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ON STABILITY OF NONCONFORMABLE HOPFIELD NEURAL NETWORKS

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Abstract. This work focuses on a particular type of Hopfield neural network that generalizes classical fractional derivatives and is distinguished by nonconformable fractional derivatives. The main goal is to determine the basic characteristics of these networks, such as the circumstances under which equilibrium points are present and distinct. By demonstrating the exponential stability of the network, we further investigate its behavior and rigorously deduce these criteria. This is accomplished by constructing a Lyapunov function, a potent instrument frequently used in stability studies. In addition to verifying the obtained stability constraints, the theoretical conclusions are supported by comprehensive numerical simulations that show the dynamics of the neural network in a variety of circumstances. These simulations provide specific illustrations of how the network reacts to various parameter combinations and inputs. Overall, this work contributes to the understanding of neural networks with fractional-order dynamics, offering insights into their mathematical properties and potential applications in areas requiring robust and stable systems.

Keywords: exponential stability; neural network; nonconformable derivative

MSC 2020: 34A08, 92B20, 93D23, 34D20

1. Introduction

Neural networks, inspired by the human brain, are fundamental in artificial intelligence and machine learning. Consisting of interconnected nodes, these networks process information similarly to biological neural networks. They gained prominence in the 1980s with the development of backpropagation, enabling the training of multi-layered networks [20], [21].

The Hopfield neural network (HNN), proposed by John Hopfield in 1982, functions as an associative memory. HNNs use interconnected neurons with symmetric weights to store and recall patterns. Through energy minimization, they can recognize stored

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patterns from noisy or incomplete inputs and solve complex optimization problems [9], [10], [2].

The resurgence of neural networks in the 2010s, fueled by advances in computational power and large datasets, led to deep learning. Despite their success, traditional neural networks and deep learning models face limitations in modeling complex, nonlinear, and nonlocal dynamics. This has prompted research into alternative frameworks, such as integrating fractional calculus into neural network design [14].

Fractional calculus, a generalization of classical calculus, deals with noninteger order derivatives and integrals. Its application to neural networks and complex system modeling is relatively recent. Fractional derivatives describe processes with memory and hereditary properties common in physical and biological systems. However, traditional fractional derivatives have limitations in preserving integer-order derivative properties and accurately capturing certain system dynamics [3], [4], [5], [7], [22], [8], [12], [1], [19].

To address these limitations, researchers have proposed nonconformable fractional derivatives, extending fractional calculus while better preserving integer-order properties. This new definition offers a comprehensive framework for modeling systems with fractional-order dynamics, capturing nonlocal and nonlinear behaviors in complex systems [5], [7], [22], [8].

In the mathematical analysis of neural networks, demonstrating the existence of a unique equilibrium point and solution stability is crucial. Researchers have focused on obtaining stability conditions for neural networks, with a particular emphasis on exponential stability to ensure a desired convergence rate. The Lyapunov function method, widely used in integer-order nonlinear systems, is also the primary tool for analyzing fractional-order nonlinear systems [3], [4], [5].

To further substantiate our discussion, we will reference several key works in the field. Notably, Rosenblatt's foundational contributions to neural networks [20], [21] and Hopfield's seminal research on associative memory besides works on neural networks [17], [9], [10], [6] provide crucial context. Additionally, studies on fractional calculus and its applications to neural networks [13], [12], [1], [19], [18], [16] highlight the evolving understanding of these mathematical frameworks in modeling complex systems.

Recent studies have further advanced the analysis of exponential stability in fractional systems, including the comprehensive works on homogeneous conformance [15], [16] and the investigation of bilinear systems in [11], which collectively enhance our understanding of stabilization techniques in this emerging field.

While existing research on neural networks with fractional derivatives typically covers uniform stability and Mittag-Leffler stability, to our knowledge, no work has addressed the stability of neural networks with nonconformable derivatives. This paper contributes to the field by studying Hopfield neural networks with nonconformable fractional derivatives and investigating the exponential stability of their equilibrium points. Consider Hopfield-type neural networks of the form

$$(1.1)\qquad \begin{aligned} (N_1^\alpha u_i)(t) &= I_i - \lambda_i u_i(t) + \sum_{j=1}^{n} \beta_{ij} A_j(u_j(t)), \quad i = 1, 2, \dots, n \\ u(t_0) &= u_0 \in \mathbb{R}^n, \end{aligned}$$

where N_1^α denotes the nonconformable fractional derivative of order α, with $0 < \alpha < 1$. The state variable of the i-th unit at time t is represented by $u_i(t)$. The parameters β_{ij} and I_i are real constants, where β_{ij} corresponds to the connection weight of unit j on unit i at time t, and I_i represents the input from outside the network to unit i. The function A_j indicates the activation of the j-th neuron, and $\lambda_i > 0$ denotes the rate at which unit i resets its potential to the resting state when disconnected from the network. For $u = (u_1, u_2, \dots, u_n)^\top \in \mathbb{R}^n$, we make use of the following vector norms:

$$\|u\|_1 = \sum_{i=1}^{n} |u_i| \quad \text{and} \quad \|u\|_2 = \sqrt{\sum_{i=1}^{n} u_i^2}.$$

The remainder of this paper is organized as follows: Section 2 presents key properties of nonconformable fractional derivatives. Section 3 contains the main results, first proving the existence and uniqueness of the equilibrium point for system (1.1), then deriving sufficient conditions for the exponential stability of the equilibrium using a Lyapunov function. Section 4 provides two numerical examples to verify our results. Lastly, Section 5 presents the conclusion of the paper.

2. Nonconformable derivative concepts

In this section, we recall the definitions and properties of nonconformable fractional derivatives, along with the stability theory within this framework.

2.1. Definitions and properties.

Definition 2.1 ([7]). Given a function $h\colon [0, \infty) \to \mathbb{R}$, the nonconformable fractional derivative $N_1^\alpha h(t)$ of order $\alpha \in (0, 1)$ of h at $t \in [0, \infty]$ is defined by

$$(2.1)\qquad N_1^\alpha h(t) = \lim_{\varepsilon \to 0} \frac{h(t + \varepsilon \mathrm{e}^{t^{-\alpha}}) - h(t)}{\varepsilon}.$$

In the case where the limit of equation (2.1) exists, we say that h is N_1^α-differentiable at the point $t \in (0, \infty)$.

R e m a r k 2.1. If h is N_1^α-differentiable over the interval $(0,t)$ and $\lim_{t\to 0^+} N_1^\alpha h(t)$ exists, then we have

$$N_1^\alpha h(0) = \lim_{t\to 0^+} N_1^\alpha h(t). \tag{2.2}$$

Definition 2.2 ([22]). Let $\alpha \in (0,1]$ and $0 \leqslant a \leqslant b$. We say that a function $h\colon [a,b] \to \mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral

$$_{N_1}J_a^\alpha h(x) = \int_a^x \mathrm{e}^{-t^{-\alpha}} h(t)\,\mathrm{d}t$$

exists and is finite.

Theorem 2.1 ([5]). *Let h be N_1^α-differentiable function in (t_0,∞) with $\alpha \in (0,1]$. Then for all $t > t_0$ we have:*

(1) *if h is differentiable, then $_{N_1}J_{t_0}^\alpha(N_1^\alpha h(t)) = h(t) - h(t_0)$;*
(2) $N_1^\alpha({}_{N_1}J_{t_0}^\alpha h(t)) = h(t)$.

Theorem 2.2 ([5]). *Let $\alpha \in (0,1]$ and f, g be N_1^α-differentiable functions at any point $t > 0$. Then*

(1) $N_1^\alpha(fg) = fN_1^\alpha(g) + gN_1^\alpha(f)$,
(2) $N_1^\alpha f(t) = \exp(t^{-\alpha})f'(t)$, *where f' is the ordinary derivative.*

Lemma 2.1 ([5]). *Let $h\colon [a,\infty) \to \mathbb{R}$ such that $N_1^\alpha h(t)$ exists on (a,∞). Then $N_1^\alpha h^2(t)$ exists on (a,∞) and*

$$N_1^\alpha h^2(t) = 2h(t)N_1^\alpha h(t) \quad \text{for all } t > a > 0.$$

2.2. Stability theory. Let us now consider nonconformable fractional-order systems of the form

$$N_1^\alpha x(t) = f(t,x), \tag{2.3}$$

where $x \in \mathbb{R}^n$, $f\colon \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $N_1^\alpha x$ is the nonconformable fractional derivative of order $0 < \alpha < 1$.

Definition 2.3 ([22]). Let $\alpha \in (0,1)$ and c be a real number. We define the nonconformable exponential function in the following way

$$E^N_{\alpha,c}(t) = \exp\left(c\int_{t_0}^{t} \exp(-u^{-\alpha})\,\mathrm{d}u\right).$$

Definition 2.4 ([22]). The origin of system (2.3) is defined to be stable if for $\varepsilon > 0$, there exists $\delta > 0$ such that the solution of system (2.3)) satisfies $\|x(t)\| < \varepsilon$ for all $t > t_0$ when $\|x_0\| < \delta$. The origin of system (2.3) is asymptotically stable if it is stable and it satisfies $\lim\limits_{t\to\infty} x(t) = 0$.

Definition 2.5 ([22]). The origin of system (2.3) is said to be fractional exponentially stable if

$$\|x(t)\| \leqslant M\|x_0\|E^N_{\alpha,-c}(t), \quad t \geqslant t_0 \geqslant 0, \tag{2.4}$$

where $c, M > 0$.

Lemma 2.2 ([5]). *Let $0 < \alpha < 1$ and $g\colon [t_0,\infty) \to \mathbb{R}^+$, $t_0 \geqslant 0$ be a continuous function and N_1^α-differentiable on (t_0,∞) such that $N_1^\alpha g(t) \leqslant -cg(t)$, where c is a positive constant. Then*

$$g(t) \leqslant E^N_{\alpha,-c}(t)g(t_0). \tag{2.5}$$

Theorem 2.3 ([5]). *Let $x = 0$ be the equilibrium point of the fractional-order system (2.3). Assume that there exists an N_1^α-differentiable Lyapunov function $V\colon [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^+$ and positive constants ε_i, $i = 1,2,3$ satisfying:*

(1) $\varepsilon_1\|x\|^2 \leqslant V(t,x(t)) \leqslant \varepsilon_2\|x\|^2$,
(2) $N_1^\alpha V(t,x(t)) \leqslant -\varepsilon_3\|x\|^2$.

Then the origin of system (2.3) is fractional exponentially stable.

3. Results

In this section, we first apply the Banach contraction principle to prove the existence and uniqueness of the equilibrium point. Then a sufficient condition for the exponential stability of system (1.1) is provided. Note that an equilibrium of (1.1) is a constant vector

$$u^* = (u_1^*, u_2^*, \dots, u_n^*)^\top \in \mathbb{R}^n,$$

which satisfies the system

$$\lambda_i u_i^* = I_i + \sum_{j=1}^{n} \beta_{ij} A_j(u_j^*), \quad i = 1, 2, \dots, n. \tag{3.1}$$

To improve readability, we denote

$$M = \max_{1\leqslant i\leqslant n} \left(\frac{K_i}{\lambda_i} \sum_{j=1}^{n} |\beta_{ji}| \right),$$

and

$$m = \min_{1\leqslant i\leqslant n} \left(\lambda_i - \frac{1}{2} \sum_{j=1}^{n} (|\beta_{ij}|K_j + |\beta_{ji}|K_i) \right).$$

3.1. Assumptions. Throughout this paper, the following assumptions will be used:

(H_0) There exist positive constants K_j, $j = 1, 2, \dots, n$ such that the functions A_j are Lipschitz continuous, that is,

$$|A_j(x_i) - A_j(y_i)| \leqslant K_j |x_i - y_i| \quad \text{for all } x_i, y_i \in \mathbb{R},\ j = 1, 2, \dots, n.$$

(H_1) For $i = 1, 2, \dots, n$, λ_i, K_i and β_{ij} satisfy the inequality

$$1 > \frac{K_i}{\lambda_i} \sum_{j=1}^{n} |\beta_{ji}|. \tag{3.2}$$

(H_2) For $i = 1, 2, \dots, n$, λ_i, K_i and β_{ij} fulfill the inequality

$$1 > \frac{1}{\lambda_i} \sum_{j=1}^{n} K_j |\beta_{ji}|. \tag{3.3}$$

R e m a r k 3.1. Advantages of Conditions (H_1) and (H_2):

- ▷ (H_1) ensures *self-stabilization*: By bounding $K_i\lambda_i^{-1}\sum\limits_{j=1}^{n}|\beta_{ji}|$, it guarantees that neuron i's reset rate (λ_i) dominates the total weighted incoming signals scaled by its own activation sensitivity (K_i), preventing destabilizing feedback.
- ▷ (H_2) enforces *input control*: The inequality $\lambda_i^{-1}\sum\limits_{j=1}^{n}K_j|\beta_{ji}|<1$ limits the collective influence of presynaptic neurons' sensitivities (K_j) and connection weights ($|\beta_{ji}|$), ensuring external inputs do not overpower the neuron's reset mechanism.
- ▷ Together, these conditions simplify stability verification and guarantee the existence of a unique equilibrium point for the network.

(H_1) and (H_2) ensure robust stability by balancing synaptic weights, activation sensitivities, and reset dynamics.

In the following subsection, we provide sufficient conditions for the existence and uniqueness of the equilibrium point. The proof is based on the Banach contraction principle.

3.2. Existence.

Theorem 3.1. *Suppose conditions* (H_0) *and* (H_1) *an satisfied, then there exists a unique equilibrium of system* (1).

P r o o f. Let $\lambda_i u_i^* = x_i$, $i=1,2,\ldots,n$. Then from (3.1) we obtain that

$$x_i = I_i + \sum_{j=1}^{n}\beta_{ij}A_j\left(\frac{x_j}{\lambda_j}\right), \quad i=1,2,\ldots,n.$$

Consider the mapping $\Psi\colon \mathbb{R}^n\to\mathbb{R}^n$, $\Psi(x)=(\Psi_1(x),\Psi_2(x),\ldots,\Psi_n(x))^\top$ defined by

$$\Psi_i(x) = I_i + \sum_{j=1}^{n}\beta_{ij}A_j\left(\frac{x_j}{\lambda_j}\right), \quad i=1,2,\ldots,n,$$

where $x=(x_1,x_2,\ldots,x_n)^\top\in\mathbb{R}^n$. Considering the norm $\|\cdot\|_1$, we will show that the mapping Ψ is a contraction. For $x,y\in\mathbb{R}^n$ we have

$$\|\Psi(x)-\Psi(y)\|_1 = \sum_{i=1}^{n}|\Psi_i(x)-\Psi_i(y)| = \sum_{i=1}^{n}\left|\sum_{j=1}^{n}\beta_{ij}\left(A_j\left(\frac{x_j}{\lambda_j}\right)-A_j\left(\frac{y_j}{\lambda_j}\right)\right)\right|,$$

and by condition (H_0), we obtain

$$\|\Psi(x)-\Psi(y)\|_1 \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|\beta_{ij}|K_j\left|\frac{x_j-y_j}{\lambda_j}\right| = \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\frac{|\beta_{ij}|K_j}{\lambda_j}\right)|x_j-y_j|. \tag{3.4}$$

It is clear from condition (H_1) that

$$\sum_{j=1}^{n} \frac{|\beta_{ji}|K_i}{\lambda_i} < 1, \quad i = 1, 2, \ldots, n.$$

Consequently, from inequality (3.4) we get

$$\|\Psi(x) - \Psi(y)\|_1 \leqslant M \sum_{i=1}^{n} |x_i - y_i| = M\|x - y\|_1.$$

Since $M < 1$, we have that $\Psi\colon \mathbb{R}^n \to \mathbb{R}^n$ is a contraction. Therefore, there exists a unique fixed point of the map Ψ which is the unique equilibrium of system (1.1). □

In the next subsection, we provide sufficient conditions for the exponential stability of the equilibrium point. The proof is based on the use of the Lyapunov function.

3.3. Exponential stability.

Theorem 3.2. *In addition to the hypotheses of Theorem* 3.1 *suppose that condition* (H_2) *is satisfied. Then the equilibrium x^* of system* (1.1) *is exponentially stable.*

P r o o f. Under the hypotheses of Theorem 3.1, system (1.1) has a unique equilibrium $u^* = (u_1^*, u_2^*, \ldots, u_n^*) \in \mathbb{R}^n$. Substituting $w(t) = u(t) - u^*$ into (1.1) leads to

$$(N_1^\alpha w_i)(t) = -\lambda_i w_i(t) + \sum_{j=1}^{n} \beta_{ij} B_j(w_j(t)), \quad i = 1, 2, \ldots, n, \tag{3.5}$$

where $B_j(w_j(t)) = A_j(w_j(t) + u_j^*) - A_j(u_j^*)$. It is observed that $B_j(0) = 0$ and B_j satisfies Lipschitz condition with K_j for $j = 1, 2, \ldots, n$. Now, it is seen that the exponential stability of the zero solution of (3.5) is equivalent to the stability of the equilibrium x^* of (1.1). Consider the Lyapunov function

$$V(t) = V(t, w(t)) = \frac{1}{2} \sum_{i=1}^{n} w_i^2(t).$$

Then nonconformable derivative of the function V along the trajectories of (3.5) is obtained as

$$\begin{aligned}(N_1^\alpha V)(t) = \sum_{i=1}^{n} w_i(t)(N_1^\alpha w_i)(t) &= \sum_{i=1}^{n} w_i(t)\Big(-\lambda_i w_i(t) + \sum_{j=1}^{n} \beta_{ij} B_j(w_j(t))\Big) \\ &\leqslant \sum_{i=1}^{n} \Big(-\lambda_i w_i^2(t) + \Big|\sum_{j=1}^{n} \beta_{ij} w_i(t) B_j(w_j(t))\Big|\Big).\end{aligned}$$

Since B_j, $j = 1, 2, \ldots, n$, satisfies Lipschitz condition with K_j, we have

$$(N_1^\alpha V)(t) \leqslant \sum_{i=1}^n \left(-\lambda_i w_i^2(t) + \sum_{j=1}^n |\beta_{ij}||w_i(t)|K_j|w_j(t)| \right),$$

and using inequality $|w_i w_j| \leqslant \frac{1}{2}(w_i^2 + w_j^2)$, we obtain

$$\begin{aligned} (N_1^\alpha V)(t) &\leqslant \sum_{i=1}^n \left(-\lambda_i w_i^2(t) + \frac{1}{2} \sum_{j=1}^n |\beta_{ij}|K_j(w_i^2(t) + w_j^2(t)) \right) \\ &= \sum_{i=1}^n \left(-\lambda_i + \frac{1}{2} \sum_{j=1}^n |\beta_{ij}|K_j \right) w_i^2 + \frac{1}{2} \sum_{i,j} |\beta_{ij}|K_j w_j^2 \\ &= \sum_{i=1}^n \left(-\lambda_i + \frac{1}{2} \sum_{j=1}^n |\beta_{ij}|K_j \right) w_i^2 + \frac{1}{2} \sum_{i,j} |\beta_{ji}|K_i w_i^2 \\ &= \sum_{i=1}^n \left(-\lambda_i + \frac{1}{2} \sum_{j=1}^n (|\beta_{ij}|K_j + |\beta_{ji}|K_i) \right) w_i^2(t). \end{aligned} \tag{3.6}$$

By the use of conditions (H_1) and (H_2), $m > 0$. That is, inequality (3.6) becomes

$$(N_1^\alpha V(t)) \leqslant -2mV(t).$$

Applying Lemma 2.2, it is observed that

$$V(t) \leqslant E^N_{\alpha,-2m}(t)V(t_0).$$

So, we have

$$\frac{1}{2} \sum_{i=1}^n w_i^2(t) \leqslant \frac{1}{2} \sum_{i=1}^n E^N_{\alpha,-2m}(t) w_i^2(t_0)$$

and

$$\|w(t)\| \leqslant E^N_{\alpha,-m}(t)\|w(t_0)\|,$$

which implies, by Definition 2.5 and Theorem 2.3, that the zero solution of (3.5) is exponentially stable and the proof is completed. □

4. Numerical simulations

In this section, examples are provided to illustrate the results. Firstly, we remark from Theorem 2.2 that if x is N_1^α-differentiable, then

$$(N_1^\alpha x)(t) = \mathrm{e}^{t^{-\alpha}} x'(t).$$

Based on the previous remark, we make use of some python script to carry out the numerical simulations.

Example 4.1. For this example, we take

$$A_i(x) = \frac{1}{2}x, \quad i = 1, 2.$$

The function A_i is Lipschitz continuous with constant $K_i = \frac{1}{2}$ for $i = 1, 2$. We then consider the following Hopfield-type neural network model with a nonconformable fractional derivative:

$$N_1^\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -0.3 & 0 \\ 0 & -0.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -0.07A_1(x_1) \\ -0.04A_1(x_1) \end{pmatrix} + \begin{pmatrix} 0.01A_2(x_2) \\ -0.03A_2(x_2) \end{pmatrix} + \begin{pmatrix} 0.01 \\ 0.04 \end{pmatrix}. \tag{4.1}$$

To verify the existence, uniqueness, and stability of the equilibrium point, we apply conditions (H_0)–(H_1)–(H_2) established earlier in the paper. These conditions ensure that the influence of the interconnection weights and the Lipschitz constants of the activation functions remain controlled, preventing instability or divergence in the system.

▷ For $i = 1$: We compute

$$\frac{L_1}{\lambda_1} \sum_{j=1}^{2} |\beta_{j1}| = 0.18 < 1,$$

and

$$\frac{1}{\lambda_1} \sum_{j=1}^{2} K_j |\beta_{j1}| = 0.18 < 1.$$

These inequalities confirm that the contraction condition holds for the first unit, satisfying the criteria for existence and uniqueness of the equilibrium point.

▷ For $i = 2$: Similarly, we find

$$\frac{L_2}{\lambda_2} \sum_{j=1}^{2} |\beta_{j2}| = 0.2 < 1,$$

and

$$\frac{1}{\lambda_2}\sum_{j=1}^{2} K_j|\beta_{j2}| = 0.2 < 1.$$

Again, these values meet the required conditions, confirming that the system remains within the bounds ensuring both existence and uniqueness of the equilibrium for the second unit.

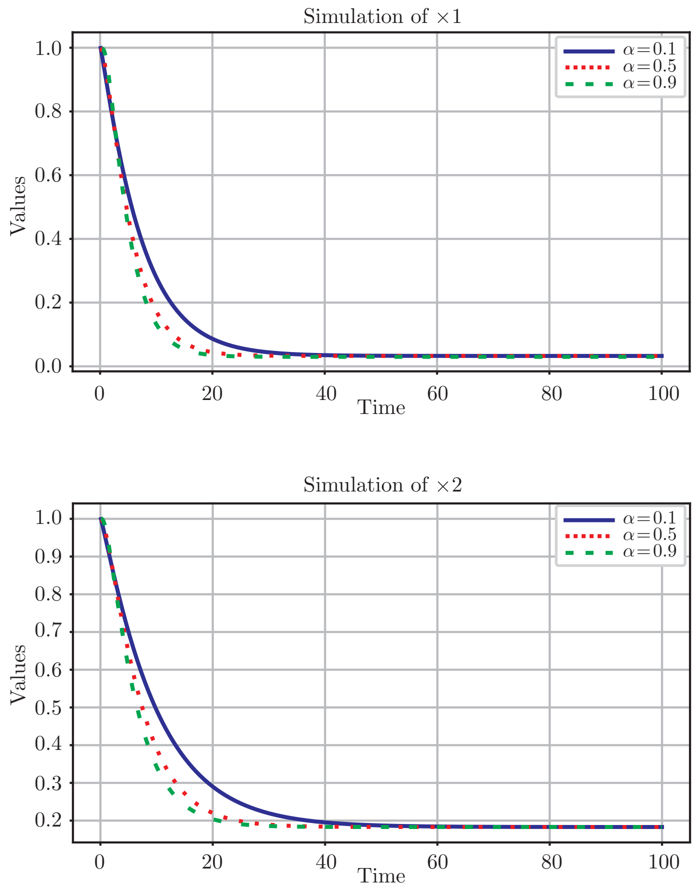

Figure 1. Dynamics of state variables in system (4.1).

This figure depicts the simulation results for the state variables x_1 (top) and x_2 (bottom) in the system defined by equation (4.1). The graphs illustrate the trajectories of these state variables over time for different values of the parameter α: 0.1, 0.5, and 0.9. Each curve represents the behavior of the corresponding variable as it approaches its equilibrium state. The convergence patterns shown in both panels indicate how varying α affects the stability and speed of the system's response, providing insights into the dynamics of the underlying model.

Example 4.2. For this example we take

$$A_i(x) = \frac{1}{1+x^2}, \quad i = 1,2,3,4.$$

Again, the function A_i is Lipshitz for $i = 1,2,3,4$. We consider the following Hopfield type neural network model with nonconformable fractional derivative

$$(4.2) \qquad N_1^\alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -0.7 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & 0 \\ 0 & 0 & -0.4 & 0 \\ 0 & 0 & 0 & -0.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$+ \begin{pmatrix} -0.05A_1(x_1) \\ -0.05A_1(x_1) \\ -0.05A_1(x_1) \\ -0.05A_1(x_1) \end{pmatrix} + \begin{pmatrix} 0.01A_2(x_2) \\ 0.01A_2(x_2) \\ 0.01A_2(x_2) \\ 0.01A_2(x_2) \end{pmatrix}$$

$$+ \begin{pmatrix} 0.01A_3(x_3) \\ 0.01A_3(x_3) \\ 0.01A_3(x_3) \\ 0.01A_3(x_3) \end{pmatrix} + \begin{pmatrix} -0.04A_4(x_4) \\ -0.04A_4(x_4) \\ -0.04A_4(x_4) \\ -0.04A_4(x_4) \end{pmatrix} + \begin{pmatrix} 0.3 \\ 0.04 \\ 0.03 \\ 0.25 \end{pmatrix}.$$

Straightforward computations show that conditions (H_0), (H_1) and (H_2) hold also for this case.

The following figure illustrates the trajectories of the state variables x_1 and x_2 on the left and x_3 and x_4 on the right, demonstrating their behavior over time for various parameter values α in the model defined by equation (4.2).

The left panel shows the convergence of x_1 and x_3 towards a unique interior equilibrium point, indicating exponential stability. This stability is quantitatively assessed with the following values for α: 0.1, 0.5 and 0.9. The equilibrium values are: $(0.1715, 0.3799, 0.1649, 0.5301)$.

The right panel presents the trajectories of x_2 and x_4, further confirming the system's stability as these variables also approach their equilibrium points. Overall, the figures provide a visual representation of the system's dynamics and the stability characteristics of the equilibrium state.

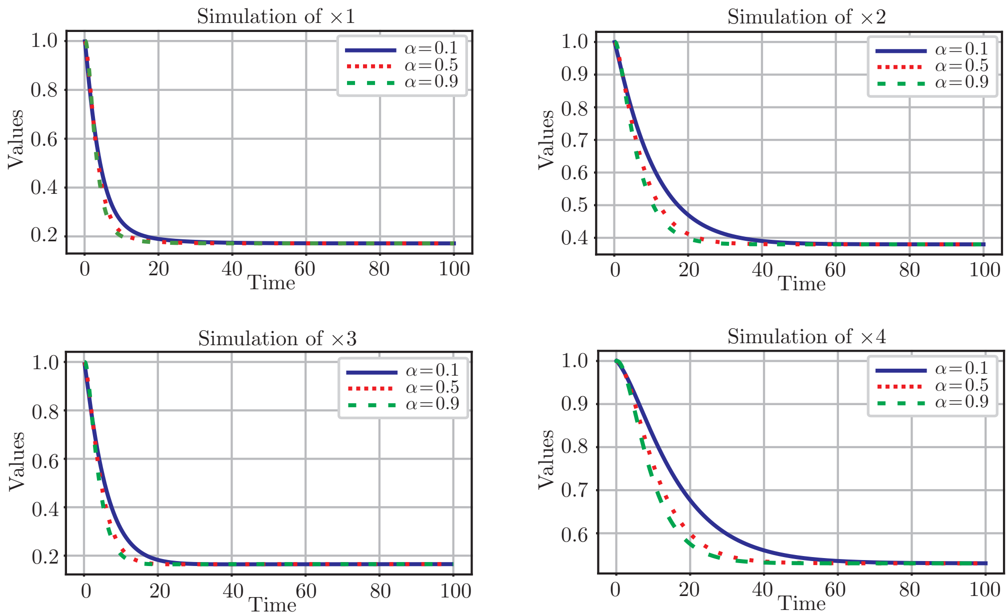

Figure 2. Dynamics of state variables in system (4.2).

5. Conclusion

The exponential stability of a fractional-order Hopfield-type neural network involving a nonconformable derivative is investigated. First, the existence and uniqueness of the equilibrium point are demonstrated. Then, conditions ensuring the exponential stability of the equilibrium are established using a Lyapunov function. The effectiveness of these conditions for different values of α is validated through numerical examples.

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