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ON $\mathcal{Z}$-REFLEXIVE RINGS

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Abstract. We introduce the notion of $\mathcal{Z}$-reflexive rings to describe reflexivity of rings in terms of their singular ideals. We show that $\mathcal{Z}$-reflexive ring is proper common generalization of a central reflexive ring, $\mathcal{Z}$-reversible ring, and singular clean ring. We discuss some its properties, characterizations, and relations with some extension rings. We show that a ring R is right $\mathcal{Z}$-reflexive if and only if $M_n(R)$ is right $\mathcal{Z}$-reflexive for every positive integer n. Also, we share the connection of right $\mathcal{Z}$-reflexive rings with J-reflexive rings.

Keywords: reflexive ring; $\mathcal{Z}$-reflexive ring; J-reflexive ring; central reflexive ring; singular ideal

MSC 2020: 13C99, 16U99, 16N20, 16D80

1. Introduction

Throughout the work, all rings are associative and have identity. In a ring R, we denote the two-sided ideal generated a, set of all idempotent elements, set of all central elements, set of all unit elements, set of all nilpotent elements, and Jacobson radical by $\langle a\rangle$, $I(R)$, $C(R)$, $U(R)$, $\mathrm{Nil}(R)$, and $J(R)$, respectively. We denote the ring of square matrices of order n over a field F by $M_n(F)$; and a square matrix whose (i,j)-entry is 1 and zero elsewhere by E_{ij}.

Recalling [9], a right ideal I of a ring R is called an *essential right ideal* of R if $I\cap J\neq 0$ for every nonzero right ideal J of R. An element $a\in R$ is called *right singular* if $\mathrm{ann}_r(a)=\{r\in R\colon ar=0\}$ is an essential right ideal, or equivalently, there exists an essential right ideal I of R such that $aI=0$. The set of all right singular elements of R is denoted by $\mathcal{Z}_r(R)$ or $\mathcal{Z}(R_R)$. $\mathcal{Z}_r(R)$, a two-sided ideal of R, is called right *singular ideal* of R. A ring R is said to be right *nonsingular* if $\mathcal{Z}_r(R)=0$. The notion of left singular ideal $\mathcal{Z}_l(R)$ is defined analogously.

The notions of reflexivity and reversibility of rings are very well-known in the ring theory. The first was introduced by Meson [11] in 1981 and the second by

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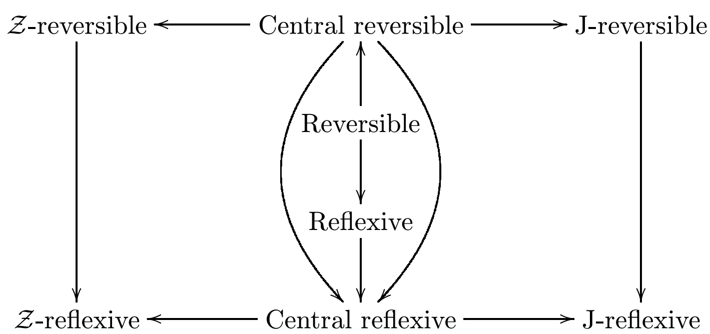
Cohn [7] in 1999. Meson called an ideal I of a ring R *reflexive* if $aRb \in I$ implies $bRa \in I$ for any $a, b \in R$; be called a ring R *reflexive* if 0 is a reflexive ideal, that is, $aRb = 0$ implies $bRa = 0$ for any $a, b \in R$. Cohn called a ring R *reversible* if whenever $ab = 0$, then $ba = 0$ for any $a, b \in R$. Clearly, every reversible ring is reflexive. In 2014, Kose et al. [8] introduced the notion of central reversible rings as a generalization of reversible rings. They called a ring R *central reversible* if whenever $ab = 0$, then $ba \in C(R)$ for any $a, b \in R$. In 2015, Chakraborty [5] introduced the notion of central reflexive rings as a common generalization of reflexive rings and central reversible rings. He called a ring R *central reflexive* if $aRb = 0$ implies $bRa \subseteq C(R)$ for any $a, b \in R$. In 2017, Calci et al. [4] introduced the notion of J-reversible rings as a generalization of central reversible rings. They called a ring R *J-reversible* if whenever $ab = 0$, then $ba \in C(R)$ for any $a, b \in R$. In 2024, Calci et al. [3] introduced the notion of J-reflexive rings as a common generalization of central reflexive rings and J-reversible rings. They called a ring R *J-reflexive* if $aRb = 0$ implies $bRa \subseteq J(R)$ for any $a, b \in R$. In 2022, Chaturvedi et al. [6] introduced the notion of right $\mathcal{Z}$-reversible rings as a generalization of central reversible rings. They called a ring R *right $\mathcal{Z}$-reversible* if for any $a, b \in R$, $ab = 0$ implies $ba \in \mathcal{Z}_r(R)$. By the motivation from these studies, we introduce the notion of $\mathcal{Z}$-reflexive rings as a common generalization of central reflexive rings and $\mathcal{Z}$-reversible rings (Definition 2.1). With this definition, we get the following implication diagram:



We summarize the contents of this paper. In Section 2, we first give the definition of right (left) $\mathcal{Z}$-reflexive rings. Then we show that the $\mathcal{Z}$-reflexive ring is not left-right symmetric (Remark 2.2). Then we give some characterizations of the right $\mathcal{Z}$-reflexive ring (Theorems 2.4 and 2.5). Further, we show that right $\mathcal{Z}$-reversible rings, right singular clean rings, and central reflexive rings are examples of right $\mathcal{Z}$-reflexive rings (Propositions 2.6, 2.9 and 2.10). Finally, we share the connection of right $\mathcal{Z}$-reflexive rings with J-reflexive rings (see Remark 2.14, Propositions 2.16 and 2.18).

In Section 3, we study $\mathcal{Z}$-reflexivity of some extension rings. In Proposition 3.1, we show that right $\mathcal{Z}$-reflexive rings are closed under arbitrary direct product of rings; and in Proposition 3.4, we show that right $\mathcal{Z}$-reflexive rings are closed under finite subdirect product of rings. In Proposition 3.6, we prove that a ring is right $\mathcal{Z}$-reflexive if and only if some its specific localization ring is so. Consequently, we find that a polynomial ring of a ring is right $\mathcal{Z}$-reflexive if and only if its Laurent polynomial ring is so (see Corollary 3.7). In Theorem 3.10, we show that a ring is right $\mathcal{Z}$-reflexive if and only if its all matrix rings are so.

2. Examples and properties of $\mathcal{Z}$-reflexive rings

Definition 2.1. We call a ring R right (or left) $\mathcal{Z}$-reflexive if for any $a, b \in R$, $aRb = 0$ implies $bRa \subseteq \mathcal{Z}_r(R)$ (or $bRa \subseteq \mathcal{Z}_l(R)$, respectively); and we call a ring R $\mathcal{Z}$-reflexive if it is both left as well as right $\mathcal{Z}$-reflexive.

Remark 2.2. The notion of $\mathcal{Z}$-reflexive rings is not left-right symmetric. For example, let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. Then by [9], Example 7.6 (5), $\mathcal{Z}_l(R) = J(R) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ and $\mathcal{Z}_r(R) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Let $A, B \in R$ be such that $ARB = 0$. Then $BRA = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \subseteq \mathcal{Z}_l(R)$. Hence R is left $\mathcal{Z}$-reflexive. However, R is not right $\mathcal{Z}$-reflexive as if we take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in R$, then $ARB = 0$ but $BRA = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix} \nsubseteq \mathcal{Z}_r(R)$.

Next, let $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_4 \end{bmatrix}$. Then by [9], Theorem 7.15, $\mathcal{Z}_r(R) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 2\mathbb{Z}_4 \end{bmatrix}$ and $\mathcal{Z}_l(R) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Let $A, B \in R$ be such that $ARB = 0$. Then $BRA = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \subseteq \mathcal{Z}_r(R)$. Hence R is right $\mathcal{Z}$-reflexive. However, R is not left $\mathcal{Z}$-reflexive as if we take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in R$, then $ARB = 0$ but $BRA = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix} \nsubseteq \mathcal{Z}_l(R)$.

Remark 2.3. Every reflexive ring is $\mathcal{Z}$-reflexive. However, the converse need not be true in general. For example, let $R = \begin{bmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{bmatrix}$. Then by [9], Example 7.6 (6), $\mathcal{Z}_l(R) = \mathcal{Z}_r(R) = J(R) = \begin{bmatrix} 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & 2\mathbb{Z}_4 \end{bmatrix}$. Let $A, B \in R$ be such that $ARB = 0$. Then $BRA = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \subseteq \mathcal{Z}_l(R) = \mathcal{Z}_r(R)$. Hence R is $\mathcal{Z}$-reflexive. However, R is not reflexive as if we take $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in R$, then $ARB = 0$ but $BRA = \begin{bmatrix} 0 & 2\mathbb{Z}_4 \\ 0 & 0 \end{bmatrix} \neq 0$.

Since all the results are symmetric for left $\mathcal{Z}$-reflexive rings and right $\mathcal{Z}$-reflexive rings, we will consider only right $\mathcal{Z}$-reflexive rings throughout the work. In the following two results, we give characterizations of right $\mathcal{Z}$-reflexive rings.

Theorem 2.4. *The following are equivalent for a ring R.*

(1) *R is right $\mathcal{Z}$-reflexive.*
(2) *For every $a \in R$, $\operatorname{ann}_r(aR)Ra \subseteq \mathcal{Z}_r(R)$ and $aR\operatorname{ann}_l(Ra) \subseteq \mathcal{Z}_r(R)$.*
(3) *For every nonempty subsets I, K of R, $IRK = 0 \Rightarrow KRI \subseteq \mathcal{Z}_r(R)$.*
(4) *For every $a, b \in R$, $\langle a \rangle \langle b \rangle = 0 \Rightarrow \langle b \rangle \langle a \rangle \subseteq \mathcal{Z}_r(R)$.*
(5) *For every right (left) ideals I, K of R, $IK = 0 \Rightarrow KI \subseteq \mathcal{Z}_r(R)$.*
(6) *For every ideals I, K of R, $IK = 0 \Rightarrow KI \subseteq \mathcal{Z}_r(R)$.*

Proof. (1) $\Rightarrow$ (2). Let $a \in R$ and $b \in \operatorname{ann}_r(aR)$. Then $aRb = 0$. Hence by (1), $bRa \subseteq \mathcal{Z}_r(R)$. This implies that $\operatorname{ann}_r(aR)Ra \subseteq \mathcal{Z}_r(R)$. Similarly, $aR\operatorname{ann}_l(Ra) \subseteq \mathcal{Z}_r(R)$.

(2) $\Rightarrow$ (3). Let I and K be two nonempty subsets of R such that $IRK = 0$. Let $a \in I$ and $b \in K$. Then $aRb = 0$ and so $b \in \operatorname{ann}_r(aR)$. Hence by (2), $bRa \subseteq \mathcal{Z}_r(R)$. This implies that $KRI \subseteq \mathcal{Z}_r(R)$.

(3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6). Clear.

(6) $\Rightarrow$ (1). Let $a, b \in R$ be such that $aRb = 0$. Then $RaRRbR = 0$. Hence by (6), $RbRRaR \subseteq \mathcal{Z}_r(R)$. This implies that $bRa \subseteq \mathcal{Z}_r(R)$ because $bRa \subseteq RbRRaR$. $\square$

Theorem 2.5. *The following statements are equivalent for a ring R.*

(1) *R is right $\mathcal{Z}$-reflexive.*
(2) *The subring $S = \{(a, b) \in R \times R\colon\ a - b \in \mathcal{Z}_r(R)\}$ of $R \times R$ is right $\mathcal{Z}$-reflexive.*

Proof. (1) $\Rightarrow$ (2). Suppose that R is right $\mathcal{Z}$-reflexive and let $(a_1, b_1), (a_2, b_2) \in S$ be such that $(a_1, b_1)S(a_2, b_2) = 0$. Then for any $r \in R$, $(a_1, b_1)(r, r)(a_2, b_2) = 0$ and so $a_1ra_2 = 0 = b_1rb_2$. This implies that $a_1Ra_2 = 0$ and $b_1Rb_2 = 0$. Hence $a_2Ra_1 \subseteq \mathcal{Z}_r(R)$ and $b_2Rb_1 \subseteq \mathcal{Z}_r(R)$. Therefore $(a_2, b_2)S(a_1, b_1) \subseteq (a_2, b_2)(R \times R)(a_1, b_1) = a_2Ra_1 \times b_2Rb_1 \subseteq \mathcal{Z}_r(R) \times \mathcal{Z}_r(R)$. Since by [12], Proposition 3.5, $\mathcal{Z}_r(R) \times \mathcal{Z}_r(R) \subseteq \mathcal{Z}_r(S)$, we have $(a_2, b_2)S(a_1, b_1) \subseteq \mathcal{Z}_r(S)$. Thus S is right $\mathcal{Z}$-reflexive.

(2) $\Rightarrow$ (1). Suppose that S is right $\mathcal{Z}$-reflexive and let $a, b \in R$ be such that $aRb = 0$. Then clearly $(a, a), (b, b) \in S$ are such that $(a, a)S(b, b) = 0$ and so $(b, b)S(a, a) \subseteq \mathcal{Z}_r(S)$. Let $r \in R$. Then $(r, r) \in S$ and so $(bra, bra) = (b, b)(r, r)(a, a) \in \mathcal{Z}_r(S)$. This

implies that $\operatorname{ann}_r((bra, bra))$ is an essential right ideal of S. Now let $0 \neq x \in R$. Then $0 \neq (x, x) \in S$. So there exists $0 \neq (c, d) \in S$ such that $(c, d)(x, x) \neq 0$ and $(c, d)(x, x) \in \operatorname{ann}_r((bra, bra))$ since we assume that S is right $\mathcal{Z}$-reflexive. Hence $(bra)cx = 0 = (bra)dx$ and so $cx, dx \in \operatorname{ann}_r(bra)$. Since $(cx, dx) \neq 0$, we get that for any $0 \neq x \in R$, there exists $e (= c$ or $d)$ in R such that $0 \neq ex \in \operatorname{ann}_r(bra)$. This implies that $\operatorname{ann}_r(bra)$ is an essential right ideal of R and so $bra \in \mathcal{Z}_r(R)$. Thus $bRa \subseteq \mathcal{Z}_r(R)$. So, R is right $\mathcal{Z}$-reflexive. □

Proposition 2.6. *Every right $\mathcal{Z}$-reversible ring is right $\mathcal{Z}$-reflexive.*

Proof. Let R be a right $\mathcal{Z}$-reversible ring and let $a, b \in R$ be such that $aRb = 0$. Then $ab = 0$ and so $abr = 0$ for any $r \in R$. Hence $bra \in \mathcal{Z}_r(R)$ for any $r \in R$. This implies that $bRa \subseteq \mathcal{Z}_r(R)$. □

Remark 2.7. A right $\mathcal{Z}$-reflexive ring need not be right $\mathcal{Z}$-reversible. For example, let $R = M_3(\mathbb{Z})$. Then R is right $\mathcal{Z}$-reflexive by Proposition 3.10. However, R is not right $\mathcal{Z}$-reversible by [6], Example 6.

Recalling [9], a ring R is called Baer if the right (left) annihilator of every nonempty subset of R is generated by an idempotent. Recalling [6], a ring R is called semicommutative if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. In the following, we give some sufficient conditions under which the notion of right $\mathcal{Z}$-reversible rings and right $\mathcal{Z}$-reflexive rings are equivalent.

Proposition 2.8. *Let R be a Baer ring or a semicommutative ring. Then R is right $\mathcal{Z}$-reversible if and only if R is right $\mathcal{Z}$-reflexive.*

Proof. Suppose that R is a Baer ring. Let R be a right $\mathcal{Z}$-reflexive ring and $a, b \in R$ be such that $ab = 0$. Then $Rab = 0$ and so $b \in \operatorname{ann}_r(Ra)$. Since R is a Baer ring, there exists $e \in I(R)$ such that $\operatorname{ann}_r(Ra) = Re$. This implies that $RaRe = 0$ and so $ReRa \subseteq \mathcal{Z}_r(R)$ as R is right $\mathcal{Z}$-reflexive. Hence $ba \in \mathcal{Z}_r(R)$ as $b \in \operatorname{ann}_r(Ra) = Re$. Thus R is right $\mathcal{Z}$-reversible. The converse follows from Proposition 2.6.

Suppose that R is a semicommutative ring. Let R be a right $\mathcal{Z}$-reflexive ring and $a, b \in R$ such that $ab = 0$. Then $aRb = 0$ by semicommutativity of R. Hence by the $\mathcal{Z}$-reflexivity of R, $bRa \subseteq \mathcal{Z}_r(R)$. This implies that $ba \in \mathcal{Z}_r(R)$. Thus R is right $\mathcal{Z}$-reversible. The converse follows from Proposition 2.6. □

Recalling [1], a ring R is called right singular clean, if every element of R can be expressed as a sum of a right singular element and an idempotent. A ring R is called a Boolean ring if $r^2 = r$ for every $r \in R$. A right $\mathcal{Z}$-reflexive ring need not be right singular clean. For example, $\mathbb{Z}_3$ is right $\mathcal{Z}$-reflexive but not right singular clean.

Proposition 2.9. *Every right singular clean ring is right $\mathcal{Z}$-reflexive.*

P r o o f. Let R be a right singular clean ring. Then, by [1], Proposition 2.3, $\overline{R} = R/\mathcal{Z}_r(R)$ is a Boolean ring and so $\overline{R}$ is commutative. Let $a, b \in R$ be such that $aRb = 0$. Then $\bar{a}\overline{R}\bar{b} = \bar{0}$. Since $\overline{R}$ is commutative, $\bar{b}\overline{R}\bar{a} = \bar{a}\overline{R}\bar{b} = \bar{0}$. This implies that $bRa \subseteq \mathcal{Z}_r(R)$. Thus R is right $\mathcal{Z}$-reflexive. □

Proposition 2.10. *Every central reflexive ring is $\mathcal{Z}$-reflexive.*

P r o o f. Suppose that R is a central reflexive ring. Let $a, b \in R$ be such that $aRb = 0$. Then $(bRa)^2 = (bRa)(bRa) = bR(ab)Ra = 0$ and so $bRa \subseteq \mathrm{Nil}(R)$. Again since R is central reflexive and $aRb = 0$, $bRa \subseteq C(R)$. Hence $bRa \subseteq \mathrm{Nil}(R) \cap C(R)$. Therefore $bRa \subseteq \mathcal{Z}_r(R) \cap \mathcal{Z}_l(R)$ by [9], Lemma 7.1. Thus R is $\mathcal{Z}$-reflexive. □

R e m a r k 2.11. A $\mathcal{Z}$-reflexive ring need not to be central reflexive. For example, let $R = \begin{bmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{bmatrix}$. Then R is $\mathcal{Z}$-reflexive by Remark 2.3. However, R is not central reflexive as if we take $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in R$, then $ARB = 0$ but $BRA = \begin{bmatrix} 0 & 2\mathbb{Z}_4 \\ 0 & 0 \end{bmatrix} \nsubseteq C(R)$.

In the following, we show that the notions of right $\mathcal{Z}$-reflexive rings, central reflexive rings, and reflexive rings are equivalent for a right nonsingular ring.

Proposition 2.12. *The following statements are equivalent for a right nonsingular ring R.*

(1) *R is right $\mathcal{Z}$-reflexive.*
(2) *R is reflexive.*
(3) *R is central reflexive.*

P r o o f. (1) $\Rightarrow$ (2). It follows from the fact that $\mathcal{Z}_r(R) = 0$. (2) $\Rightarrow$ (3). It follows directly from definitions. (3) $\Rightarrow$ (1). It follows from Proposition 2.10. □

R e m a r k 2.13. Homomorphic image (quotient) of a right $\mathcal{Z}$-reflexive ring need not be right $\mathcal{Z}$-reflexive ring. For example, let $R = F\langle x, y\rangle$ be a free algebra over a field F in two noncommutating variables x and y. Then clearly R is a right $\mathcal{Z}$-reflexive ring being an integral domain. Let I be the ideal $\langle x^2, xy, y^2\rangle$ of R and $\overline{R} = R/I$. Then, $\overline{R} = \{(a + bx + cy + dyx) + I\colon\ a, b, c, d \in F\}$. Since $r_{\overline{R}}(\overline{yx}) \cap \bar{x}\overline{R} = \{ayx + I\colon\ a \in F\} \cap \{ax + I\colon\ a \in F\} = \{I\}$, $\overline{yx} \notin \mathcal{Z}_r(\overline{R})$. Now since $\bar{x}\overline{R}\bar{y} = \bar{0}$ but $\bar{y}\overline{R}\bar{x} = \{ayx + I\colon\ a \in F\} \nsubseteq \mathcal{Z}_r(\overline{R})$. Hence $\overline{R}$ is not right $\mathcal{Z}$-reflexive.

Remark 2.14. A J-reflexive ring need not be right $\mathcal{Z}$-reflexive. For example, let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. Then by Remark 2.2, R is not right $\mathcal{Z}$-reflexive and $J(R) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$. Let $A, B \in R$ be such that $ARB = 0$. Then $BRA = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \subseteq J(R)$. Hence R is J-reflexive but not right $\mathcal{Z}$-reflexive. However, we do not know whether right $\mathcal{Z}$-reflexive rings are J-reflexive or not. Hence we raise the following question:

Question 2.15. Is every right $\mathcal{Z}$-reflexive ring J-reflexive?

In the following three results, we give partial answer to above question.

Proposition 2.16. *Let R be a right $\mathcal{Z}$-reflexive ring. If every right regular element is left invertible, then R is J-reflexive.*

Proof. Let $a, b \in R$ be such that $aRb = 0$. Then $bRa \subseteq \mathcal{Z}_r(R)$. This implies that for any $z, r \in R$, $zbra \in \mathcal{Z}_r(R)$. Hence $\mathrm{ann}_r(zbra)$ is an essential right ideal of R for any $z, r \in R$. Since $\mathrm{ann}_r(zbra) \cap \mathrm{ann}_r(1 - zbra) = 0$, so we have $\mathrm{ann}_r(1 - zbra) = 0$ for any $z, r \in R$. Hence by assumption, $1 - zbra$ is left invertible for any $z, r \in R$. It follows that $bra \in J(R)$ for any $r \in R$ and so $bRa \subseteq J(R)$. Thus R is J-reflexive. □

Recalling [15], a ring R is called *right np-injective* if, for any non-nilpotent element c of R, any right R-homomorphism $g\colon cR \to R$, there exists $b \in R$ such that $g(ca) = bca$ for all $a \in R$. In view of [15], Proposition 5 (1) and Proposition 2.16, we have the following result.

Corollary 2.17. *Let R be a right $\mathcal{Z}$-reflexive ring. If R is right np-injective, then R is J-reflexive.*

Recalling [13], a ring R is called an *exchange ring*, if for every $a \in R$, there exists $e \in I(R)$ such that $e \in aR$ and $(1 - e) \in (1 - a)R$, or equivalently, $e \in Ra$ and $(1 - e) \in R(1 - a)$.

Proposition 2.18. *Let R be a right $\mathcal{Z}$-reflexive ring. If R is an exchange ring, then R is J-reflexive.*

Proof. Let $a, b \in R$ be such that $aRb = 0$. Then $bRa \subseteq \mathcal{Z}_r(R)$. This implies that for any $r \in R$, $bra \in \mathcal{Z}_r(R)$. Since R is an exchange ring, for $bra \in R$, there exists an idempotent $e \in I(R)$ such that $e \in braR \subseteq \mathcal{Z}_r(R)$ and $(1 - e) \in (1 - bra)R$. Since we know that singular ideals contain no nonzero idempotent element, so we have $e = 0$ and $1 \in (1 - bra)R$. Similarly, we get $1 \in R(1 - bra)$. This implies that $1 - bra \in U(R)$ and so $bra \in J(R)$. Hence $bRa \subseteq J(R)$. Thus R is a J-reflexive ring. □

3. Some extension rings of $\mathcal{Z}$-reflexive rings

In the following, we show that right $\mathcal{Z}$-reflexive rings are closed under arbitrary direct product of rings.

Proposition 3.1. *Let $\{R_\alpha\}_{\alpha\in A}$ be a family of rings. Then $\prod\limits_{\alpha\in A} R_\alpha$ is right $\mathcal{Z}$-reflexive if and only if every R_α is right $\mathcal{Z}$-reflexive.*

Proof. Suppose that every R_α is right $\mathcal{Z}$-reflexive. Let $a=(a_\alpha)_{\alpha\in A}$, $b=(b_\alpha)_{\alpha\in A}\in R=\prod\limits_{\alpha\in A} R_\alpha$ be such that $aRb=0$. Then for any $\alpha\in A$, $a_\alpha R_\alpha b_\alpha=0$ and so $b_\alpha R_\alpha a_\alpha\subseteq \mathcal{Z}_r(R_\alpha)$ as R_α is right $\mathcal{Z}$-reflexive. This implies that $bRa\subseteq \prod\limits_{\alpha\in A}\mathcal{Z}_r(R_\alpha)=\mathcal{Z}_r(R)$. Thus R is right $\mathcal{Z}$-reflexive.

Conversely, suppose that R is right $\mathcal{Z}$-reflexive. Let $a_\alpha, b_\alpha\in R_\alpha$ be such that $a_\alpha R_\alpha b_\alpha=0$. Then $a=(a_\alpha)_{\alpha\in A}$ and $b=(b_\alpha)_{\alpha\in A}$ are two elements of R such that $aRb=0$. Since R is right $\mathcal{Z}$-reflexive, $bRa\subseteq\mathcal{Z}_r(R)=\prod\limits_{\alpha\in A}\mathcal{Z}_r(R_\alpha)$. This implies that $b_\alpha R_\alpha a_\alpha\subseteq\mathcal{Z}_r(R_\alpha)$. Thus every R_α is right $\mathcal{Z}$-reflexive. □

Corollary 3.2. *Let e be a central idempotent of a ring R. Then R is right $\mathcal{Z}$-reflexive if and only if eRe and $(1-e)R(1-e)$ are right $\mathcal{Z}$-reflexive.*

Lemma 3.3 ([12], Lemma 2.6)**.** *Let $J_1, J_2,\dots,J_n$ be ideals of a ring R such that $\bigcap\limits_{i=1}^{n} J_i=0$. If $r\in R$ and $r+J_i\in\mathcal{Z}_r(R/J_i)$ for all $1\leqslant i\leqslant n$, then $r\in\mathcal{Z}_r(R)$.*

In the following, we show that right $\mathcal{Z}$-reflexive rings are closed under finite subdirect product of rings.

Proposition 3.4. *A finite subdirect product of right $\mathcal{Z}$-reflexive rings is right $\mathcal{Z}$-reflexive.*

Proof. Let $J_1, J_2,\dots,J_n$ be ideals of a ring R such that each R/J_i is right $\mathcal{Z}$-reflexive and $\bigcap\limits_{i=1}^{n} J_i=0$. Let $a,b\in R$ be such that $aRb=0$. Then for any i, $(a+J_i)(R/J_i)(b+J_i)=\{J_i\}$ and so $(b+J_i)(R/J_i)(a+J_i)\subseteq\mathcal{Z}_r(R/J_i)$ as R/J_i is right $\mathcal{Z}$-reflexive. This implies that for any $r\in R$, $(bra+J_i)=(b+J_i)(r+J_i)(a+J_i)\in\mathcal{Z}_r(R/J_i)$ for all $1\leqslant i\leqslant n$. Hence by Lemma 3.3, $bra\in\mathcal{Z}_r(R)$ for any $r\in R$. Thus $bRa\subseteq\mathcal{Z}_r(R)$ and so R is right $\mathcal{Z}$-reflexive. □

Corollary 3.5. *Let $J_1, J_2,\dots,J_n$ be ideals of a ring R such that R/J_1, $R/J_2,\dots$, R/J_n are right $\mathcal{Z}$-reflexive. Then $R/\bigcap\limits_{i=1}^{n} J_i$ is right $\mathcal{Z}$-reflexive.*

Proposition 3.6. *Let S be a multiplicatively closed subset of a ring R consisting of central regular elements only. Then R is right $\mathcal{Z}$-reflexive if and only if the localization ring $S^{-1}R$ is right $\mathcal{Z}$-reflexive.*

Proof. Suppose that $S^{-1}R$ is right $\mathcal{Z}$-reflexive. Let $a, b \in R$ be such that $aRb = 0$. Then for any $s, t \in S$, $(s^{-1}a)(S^{-1}R)(t^{-1}b) = (s^{-1}S^{-1}t^{-1})(aRb) = 0$ and so $(t^{-1}b)(S^{-1}R)(s^{-1}a) = (t^{-1}S^{-1}s^{-1})(bRa) \subseteq \mathcal{Z}_r(S^{-1}R)$. Since, by [12], $\mathcal{Z}_r(S^{-1}R) = S^{-1}\mathcal{Z}_r(R)$, so $(t^{-1}S^{-1}s^{-1})(bRa) \subseteq \mathcal{Z}_r(R)$. Hence it follows that for each $r \in R$, there is some $u \in S$ such that $ubra \in \mathcal{Z}_r(R)$. Now since $ubra \in \mathcal{Z}_r(R)$, so there exists an essential right ideal I of R such that $ubraI = 0$ which implies that $braI = 0$ as u is regular. Hence for any $r \in R$, $bra \in \mathcal{Z}_r(R)$. Thus R is right $\mathcal{Z}$-reflexive.

Conversely, suppose that R is right $\mathcal{Z}$-reflexive. Let $s^{-1}a, t^{-1}b \in S^{-1}R$ be such that $(s^{-1}a)(S^{-1}R)(t^{-1}b) = (s^{-1}S^{-1}t^{-1})(aRb) = 0$. Then $a, b \in R$ are such that $aRb = 0$ and so $bRa \subseteq \mathcal{Z}_r(R)$. Hence $(t^{-1}b)(S^{-1}R)(s^{-1}a) = (t^{-1}S^{-1}s^{-1})(bRa) \subseteq S^{-1}\mathcal{Z}_r(R) = \mathcal{Z}_r(S^{-1}R)$. Thus $S^{-1}R$ is right $\mathcal{Z}$-reflexive. □

Corollary 3.7. *For any ring R, polynomial ring $R[x]$ is right $\mathcal{Z}$-reflexive if and only if the Laurent polynomial ring $R[x, x^{-1}]$ is right $\mathcal{Z}$-reflexive.*

Proof. Let $S = \{1, x, x^2, \ldots\} \subseteq R[x]$. Then S is a multiplicatively closed subset of $R[x]$ consisting of central regular elements only such that $S^{-1}R[x] = R[x, x^{-1}]$. Hence the result follows directly from Proposition 3.6. □

The following is a well-known result which gives complete classification of right ideals and ideals of matrix rings.

Lemma 3.8. *Let R be a ring and n be a positive integer.*

(1) *Each ideal of $M_n(R)$ is of the form $M_n(I)$ for some ideal I of R.*
(2) *Each right ideal of $M_n(R)$ is of the form $\{[c_1, c_2, \ldots, c_n] \in M_n(R)\colon c_1, c_2, \ldots, c_n \in N\}$ for some submodule N of right R-module R^n. So each right ideal can be denoted by $C(N)$.*

Lemma 3.9. *Let R be a ring and n be a positive integer. Then $M_n(\mathcal{Z}_r(R)) = \mathcal{Z}_r(M_n(R))$.*

Proof. Let $A = [a_{ij}] \in M_n(\mathcal{Z}_r(R))$. Then $a_{ij} \in \mathcal{Z}_r(R)$ for all $1 \leqslant i, j \leqslant n$. So there exists an essential right ideal I_{ij} of R such that $a_{ij}I_{ij} = 0$ for all $1 \leqslant i, j \leqslant n$. Let $I = \bigcap_{1 \leqslant i,j \leqslant n} I_{ij}$. Then I is an essential right ideal of R such that $a_{ij}I = 0$ for all $1 \leqslant i, j \leqslant n$. Hence if we take $J = M_n(I) = C(I^n)$, then by Lemma 3.8 (2), J is a right ideal of $M_n(R)$ such that $AJ = 0$. Now we show that J is essential. Let K be a nonzero right ideal of $M_n(R)$. Then by Lemma 3.8 (2), there exists a nonzero submodule N of R-module R^n such that $K = C(N)$. Since I is an essential right ideal of R, hence by [2], Proposition 5.20, I^n is an essential submodule of right R-module R^n and so $I^n \cap N \neq 0$. It follows that $J \cap K = C(I^n) \cap C(N) \neq 0$. Therefore J is an essential right ideal of $M_n(R)$ and so $A \in \mathcal{Z}_r(M_n(R))$. Thus $M_n(\mathcal{Z}_r(R)) \subseteq \mathcal{Z}_r(M_n(R))$.

Now, since $\mathcal{Z}_r(M_n(R))$ is an ideal of $M_n(R)$, then by Lemma 3.8 (1), there exists an ideal I of R such that $\mathcal{Z}_r(M_n(R)) = M_n(I)$. Now we need to prove that $I = \mathcal{Z}_r(R)$. From the above argument, $\mathcal{Z}_r(R) \subseteq I$. Next let $a \in I$. Then $A = aI_n \in M_n(I) = \mathcal{Z}_r(M_n(R))$. So, there exists an essential right ideal $K = C(N)$ of $M_n(R)$ such that $AK = (aI_n)K = aK = aC(N) = C(aN) = 0$, where N is some nonzero submodule of the right R-module R^n. This implies that $aN = 0$. Let $J_i \neq 0$ be a subset of R whose elements are ith-coordinate of some element of N. Then J_i is a nonzero right ideal of R such that $aJ_i = 0$. Let E be a nonzero right ideal of R. Then $E_1 = 0 \oplus \ldots \oplus E$ (ith coordinate) $\oplus \ldots \oplus 0$ is a nonzero submodule of the right R-module R^n. So by Lemma 3.8(2), $C(E_1)$ is a nonzero right ideal of $M_n(R)$. Hence $C(N) \cap C(E_1) \neq 0$ as $K = C(N)$ is an essential right ideal of $M_n(R)$. It follows that $J_i \cap E \neq 0$. Thus J_i is an essential right ideal of R such that $aJ_i = 0$ and so $a \in \mathcal{Z}_r(R)$. Hence $I \subseteq \mathcal{Z}_r(R)$. This completes the proof. □

Theorem 3.10. *A ring R is right $\mathcal{Z}$-reflexive if and only if $M_n(R)$ is right $\mathcal{Z}$-reflexive for every positive integer n.*

Proof. Suppose that R is right $\mathcal{Z}$-reflexive. Let $I = M_n(I_1)$, $J = M(J_1)$ be two ideals of $M_n(R)$ such that $IJ = 0$. Then we have $M_n(I_1J_1) = M_n(I_1)M_n(J_1) = IJ = 0$ and so $I_1J_1 = 0$. Hence by Theorem 2.4, $J_1I_1 \subseteq \mathcal{Z}_r(R)$. Therefore by Lemma 3.9, $JI = M_n(J_1)M_n(I_1) = M_n(J_1I_1) \subseteq M_n(\mathcal{Z}_r(R)) = \mathcal{Z}_R(M_n(R))$. Thus $M_n(R)$ is right $\mathcal{Z}$-reflexive by Theorem 2.4.

Conversely, suppose that $M_n(R)$ is right $\mathcal{Z}$-reflexive. Let I, J be two ideals of R such that $IJ = 0$. Then $I_1 = M_n(I), J_1 = M(J)$ are two ideals of $M_n(R)$ such that $I_1J_1 = M_n(I)M_n(J) = M_n(IJ) = 0$. Hence $J_1I_1 = M_n(J)M_n(I) = M_n(JI) \subseteq \mathcal{Z}_r(M_n(R))$ by Theorem 2.4. Therefore by Lemma 3.9, $JI \subseteq \mathcal{Z}_r(R)$. Thus R is right $\mathcal{Z}$-reflexive by Theorem 2.4. □

Remark 3.11. Recall that a ring R is called *directly finite* if for any $a, b \in R$, $ab = 1$ implies that $ba = 1$. A right $\mathcal{Z}$-reflexive ring need not be directly finite. For example, let R be a domain such that $M_2(R)$ is not directly finite (e.g., [10], Exercise 1.18). Since R is a domain, R is right $\mathcal{Z}$-reflexive and so $M_2(R)$ is right $\mathcal{Z}$-reflexive by Proposition 3.10. However, $M_2(R)$ is not directly finite.

Recall that for any ring R and for any positive integer n, the subset

$$V_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ 0 & a_1 & a_2 & \dots & a_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_1 \end{pmatrix} : a_1, \dots, a_n \in R \right\}$$

forms a subring of $M_n(R)$. For convenience, we denote the elements of $V_n(R)$ by an n-tuple $(a_1, a_2, \dots, a_n)$.

Proposition 3.12. *A ring R is right $\mathcal{Z}$-reflexive if and only if $V_n(R)$ is right $\mathcal{Z}$-reflexive.*

Proof. By [12], Proposition 3.4, we have $\mathcal{Z}_r(V_n(R)) = \{(a_1, a_2, \dots, a_n) : a_1 \in \mathcal{Z}_r(R), a_2, \dots, a_n \in R\}$. Suppose that R is right $\mathcal{Z}$-reflexive. Let $A = (a_1, a_2, \dots, a_n)$, $B = (b_1, b_2, \dots, b_n) \in V_n(R)$ be such that $AV_n(R)B = 0$. Then $a_1Rb_1 = 0$ and so $b_1Ra_1 \subseteq \mathcal{Z}_r(R)$. Hence $BV_n(R)A = \{(b_1ra_1, *, *, \dots, *) \colon r \in R\} \subseteq \mathcal{Z}_r(V_n(R))$. Thus $V_n(R)$ is right $\mathcal{Z}$-reflexive.

Conversely, suppose that $V_n(R)$ is right $\mathcal{Z}$-reflexive. Let $a, b \in R$ be such that $aRb = 0$. Then there are $A = aI_n$, $B = bI_n \in V_n(R)$ such that $AV_n(R)B = 0$ and so $BV_n(R)A \subseteq \mathcal{Z}_r(V_n(R))$. It follows that $bRa \subseteq \mathcal{Z}_r(R)$. Thus R is right $\mathcal{Z}$-reflexive. □

Corollary 3.13. *Let n be a positive integer greater than* 1. *Then, R is right $\mathcal{Z}$-reflexive if and only if $R[x]/\langle x^n \rangle$ is right $\mathcal{Z}$-reflexive.*

Proof. Define a map $\theta\colon R[x]/\langle x^n \rangle \to V_n(R)$ given by

$$\theta(a_1 + a_2x + \dots + a_nx^{n-1} + \langle x^n \rangle) = (a_1, a_2, \dots, a_n).$$

Then θ is an isomorphism. So, the result follows from Proposition 3.12. □

Recalling [6], let R be a ring and M be an (R, R)-bimodule. Then the set $R \times M$ forms a ring with the addition $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$ and multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_2m_1 + r_1m_2)$, where $r_i \in R$ and $m_i \in M$. This extension is known as *trivial extension* of R by M and denoted by $T(R, M)$. Since $T(R, R)$ is isomorphic to $V_2(R)$, hence by Proposition 3.12, we have the following result.

Corollary 3.14. *A ring R is right $\mathcal{Z}$-reflexive if and only if the trivial extension $T(R, R)$ is right $\mathcal{Z}$-reflexive.*

Recalling [14], a ring R is called an *Armendariz ring* if for any $f(x)=\sum_i a_ix^i$ and $g(x)=\sum_j b_jx^j\in R[x]$, $f(x)g(x)=0$ implies that $a_ib_j=0$ for all i, j.

Proposition 3.15. *The following statements are equivalent for an Armendariz ring R.*

(1) *R is right $\mathcal{Z}$-reflexive.*

(2) *$R[x]$ is right $\mathcal{Z}$-reflexive.*

(3) *$R[x,x^{-1}]$ is right $\mathcal{Z}$-reflexive.*

Proof. (1) $\Rightarrow$ (2). Suppose that R is right $\mathcal{Z}$-reflexive. Let $f(x)=\sum_i a_ix^i$, $g(x)=\sum_j b_jx^j\in R[x]$ be such that $f(x)R[x]g(x)=0$. Then for any $r\in R$, $f(x)rg(x)=0$ and so $a_irb_j=0$ for all i, j, as R is an Armendariz ring. Hence for all i, j, $a_iRb_j=0$ and so $b_jRa_i\in\mathcal{Z}_r(R)$ as R is right $\mathcal{Z}$-reflexive. This implies that $g(x)R[x]f(x)\subseteq(\mathcal{Z}_r(R))[x]$. Since by [10], Exercise 7.35, $\mathcal{Z}_r(R[x])=(\mathcal{Z}_r(R))[x]$, so $g(x)R[x]f(x)\subseteq\mathcal{Z}_r(R[x])$. Thus, $R[x]$ is right $\mathcal{Z}$-reflexive.

(2) $\Rightarrow$ (1). Suppose that $R[x]$ is right $\mathcal{Z}$-reflexive. Let $a,b\in R$ be such that $aRb=0$. Then $aR[x]b=0$ and so $bR[x]a\subseteq\mathcal{Z}_r(R[x])=(\mathcal{Z}_r(R))[x]$. This implies that $bRa\subseteq\mathcal{Z}_r(R)$. Thus, R is right $\mathcal{Z}$-reflexive.

(2) $\Leftrightarrow$ (3). It follows from Corollary 3.7. □

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