrow$) It is clear.

($\Rightarrow$) Let W belong to the class $\mathcal{S}_{\text{Rad-cc}}$ and $A \leqslant W$. Since W is semilocal, then $\frac{W}{\operatorname{Rad}(W)}$ is semisimple and so it is coatomic. Additionally,

$$\frac{\frac{W}{\operatorname{Rad}(W)}}{\frac{A+\operatorname{Rad}(W)}{\operatorname{Rad}(W)}} \cong \frac{W}{A+\operatorname{Rad}(W)}$$

is coatomic as it is isomorphic to a factor module of a coatomic module which is coatomic. Hence, $A+\operatorname{Rad}(W) \leqslant W$ is co-coatomic. Based on the hypothesis, a Rad-supplement $T \leqslant A+\operatorname{Rad}(W)$ in W exists satisfying that $W = [A+\operatorname{Rad}(W)]+T$ and $[A+\operatorname{Rad}(W)] \cap T \leqslant \operatorname{Rad}(T)$. Following that, $W = A+T$ is obtained as $\operatorname{Rad}(W) \ll W$. Additionally, we have $A \cap T \leqslant [A+\operatorname{Rad}(W)] \cap T \leqslant \operatorname{Rad}(T)$. Hence, W is a Rad-supplemented module. □

As mentioned in [6], a module W is called *w-local* if it possesses only one exact maximal submodule. It is noteworthy that every local module is w-local. Both of these definitions coincide on modules over a left max ring.

Lemma 2.1. *Any w-local module belongs to the class $\mathcal{S}_{\text{Rad(cc)}}$.*

P r o o f. Let W be a w-local module, and let $A \leqslant W$ be co-coatomic. Since $\frac{W}{A}$ is coatomic, there exists a maximal submodule $\frac{P}{A} \leqslant \frac{W}{A}$ containing every proper submodule of $\frac{W}{A}$. Consequently, $P \leqslant W$ is maximal such that $A \leqslant P$. Following this, we have that $A \leqslant \operatorname{Rad}(W)$ and W itself is a Rad-supplement of A in W. This implies that $W \in \mathcal{S}_{\text{Rad-cc}}$. □

Proposition 2.2. *The class $\mathcal{S}_{\text{Rad-cc}}$ is closed under taking quotients.*

P r o o f. Let W be a module belonging to the class $\mathcal{S}_{\text{Rad-cc}}$ and assume that $A \leqslant W$. For any co-coatomic submodule $\frac{N}{A} \leqslant \frac{W}{A}$,

$$\frac{W}{N} \cong \frac{\frac{W}{A}}{\frac{N}{A}}$$

is coatomic. Thus, $N \leqslant W$ is co-coatomic. Consequently, there exists a Rad-supplement submodule $T \leqslant W$ of N. By applying Lemma 2.3 in [18], $\frac{T+A}{A}$ is a Rad-supplement of $\frac{N}{A}$ in $\frac{W}{A}$. □

Proposition 2.3. *Let W belong to the class $\mathcal{S}_{\text{Rad-cc}}$, let $A \leqslant W$ such that* $\operatorname{Rad}(W) \leqslant A$. *If* $\operatorname{Rad}\left(\frac{W}{A}\right) = 0$, *then each co-coatomic submodule of $\frac{W}{A}$ is a direct summand.*

P r o o f. Consider $\frac{N}{A} \leqslant \frac{W}{A}$ as co-coatomic. Then

$$\frac{\frac{W}{A}}{\frac{N}{A}} \cong \frac{W}{N}$$

is coatomic. Thus, $N \leqslant W$ is co-coatomic. Under the given assumption, consider a submodule T of W where $W = N + T$ and $N \cap T \leqslant \mathrm{Rad}(T)$. From [18], Lemma 2.3, $\frac{T+A}{A}$ is a Rad-supplement of $\frac{N}{A}$ in $\frac{W}{A}$. Since $\mathrm{Rad}(W) \leqslant A$, it follows that $\left(\frac{T+A}{A}\right) \cap \frac{N}{A} = 0$. Hence, $\frac{N}{A} \leqslant_{\oplus} \frac{W}{A}$. □

Corollary 2.1. *Let W belong to the class $\mathcal{S}_{\text{Rad-cc}}$. It can be stated that each co-coatomic submodule of $\frac{W}{\mathrm{Rad}(W)}$ is a direct summand of $\frac{W}{\mathrm{Rad}(W)}$.*

Now, we present the following useful lemma, which allows us to prove that the finite sum of modules belonging to the class $\mathcal{S}_{\text{Rad-cc}}$ also belongs to the class $\mathcal{S}_{\text{Rad-cc}}$.

Lemma 2.2. *Let W belong to the class $\mathcal{S}_{\text{Rad-cc}}$ with $A \leqslant W$. If T is a co-coatomic submodule of W such that $A + T$ is a Rad-supplement in W, then T has a Rad-supplement in W.*

P r o o f. If Y is a Rad-supplement of $A + T$ in W, then we can write $W = (A + T) + Y$ and $(A + T) \cap Y \leqslant \mathrm{Rad}(Y)$. $A \cap (T + Y)$ is a co-coatomic submodule of A, because

$$\frac{\frac{W}{T}}{\frac{T+Y}{T}} \cong \frac{A + (T + Y)}{T + Y} \cong \frac{A}{A \cap (T + Y)}$$

is coatomic. Let U be a Rad-supplement of $A \cap (T + Y)$ in A. Then

$$A = [A \cap (T + Y)] + U$$

and

$$[A \cap (T + Y)] \cap U = U \cap (T + Y) \leqslant \mathrm{Rad}(U).$$

Further, we have

$$W = (A + T) + Y = [A \cap (T + Y)] + U + T + Y = T + (Y + U)$$

and

$$\begin{aligned} T \cap (Y + U) &\leqslant [Y \cap (T + U)] + [U \cap (T + Y)] \\ &\leqslant [Y \cap (T + A)] + [U \cap (T + Y)] \\ &\leqslant \mathrm{Rad}(Y) + \mathrm{Rad}(U) \\ &\leqslant \mathrm{Rad}(Y + U). \end{aligned}$$

As desired, $Y + U$ is a Rad-supplement of T in W. □

Theorem 2.1. *The class $\mathcal{S}_{\text{Rad-cc}}$ is closed under taking finite sums.*

Proof. Let W_1 and W_2 belong to the class $\mathcal{S}_{\text{Rad-cc}}$. It suffices to prove that $W = W_1 + W_2$ belongs to the class $\mathcal{S}_{\text{Rad-cc}}$ when both modules W_1 and W_2 belong to the class $\mathcal{S}_{\text{Rad-cc}}$. Consider A as a co-coatomic submodule of W. Then $W = W_1 + W_2 + A$ can be written, and clearly 0 is a trivial Rad-supplement of W. Additionally, since $W_2 + A$ is co-coatomic in W and W_1 belongs to the class $\mathcal{S}_{\text{Rad-cc}}$, it follows that $W_2 + A$ has a Rad-supplement in W by Lemma 2.2. By applying Lemma 2.2 again, it is obvious that A has a Rad-supplement in W, since A is co-coatomic and W_2 belongs to the class $\mathcal{S}_{\text{Rad-cc}}$. Finally, we conclude that $W \in \mathcal{S}_{\text{Rad-cc}}$. □

Recall from [21] that a module N is called *finite W-generated* if an epimorphism from $W^{(I)}$ to N exists for some finite set I. The next corollary is obtained directly from Proposition 2.2 and Theorem 2.1.

Corollary 2.2. *The class $\mathcal{S}_{\text{Rad-cc}}$ is closed under taking W-generated modules for each W belonging to the class.*

Remember that a ring S is a *left V-ring* if each simple S-module is injective. This condition is equivalent to every S-module W having a zero radical and it is also equivalent to each proper submodule being the intersection of maximal submodules as noted in [14].

Proposition 2.4. *If S is a left V-ring, then the class $\mathcal{S}_{\text{Rad-cc}}$ consists of semisimple modules if and only if W is semisimple.*

Proof. Since left V-rings are characterized by the condition that the radical of every module is zero, then every nonzero module is coatomic.

($\Rightarrow$) Thus, $W \in \mathcal{S}_{\text{Rad-cc}}$ if and only if each submodule of W has a Rad-supplement. Since the radical of every module is zero, Rad-supplements are direct summands, meaning every submodule of W is a direct summand.

($\Leftarrow$) Let W be a semisimple module. Since each submodule of W is a direct summand, each submodule has a Rad-supplement. □

Corollary 2.3. *If S is a left V-ring, then the class $\mathcal{S}_{\text{Rad-cc}}$ is closed under taking direct sums.*

Proof. Let S be a left V-ring and $\{W_i;\ i \in I\}$ be a class of S-modules which also belongs to the class $\mathcal{S}_{\text{Rad-cc}}$ for any index set I. It is a fact that any direct sum of semisimple modules is semisimple, as stated in [11], Corollary 8.1.5. On the other hand, semisimple modules and those modules belonging to the class $\mathcal{S}_{\text{Rad-cc}}$ coincide over left V-rings. Therefore, the claim is clear. □

Corollary 2.4. *Let W be a module over a left V-ring. Then $W \in \mathcal{S}_{\text{Rad-cc}}$ if and only if W is the sum of local submodules of itself.*

Proof. This follows from [1], Proposition 3.6, as $\text{Soc}(W) = \text{Loc}(W)$ where $\text{Loc}(W)$ is the sum of local submodules of W. □

Let us consider the $\mathbb{Z}$-module $\frac{\mathbb{Z}}{8\mathbb{Z}}$ which belongs to the class $\mathcal{S}_{\text{Rad-cc}}$. In contrast, ${}_{\mathbb{Z}}\mathbb{Z}$ does not belong to the class $\mathcal{S}_{\text{Rad-cc}}$. Hence, this example emphasizes that a module W need not belong to the class $\mathcal{S}_{\text{Rad-cc}}$ while a factor module of W can still belong to the class $\mathcal{S}_{\text{Rad-cc}}$.

Now, we investigate the suitable conditions that make the implication given above be valid.

Theorem 2.2. *Let W be a module and let A be a submodule such that $A \in \mathcal{S}_{\text{Rad-cc}}$ of W where $\frac{W}{A}$ is radical. Then $W \in \mathcal{S}_{\text{Rad-cc}}$.*

Proof. Let $N \leqslant W$ be co-coatomic. Thus, so is $\frac{W}{N+A} \cong \frac{\frac{W}{N}}{\frac{N+A}{N}}$. By the assumption, $\frac{W}{N+A}$ is radical, as $\frac{W}{A}$ is radical. Therefore, we can express $W = N + A$. Since $A \in \mathcal{S}_{\text{Rad-cc}}$ and W has the trivial Rad-supplement $\{0\}$, N has a Rad-supplement in W by Lemma 2.2. □

Corollary 2.5. *Let W be a module. Suppose that $\frac{W}{\text{Soc}(W)}$ has no nonzero maximal submodule. Then $W \in \mathcal{S}_{\text{Rad-cc}}$.*

It is evident that each Rad-supplemented module belongs to the class $\mathcal{S}_{\text{Rad-cc}}$. However, the converse is not universally true. To illustrate this, let us provide an example confirming this assertion.

Example 2.1. Consider the non-Noetherian commutative ring formed by the direct product $\prod_{i\geqslant 1}^{\infty} S_i$, where each S_i is a field denoted as S. Suppose that M is a subring of the given non-Noetherian commutative ring, consisting of all sequences $(s_i)_{i\in\mathbb{N}}$ where there exist $s \in S$ and $j \in \mathbb{N}$ with $s_i = s$ for all $i \geqslant j$. Let $W = {}_M M$. Given that $\text{Soc}(W)$ is a maximal submodule of W, it follows that the quotient module $\frac{W}{\text{Soc}(W)}$ does not possess any nonzero maximal submodule. According to Corollary 2.5, this leads to the conclusion that $W \in \mathcal{S}_{\text{Rad-cc}}$. However, it is worth noting that W is not Rad-supplemented, as demonstrated in [17], Example 2.5.

Proposition 2.5. *Let $W \in \mathcal{S}_{\text{Rad-cc}}$. If W includes a maximal submodule, then it also contains a w-local submodule.*

Proof. Consider a maximal submodule P of W. It is evident that P is also a co-coatomic submodule of W. Based on the assumption, there exists $T \leqslant W$ where $W = P + T$ and $P \cap T \subseteq \text{Rad}(T)$. Hence, T is a w-local by [6], Lemma 3.3. □

Now, we give a characterization of the module class $\mathcal{S}_{\text{Rad-cc}}$ based on radical submodules.

Theorem 2.3. *Let W be a module and $A \leqslant W$ be radical. If $\frac{W}{A} \in \mathcal{S}_{\text{Rad-cc}}$, then $W \in \mathcal{S}_{\text{Rad-cc}}$.*

P r o o f. Consider any co-coatomic submodule N of W. It follows that $\frac{W}{N+A} \cong \frac{\frac{W}{N}}{\frac{N+A}{N}} \cong \frac{\frac{W}{A}}{\frac{N+A}{A}}$ is coatomic, and so $\frac{N+A}{A} \leqslant \frac{W}{A}$ is co-coatomic. By the assumption, there is a Rad-supplement submodule of $\frac{N+A}{A}$, say $\frac{T}{A}$, in $\frac{W}{A}$ where $\frac{W}{A} = \frac{N+A}{A} + \frac{T}{A}$ and $\frac{(N+A)\cap T}{A} \leqslant \text{Rad}(\frac{T}{A}) = \frac{\text{Rad}(T)}{A}$ as $A = \text{Rad}(A) \leqslant \text{Rad}(T)$. Therefore, $W = N+T$ and $N \cap T \leqslant \text{Rad}(T)$. Hence, T is a Rad-supplement of N in W. □

Recall that $P(W)$ denotes the sum of all radical submodules of W. Thus, $P(W)$ represents the largest radical submodule contained within W.

Corollary 2.6. *Let W be a module, and consider submodules N and K where $N \subseteq K \subseteq W$. Then $K \in \mathcal{S}_{\text{Rad-cc}}$ if and only if $\frac{K}{P(N)} \in \mathcal{S}_{\text{Rad-cc}}$.*

P r o o f. The conclusion follows from Theorem 2.3. □

Corollary 2.7. *Let W be a module and $N \subseteq W$. Then $W \in \mathcal{S}_{\text{Rad-cc}}$ if and only if $\frac{W}{P(N)} \in \mathcal{S}_{\text{Rad(cc)}}$. In particular, $W \in \mathcal{S}_{\text{Rad-cc}}$ if and only if $\frac{W}{P(W)} \in \mathcal{S}_{\text{Rad-cc}}$.*

P r o o f. This is clear by Corollary 2.6. □

Let us give another characterization for a ring R to belong to the class $\mathcal{S}_{\text{Rad-cc}}$, using the fact that every finitely generated module is coatomic.

Theorem 2.4. *For a ring S, the following statements are equivalent.*

(1) *${}_SS$ is Rad-supplemented.*
(2) *${}_SS \in \mathcal{S}_{\text{Rad-cc}}$.*
(3) *${}_SS$ is cofinitely Rad-supplemented.*

P r o o f. The proof is clear. □

For a finitely generated module W, we can extend the previous theorem in the following manner.

Theorem 2.5. *Let W be a finitely generated module. The following statements are equivalent.*

(1) *W is Rad-supplemented.*
(2) *$W \in \mathcal{S}_{\text{Rad-cc}}$.*
(3) *W is cofinitely Rad-supplemented.*

P r o o f. The proof is straightforward. □

Proposition 2.6. *The following statements are equivalent for a coatomic module W:*

(1) $W \in \mathcal{S}_{\text{Rad-cc}}$.
(2) W *is Rad-supplemented.*

P r o o f. (1) $\Rightarrow$ (2) Let W be a coatomic module. Every submodule of W is co-coatomic, as each factor module of W is coatomic. By the hypothesis, each submodule of W has a Rad-supplement in W. Consequently, W is Rad-supplemented.

(2) $\Rightarrow$ (1) This is clear. □

In accordance with [22], an epimorphism $\varphi\colon F \to W$ is called a *generalized cover* if $\ker(\varphi) \subseteq \text{Rad}(F)$, and a generalized cover $\varphi\colon F \to W$ is referred to as a *projective cover* if F is projective. A *generalized semiperfect* module is a module whose factor modules have a generalized projective cover. It has been established in [22], Theorem 2.2 that each generalized semiperfect module is Rad-supplemented. Given that every Rad-supplemented module is Rad-cc-supplemented, it follows that every semiperfect module belongs to the class $\mathcal{S}_{\text{Rad-cc}}$.

Proposition 2.7. *Let S be a semilocal ring. Every left S-module, which belongs to the class $\mathcal{S}_{\text{Rad-cc}}$, is generalized semiperfect if and only if S is left perfect.*

P r o o f. ($\Leftarrow$) This is evident.

($\Rightarrow$) Conversely, consider W as a left S-module with $W = \text{Rad}(W)$. Since $W \in \mathcal{S}_{\text{Rad(cc)}}$, it is generalized semiperfect by the hypothesis. So there is an onto homomorphism $\varphi\colon P \to W$ where P is projective. Since $\ker(\varphi) \subseteq \text{Rad}(P) \neq P$, $\frac{P}{\ker(\varphi)} \cong W$ has a maximal submodule. Since $W = \text{Rad}(W)$, this implies that $W = 0$. Thus, S is left perfect by [21], 42.9. □

From [21], we retrieve the definition that an S-module W is called *reduced* if every submodule of W contains a maximal submodule, denoted as $P(W) = 0$. It is important to note that for every coatomic S-module W, $\text{Rad}(W) \ll W$. Additionally, it is worth mentioning that every coatomic S-module is reduced over commutative Noetherian rings.

Proposition 2.8. *Let W be a coatomic module over a commutative Noetherian ring. Then $W \in \mathcal{S}_{\text{Rad-cc}}$ if and only if W is supplemented.*

P r o o f. The implication can be deduced from Lemma 4 in reference [19]. □

Proposition 2.9. *Let W be a module over a commutative Noetherian ring and A be a co-coatomic submodule of W. If $W \in \mathcal{S}_{\text{Rad(cc)}}$, then $A \in \mathcal{S}_{\text{Rad-cc}}$.*

P r o o f. Consider $W \in \mathcal{S}_{\text{Rad-cc}}$. We can deduce from Proposition 2.2 that $\frac{W}{A} \in \mathcal{S}_{\text{Rad-cc}}$. Since $A \leqslant W$ is co-coatomic, the factor module $\frac{W}{A}$ is coatomic. According to Proposition 2.8, $\frac{W}{A}$ is Rad-supplemented. Furthermore, due to the coatomic $\frac{W}{A}$, it is also reduced. Consequently, by [19], Proposition 2, we can infer that A is Rad-supplemented. Hence, $A \in \mathcal{S}_{\text{Rad-cc}}$. □

R e m a r k 2.1. In view of Theorem 2.2 and Proposition 2.9, we obtain that the class of the module class $\mathcal{S}_{\text{Rad-cc}}$ is not closed under extensions, in general. Let $0 \to K \to W \to A \to 0$ be a short exact sequence. If $K \leqslant W$ is co-coatomic and A is radical, then the module class $\mathcal{S}_{\text{Rad-cc}}$ is closed under extensions.

Corollary 2.8. *For a module W over a commutative Noetherian ring, the following statements are equivalent:*

(1) $W \in \mathcal{S}_{\text{Rad-cc}}$.
(2) *Each maximal submodule of W belongs to the class $\mathcal{S}_{\text{Rad-cc}}$.*
(3) *Each cofinite submodule of W belongs to the class $\mathcal{S}_{\text{Rad-cc}}$.*

P r o o f. (1) ⇒ (3) Consider a cofinite submodule A of W. Thus, the quotient module $\frac{W}{A}$ is finitely generated. Since $\frac{W}{A}$ is coatomic, A is a co-coatomic submodule of W. The proof follows from Proposition 2.9.

(3) ⇒ (2) This is clear.

(2) ⇒ (1) Consider W as the sum of A and B, where A and B are maximal submodules of W. By the hypothesis, we have that A and B belong to the class $\mathcal{S}_{\text{Rad-cc}}$. Thus, by Theorem 2.2, $W \in \mathcal{S}_{\text{Rad-cc}}$. If W is w-local, $W \in \mathcal{S}_{\text{Rad-cc}}$ by Lemma 2.1. □

Definition 2.2. We define $\mathcal{TS}_{\text{Rad-cc}} = \{M;\ \text{for all } N \leqslant M, N \in \mathcal{S}_{\text{Rad-cc}}\}$.

It is obvious that any semisimple module belongs to the class $\mathcal{S}_{\text{Rad-cc}}$. In the following proposition, we explore a hereditary class and cohereditary class of a module.

Proposition 2.10. *The class $\mathcal{TS}_{\text{Rad-cc}}$ is closed under taking submodules and quotients.*

P r o o f. Let $W \in \mathcal{TS}_{\text{Rad-cc}}$ and $K \leqslant A \leqslant W$. Then $K \in \mathcal{S}_{\text{Rad(cc)}}$ by hypothesis. Therefore, A belongs to the class $\mathcal{TS}_{\text{Rad-cc}}$. Consider any factor module $\frac{W}{A}$ of W where $\frac{K}{A} \leqslant \frac{W}{A}$. So $K \leqslant W$. Since $W \in \mathcal{TS}_{\text{Rad-cc}}$, we have $K \in \mathcal{S}_{\text{Rad-cc}}$. Consequently $\frac{W}{A} \in \mathcal{TS}_{\text{Rad-cc}}$. □

3. Module class $\mathcal{S}_{\text{Rad-cc}}$ over commutative domains

In this part, we will take all rings as commutative domains, unless specified otherwise.

Remember that $T(W)$ is represented by the set of whole elements $u \in W$ for which there is a nonzero element $s \in S$ with $su = 0$, i.e., $\text{Ann}(u) \neq 0$. Here, $T(W)$ is called the *torsion submodule* of W. If $W = T(W)$, W is called a *torsion module*, while W is referred to as *torsion-free* if $T(W) = 0$.

Proposition 3.1. *Let W be an S-module over a nonsemilocal commutative domain. If $W \in \mathcal{TS}_{\text{Rad-cc}}$, then it follows that W is a torsion module.*

Proof. Assume that $0 \neq x \in W$. We show that $\text{Ann}(x) \neq 0$. Suppose that $\text{Ann}(x) = 0$, i.e., $S \cong Sx$. Since $W \in \mathcal{TS}_{\text{Rad-cc}}$, the S-module $Sx \in \mathcal{TS}_{\text{Rad-cc}}$. Then ${}_SS \in \mathcal{S}_{\text{Rad-cc}}$. Thus, Theorem 2.4 establishes that ${}_SS$ is Rad-supplemented. Additionally, according to [19], Proposition 4, it can be deduced that S is semilocal. This contradicts our assumption. Consequently, W is a torsion module. □

Corollary 3.1. *Let $W \in \mathcal{TS}_{\text{Rad-cc}}$ over a nonsemilocal Dedekind domain. Then W is torsion.*

It is obvious that there is a decomposition $W = \bigoplus_{P\in\Omega} T_P(W)$ for a torsion module W over a Dedekind domain S. Here, Ω denotes the set of all maximal (prime) ideals of S, and $T_P(W)$, a submodule of W, represents the set of all elements $x \in W$ for which there exists a positive integer n such that $p^n x = 0$.

Lemma 3.1. *Let W be a coatomic module over a nonsemilocal Dedekind domain. Then W belongs to $\mathcal{S}_{\text{Rad-cc}}$ if and only if $\frac{W}{P(W)}$ is torsion and each p-primary part of $\frac{W}{P(W)} \in \mathcal{S}_{\text{Rad-cc}}$.*

Proof. It clear by Proposition 2.26 and Theorem 3.1 in [23]. □

Corollary 3.2. *Let $W \in \mathcal{S}_{\text{Rad-cc}}$ be a coatomic torsion-free over a nonlocal Dedekind domain. Then it is radical.*

Proof. Given that S is a Dedekind domain, there exists $K \leqslant_\oplus W$ where $\frac{W}{P(W)} \cong K$. According to [1], Lemma 4.4, $P(W)$ is injective. As W is a torsion-free S-module, W is radical by Lemma 3.1. □

Theorem 3.1. *If the ring S is a Dedekind domain, then $\mathcal{TS}_{\text{Rad-cc}} = \{M;\ M$ is a torsion module whose all of its p-primary parts belonging to $\mathcal{S}_{\text{Rad-cc}}\}$.*

P r o o f. Let W be a module over a nonsemilocal Dedekind domain and $W \in \mathcal{TS}_{\text{Rad-cc}}$. ($\Rightarrow$) This is obvious by Corollary 2.7.

($\Leftarrow$) Consider W as a torsion module where each p-primary part of W belongs to the class $\mathcal{TS}_{\text{Rad(cc)}}$. Let $K \leqslant N \leqslant W$ where $K \leqslant N$ is co-coatomic. Since $W = \bigoplus_{P\in\Omega} T_P(W)$, $N = \bigoplus_{P\in\Omega} (N \cap T_P(W))$ and $K = \bigoplus_{P\in\Omega} (K \cap T_P(W))$. It follows from $\frac{N}{K} \cong \bigoplus_{P\in\Omega} \big(\frac{N\cap T_P(W)}{K\cap T_P(W)}\big)$ that $K \cap T_P(W)$ is co-coatomic submodule of $N \cap T_P(W)$ for every $P \in \Omega$. By the hypothesis, $K \cap T_P(W)$ has a Rad-supplement X_P in $N \cap T_P(W)$. Thus,

$$N \cap T_P(W) = [K \cap T_P(W)] + X_P$$

and

$$[K \cap T_P(W)] \cap X_P = K \cap X_P \subseteq \text{Rad}(X_P).$$

Let $X = \bigoplus_{P\in\Omega} X_P$. Then $N = K + X$. Since $K \cap X_P \subseteq \text{Rad}(X_P)$ for every $P \in \Omega$,

$$K \cap X = \Big[\bigoplus_{P\in\Omega} K \cap T_P(M)\Big] \cap \Big(\bigoplus_{P\in\Omega} X_P\Big) \subseteq \text{Rad}(X)$$

by [11], Corollary 9.1.5. So, $N \in \mathcal{S}_{\text{Rad(cc)}}$. Therefore, $W \in \mathcal{TS}_{\text{Rad-cc}}$. □

In the example provided below, it is demonstrated that the module classes $\mathcal{TS}_{\text{Rad-cc}}$ and $\mathcal{S}_{\text{Rad-cc}}$ are not necessarily closed under extension in general.

E x a m p l e 3.1. In Example 2.1, the modules $\text{Soc}(W)$ and $\frac{W}{\text{Soc}(W)} \in \mathcal{S}_{\text{Rad-cc}}$ but W does not.

4. Module class $\mathcal{S}_{\oplus\text{-Rad-cc}}$

Definition 4.1. $\mathcal{S}_{\oplus\text{-Rad-cc}}$ is defined as the class $\{W;$ for all co-coatomic $A \leqslant W$, A has a Rad-supplement that is a direct summand of $W\}$.

With this definition, we generalize the concept of the module class $\mathcal{S}_{\oplus\text{-Rad-cc}}$ which is introduced in [3].

It is obvious that the module class $\mathcal{S}_{\oplus\text{-Rad-cc}}$ is a subclass of the module class $\mathcal{S}_{\text{Rad-cc}}$.

Furthermore, we have the following relations between some specific variations of supplemented modules and our modules from the first part of the paper, owing to the definitions given above.

Proposition 4.1. *Let* $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ *with* $\text{Rad}(W) \ll W$. *Then* W *is* $\oplus$*-co-coatomically supplemented.*

P r o o f. Consider any co-coatomic submodule A of W. By the assumption, there exists $B \leqslant_{\oplus} W$ such that $W = A + B$, $A \cap B \leqslant \mathrm{Rad}(B)$. Since $\mathrm{Rad}(B) \leqslant \mathrm{Rad}(W) \ll W$, $B \leqslant_{\oplus} W$, we have also $A \cap B \ll B$ by [21], 19.3 (5). Hence, W is $\oplus$-co-coatomically supplemented. □

Corollary 4.1. *Assume W is a coatomic module. Then, $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ if and only if W is $\oplus$-co-coatomically supplemented.*

P r o o f. Since every coatomic module has a small radical, the proof follows from Proposition 4.1. □

Proposition 4.2. *Let W be a projective module. The following statements are equivalent:*

(1) $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.
(2) *For each co-coatomic submodule A of W, there exist $Y \leqslant A$ and $B \leqslant W$ such that $W = Y \oplus B$ and $A \cap B \leqslant \mathrm{Rad}(B)$.*

P r o o f. (1) ⇒ (2) If A is a co-coatomic submodule of W, then there exists a Rad-supplement B of A such that is a direct summand of W. Then $A + B = W$, $A \cap B \leqslant \mathrm{Rad}(B)$ and $W = Z \oplus B$ for some submodule Z. Since $\frac{A}{A\cap B} \cong \frac{A+B}{B} = \frac{W}{B} \cong Z$, then $\frac{A}{A\cap B}$ is a projective module. Hence the exact sequence $0 \to A \cap B \to A \to \frac{A}{A\cap B} \to 0$ splits, and so $A \cap B$ is a direct summand of A. Let $A = (A \cap B) \oplus Y$. Therefore, $Y \oplus B = Y + (A \cap B) + B = A + B = W$.

(2) ⇒ (1) If A and B are as in (2), then B is a Rad-supplement of A, and a direct summand of W. □

Theorem 4.1. *The following statements are equivalent for a finitely generated module W.*

(1) *W is cofinitely $\oplus$-Rad-supplemented.*
(2) *W is $\oplus$-Rad-supplemented.*
(3) $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.

P r o o f. (1) ⇒ (2) and (2) ⇒ (3) are clear.

(3) ⇒ (1) is clear from [9], Proposition 2.2. □

Lemma 4.1. *If W is a radical module, then $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.*

P r o o f. Consider a co-coatomic submodule A of W. Since $\mathrm{Rad}(W) = W$, $\mathrm{Rad}(W)$ is a Rad-supplement of A in W which is a direct summand of W, as desired. □

Proposition 4.3. *Let $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. If W has a maximal submodule, then W includes a w-local direct summand.*

Proof. This follows from [16], Proposition 2.4. □

Corollary 4.2. *If $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ has a small radical, then W has a local direct summand.*

Lemma 4.2. *If W is a w-local module, then $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.*

Proof. Assume that A is a co-coatomic submodule of W. Consequently, the factor module $\frac{W}{A}$ is also coatomic. Therefore, there exists a maximal submodule $\frac{P}{A} \leqslant \frac{W}{A}$ containing all the proper submodules of $\frac{W}{A}$. Since W is w-local, the radical $P = \operatorname{Rad}(W)$ stands out as the unique maximal submodule of W that encompasses A. Consequently, W is as a Rad-supplement of A in W, constituting a direct summand. This concludes the proof. □

Corollary 4.3. *Every direct summand of an w-local module belongs to the class $\mathcal{S}_{\oplus\text{-Rad(cc)}}$.*

Proof. The clarity of the statement arises from the implications of Lemma 4.1 and Lemma 4.2, which establish that every direct summand of a w-local module is to either a radical or w-local. This observation aligns with the proposition stated in [16], Proposition 2.6. □

Proposition 4.4. *The following statements are equivalent for an indecomposable module W:*

(1) $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.
(2) *Each maximal submodule of W has a Rad-supplement, that is a direct summand of W.*
(3) $\operatorname{Rad}(W) = W$ *or W is w-local.*

Proof. (1) ⇒ (2) Consider any maximal submodule P of W. Since $\frac{W}{P}$ is cyclic, and thus coatomic, then $P \leqslant W$ is co-coatomic. Therefore, by the assumption, P has a Rad-supplement which is a direct summand of W.

(2) ⇒ (3) Assume that $\operatorname{Rad}(W) \neq W$. Then there exists a maximal submodule $P \leqslant W$ having a Rad-supplement $T \leqslant_{\oplus} W$, such that $W = P + T$, $P \cap T \leqslant \operatorname{Rad}(T)$ and $W = T \oplus T_1$ for some submodule T_1 of W. Then, $T = 0$ or $T = W$ as W is indecomposable. Suppose that $T = 0$. This case contradicts that P is a maximal submodule of W. So, $T = W$. Thus, $P \leqslant \operatorname{Rad}(W)$ is obtained, and this implies that $P = \operatorname{Rad}(W)$. Hence, W is an w-local module.

(3) ⇒ (1) This is clear from Lemma 4.1 and Lemma 4.2. □

Corollary 4.4. *Let W be an indecomposable module that is not radical. Then W is w-local if and only if $W \in \mathcal{S}_{\oplus\text{-Rad(cc)}}$.*

Lemma 4.3. *Let W be a module and $A \leqslant W$. If A is an w-local, then X is a Rad-supplement of a proper co-coatomic submodule T of W such that $W = A + T$.*

Proof. Consider T as a co-coatomic submodule of W such that $W = A + T$. Then $\frac{W}{T} \cong \frac{A}{A \cap T}$ is coatomic and so $\frac{A}{A \cap T}$ has a maximal submodule. As A is w-local, $\text{Rad}(A)$ is the only maximal submodule of A containing $A \cap T$. Hence, A serves as a Rad-supplement of T in W. □

Proposition 4.5. *Let W be a module and $A \leqslant W$. If A is a co-coatomic submodule of W and $\{S_i\}_{i=1}^n$ is the family of w-local submodules of W where T is a Rad-supplement of $A + \sum\limits_{i=1}^{n} S_i$ in W, then $T + \sum\limits_{i \in I} S_i$ is a Rad-supplement of A in W where $I \subseteq \{1, 2, \ldots, n\}$.*

Proof. Let us point out that each submodule of W containing A is also co-coatomic in W. Using this fact, the proposition can be proved via Lemma 4.3 and [9], Lemma 2.13 by repeated operations. □

Now, we investigate the properties of direct sums of modules which belong to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$.

Proposition 4.6. *Let A and B belong to $\mathcal{S}_{\oplus\text{-Rad-cc}}$. Then so does $W = A \oplus B$.*

Proof. Consider Y as a co-coatomic submodule of W. By the isomorphism,

$$\frac{\frac{W}{Y}}{\frac{Y \cap A}{Y}} \cong \frac{W}{(Y + A)} = \frac{(Y + A) + B}{Y \cap A} \cong \frac{B}{B \cap (Y + A)}$$

is coatomic. Thus, $B \cap (Y + A)$ is a co-coatomic submodule of B. From the assumption, there is a direct summand H of B where

$$B = [B \cap (Y + A)] + H$$

and

$$[B \cap (Y + A)] \cap H = (Y + A) \cap H \leqslant \text{Rad}(H).$$

Now following this, we have

$$\begin{aligned} W = A + B = A + [B \cap (Y + A)] + H &= [(A + B) \cap (Y + A)] + H \\ &= [W \cap (Y + A)] + H = Y + A + H. \end{aligned}$$

Since $Y \leqslant W$ is co-coatomic, $\frac{W}{Y}$ is coatomic. As being coatomic is a property preserved under factor modules, we have that $A \cap (Y + H) \leqslant A$ is co-coatomic by using the isomorphism

$$\frac{A}{A \cap (Y + H)} \cong \frac{W}{Y + H} \cong \frac{\frac{W}{Y}}{\frac{Y+H}{Y}}.$$

Thus, there exists $T \leqslant_{\oplus} A$ where

$$A = [A \cap (Y + H)] + T,$$
$$[A \cap (Y + H)] \cap T = (Y + H) \cap T \leqslant \mathrm{Rad}(T).$$

Therefore,

$$W = A + (Y + H) = [A \cap (Y + H)] + T + (Y + H) = Y + (H \oplus T).$$

Furthermore

$$\begin{aligned} Y \cap (H + T) &\leqslant [H \cap (Y + T)] + [T \cap (Y + H)] \\ &\leqslant \mathrm{Rad}(H) \oplus \mathrm{Rad}(T) = \mathrm{Rad}(H \oplus T). \end{aligned}$$

Therefore, $H \oplus T$ is a Rad-supplement of Y in W which is a direct summand of W. Consequently, $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. □

Corollary 4.5. *A finite direct sum of modules which belongs to the class* $\mathcal{S}_{\oplus\text{-Rad-cc}}$ *remains in the class* $\mathcal{S}_{\oplus\text{-Rad-cc}}$.

Corollary 4.6. *A finite direct sum of w-local modules also belongs to the class* $\mathcal{S}_{\oplus\text{-Rad-cc}}$.

P r o o f. The clarity of this statement is derived from the implications presented in Lemma 4.2 and Corollary 4.5. □

Consider direct summands A and B of W such that $W = A + B$. If $A \cap B \leqslant_{\oplus} W$, then W is referred to as a module with the property (D_3) [15].

Proposition 4.7. *Let* $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ *with the condition* (D_3). *In that case, each co-coatomic direct summand of* W *belongs to the class* $\mathcal{S}_{\oplus\text{-Rad-cc}}$.

P r o o f. Consider $A \leqslant_{\oplus} W$ as co-coatomic. Thus, $W = A \oplus X_1$ for some $X_1 \leqslant W$ where $\frac{W}{A} \cong X_1$ is coatomic. For any co-coatomic submodule N of A, we have $\frac{W}{N} = \frac{A \oplus X_1}{N} \cong \frac{A}{N} \oplus \frac{X_1}{X_1 \cap N}$. Since X_1 is coatomic, then $\frac{X_1}{X_1 \cap N}$ is coatomic. Therefore, $\frac{W}{N}$ is coatomic as a direct sum of two coatomic modules. Based on the hypothesis, there is a direct summand D of W such that $W = N + D$ and $N \cap D \leqslant \mathrm{Rad}(D)$. It follows that $A \cap D$ is a direct summand of W since it has (D_3). From the modularity we get $A = N + (A \cap D)$. Moreover, we have

$$N \cap (A \cap D) = N \cap D \leqslant \mathrm{Rad}(D) \cap (A \cap D) = \mathrm{Rad}(A \cap D)$$

from [8], 3.7 (3). Hence, $A \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. □

A submodule A of W is called *closed* in W if it has no proper essential extension in W. According to [8], an S-module W is referred to as a (UC)*-module* if every submodule has a unique closure in W. W is called an *extending module* if each closed submodule of W is a direct summand of W.

Corollary 4.7. *Let W be a UC-extending module. If $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$, then so is every co-coatomic direct summand of W.*

Now, we explore adequate conditions under which a factor module of modules, that belong to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$, also belongs to this class $\mathcal{S}_{\oplus\text{-Rad-cc}}$.

E x a m p l e 4.1. Let S be a local commutative ring whose ideals are not totally ordered with respect to inclusion. For $n \geqslant 2$, the finitely presented indecomposable module $W \cong \frac{S^{(n)}}{K}$ cannot be generated by fewer than n elements [20], Theorem 2. Here, $S^{(n)}$ belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$ by [9], Example 4.1 but W does not belong to $\mathcal{S}_{\oplus\text{-Rad-cc}}$.

From the provided example, we observe that a submodule of a module which belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$ may not belong to this class itself.

According to the information presented in reference [21], a submodule A of an S-module W is called *fully invariant* if $g(A) \leqslant A$ for all g in $\mathrm{End}_S(W)$. Notably, $\mathrm{Rad}(W)$ qualifies as a fully invariant submodule of W.

Now, we show that the implication given above is valid for the factor modules of a module constructed from fully invariant submodules.

Proposition 4.8. *Let W be a module and $A \leqslant W$ be a fully invariant submodule. If $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$, then so does $\frac{W}{A}$.*

P r o o f. Let $\frac{N}{A} \leqslant \frac{W}{A}$ be co-coatomic. Thus, $\frac{W}{N} \cong \frac{\frac{W}{A}}{\frac{N}{A}}$ is coatomic, meaning that N is co-coatomic in W. By the hypothesis, there is a Rad-supplement K of N

in W which is a direct summand, that is $W = N + K$, $N \cap K \leqslant \mathrm{Rad}(K)$ and $W = K \oplus K_1$. By [7], Lemma 3.4, $\frac{K+A}{A}$ is a Rad-supplement of $\frac{N}{A}$ in $\frac{W}{A}$. Moreover, as A is fully invariant, we have $A = (K \cap A) \oplus (K_1 \cap A)$ for some $K_1 \leqslant W$. Using this fact, we obtain

$$\begin{aligned}\frac{(K+A)}{X} \cap \frac{(K_1+A)}{X} &= \frac{K \oplus (K_1 \cap X)}{A} \cap \frac{K_1 \oplus (K \cap A)}{A} \\ &= \frac{(K \cap A) \oplus [((K_1 \cap A) + K) \cap K_1]}{A} \\ &= \frac{(K \cap A) \oplus (K_1 \cap A)}{A} = \frac{(K \oplus K_1) \cap A}{A} = 0.\end{aligned}$$

Therefore, $\frac{W}{A} = \frac{(K+A)}{A} \oplus \frac{(K_1+A)}{A}$ is obtained. Hence, $\frac{W}{A} \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. □

Corollary 4.8. *Let $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. Then, $\frac{W}{\mathrm{Rad}(W)}$ and $\frac{W}{\mathrm{Soc}(M)}$ belong to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$.*

Proposition 4.9. *Let $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ and $A \leqslant W$ be a fully invariant submodule. If $A \leqslant_{\oplus} W$ is co-coatomic, then $A \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.*

Proof. Let $Y \leqslant A$ be co-coatomic. Then $\frac{A}{Y}$ is coatomic. As A is co-coatomic direct summand of W, there is a submodule X_1 of W such that $W = A \oplus X_1$. It is clear that $\frac{W}{A} \cong X_1$ is also coatomic. Further, $\frac{W}{Y} = \frac{A \oplus X_1}{Y} = \frac{A}{Y} \oplus \frac{X_1+Y}{Y} \cong \frac{A}{Y} \oplus \frac{X_1}{X_1 \cap Y}$. As X_1 is co-coatomic, $\frac{X_1}{X_1 \cap Y}$ is coatomic. Since $\frac{A}{Y}$ and $\frac{X_1}{X_2 \cap Y}$ are coatomic, $\frac{W}{Y}$ is coatomic. So, Y is a co-coatomic submodule of W. From the assumption, there is a direct summand D of W such that $W = Y + D$, $Y \cap D \leqslant \mathrm{Rad}(D)$ and $W = D \oplus D_1$. Since A is fully invariant submodule of W, we have $A = (A \cap D) \oplus (A \cap D_1)$. From modularity, we obtain $A = Y + (X \cap D)$ and $Y \cap (A \cap D) = Y \cap D \leqslant \mathrm{Rad}(D) \leqslant \mathrm{Rad}(W)$. As $A \cap D$ is a direct summand of W, we have $Y \cap (A \cap D) \leqslant \mathrm{Rad}(A \cap D)$. Thus $A \cap D$ is a Rad-supplement of Y in A which is a direct summand. Hence, $A \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. □

Proposition 4.10. *For a left V-ring, the class of $\mathcal{S}_{\oplus\text{-Rad-cc}}$ consists of semisimple modules.*

Proof. This is a consequence of Proposition 2.4. □

Theorem 4.2. *S is a semiperfect ring if and only if each finitely generated free S-module belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$.*

P r o o f. ($\Rightarrow$) Assume that S is a semiperfect ring. Then by [15], Corollary 4.2, ${}_SS$ is $\oplus$-supplemented. It follows that ${}_SS$ is also $\oplus$-Rad-supplemented. Thus, $S \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. Hence, any finitely generated free S-module $F = S^{(n)}$ belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$ by Corollary 4.5.

($\Leftarrow$) By the hypothesis, ${}_SS \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ as a finitely generated free S-module. As each submodule of ${}_SS$ is co-coatomic, ${}_SS$ is $\oplus$-Rad-supplemented and therefore, it is $\oplus$-supplemented by [9], Proposition 2.2. Therefore, ${}_SS$ is supplemented. Hence, S is a semiperfect ring by [15], Corollary 4.42. □

Corollary 4.9. *A ring S is semiperfect if and only if ${}_SS \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.*

Proposition 4.11. *If $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ and W is coatomic, then W is an irredundant sum of local direct summands of W.*

P r o o f. This is clear from [9], Proposition 2.18. □

The following example shows that a module, which belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$, need not be $\oplus$-Rad-supplemented.

E x a m p l e 4.2. Let us consider the $\mathbb{Z}$-module $\mathbb{Z}_{(p)} = \{\frac{x}{y} \in \mathbb{Q} \mid p \nmid y\}$ where p is a prime number. Since $p\mathbb{Z}_{(p)}$ is the unique maximal of $\mathbb{Z}_{(p)}$ then $\mathbb{Z}_{(p)} \in \mathcal{S}_{\oplus\text{-Rad(cc)}}$ by Lemma 4.2. However, it can be seen from [4], Example 3.1 that $\mathbb{Z}_{(p)}$ is not a Rad-supplemented module. Therefore, $\mathbb{Z}_{(p)}$ is not $\oplus$-Rad-supplemented.

Proposition 4.12. *If $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$, then $\frac{W}{P(W)}$ is $\oplus$-co-coatomically supplemented.*

P r o o f. Since $f(P(W)) \leqslant P(W)$ for every $f \in \text{End}_R(W)$, $P(W)$ is a fully invariant submodule of W. Then, $\frac{W}{P(W)} \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ by Proposition 4.9. Consider K as a submodule of W that is both co-coatomic and contains $P(W)$. Then, there is a direct summand $\frac{T}{P(W)}$ of $\frac{W}{P(W)}$ such that $\frac{W}{P(W)} = \frac{K}{P(W)} + \frac{T}{P(W)}$ and $\frac{K\cap T}{P(W)} \subseteq \text{Rad}\big(\frac{T}{P(W)}\big)$. Since $\frac{W}{P(W)}$ is reduced, $\frac{W}{P(W)}$ is coatomic by [5], Theorem 2.6. So $\text{Rad}\big(\frac{W}{P(W)}\big) \ll \frac{W}{P(W)}$. By Proposition 4.9, we have that $\frac{W}{P(W)}$ is $\oplus$-co-coatomically supplemented. □

Theorem 4.3. *Let W be a module over a Dedekind domain. The following statements are equivalent:*

(1) $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.
(2) $\frac{W}{P(W)} \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.
(3) $\frac{W}{P(W)}$ *is $\oplus$-co-coatomically supplemented.*

Proof. (1) $\Rightarrow$ (3) This conclusion follows from Proposition 4.12.

(3) $\Rightarrow$ (2) This implication is clear.

(2) $\Rightarrow$ (1) By our assumption, there exists a decomposition $W = P(W) \oplus N$ for some submodule N of W. Since $\frac{W}{P(W)} \cong N$ belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$ and $P(W)$ is a radical module, $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ by Lemma 4.1 and Corollary 4.5. □

Theorem 4.4. *Over a Dedekind ring, $\mathcal{S}_{\text{Rad-cc}}$ is a subclass of $\mathcal{S}_{\oplus\text{-Rad-cc}}$.*

Proof. Assume that $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. The fact that $\frac{W}{P(W)}$ is co-coatomically supplemented is readily apparent. Since $\frac{W}{P(W)}$ is reduced, $\frac{W}{P(W)}$ is $\oplus$-co-coatomically supplemented. By Theorem 4.3, $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. □

5. Module class $\mathcal{SP}_{\text{Rad-cc}}$

In [13], the author introduced generalized cofinitely semiperfect modules using generalized projective covers of Rad-supplement modules. We now refine this term via Rad_{cc}-semiperfect modules as follows.

Definition 5.1. We define $\mathcal{SP}_{\text{Rad-cc}}$ as the class $\{W;$ each isomorphic factor of W has a generalized projective cover$\}$.

A ring S is called a *left max ring* when every nonzero left S-module possesses a maximal submodule. This is equivalent to the claim that each nonzero left S-module is coatomic [21]. Consequently, any module which belongs to the class $\mathcal{SP}_{\text{Rad-cc}}$ over a left max ring is considered to be generalized semiperfect.

Recall from [8] that a ring S is called *left Bass* if each nonzero S-module possesses a maximal submodule. Left Bass rings are characterized by $\text{Rad}(W) \ll W$ for each nonzero S-module W.

Proposition 5.1. *Every module which belongs to the class $\mathcal{SP}_{\text{Rad-cc}}$ over a left Bass ring is co-coatomically semiperfect.*

Proof. The proof is obvious since $\text{Rad}(W) \ll W$ holds for every nonzero module W over a left Bass ring. □

Proposition 5.2. *Let S be a regular ring. Then every S-module which belongs to the class $\mathcal{SP}_{\text{Rad-cc}}$ is co-coatomically semiperfect.*

Proof. Consider A as a co-coatomic submodule within a generalized co-coatomically semiperfect S-module denoted by W. In such a case, there is a generalized cover f from the projective module P to the factor module $\frac{W}{A}$. Given that S is regular and $\text{Rad}(P) = \text{Rad}(S)P = 0$, then $\ker(f) \ll P$. Consequently, f is a projective cover of $\frac{W}{A}$. This implies that $W \in \mathcal{SP}_{\text{Rad-cc}}$. □

Corollary 5.1. *Let W be an S-module over a left V-ring. W is co-coatomically semiperfect if and only if $W \in \mathcal{SP}_{\text{Rad-cc}}$.*

Proof. This follows directly from Proposition 5.2 as every module over a left V-ring possesses a zero radical. □

Theorem 5.1. *For a module W, the following statements are equivalent:*

(1) $W \in \mathcal{SP}_{\text{Rad-cc}}$.

(2) $W \in \mathcal{AS}_{\text{Rad-cc}}$ *with supplements that have generalized projective covers.*

(3) $W \in \mathcal{S}_{\text{Rad-cc}}$ *with Rad-supplements that have generalized projective covers.*

Proof. (1) $\Rightarrow$ (2) Let A be a co-coatomic submodule of W, where W can be expressed as the sum of A and a submodule Y such that $W = A + Y$. By the hypothesis, there exists an epimorphism $f\colon P \to \frac{W}{A}$ for a projective module P with $\ker(f) \leqslant \text{Rad}(P)$. As P is projective, there is a homomorphism $g\colon P \to Y$. Consequently, the image of g, denoted as $\text{Im}(g)$, serves as a Rad-supplement of A in W according to [13], Lemma 4.1, and it is contained in Y. Thus, P characterizes as a generalized projective cover for $\text{Im}(g)$, given by $\ker(g) \leqslant \ker(f) \leqslant \text{Rad}(P)$.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) Consider a co-coatomic submodule A of the module W. Under the given hypothesis, there is a Rad-supplement Y of A, and this Rad-supplement has a generalized projective cover denoted by f, specifically, the map $f\colon P \to Y$ where P is the generalized projective cover. Then, the map $\pi\colon Y \to \frac{Y}{A \cap Y}$ is a generalized cover. Consequently, the composition $\pi \circ f\colon P \to \frac{Y}{A\cap Y} \cong \frac{W}{A}$ serves as a generalized projective cover. This result can be deduced from [22], Lemma 1.1 and Lemma 1.2. Hence, $W \in \mathcal{SP}_{\text{Rad-cc}}$. □

Proposition 5.3. *$\mathcal{SP}_{\text{Rad-cc}}$ is closed under taking quotients.*

Proof. Consider $W \in \mathcal{SP}_{\text{Rad-cc}}$, and $f\colon W \to W_1$ as a homomorphism. Assume that $A \leqslant f(W)$ is co-coatomic. The homomorphism $\pi\colon W \to \frac{f(W)}{A}$ such that $\pi(m) = f(m) + A$ is surjective for every $m \in W$. Since $W \in \mathcal{SP}_{\text{Rad-cc}}$ and $\frac{W}{f^{-1}(A)} \cong \frac{f(W)}{A}$, $\frac{f(W)}{A}$ is a generalized projective cover. Consequently, $f(W) \in \mathcal{SP}_{\text{Rad-cc}}$. □

Corollary 5.2. *Each factor module of a generalized co-coatomically semiperfect module inherits the property of belonging to the class $\mathcal{SP}_{\text{Rad-cc}}$.*

Proposition 5.4. *$\mathcal{SP}_{\text{Rad-cc}}$ is closed under taking generalized projective covers.*

Proof. Consider $N \in \mathcal{SP}_{\text{Rad-cc}}$, and let $f\colon W \to N$ be a generalized cover. Additionally, let $A \leqslant W$ be co-coatomic. The homomorphism $\overline{f}\colon \frac{W}{A} \to \frac{N}{f(A)}$ defined by $\overline{f}(m+A) = f(m)+f(A)$ is an epimorphism. The kernel of $\overline{f}$ is given by $\ker(\overline{f}) = \frac{\ker(f)+L}{L} \leqslant \frac{\text{Rad}(W)+L}{L} \leqslant \text{Rad}\bigl(\frac{W}{L}\bigr)$, as f is a generalized cover. Thus, $\overline{f}$ is a generalized cover. On the other hand, $\frac{N}{f(A)}$ is coatomic as it is homomorphic image of a coatomic factor module $\frac{W}{A}$. By our assumption, there is a generalized cover $\pi\colon P \to \frac{N}{f(A)}$ where P is projective. Therefore, a homomorphism $g\colon P \to \frac{W}{A}$ exists such that $\overline{f} \circ g = h$. As h is an epimorphism and $\overline{f}$ is a generalized cover, it follows that g is an epimorphism. Furthermore, $\ker(g) \leqslant \ker(\overline{f} \circ g) = \ker(h) \leqslant \text{Rad}(P)$. Consequently, P is a generalized projective cover of $\frac{W}{A}$. □

Corollary 5.3. *If $A \leqslant \text{Rad}(W)$ and $\frac{W}{A}$ belong to the class $\mathcal{SP}_{\text{Rad-cc}}$, then $W \in \mathcal{SP}_{\text{Rad-cc}}$.*

A generalized cover $f\colon P \to W$ is called a *generalized W-projective cover* whenever P is W-projective.

Now, we give a characterization theorem for a *generalized W-projective cover.*

Theorem 5.2. *The following statements equivalent for a module W:*

(1) *Each coatomic factor module W has a generalized W-projective cover.*
(2) *$W \in \mathcal{AS}_{\text{Rad-cc}}$ which have generalized W-projective covers.*
(3) *$W \in \mathcal{S}_{\text{Rad-cc}}$ which have generalized W-projective covers.*

Proof. The proof is analogous to the proofs of Theorem 5.1 and Proposition 5.4. □

Proposition 5.5. *Let $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ and W be projective. Then $W \in \mathcal{SP}_{\text{Rad-cc}}$.*

Proof. Let A be a co-coatomic submodule of W. Under the given hypothesis, there exists a Rad-supplement T that serves as a direct summand in W, meaning that $W = A + T$ where $A \cap T \leqslant \text{Rad}(T)$, and $W = T \oplus T_1$ for some submodule $T_1 \leqslant W$. Consider the inclusion map $i\colon T \to W$ and the natural homomorphism $\pi\colon W \to \frac{W}{A}$. Note that T is projective. The composition map $\pi \circ i\colon T \to \frac{W}{A}$ is an epimorphism and $\ker(\pi \circ i) = A \cap T \leqslant \text{Rad}(T)$. Hence, T is a generalized projective cover of $\frac{W}{A}$ and so $W \in \mathcal{SP}_{\text{Rad-cc}}$. □

Proposition 5.6. *Let $P \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ and $f\colon P \to W$ be a generalized projective cover of W. Then $P \in \mathcal{SP}_{\text{Rad-cc}}$ if and only if $W \in \mathcal{SP}_{\text{Rad-cc}}$.*

Proof. ($\Rightarrow$) This is clear by Corollary 5.2 and Proposition 5.5.

($\Leftarrow$) It follows from Proposition 5.4. □

Lemma 5.1. *Let W be a projective module. If W is semiperfect, then each finitely W-generated module belongs to the class $\mathcal{SP}_{\text{Rad-cc}}$. Conversely, if W is finitely generated, the converse statement holds true.*

P r o o f. Clear by [3], Lemma 3.10. □

Since every finitely generated module has small radical, the notions of co-coatomically semiperfect modules and modules that belong to the class $\mathcal{SP}_{\text{Rad-cc}}$ arise in the same situation for finitely generated free modules.

Corollary 5.4. *The following statements are equivalent for a ring S:*

(1) *S is semiperfect.*
(2) *Each finitely generated free S-module is semiperfect.*
(3) *Each finitely generated free S-module is co-coatomically semiperfect.*
(4) *Each finitely generated free S-module belongs to the class $\mathcal{SP}_{\text{Rad-cc}}$.*

P r o o f. (1) ⇔ (2) This is clear from [10], Lemma 1.2 and [12], Theorem 2.1.
(1) ⇔ (3) This is clear by [3], Proposition 3.11.
(1) ⇔ (4) This is clear from Lemma 5.1. □

Acknowledgements. The authors would like to thank the referees for their contributions during the preparation of the article for publication.

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