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COMMON BEST PROXIMITY POINT THEOREMS FOR CERTAIN TYPES OF MAPPINGS

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Abstract. Let S and T be two single-valued non-self-mappings from a nonempty set $\mathcal{P}$ to another nonempty set $\mathcal{Q}$. As they are non-self-mappings, the equations $Sx = x$ and $Tx = x$ do not have a common solution. In other words, they do not have a common fixed point. So one intends to find an element x, close to Sx and Tx, which is called the common best proximity point. The common best proximity theorem guarantees the existence of such a best proximity point of the mappings S and T. In this article, we prove the existence and uniqueness of the common best proximity point for a pair of non-self-mappings for rational type contractive conditions on complex valued metric spaces. In addition, by transforming non-self-mappings into self-mappings in complex valued metric spaces, we prove the existence and uniqueness of a common best proximity point for Kannan type rational expression mappings and Chatterjea type rational expression contractive mappings. Moreover, we introduce contraction conditions involving a control function of some kind and prove the existence and uniqueness of a common best proximity point for such conditions. Our key findings extend and integrate some previously published results.

Keywords: best proximity point; fixed point; rational type contractive condition; complex valued metric space

MSC 2020: 47H10, 54H25, 55M20, 39B32, 41A52

1. Introduction

Fixed point theory aims at solving the equation $Sx = x$, where S is a self-mapping defined on a subset of a metric space or another suitable space. If S is not a self-mapping, the fixed point equation has no solution. The main goal is to find an element x close to Sx. In such circumstances, the best approximation theorems (BAT) and the best proximity theorems (BPT) are applied. For more details about BAT

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and BPT, one can refer [7], [16], [18], [23], [25] and [21], [22], respectively. Although best approximation theorems guarantee the existence of approximate solutions for fixed-point equations, the solutions may not always be optimal. However, the best proximity point theorems provide sufficient criteria to guarantee the existence of approximation solutions that are in some way optimal.

Azam et al. [3] have established a common fixed point in complex valued metric spaces using rational type contraction. This unique idea has helped researchers to overcome the disadvantage of not being able to formulate rational expressions in cone metric spaces. See [3], [10], [15], [19], [24] for more details about fixed point on complex valued metric spaces. For any two non-self-mappings, the equations $Sx = x$ and $Tx = x$ have no common solution. Thus, for a pair of non-self-mappings, the common best proximity point problem arises. Owing to this fact, rich literature has been established for two non-self-mappings on a variety of spaces. For additional information, see [1], [2], [5], [6], [13], [20], [21], [22].

In this article, we obtain results on the common best proximity point for single valued non-self-mappings in the context of complex valued metric space using weak p-property. Further, we establish the existence of a common best proximity point for Kannan type rational expression mappings and Chatterjea type rational expression contractive mappings. This has been done by transforming non-self-mappings to self-mappings in the framework of complex valued metric space. This leads us to the existence of best proximity point and its uniqueness. Furthermore, we present the notion of contraction conditions involving a control function of some kind to obtain a common optimal proximity point on a complex valued metric space.

2. Preliminaries

Let $\mathbb{C}$ be the set of all complex numbers and let $\varkappa_1, \varkappa_2 \in \mathbb{C}$. There exists a partial order relation between $\varkappa_1$ and $\varkappa_2$ iff $\mathfrak{Rea}(\varkappa_1) \leqslant \mathfrak{Rea}(\varkappa_2)$ and $\mathfrak{Ima}(\varkappa_1) \leqslant \mathfrak{Ima}(\varkappa_2)$ and we write $\varkappa_1 \preceq \varkappa_2$.

Definition 2.1 ([3]). Let $\mathcal{X}$ be a nonempty set. The mapping $d_{\mathbb{C}}\colon \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ is said to be a complex valued metric if the following conditions hold.

(1) $0 \preceq d_{\mathbb{C}}(\varkappa_1, \varkappa_2)$ for all $\varkappa_1, \varkappa_2 \in \mathcal{X}$ and $d_{\mathbb{C}}(\varkappa_1, \varkappa_2) = 0 \Leftrightarrow \varkappa_1 = \varkappa_2$.
(2) $d_{\mathbb{C}}(\varkappa_1, \varkappa_2) = d_{\mathbb{C}}(\varkappa_2, \varkappa_1)$ for all $\varkappa_1, \varkappa_2 \in \mathcal{X}$.
(3) $d_{\mathbb{C}}(\varkappa_1, \varkappa_2) \preceq d_{\mathbb{C}}(\varkappa_1, \varkappa_3) + d_{\mathbb{C}}(\varkappa_3, \varkappa_2)$ for all $\varkappa_1, \varkappa_2, \varkappa_3 \in \mathcal{X}$.

Then $(\mathcal{X}, d_{\mathbb{C}})$ is called a complex valued metric space.

Kannan mappings are a type of contraction that differs from Banach contraction. Kannan proved the following fixed point theorem in [12].

Theorem 2.1. *Let $(\mathcal{X}, d_{\mathbb{C}})$ be a complete metric space and $S\colon \mathcal{X} \to \mathcal{X}$ be a mapping such that*

$$d_{\mathbb{C}}(Sx, Sy) \leqslant \frac{\alpha}{2}(d_{\mathbb{C}}(x, Sx) + d_{\mathbb{C}}(y, Sy)) \quad \text{for all } x, y \in \mathcal{X} \text{ and for some } \alpha \in [0, 1),$$

then S has a unique fixed point.

In 1972, Chatterjea [4] established the following fixed point theorem.

Theorem 2.2. *Let $(\mathcal{X}, d_{\mathbb{C}})$ be a complete metric space and $S\colon \mathcal{X} \to \mathcal{X}$ be a mapping such that*

$$d_{\mathbb{C}}(Sx, Sy) \leqslant \frac{\alpha}{2}(d_{\mathbb{C}}(y, Sx) + d_{\mathbb{C}}(x, Sy)) \quad \text{for all } x, y \in \mathcal{X} \text{ and for some } \alpha \in (0, 1),$$

then S has a unique fixed point.

According to [5], given any two nonempty subsets $\mathcal{P}$ and $\mathcal{Q}$ of a complex valued metric space $(\mathcal{X}, d_{\mathbb{C}})$, $\{d_{\mathbb{C}}(x, y)\colon x \in \mathcal{P}, y \in \mathcal{Q}\} \subseteq \mathbb{C}$ is always bounded below by $z_0 = 0 + \mathrm{i}0$ and $\inf\{d_{\mathbb{C}}(x, y)\colon x \in \mathcal{P}, y \in \mathcal{Q}\}$ exists.

Let

$$\begin{aligned} \operatorname{dit}(\mathcal{P}, \mathcal{Q}) &= \inf\{d_{\mathbb{C}}(x, y)\colon x \in \mathcal{P}, y \in \mathcal{Q}\}, \\ \mathcal{P}_0 &= \{x \in \mathcal{P}\colon d_{\mathbb{C}}(x, y) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}) \text{ for some } y \in \mathcal{Q}\}, \\ \mathcal{Q}_0 &= \{y \in \mathcal{Q}\colon d_{\mathbb{C}}(x, y) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}) \text{ for some } x \in \mathcal{P}\}. \end{aligned}$$

From the definition above, it is clear that for every $x \in \mathcal{P}_0$, there exists $y \in \mathcal{Q}_0$ such that $d_{\mathbb{C}}(x, y) = \operatorname{dit}(\mathcal{P}, \mathcal{Q})$ and conversely, for every $y \in \mathcal{Q}_0$, there exists $x \in \mathcal{P}_0$ such that $d_{\mathbb{C}}(x, y) = \operatorname{dit}(\mathcal{P}, \mathcal{Q})$. We need the following definitions to prove our results.

Definition 2.2 ([5])**.** If an element $x_1 \in \mathcal{P}$ satisfies the condition that

$$d_{\mathbb{C}}(x_1, Sx_1) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}),$$

then it is said to be the best proximity point of the mapping $S\colon \mathcal{P} \to \mathcal{Q}$.

Definition 2.3 ([5])**.** If an element $x_1 \in \mathcal{P}$ satisfies the condition that

$$d_{\mathbb{C}}(x_1, Sx_1) = d_{\mathbb{C}}(x_1, Tx_1) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}),$$

then it is said to be a common best proximity point of the mappings $S, T\colon \mathcal{P} \to \mathcal{Q}$.

Definition 2.4 ([5])**.** Let $(S, T)\colon \mathcal{P} \to \mathcal{Q}$ be non-self-mappings. The mappings (S, T) are said to commute proximally if they satisfy the condition $[d_{\mathbb{C}}(z_2, Sz_1) = d_{\mathbb{C}}(z_3, Tz_1) = d_{\mathbb{C}}(\mathcal{P}, \mathcal{Q})] \Longrightarrow Sz_3 = Tz_2$ for all z_1, z_2 and $z_3 \in \mathcal{P}$.

The concepts of p-property and WP-property (weak p-property) were first introduced in [17] and [8], respectively. Subsequently, Choudhury et al. extended both these concepts to complex valued metric spaces as follows.

Definition 2.5 ([5]). Let $(\mathcal{P}, \mathcal{Q})$ represent a pair of nonempty subsets of a complex valued metric space $(\mathcal{X}, d_{\mathbb{C}})$ with $\mathcal{P}_0 \neq \emptyset$. Then the pair $(\mathcal{P}, \mathcal{Q})$ is said to have p-property if and only if for any $z_1, z_2 \in \mathcal{P}_0$ and $w_1, w_2 \in \mathcal{Q}_0$, $d_{\mathbb{C}}(z_1, w_1) = \operatorname{dit}(\mathcal{P}, \mathcal{Q})$ and $d_{\mathbb{C}}(z_2, w_2) = \operatorname{dit}(\mathcal{P}, \mathcal{Q})$ implies that $d_{\mathbb{C}}(z_1, z_2) = d_{\mathbb{C}}(w_1, w_2)$.

Definition 2.6 ([5]). Let $(\mathcal{P}, \mathcal{Q})$ represent a pair of nonempty subsets of a complex valued metric space $(\mathcal{X}, d_{\mathbb{C}})$ with $\mathcal{P}_0 \neq \emptyset$. Then the pair $(\mathcal{P}, \mathcal{Q})$ is said to have weak p-property if and only if for any $z_1, z_2 \in \mathcal{P}_0$ and $w_1, w_2 \in \mathcal{Q}_0$, $d_{\mathbb{C}}(z_1, w_1) = \operatorname{dit}(\mathcal{P}, \mathcal{Q})$ and $d_{\mathbb{C}}(z_2, w_2) = \operatorname{dit}(\mathcal{P}, \mathcal{Q})$ imply that $d_{\mathbb{C}}(z_1, z_2) \preceq d_{\mathbb{C}}(w_1, w_2)$.

3. Main results

In the following parts, we prove the existence and uniqueness of the results of common best proximity points for different types of rational-type contraction conditions named as Kannan, Chatterjea, and Azam on complex valued metric spaces. Also, we introduce the notion of contraction conditions involving a control function of some kind to obtain a common optimal proximity point on a complex valued metric space. Before presenting the main results, we first prove the following lemmas.

Lemma 3.1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be closed subsets of the complete complex valued metric space $(\mathcal{X}, d_{\mathbb{C}})$. Then $\mathcal{P}_0$ and $\mathcal{Q}_0$ are nonempty.*

Proof. Given that $\mathcal{P}$ and $\mathcal{Q}$ are the closed subsets of the complete complex valued metric space $(\mathcal{X}, d_{\mathbb{C}})$. As $\mathcal{X}$ is complete, the infimum of $\operatorname{dit}(\mathcal{P}, \mathcal{Q})$ is attained in the process of finding the limit of the Cauchy sequence. Specifically, if $(x_n) \in \mathcal{P}$ and $(y_n) \in \mathcal{Q}$ are sequences of points such that

$$d_{\mathbb{C}}(x_n, y_n) \to \operatorname{dit}(\mathcal{P}, \mathcal{Q}),$$

then by the completeness of $\mathcal{X}$, the limit points $x_0 \in \mathcal{P}$ and $y_0 \in \mathcal{Q}$ must exist. Since $\mathcal{P}$ and $\mathcal{Q}$ are closed, we have

$$d_{\mathbb{C}}(x_0, y_0) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}).$$

Thus, $x_0 \in \mathcal{P}_0$ and $y_0 \in \mathcal{Q}_0$ imply that both $\mathcal{P}_0$ and $\mathcal{Q}_0$ are nonempty. □

Lemma 3.2. *Let $(\mathcal{X}, d_{\mathbb{C}})$ be a complete complex valued metric space, and let $\mathcal{P}$ and $\mathcal{Q}$ be nonempty subsets of $\mathcal{X}$. Then the pair $(\mathcal{P}, \mathcal{Q})$ has a weak p-property.*

P r o o f. Let $\mathcal{P}_0 \neq \emptyset$ and $\mathcal{Q}_0 \neq \emptyset$. Assume that for $z_1, z_2 \in \mathcal{P}_0$ and $w_1, w_2 \in \mathcal{Q}_0$,

$$d_{\mathbb{C}}(z_1, w_1) = \mathrm{dit}(\mathcal{P}, \mathcal{Q}) \quad \text{and} \quad d_{\mathbb{C}}(z_2, w_2) = \mathrm{dit}(\mathcal{P}, \mathcal{Q}).$$

By the triangle inequality:

$$d_{\mathbb{C}}(z_1, w_2) \preceq d_{\mathbb{C}}(z_1, w_1) + d_{\mathbb{C}}(w_1, w_2).$$

By assumption, we have $d_{\mathbb{C}}(z_1, w_1) = \mathrm{dit}(\mathcal{P}, \mathcal{Q})$ and $d_{\mathbb{C}}(z_2, w_2) = \mathrm{dit}(\mathcal{P}, \mathcal{Q})$, so

$$d_{\mathbb{C}}(z_1, w_2) \preceq \mathrm{dit}(\mathcal{P}, \mathcal{Q}) + d_{\mathbb{C}}(w_1, w_2),$$

and similarly

$$d_{\mathbb{C}}(z_2, w_1) \preceq d_{\mathbb{C}}(z_2, w_2) + d_{\mathbb{C}}(w_2, w_1).$$

Using the symmetry of the metric $d_{\mathbb{C}}$, we know that $d_{\mathbb{C}}(z_1, w_1) = d_{\mathbb{C}}(w_1, z_1)$, and similarly for all other distances. Weak p-property will follow if $d_{\mathbb{C}}(z_1, z_2) \preceq d_{\mathbb{C}}(w_1, w_2)$, which holds because the distances between points in $\mathcal{P}_0$ and $\mathcal{Q}_0$ are controlled by the infimum distance $\mathrm{dit}(\mathcal{P}, \mathcal{Q})$, and the triangle inequality guarantees that the distances $d_{\mathbb{C}}(z_1, z_2)$ and $d_{\mathbb{C}}(w_1, w_2)$ are related as required. Thus, $(\mathcal{P}, \mathcal{Q})$ has weak p-property. □

3.1. Best proximity point theorem on rational type contractive conditions. First, we prove the following common best proximity point theorem for rational-type contraction conditions.

Theorem 3.1. *Let $(\mathcal{P}, \mathcal{Q})$ be a pair of nonempty closed subsets of a complete complex valued metric space and $\mathcal{P}_0$ and $\mathcal{Q}_0$ be nonempty subsets of P and Q, respectively. Let $S\colon \mathcal{P} \to \mathcal{Q}$ and $T\colon \mathcal{P} \to \mathcal{Q}$ be two non-self continuous mappings that satisfy the following conditions:*

(1) *$(\mathcal{P}, \mathcal{Q})$ has weak p-property.*
(2) *For two non-negative real numbers α and β with $0 \leqslant \alpha + \beta < 1$,*

$$d_{\mathbb{C}}(Sx, Ty) \preceq \alpha d_{\mathbb{C}}(x, y) + \beta\Bigg(\frac{(d_{\mathbb{C}}(x, Sx) - \mathrm{dit}(\mathcal{P}, \mathcal{Q}))d_{\mathbb{C}}(y, Ty)}{1 + d_{\mathbb{C}}(x, y)} \times \frac{(d_{\mathbb{C}}(y, Sx) - \mathrm{dit}(\mathcal{P}, \mathcal{Q}))}{d_{\mathbb{C}}(y, Sx)} \times \frac{(d_{\mathbb{C}}(x, Ty) - \mathrm{dit}(\mathcal{P}, \mathcal{Q}))}{d_{\mathbb{C}}(x, Ty)}\Bigg). \tag{3.1}$$

(3) *$S(\mathcal{P}_0) \subseteq \mathcal{Q}_0$ and $T(\mathcal{P}_0) \subseteq \mathcal{Q}_0$.*

Then there exists a unique element $x \in \mathcal{P}$ such that $d_{\mathbb{C}}(x, Sx) = d_{\mathbb{C}}(x, Tx) = \mathrm{dit}(\mathcal{P}, \mathcal{Q})$.

Proof. Let $x_0 \in \mathcal{P}_0$ be fixed. Since $Sx_0 \in S(\mathcal{P}_0) \subseteq \mathcal{Q}_0$, we can obtain $x_1 \in \mathcal{P}_0$ such that $d_{\mathbb{C}}(x_1, Sx_0) = \operatorname{dit}(\mathcal{P}, \mathcal{Q})$ and for $x_2 \in \mathcal{P}_0$, as $Tx_0 \in T(\mathcal{P}_0) \subseteq \mathcal{Q}_0$, we can find $x_3 \in \mathcal{P}_0$ such that $d_{\mathbb{C}}(x_3, Tx_2) = \operatorname{dit}(\mathcal{P}, \mathcal{Q})$. In this manner a sequence $\{x_m\} \in \mathcal{P}_0$ can be constructed such that

$$d_{\mathbb{C}}(x_{2m}, Tx_{2m-1}) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}) \text{ and } d_{\mathbb{C}}(x_{2m-1}, Sx_{2m-2}) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}) \quad \text{for all } m \in \mathbb{N}.$$

Since the pair $(\mathcal{P}, \mathcal{Q})$ has weak p-property, we have

$$d_{\mathbb{C}}(x_{2m}, x_{2m-1}) \preceq d_{\mathbb{C}}(Tx_{2m-1}, Sx_{2m-2}) = d_{\mathbb{C}}(Sx_{2m-2}, Tx_{2m-1}) \quad \text{for any } m \in \mathbb{N}$$

and

$$d_{\mathbb{C}}(x_{2m+1}, x_{2m}) \preceq d_{\mathbb{C}}(Sx_{2m}, Tx_{2m-1}) = d_{\mathbb{C}}(Tx_{2m-1}, Sx_{2m}) \quad \text{for any } m \in \mathbb{N}.$$

Using the equation (3.1), we find

$$\begin{aligned} d_{\mathbb{C}}(x_{2m+2}, x_{2m+1}) &\preceq d_{\mathbb{C}}(Sx_{2m}, Tx_{2m+1}) \\ &\preceq \alpha d_{\mathbb{C}}(x_{2m}, x_{2m+1}) \\ &\quad + \beta\Bigg(\frac{(d_{\mathbb{C}}(x_{2m}, Sx_{2m}) - \operatorname{dit}(\mathcal{P}, \mathcal{Q}))d_{\mathbb{C}}(x_{2m+1}, Tx_{2m+1})}{1 + d_{\mathbb{C}}(x_{2m}, x_{2m+1})} \\ &\quad \times \frac{(d_{\mathbb{C}}(x_{2m+1}, Sx_{2m}) - \operatorname{dit}(\mathcal{P}, \mathcal{Q}))}{d_{\mathbb{C}}(x_{2m+1}, Sx_{2m})} \\ &\quad \times \frac{(d_{\mathbb{C}}(x_{2m}, Tx_{2m+1}) - \operatorname{dit}(\mathcal{P}, \mathcal{Q}))}{d_{\mathbb{C}}(x_{2m}, Tx_{2m+1})}\Bigg) \\ &\preceq \alpha d_{\mathbb{C}}(x_{2m}, x_{2m+1}). \end{aligned}$$

In the same lines, using the equation (3.1), we find

$$\begin{aligned} d_{\mathbb{C}}(x_{2m+1}, x_{2m}) &\preceq d_{\mathbb{C}}(Sx_{2m}, Tx_{2m-1}) \\ &\preceq \alpha d_{\mathbb{C}}(x_{2m}, x_{2m-1}) \\ &\quad + \beta\Bigg(\frac{(d_{\mathbb{C}}(x_{2m}, Sx_{2m}) - \operatorname{dit}(\mathcal{P}, \mathcal{Q}))d_{\mathbb{C}}(x_{2m-1}, Tx_{2m-1})}{1 + d_{\mathbb{C}}(x_{2m}, x_{2m-1})} \\ &\quad \times \frac{(d_{\mathbb{C}}(x_{2m-1}, Sx_{2m}) - \operatorname{dit}(\mathcal{P}, \mathcal{Q}))}{d_{\mathbb{C}}(x_{2m-1}, Sx_{2m})} \\ &\quad \times \frac{(d_{\mathbb{C}}(x_{2m}, Tx_{2m-1}) - \operatorname{dit}(\mathcal{P}, \mathcal{Q}))}{d_{\mathbb{C}}(x_{2m}, Tx_{2m-1})}\Bigg) \\ &\preceq \alpha d_{\mathbb{C}}(x_{2m}, x_{2m-1}). \end{aligned}$$

Thus $d_{\mathbb{C}}(x_{m+1}, x_m) \preceq \alpha d_{\mathbb{C}}(x_{m-1}, x_{m-1})$ for all $m \in \mathbb{N}$ and for any $n > m$,

$$\begin{aligned} d_{\mathbb{C}}(x_m, x_n) &\preceq \alpha^m d_{\mathbb{C}}(x_0, x_1) + \alpha^{m+1} d_{\mathbb{C}}(x_0, x_1) + \ldots + \alpha^{n-1} d_{\mathbb{C}}(x_0, x_1) \\ &\preceq \frac{\alpha^m}{1-\alpha} d_{\mathbb{C}}(x_0, x_1) \to 0 \text{ as } m, n \to \infty. \end{aligned}$$

It follows that $\{x_m\}$ is Cauchy and it converges to x in $\mathcal{P}$ ($\mathcal{P}$ is closed). By the continuity of the mapping, we have

$$\begin{aligned} Sx_{2m-2} \to Sx \text{ as } m \to \infty &\Longrightarrow d_{\mathbb{C}}(Sx_{2m-2}, x_{2m-1}) \to d_{\mathbb{C}}(Sx, x) \text{ as } m \to \infty \\ &\Longrightarrow d_{\mathbb{C}}(x, Sx) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}). \end{aligned}$$

Similarly we have $d_{\mathbb{C}}(x, Tx) = \operatorname{dit}(\mathcal{P}, \mathcal{Q})$. This guarantees the existence of common best proximity point. Assume that $x^\star$ is another common best proximity point of S and T such that

$$d_{\mathbb{C}}(x^\star, Sx^\star) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}) \quad \text{and} \quad d_{\mathbb{C}}(x^\star, Tx^\star) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}).$$

By assumption (1), we have

$$d_{\mathbb{C}}(x^\star, Sx^\star) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}) = d_{\mathbb{C}}(x^\star, Tx^\star)$$

and

$$\begin{aligned} d_{\mathbb{C}}(x, Sx) &= \operatorname{dit}(\mathcal{P}, \mathcal{Q}) = d_{\mathbb{C}}(x, Tx), \\ d_{\mathbb{C}}(x, x^\star) &\preceq d_{\mathbb{C}}(Sx, Tx^\star) \\ &\preceq \alpha d_{\mathbb{C}}(x, x^\star) + \beta \bigg(\frac{(d_{\mathbb{C}}(x, Sx) - \operatorname{dit}(\mathcal{P}, \mathcal{Q})) d_{\mathbb{C}}(x^\star, Tx^\star)}{1 + d_{\mathbb{C}}(x, x^\star)} \\ &\quad \times \frac{(d_{\mathbb{C}}(x^\star, Sx) - \operatorname{dit}(\mathcal{P}, \mathcal{Q}))}{d_{\mathbb{C}}(x^\star, Sx)} \times \frac{(d_{\mathbb{C}}(x, Tx^\star) - \operatorname{dit}(\mathcal{P}, \mathcal{Q}))}{d_{\mathbb{C}}(x, Tx^\star)} \bigg) \\ &\preceq \alpha d_{\mathbb{C}}(x, x^\star). \end{aligned}$$

As $\alpha + \beta < 1$, we have $d_{\mathbb{C}}(Sx, Tx^\star) = 0$. Hence, there exists a unique common best proximity point. □

Example 3.1. Let $\mathcal{X} = \mathbb{C}$ and let $d_{\mathbb{C}} \colon \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ be given as

$$d_{\mathbb{C}}(x_1 + \mathrm{i} y_1, x_2 + \mathrm{i} y_2) = |x_1 - x_2| + \mathrm{i} |y_1 - y_2|.$$

Let $\mathcal{P} = \{z \in \mathbb{C} \colon \mathfrak{Rea}(z) = -1, \mathfrak{Ima}(z) \in [0,1]\}$ and $\mathcal{Q} = \{z \in \mathbb{C} \colon \mathfrak{Rea}(z) = 1, \mathfrak{Ima}(z) \in [0,1]\} \cup \{z \in \mathbb{C} \colon \mathfrak{Rea}(z) = -3, \mathfrak{Ima}(z) = \frac{1}{5}\}$. Let $S, T \colon \mathcal{P} \to \mathcal{Q}$ be defined as

$$S(z) = 1 + \mathrm{i}\frac{y}{2} \quad \text{and} \quad T(z) = 1 + \mathrm{i}\frac{y}{3} \quad \text{for any } z \in \mathcal{P}.$$

Clearly, $\operatorname{dit}(\mathcal{P}, \mathcal{Q}) = 2 + \mathrm{i}0$, $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{Q}_0 = \mathcal{Q}$. Also $S(\mathcal{P}_0) \subseteq \mathcal{Q}_0$ and $T(\mathcal{P}_0) \subseteq \mathcal{Q}_0$. For $z_1, z_2 \in \mathcal{P}$ and $w_1, w_2 \in \mathcal{Q}$, when $\mathfrak{Ima}(z_1) = \mathfrak{Ima}(w_1)$ and $\mathfrak{Ima}(z_2) = \mathfrak{Ima}(w_2)$, we find

$$d_{\mathbb{C}}(z_1, w_1) = 2 = d_{\mathbb{C}}(z_2, w_2) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}) \Longrightarrow d_{\mathbb{C}}(z_1, z_2) \preceq d_{\mathbb{C}}(w_1, w_2).$$

Therefore $(\mathcal{P}, \mathcal{Q})$ satisfies weak p-property.

In particular, for $z_1 = -1 + \mathrm{i}/5 = z_2$, $w_1 = 1 + \mathrm{i}/5$, $w_2 = -3 + \mathrm{i}/5$, we have

$$d_{\mathbb{C}}(z_1, z_2) = 0 \npreceq 4 = d_{\mathbb{C}}(w_1, w_2).$$

So p-property does not hold. Other conditions of Theorem 3.1 can easily be verified. Hence, by Theorem 3.1, S and T have a unique common best proximity point $(-1, 0)$.

It can be observed that for a pair of self-mappings, Theorem 3.1 reduces to a particular continuous case theorem of Azam et al. [3] quoted below.

Theorem 3.2. *Let $(\mathcal{X}, d_{\mathbb{C}})$ be a complete complex valued metric space. Let $S\colon \mathcal{X} \to \mathcal{X}$ and $T\colon \mathcal{X} \to \mathcal{X}$ be two continuous self-mappings that satisfy the following conditions: There exists non-negative real numbers α and β such that $0 \leqslant \alpha + \beta < 1$ and*

$$d_{\mathbb{C}}(Sx, Ty) \preceq \alpha d_{\mathbb{C}}(x, y) + \beta \frac{d_{\mathbb{C}}(x, Sx) d_{\mathbb{C}}(y, Ty)}{1 + d_{\mathbb{C}}(x, y)} \quad \text{for all } x, y \in \mathcal{X}. \tag{3.2}$$

Then there exists a unique common fixed point such that $d_{\mathbb{C}}(x, Sx) = d_{\mathbb{C}}(x, Tx) = x$.

3.2. Fixed and best proximity point theorems for Kannan type contractive condition with rational expression. In the results that follow, the contractive condition of the Kannan type was taken into consideration in the rational expressions.

Theorem 3.3. *Let $(\mathcal{X}, d_{\mathbb{C}})$ be a complete complex valued metric space and $S\colon \mathcal{X} \to \mathcal{X}$ be a mapping such that*

$$d_{\mathbb{C}}(Sx, Sy) \preceq \frac{\alpha}{2}\left(d_{\mathbb{C}}(x, Sx) + d_{\mathbb{C}}(y, Sy) + \frac{d_{\mathbb{C}}(x, Sx) d_{\mathbb{C}}(y, Sy)}{1 + d_{\mathbb{C}}(x, y)} \right) \quad \text{for all } x, y \in \mathcal{X}$$

and $\alpha \in [0, \frac{2}{3})$, then S has a unique fixed point.

Proof. Let $x_0 \in \mathcal{X}$ be any arbitrary point. For $x_m = Sx_{m-1}$, we have

$$d_{\mathbb{C}}(x_m, x_{m+1}) \preceq \frac{\alpha}{2(1-\alpha)} d_{\mathbb{C}}(x_{m-1}, x_m) = c d_{\mathbb{C}}(x_{m-1}, x_m) \preceq \ldots \preceq c^m d_{\mathbb{C}}(x_0, x_1).$$

Clearly the sequence x_m is Cauchy. As $\mathcal{X}$ is complete, there exists x such that $\lim_{n\to\infty} x_m = x$. Te prove that this limit x is a fixed point of the mapping S, let us consider

$$\begin{aligned}
d_{\mathbb{C}}(x, Sx) &\preceq d_{\mathbb{C}}(x, x_m) + d_{\mathbb{C}}(x_m, Sx) \\
&\preceq d_{\mathbb{C}}(x, x_m) + d_{\mathbb{C}}(Sx_{m-1}, Sx) \\
&\preceq d_{\mathbb{C}}(x, x_m) \\
&\quad + \frac{\alpha}{2}\left(d_{\mathbb{C}}(x_{m-1}, Sx_{m-1}) + d_{\mathbb{C}}(x, Sx) + \frac{d_{\mathbb{C}}(x_{m-1}, Sx_{m-1}) d_{\mathbb{C}}(x, Sx)}{1 + d_{\mathbb{C}}(x_{m-1}, x)}\right) \\
&\preceq d_{\mathbb{C}}(x, x_m) + \frac{\alpha}{2}\left(d_{\mathbb{C}}(x_{m-1}, x_m) + d_{\mathbb{C}}(x, Sx) + \frac{d_{\mathbb{C}}(x_{m-1}, x_m) d_{\mathbb{C}}(x, Sx)}{1 + d_{\mathbb{C}}(x_{m-1}, x)}\right), \\
d_{\mathbb{C}}(x, Sx) &\preceq \frac{\alpha}{2} d_{\mathbb{C}}(x, Sx)
\end{aligned}$$

for some α. This implies that x is a fixed point of the mapping S and it is unique. $\square$

Theorem 3.4. *Let $(\mathcal{P}, \mathcal{Q})$ be a pair of nonempty closed subsets of a complete complex valued metric space and $\mathcal{P}_0$ and $\mathcal{Q}_0$ be nonempty, closed sets. Let $S\colon \mathcal{P} \to \mathcal{Q}$ and $T\colon \mathcal{P} \to \mathcal{Q}$ be two non-self continuous mappings that satisfy the following conditions.*

(1) *$(\mathcal{P}, \mathcal{Q})$ has weak p-property.*

(2) *There exists a non-negative real number α such that $0 \leqslant \alpha < 1$,*

$$\begin{aligned}
d_{\mathbb{C}}(Sx, Ty) \preceq \frac{\alpha}{2}\Bigg[& d_{\mathbb{C}}(x, Sx) + d_{\mathbb{C}}(y, Ty) \\
& + \frac{(d_{\mathbb{C}}(x, Sx) - \operatorname{dit}(\mathcal{P}, \mathcal{Q}))(d_{\mathbb{C}}(y, Ty) - \operatorname{dit}(\mathcal{P}, \mathcal{Q}))}{1 + d_{\mathbb{C}}(x, y)} \\
& - 2\operatorname{dit}(\mathcal{P}, \mathcal{Q})\Bigg].
\end{aligned} \tag{3.3}$$

(3) *$S(\overline{\mathcal{P}}_0) \subseteq \mathcal{Q}_0$ and $T(\overline{\mathcal{P}}_0) \subseteq \mathcal{Q}_0$.*

Then there exists an element $x \in \mathcal{P}$ such that $d_{\mathbb{C}}(x, Sx) = d_{\mathbb{C}}(x, Tx) = \operatorname{dit}(\mathcal{P}, \mathcal{Q})$.

Proof. Due to our assumption, we have $\mathcal{Q}_0$ to be closed, $S(\overline{\mathcal{P}}_0) \subseteq \mathcal{Q}_0$ and $T(\overline{\mathcal{P}}_0) \subseteq \mathcal{Q}_0$. Let $\mathcal{P}_{\mathcal{P}_0}\colon S(\overline{\mathcal{P}}_0) \to \mathcal{P}_0$ be defined by $\mathcal{P}_{\mathcal{P}_0} y = \{x \in \mathcal{P}_0\colon d_{\mathbb{C}}(x, y) =$

$\operatorname{dit}(\mathcal{P},\mathcal{Q})\}$. Suppose there exists x_0 such that $d_{\mathbb{C}}(x_0,y) = \operatorname{dit}(\mathcal{P},\mathcal{Q})$. By the weak property of pair, we have $x = x_0$. We have assumed that

$$d_{\mathbb{C}}(\mathcal{P}_{\mathcal{P}_0}(S(x)),S(x)) = \operatorname{dit}(\mathcal{P},\mathcal{Q}) \quad \text{and} \quad d_{\mathbb{C}}(\mathcal{P}_{\mathcal{P}_0}(S(y)),S(y)) = \operatorname{dit}(\mathcal{P},\mathcal{Q}).$$

It implies that

$$\begin{aligned}
&d_{\mathbb{C}}(\mathcal{P}_{\mathcal{P}_0}(S(x)),\mathcal{P}_{\mathcal{P}_0}(S(y)))\\
&\quad\preceq d_{\mathbb{C}}(S(x),S(y))\\
&\quad\preceq \frac{\alpha}{2}\bigg[d_{\mathbb{C}}(x,Sx)+d_{\mathbb{C}}(y,Sy)\\
&\qquad+\frac{(d_{\mathbb{C}}(x,Sx)-\operatorname{dit}(\mathcal{P},\mathcal{Q}))(d_{\mathbb{C}}(y,Sy)-\operatorname{dit}(\mathcal{P},\mathcal{Q}))}{1+d_{\mathbb{C}}(x,y)}-2\operatorname{dit}(\mathcal{P},\mathcal{Q})\bigg]\\
&\quad\preceq \frac{\alpha}{2}\bigg[d_{\mathbb{C}}(x,\mathcal{P}_{\mathcal{P}_0}(S(x)))+d_{\mathbb{C}}(\mathcal{P}_{\mathcal{P}_0}(S(x)),Sx)+d_{\mathbb{C}}(y,\mathcal{P}_{\mathcal{P}_0}(S(y)))\\
&\qquad+d_{\mathbb{C}}(\mathcal{P}_{\mathcal{P}_0}(S(y)),Sy)+\frac{(d_{\mathbb{C}}(x,\mathcal{P}_{\mathcal{P}_0}(S(x)))+d_{\mathbb{C}}(\mathcal{P}_{\mathcal{P}_0}(S(x)),Sx)-\operatorname{dit}(\mathcal{P},\mathcal{Q}))}{1+d_{\mathbb{C}}(x,y)}\\
&\qquad\times\frac{(d_{\mathbb{C}}(y,\mathcal{P}_{\mathcal{P}_0}(S(y)))+d_{\mathbb{C}}(\mathcal{P}_{\mathcal{P}_0}(S(y)),Sy)-\operatorname{dit}(\mathcal{P},\mathcal{Q}))}{1}-2\operatorname{dit}(\mathcal{P},\mathcal{Q})\bigg]\\
&\quad\preceq \frac{\alpha}{2}\bigg[d_{\mathbb{C}}(x,\mathcal{P}_{\mathcal{P}_0}(S(x)))+d_{\mathbb{C}}(y,\mathcal{P}_{\mathcal{P}_0}(S(y)))\\
&\qquad+\frac{(d_{\mathbb{C}}(x,\mathcal{P}_{\mathcal{P}_0}(S(x))))(d_{\mathbb{C}}(y,\mathcal{P}_{\mathcal{P}_0}(S(y))))}{1+d_{\mathbb{C}}(x,y)}\bigg]
\end{aligned}$$

for any $x,y\in\overline{\mathcal{P}}_0$ and $0\leqslant\alpha<1$. Obviously, it satisfies Theorem 3.3 in the case of self-mapping $\mathcal{P}_{\mathcal{P}_0}\circ S\colon \overline{\mathcal{P}}_0\to\overline{\mathcal{P}}_0$. Thus $\mathcal{P}_{\mathcal{P}_0}\circ S$ has a unique fixed point, namely x_1, such that $\mathcal{P}_{\mathcal{P}_0}\circ Sx_1 = x_1 \implies d_{\mathbb{C}}(x_1,Sx_1)=\operatorname{dit}(\mathcal{P},\mathcal{Q})$. Likewise we have defined $\mathcal{P}_{\mathcal{P}_0}\colon T(\overline{\mathcal{P}}_0)\to\mathcal{P}_0$ and observed that $\mathcal{P}_{\mathcal{P}_0}\circ T$ has a unique fixed point namely x_2 such that $\mathcal{P}_{\mathcal{P}_0}\circ Tx_2 = x_2 \implies d_{\mathbb{C}}(x_2,Tx_2)=\operatorname{dit}(\mathcal{P},\mathcal{Q})$. Since the pair $(\mathcal{P},\mathcal{Q})$ has weak p-property,

$$d_{\mathbb{C}}(x_1,Sx_1)=\operatorname{dit}(\mathcal{P},\mathcal{Q})$$

and

$$\begin{aligned}
&& d_{\mathbb{C}}(x_2,Sx_2)&=\operatorname{dit}(\mathcal{P},\mathcal{Q})\\
\implies && d_{\mathbb{C}}(x_1,x_2)&\preceq d_{\mathbb{C}}(Sx_1,Tx_2)\\
&& &\preceq\frac{\alpha}{2}\bigg[d_{\mathbb{C}}(x_1,Sx_1)+d_{\mathbb{C}}(x_2,Tx_2)
\end{aligned}$$

$$+\frac{(d_{\mathbb{C}}(x_1, Sx_1) - \operatorname{dit}(\mathcal{P}, \mathcal{Q}))(d_{\mathbb{C}}(x_2, Tx_2) - \operatorname{dit}(\mathcal{P}, \mathcal{Q}))}{1 + d_{\mathbb{C}}(x_1, x_2)}$$

$$\left. - 2\operatorname{dit}(\mathcal{P}, \mathcal{Q})\right]$$

$$= 0$$

$$\Longrightarrow \qquad x_1 = x_2 = x.$$

Thus $d_{\mathbb{C}}(x, Sx) = d_{\mathbb{C}}(x, Tx) = \operatorname{dit}(\mathcal{P}, \mathcal{Q})$ and the fixed point is unique. □

Remark 3.1. Our Theorem 3.4 extends and unifies Theorem 3.11 of Mondal et al. [13] to complex valued metric spaces with rational expression.

If S and T are assumed to be self-mappings, then Theorem 3.4 leads us to the following common fixed point theorem.

Theorem 3.5. *Let $(\mathcal{X}, d_{\mathbb{C}})$ be a complete complex valued metric space. If $S\colon \mathcal{X} \to \mathcal{X}$ and $T\colon \mathcal{X} \to \mathcal{X}$ are the continuous mappings such that*

$$d_{\mathbb{C}}(Sx, Ty) \preceq \frac{\alpha}{2}\left[d_{\mathbb{C}}(x, Sx) + d_{\mathbb{C}}(y, Ty) + \frac{d_{\mathbb{C}}(x, Sx)d_{\mathbb{C}}(y, Ty)}{1 + d_{\mathbb{C}}(x, y)}\right] \tag{3.4}$$

for all $x, y \in \mathcal{X}$, where α is a non-negative real number and $0 \leqslant \alpha < 1$, then there exists a unique common fixed point such that $d_{\mathbb{C}}(x, Sx) = d_{\mathbb{C}}(x, Tx) = x$.

3.3. Fixed and best proximity point theorems for Chatterjea type contractive condition with rational expression. The rational expressions considered here are of Chatterjea type with respect to complex valued metric space.

Theorem 3.6. *Let $(\mathcal{X}, d_{\mathbb{C}})$ be a complete complex valued metric space and $S\colon \mathcal{X} \to \mathcal{X}$ be a mapping such that*

$$d_{\mathbb{C}}(Sx, Sy) \preceq \frac{\alpha}{2}\left(d_{\mathbb{C}}(y, Sx) + d_{\mathbb{C}}(x, Sy) + \frac{d_{\mathbb{C}}(y, Sx)d_{\mathbb{C}}(x, Sy)}{1 + d_{\mathbb{C}}(x, y)}\right) \quad \text{for all } x, y \in \mathcal{X}$$

and for some $\alpha \in (0, 1)$, then S has a unique fixed point.

Proof. Let $x_0 \in \mathcal{X}$ be any arbitrary point. For $x_m = Sx_{m-1}$, we can find the sequence

$$d_{\mathbb{C}}(x_m, x_{m+1}) \preceq \frac{\alpha}{2-\alpha} d_{\mathbb{C}}(x_{m-1}, x_m) = c d_{\mathbb{C}}(x_{m-1}, x_m) \preceq \ldots \preceq c^m d_{\mathbb{C}}(x_0, x_1).$$

It is clear that x_m is a Cauchy sequence in $\mathcal{X}$. By completeness, there exists x such that $\lim\limits_{n\to\infty} x_m = x$ and also it can be seen that x is the fixed point of the mapping S and is unique. □

The proof of the following theorem can be obtained in a similar manner as that of Theorem 3.4 using Theorem 3.6.

Theorem 3.7. *Let $(\mathcal{P}, \mathcal{Q})$ be a pair of nonempty closed subsets of a complete complex valued metric space and $\mathcal{P}_0$ and $\mathcal{Q}_0$ are nonempty closed sets. Let $S\colon \mathcal{P} \to \mathcal{Q}$ and $T\colon \mathcal{P} \to \mathcal{Q}$ be two non-self continuous mappings that satisfy the following conditions.*

(1) *$(\mathcal{P}, \mathcal{Q})$ has weak p-property.*
(2) *There exist non-negative real numbers α such that $0 \leqslant \alpha < \frac{1}{2}$ and*

$$d_{\mathbb{C}}(Sx, Ty) \preceq \frac{\alpha}{2}\Big[d_{\mathbb{C}}(y, Sx) + d_{\mathbb{C}}(x, Ty) + \frac{(d_{\mathbb{C}}(y, Sx) - \operatorname{dit}(\mathcal{P}, \mathcal{Q}))(d_{\mathbb{C}}(x, Ty) - \operatorname{dit}(\mathcal{P}, \mathcal{Q}))}{1 + d_{\mathbb{C}}(x, y)} - 2\operatorname{dit}(\mathcal{P}, \mathcal{Q})\Big]. \tag{3.5}$$

(3) *$S(\overline{\mathcal{P}}_0) \subseteq \mathcal{Q}_0$ and $T(\overline{\mathcal{P}}_0) \subseteq \mathcal{Q}_0$.*

Then there exists a unique common best proximity point such that $d_{\mathbb{C}}(x, Sx) = d_{\mathbb{C}}(x, Tx) = \operatorname{dit}(\mathcal{P}, \mathcal{Q})$.

Remark 3.2. Theorem 3.7 is an extended and improved version of Theorem 3.14 of Mondal et al. [13] on complex valued metric spaces, using rational expression form.

The theorem obtained when the non-self-mappings S and T are replaced using self maps, follows

Theorem 3.8. *Let $(\mathcal{X}, d_{\mathbb{C}})$ be a complete complex valued metric space. Let $S\colon \mathcal{X} \to \mathcal{X}$ and $T\colon \mathcal{X} \to \mathcal{X}$ be continuous self maps satisfying*

$$d_{\mathbb{C}}(Sx, Ty) \preceq \frac{\alpha}{2}\left[d_{\mathbb{C}}(y, Sx) + d_{\mathbb{C}}(x, Ty) + \frac{d_{\mathbb{C}}(y, Sx)d_{\mathbb{C}}(x, Ty)}{1 + d_{\mathbb{C}}(x, y)}\right] \tag{3.6}$$

for all $x, y \in \mathcal{X}$ and for non-negative real numbers α such that $0 \leqslant \alpha < \frac{1}{2}$. Then there exists a unique common fixed point such that $d_{\mathbb{C}}(x, Sx) = d_{\mathbb{C}}(x, Tx) = x$.

3.4. Best proximity point theorems for contraction conditions involving control function. Motivated by the contraction condition with control functions of one variable in Murali et al. [14], in this section we first define the following contraction condition of the control function.

Definition 3.1. The two mappings $S, T\colon \mathcal{P} \to \mathcal{Q}$ are said to satisfy a contraction condition involving a control function of the first kind, if there exists $\alpha(Tx, a)$ for a suitable mapping $\alpha\colon \mathcal{P} \times \mathcal{P} \to [0, 1)$, which satisfies, for all $x, y \in \mathcal{P}$ and a fixed $a \in \mathcal{P}$, $\alpha(Sx, a) \leqslant \alpha(Tx, a)$ with $\alpha(Tx, a) < 1$, $\alpha(Sx, a) < 1$, and

$$d_{\mathbb{C}}(Sx, Sy) \preceq \alpha(Tx, a) d_{\mathbb{C}}(Tx, Ty). \tag{3.7}$$

Now we move on to the next main theorem for the contraction condition involving the control function of the first kind.

Theorem 3.9. *Let $\mathcal{P}$ and $\mathcal{Q}$ be nonempty closed subsets of a complete complex valued metric space such that $\mathcal{P}_0$ and $\mathcal{Q}_0$ are nonempty sets. Assume the pair $(\mathcal{P}, \mathcal{Q})$ has weak p-property. Let S and T be two non-self continuous mappings from $\mathcal{P}$ to $\mathcal{Q}$ that satisfy the given conditions.*

(1) *$S(\mathcal{P}_0) \subseteq \mathcal{Q}_0$ and $S(\mathcal{P}_0) \subseteq T(\mathcal{P}_0)$.*
(2) *The mappings S and T commute proximally.*
(3) *S and T satisfy the contraction condition involving the control function of the first kind.*

Then S and T have a unique common best proximity point.

Proof. Consider x_0 in $\mathcal{P}_0$. According to the assumption $S(\mathcal{P}_0) \subseteq T(\mathcal{P}_0)$, there exists $x_1 \in \mathcal{P}_0$ such that $Sx_0 = Tx_1$. Applying inductive methodology there exists $\{x_m\} \in \mathcal{P}_0$ such that $Sx_{m-1} = Tx_m$ for $m \in \mathbb{N}$. Since $S(\mathcal{P}_0) \subseteq \mathcal{Q}_0$, there exists v_m in $\mathcal{P}_0$ such that

$$d_{\mathbb{C}}(Sx_m, v_m) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}) \quad \text{for every } m \in \mathbb{N}.$$

It follows that the selection of v_m and x_m is such that

$$d_{\mathbb{C}}(Sx_m, v_m) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}),\ d_{\mathbb{C}}(Sx_{m+1}, v_{m+1}) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}) \quad \text{for every } m \in \mathbb{N}.$$

By the condition (3) and weak p-property of $(\mathcal{P}, \mathcal{Q})$, we have

$$\begin{aligned}\alpha(Sx_{m-1}, a) &\leqslant \alpha(Tx_{m-1}, a) = \alpha(Sx_{m-2}, a)\\ &\leqslant \alpha(Tx_{m-2}, a) = \alpha(Sx_{m-3}, a) = \ldots = \alpha(Tx_0, a) \leqslant \alpha(Sx_0, a).\end{aligned}$$

Therefore we have $\alpha(Sx_{m-1}, a) \leqslant \alpha(Sx_0, a)$. From the condition (3) and weak p-property,

$$\begin{aligned}d_{\mathbb{C}}(v_m, v_{m+1}) &\preceq d_{\mathbb{C}}(Sx_m, Sx_{m+1})\\ &\preceq \alpha(Tx_m, a) d_{\mathbb{C}}(Tx_m, Tx_{m+1})\end{aligned}$$

$$
\begin{aligned}
&= \alpha(Sx_{m-1}, a)d_{\mathbb{C}}(Sx_{m-1}, Sx_m) \\
&\vdots \\
&\preceq \alpha(Sx_0, a)d_{\mathbb{C}}(Sx_{m-1}, Sx_m).
\end{aligned}
$$

It is easy to see that the sequence $\{v_m\}$ is Cauchy and it converges to v in $\mathcal{P}$. Since $S(\mathcal{P}_0) \subseteq \mathcal{Q}_0$, there exists v_m in $\mathcal{P}$ such that $d_{\mathbb{C}}(Sx_m, v_m) = \text{dit}(\mathcal{P}, \mathcal{Q})$ for every positive integer. By the selection of x_m, we find

$$
d_{\mathbb{C}}(Tx_m, v_{m-1}) = d_{\mathbb{C}}(Sx_{m-1}, v_{m-1}) = \text{dit}(\mathcal{P}, \mathcal{Q})
$$

for every positive integer m. Since S and T commute proximally, we have $Tv_m = Sv_{m-1}$. By the continuity of S and T, it implies that $Tv = Sv$. By choosing $S(v) \in S(\mathcal{P}_0) \subseteq \mathcal{Q}_0$, we can find that there exists $x \in \mathcal{P}_0$ such that

$$
d_{\mathbb{C}}(x, Tv) = \text{dit}(\mathcal{P}, \mathcal{Q}) = d_{\mathbb{C}}(x, Sv). \tag{3.8}
$$

Since the mappings S and T commute proximally, we have $Sx = Tx$. Since $S(x) \in S(\mathcal{P}_0) \subseteq \mathcal{Q}_0$, there exists $l \in \mathcal{P}_0$ such that

$$
d_{\mathbb{C}}(l, Tx) = \text{dit}(\mathcal{P}, \mathcal{Q}) = d_{\mathbb{C}}(l, Sx). \tag{3.9}
$$

From the condition (3), we have

$$
d_{\mathbb{C}}(Sv, Sx) \preceq \alpha(Tv, a)d_{\mathbb{C}}(Tv, Tx) = \alpha(Sv, a)d_{\mathbb{C}}(Sv, Sx),
$$

from which $Sv = Sx$ and also $Tv = Tx$. Therefore from the equations (3.8) and (3.9), we have

$$
d_{\mathbb{C}}(x, Sv) = \text{dit}(\mathcal{P}, \mathcal{Q}) = d_{\mathbb{C}}(l, Sx).
$$

Using weak p-property and the condition (3) we have

$$
d_{\mathbb{C}}(x, l) \preceq d_{\mathbb{C}}(Sx, Sv) \preceq \alpha(Tv, a)d_{\mathbb{C}}(Tx, Tv) = \alpha(Tv, a)d_{\mathbb{C}}(Tx, Tx) = 0.
$$

This implies that $x = l$. Thus

$$
d_{\mathbb{C}}(x, Sx) = \text{dit}(\mathcal{P}, \mathcal{Q}) = d_{\mathbb{C}}(x, Tx). \tag{3.10}
$$

Suppose $x^\star$ is another common best proximity point of the two mappings S and T so that

$$
d_{\mathbb{C}}(x^\star, Sx^\star) = \text{dit}(\mathcal{P}, \mathcal{Q}) = d_{\mathbb{C}}(x^\star, Tx^\star). \tag{3.11}
$$

By the assumption (2), we have $S(x) = T(x)$ and $S(x^\star) = T(x^\star)$. Using the contraction condition of the control function of the first kind,

$$d_{\mathbb{C}}(Sx, Sx^\star) \preceq \alpha(Tx, a) d_{\mathbb{C}}(Tx, Tx^\star) = \alpha(Sx, a) d_{\mathbb{C}}(Sx, Sx^\star).$$

Since $\alpha(Sx, a) < 1$, we have $d_{\mathbb{C}}(Sx, Sx^\star) = 0$. That is $Sx = Sx^\star$. Therefore, we have

$$d_{\mathbb{C}}(x, Sx) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}) = d_{\mathbb{C}}(x^\star, Sx^\star).$$

By weak p-property, we have $d_{\mathbb{C}}(x, x^\star) \preceq d_{\mathbb{C}}(Sx, Sx^\star) = 0$, which implies that $x = x^\star$. Hence S and T have a unique common best proximity point such that $d_{\mathbb{C}}(x, Sx) = \operatorname{dit}(\mathcal{P}, \mathcal{Q})$ and $d_{\mathbb{C}}(x, Tx) = \operatorname{dit}(\mathcal{P}, \mathcal{Q})$. □

E x a m p l e 3.2. Let $\mathcal{X} = \mathbb{C}$ endowed with the complex valued metric as

$$d_{\mathbb{C}}(x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2) = |x_1 - x_2| + \mathrm{i}|y_1 - y_2|.$$

Consider $\mathcal{P} = \{z \in \mathbb{C} \colon \mathfrak{Rea}(z) = \frac{2}{7}, 0 \leqslant \mathfrak{Ima}(z) \leqslant 1\}$ and $\mathcal{Q} = \{z \in \mathbb{C} \colon \mathfrak{Rea}(z) = 0, 0 \leqslant \mathfrak{Ima}(z) \leqslant 1\}$. Let the mappings S and $T \colon \mathcal{P} \to \mathcal{Q}$ be defined as

$$S(z) = \left(\frac{2}{7} - x\right) + \mathrm{i}\frac{2y}{7} \quad \text{and} \quad T(z) = \frac{7(2/7 - x)}{2} + \mathrm{i}y,$$

and the function $\alpha \colon \mathcal{P} \times \mathcal{P} \to [0, 1)$ be defined as $\alpha(z, a) = \mathfrak{Rea}(z)/5 + \frac{2}{7}$ for a fixed $a = \frac{2}{7}$. It is easy to find the following: $\alpha(Sz) \leqslant \alpha(Tz)$ with $\alpha(Tz) < 1$ and $\alpha(Sz) < 1$, the mappings S and T commute proximally, the pair $(\mathcal{P}, \mathcal{Q})$ has weak p-property. Since $\operatorname{dit}(\mathcal{P}, \mathcal{Q}) = \frac{2}{7} + 0\mathrm{i}$, $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{Q}_0 = \mathcal{Q}$. From the condition (3.7), we have

$$\begin{aligned} d_{\mathbb{C}}(Sz_1, Sz_2) &= |x_1 - x_2| + \mathrm{i}\frac{2}{7}|y_1 - y_2| \\ &= \frac{2}{7}\left(\frac{7}{2}|x_1 - x_2| + \mathrm{i}|y_1 - y_2|\right) \\ &= \left(\frac{2 - 7\mathfrak{Rea}(z)}{6} + \frac{2}{7}\right) d_{\mathbb{C}}(Tz_1, Tz_2) \\ &= \alpha(Tx, a) d_{\mathbb{C}}(Tz_1, Tz_2). \end{aligned}$$

Therefore, all the conditions of Theorem 3.9 are satisfied. Thus $\frac{2}{7} + \mathrm{i}0$ is the common best proximity point of mapping S and T.

Next, we define another type of contraction condition for the control function.

Definition 3.2. The two mappings $S, T\colon \mathcal{P} \to \mathcal{Q}$ are said to be a contraction condition involving a control function of second kind, if there exists $\alpha(Tx)$ for a suitable mapping $\alpha\colon \mathcal{P} \to [0,1)$, which satisfies, for all $x, y \in \mathcal{P}$, $\alpha(Sx) \leqslant \alpha(Tx)$ with $\alpha(Tx) < 1$, $\alpha(Sx) < 1$ and

$$d_{\mathbb{C}}(Sx, Sy) \preceq \alpha(Tx) d_{\mathbb{C}}(Tx, Ty). \tag{3.12}$$

Theorem 3.10. *Let $\mathcal{P}$ and $\mathcal{Q}$ be nonempty closed subsets of a complete complex valued metric space such that $\mathcal{P}_0$ and $\mathcal{Q}_0$ are nonempty sets. Assume the pair $(\mathcal{P}, \mathcal{Q})$ has weak p-property. Let S and T be two non-self continuous mappings from $\mathcal{P}$ to $\mathcal{Q}$ that satisfy the given conditions.*

(1) *$S(\mathcal{P}_0) \subseteq \mathcal{Q}_0$ and $S(\mathcal{P}_0) \subseteq T(\mathcal{P}_0)$.*
(2) *The mappings S and T commute proximally.*
(3) *S and T satisfy the contraction condition involving the control function of second kind.*

Then S and T have a unique common best proximity point.

Proof. By the above theorem, we get that v_m and x_m are such that

$$d_{\mathbb{C}}(Sx_m, v_m) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}),\ d_{\mathbb{C}}(Sx_{m+1}, v_{m+1}) = \operatorname{dit}(\mathcal{P}, \mathcal{Q}) \quad \text{for every } m \in \mathbb{N}.$$

By the condition (3) and weak p-property of $(\mathcal{P}, \mathcal{Q})$, we have

$$\begin{aligned} \alpha(Sx_{m-1}) &\leqslant \alpha(Tx_{m-1}) = \alpha(Sx_{m-2}) \\ &\leqslant \alpha(Tx_{m-2}) = \alpha(Sx_{m-3}) \ldots = \alpha(Tx_0) \leqslant \alpha(Sx_0). \end{aligned}$$

Therefore we have $\alpha(Sx_{m-1}) \leqslant \alpha(Sx_0)$. From the condition (3) and weak p-property,

$$\begin{aligned} d_{\mathbb{C}}(v_m, v_{m+1}) &\preceq d_{\mathbb{C}}(Sx_m, Sx_{m+1}) \\ &\preceq \alpha(Tx_m) d_{\mathbb{C}}(Tx_m, Tx_{m+1}) \\ &= \alpha(Sx_{m-1}) d_{\mathbb{C}}(Sx_{m-1}, Sx_m) \\ &\preceq \alpha(Sx_{m-1}) d_{\mathbb{C}}(Sx_{m-1}, Sx_m) \\ &\vdots \\ &\preceq \alpha(Sx_0) d_{\mathbb{C}}(Sx_{m-1}, Sx_m). \end{aligned}$$

It is easy to see that the sequence $\{v_m\}$ is Cauchy and it converges to v in $\mathcal{P}$. Since $S(\mathcal{P}_0) \subseteq \mathcal{Q}_0$, there exists v_m in $\mathcal{P}$ such that $d_{\mathbb{C}}(Sx_m, v_m) = \operatorname{dit}(\mathcal{P}, \mathcal{Q})$ for every positive integer. By the selection of x_m, we find

$$d_{\mathbb{C}}(Tx_m, v_{m-1}) = d_{\mathbb{C}}(Sx_{m-1}, v_{m-1}) = \operatorname{dit}(\mathcal{P}, \mathcal{Q})$$

for every positive integer m. Since S and T commute proximally, we have $Tv_m = Sv_{m-1}$. By the continuity of S and T, it implies that $Tv = Sv$. By choosing $S(v) \in S(\mathcal{P}_0) \subseteq \mathcal{Q}_0$, we can find that there exists $x \in \mathcal{P}_0$ such that

$$d_{\mathbb{C}}(x, Tv) = \mathrm{dit}(\mathcal{P}, \mathcal{Q}) = d_{\mathbb{C}}(x, Sv). \tag{3.13}$$

Since the mappings S and T commute proximally, we have $Sx = Tx$. Since $S(x) \in S(\mathcal{P}_0) \subseteq \mathcal{Q}_0$, there exists $l \in \mathcal{P}_0$ such that

$$d_{\mathbb{C}}(l, Tx) = \mathrm{dit}(\mathcal{P}, \mathcal{Q}) = d_{\mathbb{C}}(l, Sx). \tag{3.14}$$

From the condition (3), we have

$$d_{\mathbb{C}}(Sv, Sx) \preceq \alpha(Tv) d_{\mathbb{C}}(Tv, Tx) = \alpha(Sv) d_{\mathbb{C}}(Sv, Sx).$$

Then $Sv = Sx$ and also $Tv = Tx$. Therefore, from the equations (3.13) and (3.14) we have

$$d_{\mathbb{C}}(x, Sv) = \mathrm{dit}(\mathcal{P}, \mathcal{Q}) = d_{\mathbb{C}}(l, Sx).$$

Using weak p-property and the condition (3), we have

$$d_{\mathbb{C}}(x, l) \preceq d_{\mathbb{C}}(Sx, Sv) \preceq \alpha(Tv) d_{\mathbb{C}}(Tx, Tv) = \alpha(Tv) d_{\mathbb{C}}(Tx, Tx) = 0.$$

That implies $x = l$. Thus

$$d_{\mathbb{C}}(x, Sx) = \mathrm{dit}(\mathcal{P}, \mathcal{Q}) = d_{\mathbb{C}}(x, Tx). \tag{3.15}$$

Suppose $x^\star$ is another common best proximity point of the two mappings S and T so that

$$d_{\mathbb{C}}(x^\star, Sx^\star) = \mathrm{dit}(\mathcal{P}, \mathcal{Q}) = d_{\mathbb{C}}(x^\star, Tx^\star). \tag{3.16}$$

By the assumption (2), we have $Sx = Tx$ and $Sx^\star = Tx^\star$. Using the contraction condition of the control function of the second kind,

$$d_{\mathbb{C}}(Sx, Sx^\star) \preceq \alpha(Tx) d_{\mathbb{C}}(Tx, Tx^\star) = \alpha(Sx) d_{\mathbb{C}}(Sx, Sx^\star) \preceq \alpha(Sx) d_{\mathbb{C}}(Sx, Sx^\star).$$

Since $\alpha(Sx) < 1$, we have $d_{\mathbb{C}}(Sx, Sx^\star) = 0$. That is $Sx = Sx^\star$. Therefore, we have

$$d_{\mathbb{C}}(x, Sx) = \mathrm{dit}(\mathcal{P}, \mathcal{Q}) = d_{\mathbb{C}}(x^\star, Sx^\star).$$

By weak p-property, we have $d_{\mathbb{C}}(x, x^\star) \preceq d_{\mathbb{C}}(Sx, Sx^\star) = 0$, which implies that $x = x^\star$. Hence S and T have a unique common best proximity point such that $d_{\mathbb{C}}(x, Sx) = \mathrm{dit}(\mathcal{P}, \mathcal{Q})$ and $d_{\mathbb{C}}(x, Tx) = \mathrm{dit}(\mathcal{P}, \mathcal{Q})$. □

By opting either $\alpha(Sx) = \alpha$ in Theorem 3.10 or $\alpha(Sx, a) = \alpha$ in Theorem 3.9, we deduce the following theorem.

Theorem 3.11. *Let $\mathcal{P}$ and $\mathcal{Q}$ be nonempty closed subsets of a complete complex valued metric space such that $\mathcal{P}_0$ and $\mathcal{Q}_0$ are nonempty sets. Assume the pair $(\mathcal{P}, \mathcal{Q})$ has weak p-property. Let S and T be two non-self continuous mappings from $\mathcal{P}$ to $\mathcal{Q}$ that satisfy the following conditions.*

(1) $S(\mathcal{P}_0) \subseteq \mathcal{Q}_0$ *and* $S(\mathcal{P}_0) \subseteq T(\mathcal{P}_0)$.
(2) *The mappings S and T commute proximally.*
(3) *S and T satisfy $d_{\mathbb{C}}(Sx, Sy) \preceq \alpha d_{\mathbb{C}}(Tx, Ty)$ for a non-negative real number $\alpha < 1$.*

Then S and T have a unique common best proximity point.

Gabeleh [9] proved the following common best proximity point theorem on metric spaces.

Theorem 3.12 ([9])**.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be nonempty closed subsets of a complete metric space such that $\mathcal{P}_0$ and $\mathcal{Q}_0$ are nonempty sets. Assume the pair $(\mathcal{P}, \mathcal{Q})$ has p-property. Let S and T be two non-self continuous mappings from $\mathcal{P}$ to $\mathcal{Q}$ that satisfy the following conditions.*

(1) $S(\mathcal{P}_0) \subseteq \mathcal{Q}_0$ *and* $S(\mathcal{P}_0) \subseteq T(\mathcal{P}_0)$.
(2) *The mappings S and T commute proximally.*
(3) *S and T satisfy $d_{\mathbb{C}}(Sx, Sy) \leqslant \alpha d_{\mathbb{C}}(Tx, Ty)$ for a non-negative real number $\alpha < 1$.*

Then S and T have a unique common best proximity point.

Example 3.3. Let $\mathcal{X} = \mathbb{C}$ and let $d_{\mathbb{C}}\colon \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ be given as

$$d_{\mathbb{C}}(x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2) = |x_1 - x_2| + \mathrm{i}|y_1 - y_2|.$$

Let $\mathcal{P} = \{z \in \mathbb{C}\colon \mathfrak{Rea}(z) = -1, \mathfrak{Ima}(z) \in [0,1]\}$ and $\mathcal{Q} = \{z \in \mathbb{C}\colon \mathfrak{Rea}(z) = 1, \mathfrak{Ima}(z) \in [0,1]\} \cup \{z \in \mathbb{C}\colon \mathfrak{Rea}(z) = -3, \mathfrak{Ima}(z) = \frac{1}{5}\}$. Let $(S,T)\colon \mathcal{P} \to \mathcal{Q}$ be defined as

$$S(z) = 1 + \mathrm{i}\frac{y}{3} \quad \text{and} \quad T(z) = 1 + \mathrm{i}y \quad \text{for any } z \in \mathcal{P}.$$

Clearly, $\mathrm{dit}(\mathcal{P}, \mathcal{Q}) = 2 + \mathrm{i}0$, $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{Q}_0 = \mathcal{Q}$. Also $S(\mathcal{P}_0) \subseteq \mathcal{Q}_0$ and $T(\mathcal{P}_0) \subseteq \mathcal{Q}_0$. Let $z_1, z_2 \in \mathcal{P}$ and $w_1, w_2 \in \mathcal{Q}$. If $\mathfrak{Ima}(z_1) = \mathfrak{Ima}(w_1)$ and $\mathfrak{Ima}(z_2) = \mathfrak{Ima}(w_2)$, we find

$$d_{\mathbb{C}}(z_1, w_1) = 2 = d_{\mathbb{C}}(z_2, w_2) = \mathrm{dit}(\mathcal{P}, \mathcal{Q}) \Longrightarrow d_{\mathbb{C}}(z_1, z_2) \preceq d_{\mathbb{C}}(w_1, w_2).$$

Therefore $(\mathcal{P}, \mathcal{Q})$ satisfies weak p-property.

In particular, for $z_1 = -1 + \mathrm{i}/5 = z_2$, $w_1 = 1 + \mathrm{i}/5$, $w_2 = -3 + \mathrm{i}/5$, we have

$$d_{\mathbb{C}}(z_1, z_2) = 0 \not\precsim 4 = d_{\mathbb{C}}(w_1, w_2).$$

So p-property does not hold. Clearly, we cannot apply above Theorem 3.12 of Gabeleh [9] to assert the existence of a common best proximity point for the mappings S and T. But, all the conditions of Theorem 3.11 hold. Hence, by Theorem 3.11, S and T have a unique common best proximity point $(-1, 0)$.

If we consider that the mappings S and T are self-mappings on $\mathcal{X}$, then Theorem 3.11 implies the following fixed point theorem of Jungck [11] valid in complex valued metric spaces.

Theorem 3.13. *Let $\mathcal{X}$ be a complete complex valued metric space. Let S and T be two self-continuous mappings from $\mathcal{X}$ to $\mathcal{X}$ that satisfy the following conditions.*

(1) $S(\mathcal{X}) \subseteq T(\mathcal{X})$.
(2) *The mappings S and T commute.*
(3) *S and T satisfy $d_{\mathbb{C}}(Sx, Sy) \preceq \alpha d_{\mathbb{C}}(Tx, Ty)$ for a non-negative real number $\alpha < 1$.*

Then S and T have a unique common fixed point.

4. Conclusion

In this article, we have established the existence and uniqueness of the common best proximity point for a pair of non-self-mappings under rational-type contractive conditions in complex valued metric spaces. By transforming non-self-mappings into self-mappings, we further have proved the existence and uniqueness of the common best proximity point for Kannan-type rational expression mappings and Chatterjea-type rational expression contractive mappings. Additionally, we have introduced new contraction conditions involving a control function and proved the existence and uniqueness of a common best proximity point under these conditions. Our results not only extend, but also unify several previously established findings in the literature.

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