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SAKAGUCHI TYPE FUNCTIONS DEFINED BY BERNOULLI POLYNOMIALS

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Abstract. In this paper, the class of Sakaguchi-type functions defined by Bernoulli polynomials has been introduced as a novel subclass of bi-univalent functions. The bounds for the Fekete-Szegö inequality and the initial coefficients $|a_2|$ and $|a_3|$ have also been estimated.

Keywords: analytic function; bi-univalent function; Sakaguchi type function; Bernoulli polynomial

MSC 2020: 30C45, 30C50

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in the open unit disc $\mathfrak{U} = \{z \in \mathbb{C}\colon\ |z| < 1\}$ and consider the classes $\mathcal{P}$, $\mathcal{A}$ and $\mathcal{S}$ defined by

$$\begin{aligned} \mathcal{P} &= \{p \in \mathcal{H}\colon\ p(0) = 1 \text{ and } \Re(p(z)) > 0,\ z \in \mathfrak{U}\},\\ \mathcal{A} &= \{\mathfrak{f} \in \mathcal{H}\colon\ \mathfrak{f}(0) = \mathfrak{f}'(0) - 1 = 0\},\\ \mathcal{S} &= \{\mathfrak{f} \in \mathcal{A}\colon\ \mathfrak{f} \text{ is univalent in } \mathfrak{U}\}, \end{aligned}$$

respectively. It is clear that the function $\mathfrak{f} \in \mathcal{A}$ can be expressed as

$$\mathfrak{f}(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathfrak{U}. \tag{1.1}$$

For two functions $\mathfrak{f}, \mathfrak{g} \in \mathcal{H}$ we say that the function $\mathfrak{f}$ is subordinate to $\mathfrak{g}$ in $\mathfrak{U}$, and write

$$\mathfrak{f}(z) \prec \mathfrak{g}(z), \quad z \in \mathfrak{U}$$

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if there exists a Schwarz function

$$\omega \in \Omega := \{\omega \in \mathcal{H}\colon\ \omega(0) = 0 \text{ and } |\omega(z)| < 1,\ z \in \mathfrak{U}\}$$

such that

$$\mathfrak{f}(z) = \mathfrak{g}(\omega(z)), \quad z \in \mathfrak{U}.$$

A subclass consisting of functions $\mathfrak{f} \in \mathcal{A}$ satisfying the analytic criterion

$$\Re\biggl(\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z) - \mathfrak{f}(-z)}\biggr) > \alpha, \quad 0 \leqslant \alpha < 1$$

was introduced by Sakaguchi [20] and these functions were named after him as Sakaguchi type functions [17], [18], [22]. Sakaguchi type functions are starlike with respect to symmetric points. Frasin [10] generalized Sakaguchi type class which had functions of the form (1.1) given by

$$\Re\biggl(\frac{(s-b)z\mathfrak{f}'(z)}{\mathfrak{f}(sz) - \mathfrak{f}(bz)}\biggr) > \alpha, \quad 0 \leqslant \alpha < 1,\ s, b \in \mathbb{C} \text{ with } s \neq b,\ |s| \leqslant 1,\ |b| \leqslant 1,\ z \in \mathfrak{U}.$$

Orthogonal polynomials have been the subject of extensive research in recent years from a variety of perspectives due to their importance in probability theory, mathematical statistics, mathematical physics, and engineering. The classical orthogonal polynomials (Hermite polynomials, Laguerre polynomials, Jacobi polynomials, and Bernoulli polynomials) are the orthogonal polynomials that are most widely utilized in applications. We provide additional contemporary instances of the link between geometric function theory and classical orthogonal polynomials in [2], [4], [5], [11], [25]. One of the numerous special functions researched is fractional calculus, a traditional branch of mathematical analysis whose foundations were provided by Liouville in an 1832 work and which is today a very active study topic [14]. The Bernoulli polynomials are a field of mathematics named after Jacob Bernoulli (1654–1705). To solve fractional-order differential equations of the Lane-Emden type, a novel approximation method based on orthonormal Bernoulli polynomials has been developed [19], whereas in [8], [9], [13] Bernoulli polynomials are used to numerically resolve Fredholm fractional integrodifferential equations with right-sided Caputo derivatives.

The Bernoulli polynomials $\mathfrak{B}_n(x)$ are frequently specified (see, e.g., [15]) using the generating function

$$\mathcal{F}(x,t) = \frac{t\mathrm{e}^{xt}}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} \frac{\mathfrak{B}_n(x)}{n!} t^n, \quad |t| < 2\pi, \tag{1.2}$$

where $\mathfrak{B}_n(x)$ are polynomials in $x \in \mathbb{C}$, for each non-negative integer n (see Figures 1 and 2). Since the Bernoulli polynomials are easily computed using recursion as

$$\sum_{j=0}^{n-1} \binom{n}{j} \mathfrak{B}_j(x) = n x^{x-1}, \quad n = 2, 3, \ldots, \tag{1.3}$$

Bernoulli's initial few polynomials are

$$\mathfrak{B}_0(x) = 1,\ \mathfrak{B}_1(x) = x - \frac{1}{2},\ \mathfrak{B}_2(x) = x^2 - x + \frac{1}{6},\ \mathfrak{B}_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \ldots \tag{1.4}$$

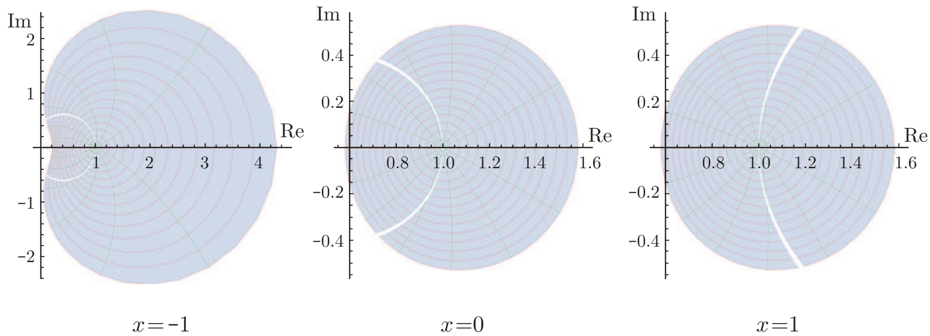

Figure 1. Image of $\mathfrak{U}$ under $\mathcal{F}(x, z)$.

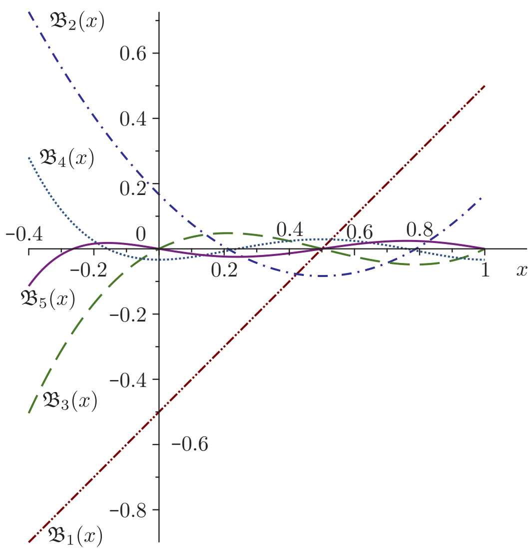

Figure 2. Graphs of $\mathfrak{B}_n(x)$, for x real.

A function $\mathfrak{f} \in \mathcal{A}$ is called bi-univalent in $\mathfrak{U}$ if $\mathfrak{f} \in \mathcal{S}$ and its inverse function has an analytic continuation to $|w| < 1$. Let $\Sigma = \{\mathfrak{f} \in \mathcal{S}\colon \mathfrak{f}$ is bi-univalent$\}$. For the function $\mathfrak{f} \in \mathcal{A}$ given by (1.1), the inverse function $\mathfrak{g} = \mathfrak{f}^{-1}$ is of the form

$$\mathfrak{g}(w) = \mathfrak{f}^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots \tag{1.5}$$

Examples of functions that belong to the class Σ include

$$\mathfrak{f}_1(z) = \frac{z}{1-z}, \quad \mathfrak{f}_2(z) = -\log(1-z), \quad \mathfrak{f}_3(z) = \frac{1}{2}\log\Big(\frac{1+z}{1-z}\Big).$$

However, the function $z/(1-z)^2$ is a member of the class $\mathcal{S}$ but does not belong to Σ.

Although Lewin [12] stated the class of bi-univalent functions, Netanyahu, Clunie, Brannan, and many others [1], [3], [4], [6], [16], [21], [23], [24], [25], [26] elevated the passion on the boundaries for the coefficients of these classes. Young scholars have been fascinated by this area of study so far.

In the light of the above definitions, we introduce a new general subclass of $\mathcal{A}$ with respect to symmetric points.

Definition 1.1. The function $\mathfrak{f} \in \Sigma$ is in the class $\mathfrak{B}\mathcal{S}_{\Sigma}^{\lambda}(s,b,x,z)$ if

$$\frac{(s-b)z\mathfrak{F}_{\lambda}'(z)}{\mathfrak{F}_{\lambda}(sz) - \mathfrak{F}_{\lambda}(bz)} \prec \mathcal{F}(x,z), \quad z \in \mathfrak{U}, \tag{1.6}$$

and

$$\frac{(s-b)w\mathfrak{G}_{\lambda}'(w)}{\mathfrak{G}_{\lambda}(sw) - \mathfrak{G}_{\lambda}(bw)} \prec \mathcal{F}(x,w), \quad w \in \mathfrak{U}, \tag{1.7}$$

where

$$\mathfrak{F}_{\lambda}(z) = (1-\lambda)\mathfrak{f}(z) + \lambda z\mathfrak{f}'(z), \quad z \in \mathfrak{U},$$
$$\mathfrak{G}_{\lambda}(w) = (1-\lambda)\mathfrak{g}(w) + \lambda w\mathfrak{g}'(w), \quad w \in \mathfrak{U},$$

$\mathfrak{g} = \mathfrak{f}^{-1}$ is given by (1.5) and $0 \leqslant \lambda \leqslant 1$, $s, b \in \mathbb{C}$ with $s \neq b$, $|s| \leqslant 1$, $|b| \leqslant 1$.

Remark 1.1. (i) For $s = 1$ and $b = -1$, we get the following new class $\mathfrak{B}\mathcal{S}_{\Sigma}^{\lambda}(x,z)$, which consists of functions $\mathfrak{f} \in \Sigma$ satisfying

$$\frac{2z\mathfrak{F}_{\lambda}'(z)}{\mathfrak{F}_{\lambda}(z) - \mathfrak{F}_{\lambda}(-z)} \prec \mathcal{F}(x,z)$$

and

$$\frac{2w\mathfrak{G}'_\lambda(w)}{\mathfrak{G}_\lambda(w)-\mathfrak{G}_\lambda(-w)} \prec \mathcal{F}(x,w).$$

(ii) For $s=1$ and $b=0$ we get the following new class $\mathfrak{B}\mathcal{H}^\lambda_\Sigma(x,z)$, which consists of functions $\mathfrak{f}\in\Sigma$ satisfying

$$\frac{z\mathfrak{F}'_\lambda(z)}{\mathfrak{F}_\lambda(z)} \prec \mathcal{F}(x,z)$$

and

$$\frac{w\mathfrak{G}'_\lambda(w)}{\mathfrak{G}_\lambda(w)} \prec \mathcal{F}(x,w).$$

(iii) For $s=1$, $b=-1$ and $\lambda=0$ we get the following new class $\mathfrak{B}\mathcal{S}_\Sigma(x,z)$, which consists of functions $\mathfrak{f}\in\Sigma$ satisfying

$$\frac{2z\mathfrak{f}'(z)}{\mathfrak{f}(z)-\mathfrak{f}(-z)} \prec \mathcal{F}(x,z)$$

and

$$\frac{2w\mathfrak{g}'(w)}{\mathfrak{g}(w)-\mathfrak{g}(-w)} \prec \mathcal{F}(x,w).$$

(iv) For $s=1$, $b=-1$ and $\lambda=1$ we get the following new class $\mathfrak{B}\mathcal{C}_\Sigma(x,z)$, which consists of functions $\mathfrak{f}\in\Sigma$ satisfying

$$\frac{2(z\mathfrak{f}'(z))'}{(\mathfrak{f}(z)-\mathfrak{f}(-z))'} \prec \mathcal{F}(x,z)$$

and

$$\frac{2(w\mathfrak{g}'(w))'}{(\mathfrak{g}(w)-\mathfrak{g}(-w))'} \prec \mathcal{F}(x,w).$$

(v) For $s=1$, $b=0$ and $\lambda=0$ we get the class $\mathfrak{B}\mathcal{H}_\Sigma(x,z)$, which consists of functions $\mathfrak{f}\in\Sigma$ satisfying

$$\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)} \prec \mathcal{F}(x,z)$$

and

$$\frac{w\mathfrak{g}'(w)}{\mathfrak{g}(w)} \prec \mathcal{F}(x,w).$$

(vi) For $s=1$, $b=0$ and $\lambda=1$ we get the class $\mathfrak{B}\mathcal{N}_\Sigma(x,z)$, which consists of functions $\mathfrak{f}\in\Sigma$ satisfying

$$1+\frac{z\mathfrak{f}''(z)}{\mathfrak{f}'(z)} \prec \mathcal{F}(x,z)$$

and

$$1+\frac{w\mathfrak{g}''(w)}{\mathfrak{g}'(w)} \prec \mathcal{F}(x,w).$$

The classes $\mathfrak{B}\mathcal{S}_\Sigma(x,z)$ and $\mathfrak{B}\mathcal{C}_\Sigma(x,z)$ are introduced by Buyankara et al. [7].

2. Coefficients estimates for the class $\mathfrak{BS}_{\Sigma}^{\lambda}(s,b,x,z)$

Throughout this paper, unless otherwise stated, we assume that

$$0 \leqslant \lambda \leqslant 1, \quad s,b \in \mathbb{C} \text{ with } s \neq b,\ |s| \leqslant 1,\ |b| \leqslant 1.$$

Theorem 2.1. *Let $\mathfrak{f}$ given by* (1.1) *be in the class $\mathfrak{BS}_{\Sigma}^{\lambda}(s,b,x,z)$, and define*

$$L = (1+2\lambda)\tau - (1+\lambda)^2(s+b)\sigma \quad \text{and} \quad M = (1+\lambda)\sigma, \tag{2.1}$$

where

$$\sigma = 2 - (s+b) \quad \text{and} \quad \tau = 3 - (s^2 + sb + b^2). \tag{2.2}$$

Then we have

$$|a_2| \leqslant \min\left\{\frac{|x-1/2|}{(1+\lambda)|\sigma|},\ \gamma\right\},$$

where

$$\gamma = \begin{cases} \dfrac{\sqrt{6}|x-1/2|^{3/2}}{\sqrt{|3(2L-M^2)(x-1/2)^2 + M^2|}}, & L \neq \dfrac{1}{2}M^2 \\ & \text{and } \left(x - \dfrac{1}{2}\right)^2 \neq -\dfrac{M^2}{3(2L-M^2)}, \\ \dfrac{\sqrt{6}|x-1/2|^{3/2}}{|M|}, & L = \dfrac{1}{2}M^2 \end{cases} \tag{2.3}$$

and

$$|a_3| \leqslant \frac{|x-1/2|^2}{(1+\lambda)^2|\sigma|^2} + \frac{|x-1/2|}{(1+2\lambda)|\tau|}. \tag{2.4}$$

Proof. Let $\mathfrak{f} \in \mathfrak{BS}_{\Sigma}^{\lambda}(s,b,x,z)$. Then there exist analytic functions $l(z)$, $m(w)$: $\mathfrak{U} \to \mathfrak{U}$, given by

$$l(z) = l_1 z + l_2 z^2 + l_3 z^3 + \dots \tag{2.5}$$

and

$$m(w) = m_1 w + m_2 w^2 + m_3 w^3 + \dots, \tag{2.6}$$

respectively, which are analytic in $\mathfrak{U}$ with $l(0) = 0$, $m(0) = 0$ and $|l(z)| < 1$, $|m(w)| < 1$, $z, w \in \mathfrak{U}$, such that

$$\frac{(s-b)z\mathfrak{F}'_\lambda(z)}{\mathfrak{F}_\lambda(sz) - \mathfrak{F}_\lambda(bz)} = \mathfrak{F}(x, l(z)) \tag{2.7}$$

and

$$\frac{(s-b)w\mathfrak{G}'_\lambda(w)}{\mathfrak{G}_\lambda(sw) - \mathfrak{G}_\lambda(bw)} = \mathfrak{F}(x, m(w)), \tag{2.8}$$

respectively. It is to be noted that since

$$|l(z)| = |l_1 z + l_2 z^2 + l_3 z^3 + \ldots| < 1, \quad z \in \mathfrak{U}$$

and

$$|m(w)| = |m_1 w + m_2 w^2 + m_3 w^3 + \ldots| < 1, \quad w \in \mathfrak{U},$$

then

$$|l_i| \leqslant 1 \quad \text{and} \quad |m_i| \leqslant 1, \quad i = 1, 2, 3, \ldots \tag{2.9}$$

Since

$$\frac{(s-b)z\mathfrak{F}'_\lambda(z)}{\mathfrak{F}_\lambda(sz) - \mathfrak{F}_\lambda(bz)} = 1 + (1+\lambda)\sigma a_2 z + \{(1+2\lambda)\tau a_3 - (1+\lambda)^2(s+b)\sigma a_2^2\} z^2 + \ldots \tag{2.10}$$

and

$$\begin{aligned}\frac{(s-b)w\mathfrak{G}'_\lambda(w)}{\mathfrak{G}_\lambda(sw) - \mathfrak{G}_\lambda(bw)} &= 1 - (1+\lambda)\sigma a_2 w \\ &\quad - \{(1+2\lambda)\tau a_3 + [(1+\lambda)^2(s+b)\sigma - 2(1+2\lambda)\tau] a_2^2\} w^2 + \ldots,\end{aligned} \tag{2.11}$$

where

$$\sigma = 2 - (s+b) \quad \text{and} \quad \tau = 3 - (s^2 + sb + b^2),$$

we have

$$\frac{(s-b)z\mathfrak{F}'_\lambda(z)}{\mathfrak{F}_\lambda(sz) - \mathfrak{F}_\lambda(bz)} = 1 + \mathfrak{B}_1(x) l_1 z + \left[\mathfrak{B}_1(x) l_2 + \frac{\mathfrak{B}_2(x)}{2!} l_1^2\right] z^2 + \ldots \tag{2.12}$$

and

$$\frac{(s-b)w\mathfrak{G}'_\lambda(w)}{\mathfrak{G}_\lambda(sw) - \mathfrak{G}_\lambda(bw)} = 1 + \mathfrak{B}_1(x) m_1 w + \left[\mathfrak{B}_1(x) m_2 + \frac{\mathfrak{B}_2(x)}{2!} m_1^2\right] w^2 + \ldots, \tag{2.13}$$

respectively. We get the following equations:

$$(1+\lambda)\sigma a_2 = \mathfrak{B}_1(x)l_1, \tag{2.14}$$

$$(1+2\lambda)\tau a_3 - (1+\lambda)^2(s+b)\sigma a_2^2 = \mathfrak{B}_1(x)l_2 + \frac{\mathfrak{B}_2(x)}{2!}l_1^2, \tag{2.15}$$

$$-(1+\lambda)\sigma a_2 = \mathfrak{B}_1(x)m_1, \tag{2.16}$$

$$[2(1+2\lambda)\tau - (1+\lambda)^2(s+b)\sigma]a_2^2 - (1+2\lambda)\tau a_3 = \mathfrak{B}_1(x)m_2 + \frac{\mathfrak{B}_2(x)}{2!}m_1^2. \tag{2.17}$$

Adding (2.14) and (2.16), we get the following equation:

$$l_1 = -m_1. \tag{2.18}$$

Further squaring and adding (2.14) and (2.16), we have

$$2(1+\lambda)^2\sigma^2 a_2^2 = \mathfrak{B}_1^2(x)(l_1^2 + m_1^2). \tag{2.19}$$

Then the addition of (2.15) and (2.17) gives

$$2[(1+2\lambda)\tau - (1+\lambda)^2(s+b)\sigma]a_2^2 = \mathfrak{B}_1(x)(l_2+m_2) + \frac{\mathfrak{B}_2(x)}{2!}(l_1^2+m_1^2). \tag{2.20}$$

From (1.4), (2.9) and (2.19) we get

$$|a_2| \leqslant \frac{|2x-1|}{2(1+\lambda)|\sigma|}.$$

Also using (2.19) in equation (2.20), we obtain

$$\left[2\{(1+2\lambda)\tau - (1+\lambda)^2(s+b)\sigma\} - \frac{\mathfrak{B}_2(x)}{\mathfrak{B}_1^2(x)}(1+\lambda)^2\sigma^2\right]a_2^2 = \mathfrak{B}_1(x)(l_2+m_2),$$

and then

$$a_2^2 = \frac{\mathfrak{B}_1^3(x)(l_2+m_2)}{2[(1+2\lambda)\tau - (1+\lambda)^2(s+b)\sigma]\mathfrak{B}_1^2(x) - (1+\lambda)^2\sigma^2\mathfrak{B}_2(x)}. \tag{2.21}$$

A small computation leads to

$$|a_2| \leqslant \frac{2|\mathfrak{B}_1(x)|\sqrt{3|\mathfrak{B}_1(x)|}}{\sqrt{|6(2L-M^2)(x^2-x)+(3L-M^2)|}}, \tag{2.22}$$

where

$$L = (1+2\lambda)\tau - (1+\lambda)^2(s+b)\sigma$$

and

$$M = (1+\lambda)\sigma.$$

Next, in order to obtain the bound for $|a_3|$, subtracting (2.17) from (2.15) we have

$$2(1+2\lambda)\tau(a_3 - a_2^2) = \mathfrak{B}_1(x)(l_2 - m_2) + \frac{\mathfrak{B}_2(x)}{2!}(l_1^2 - m_1^2). \tag{2.23}$$

Using equations (2.18) and (2.19) in (2.23), we get

$$a_3 = \frac{\mathfrak{B}_1^2(x)}{2(1+\lambda)^2\sigma^2}(l_1^2 + m_1^2) + \frac{\mathfrak{B}_1(x)}{2(1+2\lambda)\tau}(l_2 - m_2). \tag{2.24}$$

Applying (1.4) and (2.9), we have the desired bound for $|a_3|$:

$$|a_3| \leqslant \frac{|2x-1|^2}{4(1+\lambda)^2|\sigma|^2} + \frac{|2x-1|}{2(1+2\lambda)|\tau|}. \tag{2.25}$$

□

Letting $s = 1$ and $b = -1$ in Theorem 2.1, we get the following result.

Corollary 2.1. *Let $\mathfrak{f}$ given by* (1.1) *be in the class $\mathfrak{B}\mathcal{S}_\Sigma^\lambda(x,z)$. Then we have*

$$|a_2| \leqslant \min\left\{\frac{|x-1/2|}{2(1+\lambda)}, \gamma\right\},$$

where

$$\gamma = \begin{cases} \dfrac{\sqrt{6}|x-1/2|^{3/2}}{2\sqrt{|-3\lambda^2(x-1/2)^2 + (1+\lambda)^2|}}, & 0 < \lambda \leqslant 1 \text{ and } \left(x - \dfrac{1}{2}\right)^2 \neq \dfrac{(1+\lambda)^2}{3\lambda^2}, \\ \dfrac{\sqrt{6}|x-1/2|^{3/2}}{2}, & \lambda = 0 \end{cases}$$

and

$$|a_3| \leqslant \frac{|x-1/2|}{2}\left(\frac{1}{1+2\lambda} + \frac{|x-1/2|}{2(1+\lambda)^2}\right).$$

Letting $s = 1$ and $b = 0$ in Theorem 2.1, we get the following result.

Corollary 2.2. *Let $\mathfrak{f}$ given by* (1.1) *be in the class $\mathfrak{B}\mathcal{H}_\Sigma^\lambda(x,z)$. Then we have*

$$|a_2| \leqslant \min\left\{\frac{|x-1/2|}{1+\lambda}, \gamma\right\},$$

where

$$\gamma = \begin{cases} \dfrac{\sqrt{6}|x-1/2|^{3/2}}{\sqrt{|3(1+2\lambda-3\lambda^2)(x-1/2)^2+(1+\lambda)^2|}}, & 0 \leqslant \lambda < 1 \\ & \text{and } \left(x-\dfrac{1}{2}\right)^2 \neq \dfrac{(1+\lambda)^2}{3(3\lambda+1)(\lambda-1)} \\ \dfrac{\sqrt{6}|x-1/2|^{3/2}}{2}, & \lambda = 1 \end{cases}$$

and

$$|a_3| \leqslant \frac{|x-1/2|}{2}\left(\frac{1}{1+2\lambda} + \frac{2|x-1/2|}{(1+\lambda)^2}\right).$$

Letting $s=1$, $b=-1$ and $\lambda=0$ in Theorem 2.1, we get the following result.

Corollary 2.3. *Let* $\mathfrak{f}$ *given by* (1.1) *be in the class* $\mathfrak{B}\mathcal{S}_\Sigma(x,z)$. *Then we have*

$$|a_2| \leqslant \begin{cases} \dfrac{|x-1/2|}{2}, & \left|x-\dfrac{1}{2}\right| \geqslant \dfrac{1}{6}, \\ \dfrac{\sqrt{6}|x-1/2|^{3/2}}{2}, & \left|x-\dfrac{1}{2}\right| \leqslant \dfrac{1}{6} \end{cases}$$

and

$$|a_3| \leqslant \frac{|x-1/2|}{2}\left(1 + \frac{|x-1/2|}{2}\right).$$

Letting $s=1$, $b=-1$ and $\lambda=1$ in Theorem 2.1, we get the following result.

Corollary 2.4. *Let* $\mathfrak{f}$ *given by* (1.1) *be in the class* $\mathfrak{B}\mathcal{C}_\Sigma(x,z)$. *Then we have*

$$|a_2| \leqslant \min\left\{\frac{|x-1/2|}{4}, \frac{\sqrt{6}|x-1/2|^{3/2}}{2\sqrt{|-3(x-1/2)^2+4|}}\right\}, \quad \left(x-\frac{1}{2}\right)^2 \neq \frac{4}{3}$$

and

$$|a_3| \leqslant \frac{|x-1/2|}{2}\left(\frac{1}{3} + \frac{|x-1/2|}{8}\right).$$

Letting $s=1$, $b=0$ and $\lambda=0$ in Theorem 2.1, we get the following result.

Corollary 2.5. *Let* $\mathfrak{f}$ *given by* (1.1) *be in the class* $\mathfrak{B}\mathcal{H}_\Sigma(x,z)$. *Then we have*

$$|a_2| \leqslant \min\left\{\left|x-\frac{1}{2}\right|, \frac{6|x-1/2|^{3/2}}{\sqrt{|3(x-1/2)^2+1|}}\right\}, \quad \left(x-\frac{1}{2}\right)^2 \neq -\frac{1}{3}$$

and

$$|a_3| \leqslant \frac{|x-1/2|}{2}\left(1 + 2\left|x-\frac{1}{2}\right|\right).$$

Letting $s = 1$, $b = 0$ and $\lambda = 1$ in Theorem 2.1, we get the following result.

Corollary 2.6. *Let* $\mathfrak{f}$ *given by* (1.1) *be in the class* $\mathfrak{BN}_\Sigma(x, z)$. *Then we have*

$$|a_2| \leqslant \begin{cases} \dfrac{|x - 1/2|}{2}, & \left|x - \dfrac{1}{2}\right| \geqslant \dfrac{1}{6}, \\ \dfrac{\sqrt{6}|x - 1/2|^{3/2}}{2}, & \left|x - \dfrac{1}{2}\right| \leqslant \dfrac{1}{6} \end{cases}$$

and

$$|a_3| \leqslant \frac{|x - 1/2|}{2}\left(\frac{1}{3} + \frac{|x - 1/2|}{2}\right).$$

Remark 2.1. We now show that our class $\mathfrak{BS}_\Sigma^\lambda(s, b, x, z)$ is nonempty. We consider, for example, the function $f_*(z) = z + \beta z^2/4$, where

$$\beta = \min\left\{\frac{|x - 1/2|}{(1 + \lambda)|\sigma|}, \gamma\right\}$$

and σ and γ are defined as in Theorem 2.1. A simple calculation shows that the inverse function is $g_*(w) = (-2 + 2\sqrt{\beta w + 1})/\beta$.

Figure 3 shows the images of $z = w = \mathrm{e}^{\mathrm{i}\theta}$, $0 \leqslant \theta < 2\pi$ under $f_*(z)$ (red-full line) and $g_*(w)$ (blue-dashed line) for $s = 1$, $b = 0$, and $\lambda = 0$.

Figure 4 shows that the subordination holds for both $f_*(z)$ and $g_*(w)$. The green curve (dash-dotted line) represents the image of $z = \mathrm{e}^{\mathrm{i}\theta}$, $0 \leqslant \theta < 2\pi$ under $\mathcal{F}(1, z)$. The red and blue curves are, respectively, the images of $z = w = \mathrm{e}^{\mathrm{i}\theta}$, $0 \leqslant \theta < 2\pi$ under $zf_*'(z)/f_*(z)$ and $wg_*'(w)/g_*(w)$.

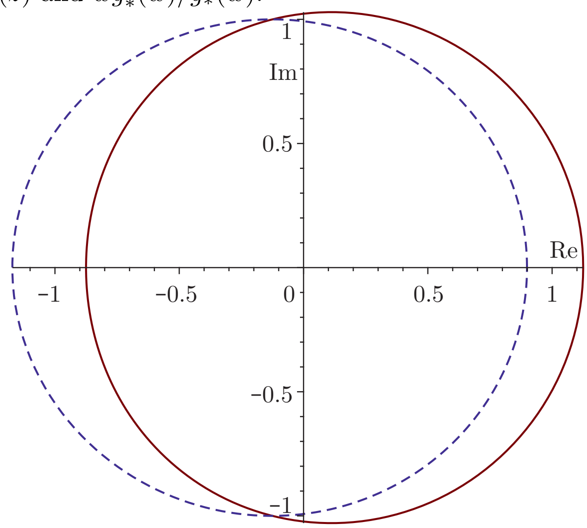

Figure 3. The images of $f_*(\mathrm{e}^{\mathrm{i}\theta})$ and $g_*(\mathrm{e}^{\mathrm{i}\theta})$, $\theta \in [0, 2\pi)$.

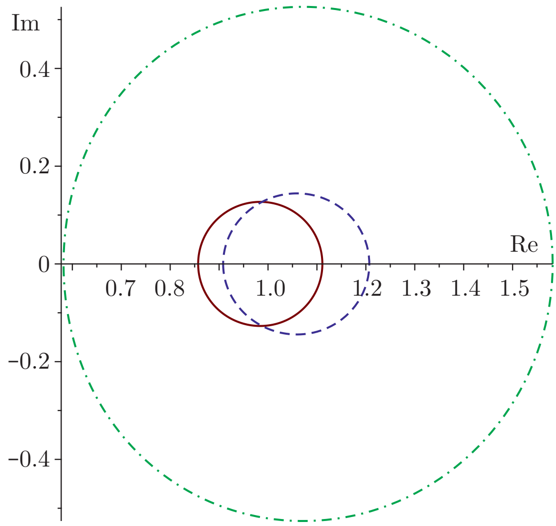

Figure 4. The images of the functions subordinated to the Bernoulli function.

Remark 2.2. It is worth noting that our results improve the results of Buyankara et al. [7].

3. Fekete-Szegö inequality for the class $\mathfrak{BS}_{\Sigma}^{\lambda}(s,b,x,z)$

Theorem 3.1. *If the function* $\mathfrak{f}$ *of the form* (1.1) *belongs to* $\mathfrak{BS}_{\Sigma}^{\lambda}(s,b,x,z)$, *then for any complex number* ρ,

$$|a_3-\rho a_2^2| \leqslant \begin{cases} \dfrac{|x-1/2|}{(1+2\lambda)|\tau|}, & 0 \leqslant |\psi(\rho)| \leqslant \dfrac{1}{2(1+2\lambda)|\tau|}, \\ |\psi(\rho)|, & |\psi(\rho)| \geqslant \dfrac{1}{2(1+2\lambda)|\tau|}, \end{cases} \tag{3.1}$$

where

$$|\psi(\rho)| = \begin{cases} \dfrac{24|x-1/2|^3}{|12(2L-M^2)(x-1/2)^2+M^2|}|1-\rho|, & L \neq \dfrac{1}{2}M^2 \\ & \text{and } \left(x-\dfrac{1}{2}\right)^2 \neq -\dfrac{M^2}{12(2L-M^2)}, \\ \dfrac{24|x-1/2|^3}{|M|^2}|1-\rho|, & L = \dfrac{1}{2}M^2 \end{cases}$$

and L *and* M *are defined by* (2.1).

P r o o f. From (2.18) and (2.23) we get

$$a_3 - \rho a_2^2 = (1-\rho)a_2^2 + \frac{\mathfrak{B}_1(x)}{2(1+2\lambda)\tau}(l_2 - m_2).$$

By using (2.21) in the above equality, we obtain

$$a_3 - \rho a_2^2 = \mathfrak{B}_1(x)\left[\left(\psi(\rho) + \frac{1}{2(1+2\lambda)\tau}\right)l_2 + \left(\psi(\rho) - \frac{1}{2(1+2\lambda)\tau}\right)m_2\right],$$

where

$$\psi(\rho) = \frac{(1-\rho)\mathfrak{B}_1^2(x)}{2[(1+2\lambda)\tau - (1+\lambda)^2(s+b)\sigma]\mathfrak{B}_1^2(x) - (1+\lambda)^2\sigma^2\mathfrak{B}_2(x)}.$$

Thus, we have

$$|a_3 - \rho a_2^2| \leqslant \begin{cases} \dfrac{|\mathfrak{B}_1(x)|}{(1+2\lambda)|\tau|}, & 0 \leqslant |\psi(\rho)| \leqslant \dfrac{1}{2(1+2\lambda)|\tau|}, \\ 2|\psi(\rho)|\,|\mathfrak{B}_1(x)|, & |\psi(\rho)| \geqslant \dfrac{1}{2(1+2\lambda)|\tau|}. \end{cases}$$

□

Letting $s = 1$ and $b = -1$ in Theorem 3.1, we get the following result.

Corollary 3.1. *If the function* $\mathfrak{f}$ *of the form* (1.1) *belongs to* $\mathfrak{B}\mathcal{S}_\Sigma^\lambda(x,z)$, *then for any complex number* ρ,

$$|a_3 - \rho a_2^2| \leqslant \begin{cases} \dfrac{|x-1/2|}{2(1+2\lambda)}, & 0 \leqslant |\psi(\rho)| \leqslant \dfrac{1}{4(1+2\lambda)}, \\ |\psi(\rho)|, & |\psi(\rho)| \geqslant \dfrac{1}{4(1+2\lambda)}, \end{cases}$$

where

$$|\psi(\rho)| = \begin{cases} \dfrac{6|x-1/2|^3}{|-12\lambda^2(x-1/2)^2 + (1+\lambda)^2|}|1-\rho|, & 0 < \lambda \leqslant 1 \\ \qquad \text{and } \left(x - \dfrac{1}{2}\right)^2 \neq \dfrac{(1+\lambda)^2}{12\lambda^2}, & \\ 6\left|x - \dfrac{1}{2}\right|^3 |1-\rho|, & \lambda = 0. \end{cases}$$

Letting $s = 1$ and $b = 0$ in Theorem 3.1, we get the following result.

Corollary 3.2. *If the function f of the form* (1.1) *belongs to $\mathfrak{B}\mathcal{H}_{\Sigma}^{\lambda}(x,z)$, then for any complex number ρ,*

$$|a_3 - \rho a_2^2| \leqslant \begin{cases} \dfrac{|x-1/2|}{2(1+2\lambda)}, & 0 \leqslant |\psi(\rho)| \leqslant \dfrac{1}{4(1+2\lambda)}, \\ |\psi(\rho)|, & |\psi(\rho)| \geqslant \dfrac{1}{4(1+2\lambda)}, \end{cases}$$

where

$$|\psi(\rho)| = \begin{cases} \dfrac{24|x-1/2|^3}{|12(1+2\lambda-3\lambda^2)(x-1/2)^2+(1+\lambda)^2|}|1-\rho|, & 0 \leqslant \lambda < 1 \\ & \text{and } \left(x-\dfrac{1}{2}\right)^2 \neq \dfrac{(1+\lambda)^2}{12(3\lambda+1)(\lambda-1)}, \\ 6\left|x-\dfrac{1}{2}\right|^3 |1-\rho|, & \lambda = 1. \end{cases}$$

Letting $s = 1$, $b = -1$ and $\lambda = 0$ in Theorem 3.1, we get the following result.

Corollary 3.3. *If the function $\mathfrak{f}$ of the form* (1.1) *belongs to $\mathfrak{B}\mathcal{S}_{\Sigma}(x,z)$, then for any complex number ρ,*

$$|a_3 - \rho a_2^2| \leqslant \begin{cases} \dfrac{|x-1/2|}{2}, & 0 \leqslant |\psi(\rho)| \leqslant \dfrac{1}{4}, \\ 6\left|x-\dfrac{1}{2}\right|^3 |1-\rho|, & |\psi(\rho)| \geqslant \dfrac{1}{4}. \end{cases}$$

Letting $s = 1$, $b = -1$ and $\lambda = 1$ in Theorem 3.1, we get the following result.

Corollary 3.4. *If the function $\mathfrak{f}$ of the form* (1.1) *belongs to $\mathfrak{B}\mathcal{C}_{\Sigma}(x,z)$, then for any complex number ρ,*

$$|a_3 - \rho a_2^2| \leqslant \begin{cases} \dfrac{|x-1/2|}{6}, & 0 \leqslant |\psi(\rho)| \leqslant \dfrac{1}{12}, \\ \dfrac{6|x-1/2|^3}{4\,|-3(x-1/2)^2+1|}|1-\rho|, & |\psi(\rho)| \geqslant \dfrac{1}{12} \quad \text{and} \quad \left(x-\dfrac{1}{2}\right)^2 \neq \dfrac{1}{3}. \end{cases}$$

Letting $s = 1$, $b = 0$ and $\lambda = 0$ in Theorem 3.1, we get the following result.

Corollary 3.5. *If the function $\mathfrak{f}$ of the form* (1.1) *belongs to $\mathfrak{B}\mathcal{H}_{\Sigma}(x, z)$, then for any complex number ρ,*

$$|a_3 - \rho a_2^2| \leqslant \begin{cases} \dfrac{|x - 1/2|}{2}, & 0 \leqslant |\psi(\rho)| \leqslant \dfrac{1}{4}, \\ \dfrac{24|x - 1/2|^3}{|12(x - 1/2)^2 + 1|}|1 - \rho|, & |\psi(\rho)| \geqslant \dfrac{1}{4} \quad \text{and} \quad \left(x - \dfrac{1}{2}\right)^2 \neq -\dfrac{1}{12}. \end{cases}$$

Letting $s = 1$, $b = 0$ and $\lambda = 1$ in Theorem 3.1, we get the following result.

Corollary 3.6. *If the function $\mathfrak{f}$ of the form* (1.1) *belongs to $\mathfrak{B}\mathcal{N}_{\Sigma}(x, z)$, then for any complex number ρ,*

$$|a_3 - \rho a_2^2| \leqslant \begin{cases} \dfrac{|x - 1/2|}{6}, & 0 \leqslant |\psi(\rho)| \leqslant \dfrac{1}{12}, \\ 6\left|x - \dfrac{1}{2}\right|^3 |1 - \rho|, & |\psi(\rho)| \geqslant \dfrac{1}{12}. \end{cases}$$

Letting $\rho = 1$ in Theorem 3.1, we get the following consequence.

Corollary 3.7. *If the function $\mathfrak{f}$ of the form* (1.1) *belongs to $\mathfrak{B}\mathcal{S}_{\Sigma}^{\lambda}(s, b, x, z)$, then*

$$|a_3 - a_2^2| \leqslant \frac{|x - 1/2|}{(1 + 2\lambda)|\tau|}.$$

4. Conclusion

In this study, we have successfully introduced a new subclass of bi-univalent functions, termed Sakaguchi-type functions, defined by Bernoulli polynomials. Through rigorous analysis, we have proved that the class is nonempty and derived bounds for the Fekete-Szegö inequality and provided estimates for the initial coefficients $|a_2|$ and $|a_3|$. These findings contribute to the broader understanding of bi-univalent functions and open new avenues for further research in this area. Future work may explore additional properties and applications of these functions, potentially leading to new insights and advancements in the field of geometric function theory.

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