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SOME CONDITIONS FOR THE TOUCHARD POLYNOMIALS TO BE IN SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract. This paper discusses certain criteria that must be satisfied for a Touchard polynomial to belong to subclasses of analytic functions. Also, we obtain some inclusion relations between the classes $\mathfrak{U}_\sigma(D,E)$ and $\widehat{\mathcal{M}}(\eta,\mu)$. We also go over the geometric characteristics of an integral operator that has to do with the Touchard polynomial. Some particular instances of our primary discoveries are also remarked upon and discussed.

Keywords: Touchard polynomial; analytic function; integral operator; Hadamard product; univalent function

MSC 2020: 30C45, 30C50

1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of all functions of the form

$$\mathrm{f}(\zeta) = \zeta + \sum_{\jmath=2}^{\infty} b_\jmath \zeta^\jmath, \tag{1.1}$$

which are univalent and analytic in $\mathcal{Y} = \{\zeta\colon\ \zeta \in \mathbb{C} \text{ and } |\zeta| < 1\}$. Also, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ that contains the function of the form

$$\mathrm{f}(\zeta) = \zeta - \sum_{\jmath=2}^{\infty} b_\jmath \zeta^\jmath, \quad b_\jmath \geqslant 0. \tag{1.2}$$

The convolution of $\mathrm{f}(\zeta)$ and $\mathrm{g}(\zeta)$ is obtained by

$$(\mathrm{f} * \mathrm{g})(\zeta) = \zeta + \sum_{\jmath=2}^{\infty} b_\jmath c_\jmath \zeta^\jmath = (\mathrm{g} * \mathrm{f})(\zeta), \quad \zeta \in \mathcal{Y},$$

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where $\mathrm{f}(\zeta) \in \mathcal{A}$, defined by (1.1), and $\mathrm{g}(\zeta) \in \mathcal{A}$ given by

$$\mathrm{g}(\zeta) = \zeta + \sum_{\jmath=2}^{\infty} c_\jmath \zeta^\jmath.$$

A new class was defined by Lashin et al. [18] as follows:

Definition 1.1 ([18]). A function $\mathrm{f}(\zeta) \in \mathfrak{X}(\Upsilon, \Phi, \hbar, \eta, \mu)$ if

$$\mathrm{Re}\left\{ \frac{(1-\eta)(\mathrm{f} * \Upsilon)(\zeta) + \eta(\mathrm{f} * \Phi)(\zeta)}{(\mathrm{f} * \hbar)(\zeta)} \right\} > \mu, \quad \mathrm{f} \in \mathcal{A},\ 0 \leqslant \eta \leqslant 1,\ 0 \leqslant \mu < 1,$$

where Υ, Φ and $\hbar$ are given by

$$\Upsilon(\zeta) = \zeta + \sum_{\jmath=2}^{\infty} \vartheta_\jmath \zeta^\jmath, \quad \Phi(\zeta) = \zeta + \sum_{\jmath=2}^{\infty} \nu_\jmath \zeta^\jmath$$

and

$$\hbar(\zeta) = \zeta + \sum_{\jmath=2}^{\infty} \gamma_\jmath \zeta^\jmath, \quad \vartheta_\jmath, \nu_\jmath, \gamma_\jmath \geqslant 0,\ \zeta \in \mathcal{Y}.$$

Also,

$$\widehat{\mathfrak{X}}(\Upsilon, \Phi, \hbar, \eta, \mu) = \mathfrak{X}(\Upsilon, \Phi, \hbar, \eta, \mu) \cap \mathcal{T}.$$

The class, defined above, includes several simpler subclasses. In the following, we present a few of these cases:

If we put $\Upsilon(\zeta) = \zeta/(1-\zeta)^2$, $\Phi(\zeta) = (\zeta + \zeta^2)/(1-\zeta)^3$ and $\hbar(\zeta) = \zeta$ in Definition 1.1, then we obtain the following subclass:

Definition 1.2 ([7]). A function $\mathrm{f}(\zeta) \in \mathcal{M}(\eta, \mu)$ if

$$\mathrm{Re}\{\mathrm{f}'(\zeta) + \eta\zeta\mathrm{f}''(\zeta)\} > \mu, \quad 0 \leqslant \eta \leqslant 1,\ 0 \leqslant \mu < 1,\ \mathrm{f} \in \mathcal{A},$$

and (see [18], [23])

$$\widehat{\mathcal{M}}(\eta, \mu) = \mathcal{M}(\eta, \mu) \cap \mathcal{T}.$$

Lemma 1.1 ([23], with $p = 1$). *A function* $\mathrm{f}(\zeta) \in \widehat{\mathcal{M}}(\eta, \mu)$ *if and only if*

$$\sum_{\jmath=2}^{\infty} \jmath(1 + \eta(\jmath - 1))b_\jmath \leqslant 1 - \mu, \quad 0 \leqslant \eta \leqslant 1,\ 0 \leqslant \mu < 1,\ \mathrm{f} \in \mathcal{T}. \tag{1.3}$$

If we put $\Upsilon(\zeta) = \zeta/(1-\zeta)^2$, $\Phi(\zeta) = (\zeta+\zeta^2)/(1-\zeta)^3$ and $\hbar(\zeta) = \zeta/(1-\zeta)$ in Definition 1.1, then we obtain the following subclass:

Definition 1.3 ([22]). A function $\mathrm{f}(\zeta) \in \mathcal{W}(\eta,\mu)$ if

$$\mathrm{Re}\left\{\frac{\zeta \mathrm{f}'(\zeta)+\eta\zeta^2\mathrm{f}''(\zeta)}{\mathrm{f}(\zeta)}\right\} > \mu, \quad 0 \leqslant \eta \leqslant 1,\ 0 \leqslant \mu < 1,\ \mathrm{f} \in \mathcal{A},$$

and (see [16], [18])

$$\widehat{\mathcal{W}}(\eta,\mu) = \mathcal{W}(\eta,\mu) \cap \mathcal{T}.$$

Lemma 1.2 ([16]). *A function* $\mathrm{f}(\zeta) \in \widehat{\mathcal{W}}(\eta,\mu)$ *if and only if*

$$\sum_{\jmath=2}^{\infty}((\jmath-1)(\eta\jmath+1)+1-\mu)b_\jmath \leqslant 1-\mu, \quad 0 \leqslant \eta \leqslant 1,\ 0 \leqslant \mu < 1,\ \mathrm{f} \in \mathcal{T}. \tag{1.4}$$

If we put $\Upsilon(\zeta) = \zeta/(1-\zeta)$, $\Phi(\zeta) = \zeta/(1-\zeta)^2$ and $\hbar(\zeta) = \zeta$ in Definition 1.1, then we obtain the following subclass:

Definition 1.4 ([8]). A function $\mathrm{f} \in \mathcal{O}(\eta,\mu)$ if

$$\mathrm{Re}\left\{\frac{(1-\eta)\mathrm{f}(\zeta)}{\zeta}+\eta\mathrm{f}'(\zeta)\right\} > \mu, \quad 0 \leqslant \eta \leqslant 1,\ 0 \leqslant \mu < 1,\ \mathrm{f} \in \mathcal{A},$$

and (see [18] and [10], with $h(z) = \zeta/(1-\zeta)$)

$$\widehat{\mathcal{O}}(\eta,\mu) = \mathcal{O}(\eta,\mu) \cap \mathcal{T}.$$

Lemma 1.3 ([18]). *A function* $\mathrm{f}(\zeta) \in \widehat{\mathcal{O}}(\eta,\mu)$ *if and only if*

$$\sum_{\jmath=2}^{\infty}(\eta(\jmath-1)+1)b_\jmath \leqslant 1-\mu, \quad 0 \leqslant \eta \leqslant 1,\ 0 \leqslant \mu < 1, \mathrm{f} \in \mathcal{T}. \tag{1.5}$$

If we put $\Upsilon(\zeta) = \zeta/(1-\zeta)$, $\Phi(\zeta) = (\zeta+\zeta^2)/(1-\zeta)^3$ and $\hbar(\zeta) = \zeta/(1-\zeta)^2$ in Definition 1.1, then we obtain the following subclass:

Definition 1.5 ([17]). A function $\mathrm{f}(\zeta) \in \mathcal{P}(\eta,\mu)$ if

$$\mathrm{Re}\left\{(1-\eta)\frac{\mathrm{f}(\zeta)}{\zeta\mathrm{f}'(\zeta)}+\eta\left(1+\frac{\zeta\mathrm{f}''(\zeta)}{\mathrm{f}'(\zeta)}\right)\right\} > \mu, \quad 0 \leqslant \eta \leqslant 1,\ 0 \leqslant \mu < 1,\ \mathrm{f} \in \mathcal{A},$$

and

$$\widehat{\mathcal{P}}(\eta,\mu) = \mathcal{P}(\eta,\mu) \cap \mathcal{T}.$$

Lemma 1.4 ([17]). *A function* $\mathrm{f}(\zeta) \in \widehat{\mathcal{P}}(\eta,\mu)$ *if and only if*

$$\sum_{\jmath=2}^{\infty}((\jmath-1)(\eta(\jmath+1)-1)+\jmath(1-\mu))b_\jmath \leqslant 1-\mu, \quad \frac{1}{3} \leqslant \eta \leqslant 1,\ 0 \leqslant \mu < 1,\ \mathrm{f} \in \mathcal{T}. \tag{1.6}$$

R e m a r k 1.1.

(i) Putting $\eta = 0$ in Definition 1.3, we have $\mathcal{S}^*(\mu)$, which contains the starlike function of order μ $(0 \leqslant \mu < 1)$, which was investigated by Robertson [32];

(ii) Putting $\eta = 1$ in Definition 1.5, we have $\mathcal{C}(\mu)$, which contains the convex function of order μ $(0 \leqslant \mu < 1)$, which was investigated by Robertson [32];

(iii) Putting $\eta = 1$ in Definition 1.4, we have $\mathcal{K}^*(\mu)$, which contains the close-to-convex function of order μ $(0 \leqslant \mu < 1)$, which was investigated by Libera [19].

Definition 1.6 ([9]). A function $\mathrm{f}(\zeta) \in \mathfrak{U}_\sigma(D, E)$ if it satisfies

$$\left| \frac{\mathrm{f}'(\zeta) - 1}{(D - E)\sigma - E(\mathrm{f}'(\zeta) - 1)} \right| < 1, \quad \sigma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\};\ -1 \leqslant E < D \leqslant 1.$$

Note that:

(1) $\mathfrak{U}_1(\alpha, -\alpha) = \mathfrak{U}(\alpha) = \{\mathrm{f} \in \mathcal{A}\colon\ |(\mathrm{f}'(\zeta) - 1)/(\mathrm{f}'(\zeta) + 1)| < \alpha,\ \zeta \in \mathcal{Y};\ 0 < \alpha \leqslant 1)\}$ (see [24]);

(2) $\mathfrak{U}_{\mathrm{e}^{-\mathrm{i}\theta}\cos\theta}(1 - 2\alpha, -1) = \mathfrak{U}_\theta(\alpha) = \{\mathrm{f} \in \mathcal{A}\colon\ \mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}(\mathrm{f}'(\zeta) - \alpha)) > 0,\ \zeta \in \mathcal{Y};\ |\theta| < \pi/2;\ 0 \leqslant \alpha < 1\}$ (see [25]).

Lemma 1.5 ([9]). *If* $\mathrm{f}(\zeta) \in \mathfrak{U}_\sigma(D, E)$, *then*

$$|b_\jmath| \leqslant \frac{(D - E)|\sigma|}{\jmath}, \quad \jmath \geqslant 2;\ \sigma \in \mathbb{C}^*;\ -1 \leqslant E < D \leqslant 1.$$

The use of special functions in geometric function theory is a contemporary and intriguing area of study. It is widely utilized in fields like mathematics, physics, and engineering. Various special functions, such as generalized hypergeometric Gaussian functions [3], [14], [20], hypergeometric functions [25], [33], Wright function [31], Fox-Wright function [6], generalized Bessel functions [4], [28]. In fact, after the appearance of the paper of Porwal [26], several researchers familiarize Poisson distribution series [11], [12]. In 2018, Porwal [27] introduced the generalized distribution series and studied its geometric properties associated with univalent functions (see also [15]) and Mittag-Leffler type Poisson distribution series [29] and give nice solicitations on certain classes of univalent functions. In 2022, Porwal and Murugusundaramoorthy [30] investigated some conditions for Touchard polynomial to be in the subclasses of analytic functions.

The Touchard polynomials, also known as Bell polynomials [2], [5], [30], are a polynomial series of binomial type that are defined by Jacques Touchard [34] in 1939 as

$$\mathcal{D}_\imath(y) = \sum_{\ell=0}^{\imath} \mathcal{G}(\imath, \ell) y^\ell = \sum_{\ell=0}^{\imath} \left(\begin{matrix} \imath \\ \ell \end{matrix} \right) y^\ell,$$

where $\mathcal{G}(\imath, \ell)$ is a stirling number of the second kind.

If X represents a random variable with a Poisson distribution with expected ρ, then its $\imath$th moment is $E(X^{\imath}) = \psi_{\imath}'(\rho)$, and can be defined as

$$\psi_{\imath}'(\rho) = \mathrm{e}^{-\rho} \sum_{\ell=0}^{\infty} \frac{\rho^{\ell} \ell^{\imath}}{\ell!}, \quad \rho > 0;\ \imath \geqslant 0,$$

where γ is expected value.

In order to study different problems with inventory of the permutations when the cycles have specific features, Jacques Touchard studied these polynomials and generalized the Bell polynomials. He also created and investigated a class of related polynomials. This new approach can be used to solve integral equations, both linear and nonlinear. Since it is difficult to solve integral equations analytically, we must often find approximations for the solutions. In this situation, the linear Volterra integro-differential equation is solved using the Touchard polynomials approach. Both linear and nonlinear Volterra integral equations have been solved using the Touchard polynomials method.

Lately, Murugusundraramoorthy and Porwal [21] introduced and defined a function $\wp_{\imath}(\rho;\zeta)$ as follows (see also [1]):

$$\wp_{\imath}(\rho;\zeta) = \zeta + \mathrm{e}^{-\rho} \sum_{\jmath=2}^{\infty} \frac{\rho^{\jmath-1}(\jmath-1)^{\imath}}{(\jmath-1)!}\zeta^{\jmath}, \quad \zeta \in \mathcal{Y};\ \rho > 0;\ \imath \geqslant 0.$$

By ratio test, the above series converge to infinity. Also, they introduced $\mathcal{A}_{\imath}(\rho;\zeta)$ as follows:

$$\mathcal{A}_{\imath}(\rho;\zeta) = 2\zeta - \wp_{\imath}(\rho;\zeta) = \zeta - \mathrm{e}^{-\rho} \sum_{\jmath=2}^{\infty} \frac{\rho^{\jmath-1}(\jmath-1)^{\imath}}{(\jmath-1)!}\zeta^{\jmath}, \quad \zeta \in \mathcal{Y};\ \rho > 0;\ \imath \geqslant 0.$$

Next, we introduce the convolution operator $\mathfrak{Q}_{\imath}(\rho;\zeta)$ for function f:

$$\mathfrak{Q}_{\imath}(\rho;\zeta) = \mathcal{A}_{\imath}(\rho;\zeta) * \mathrm{f}(\zeta) = \zeta - \mathrm{e}^{-\rho} \sum_{\jmath=2}^{\infty} \frac{\rho^{\jmath-1}(\jmath-1)^{\imath}}{(\jmath-1)!} b_{\jmath}\zeta^{\jmath}, \quad \zeta \in \mathcal{Y};\ \rho > 0;\ \imath \geqslant 0.$$

Additionally, we give the functions

$$\begin{aligned}\mathfrak{O}_{\imath}(\varrho,\rho;\zeta) &= (1-\varrho)\mathcal{A}_{\imath}(\rho;\zeta) + \varrho\zeta(\mathcal{A}_{\imath}(\rho;\zeta))' \\ &= \zeta + \mathrm{e}^{-\rho} \sum_{\jmath=2}^{\infty} [1+\varrho(\jmath-1)] \frac{\rho^{\jmath-1}(\jmath-1)^{\imath}}{(\jmath-1)!}\zeta^{\jmath}, \quad \varrho \geqslant 0,\end{aligned}$$

and

$$\widehat{\mathfrak{O}}_{\imath}(\varrho,\rho;\zeta) = 2\zeta - \mathfrak{O}_{\imath}(\varrho,\rho;\zeta) = \zeta - \mathrm{e}^{-\rho} \sum_{\jmath=2}^{\infty} [1+\varrho(\jmath-1)] \frac{\rho^{\jmath-1}(\jmath-1)^{\imath}}{(\jmath-1)!}\zeta^{\jmath}, \quad \varrho \geqslant 0.$$

Also, we define $\mathfrak{C}_\imath(\varrho,\rho;\zeta)\colon\ \mathcal{A}\to\mathcal{A}$ by the convolution as

$$\begin{aligned}(1.7)\qquad \mathfrak{C}_\imath(\varrho,\rho;\zeta) &= \widehat{\mathfrak{O}}_\imath(\varrho,\rho;\zeta) * \mathrm{f}(\zeta)\\ &= \zeta - \mathrm{e}^{-\rho}\sum_{\jmath=2}^{\infty}[1+\varrho(\jmath-1)]\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{(\jmath-1)!}b_\jmath\zeta^\jmath,\quad \varrho\geqslant 0.\end{aligned}$$

This paper discusses certain criteria that must be satisfied for a Touchard polynomial to belong to subclasses of analytic functions. Also, we obtain some inclusion relations between the classes $\mathfrak{U}_\sigma(D,E)$ and $\widehat{\mathcal{M}}(\eta,\mu)$. We also go over the geometric characteristics of an integral operator that has to do with the Touchard polynomial. Some particular instances of our primary discoveries are also remarked upon and discussed.

2. Main results

Unless otherwise mentioned we shall assume throughout the paper that $0\leqslant\eta\leqslant 1$, $0\leqslant\mu<1$, $\sigma\in\mathbb{C}^*$, $-1\leqslant E<D\leqslant 1$, $\rho>0$, $\imath\geqslant 0$, $\zeta\in\mathcal{Y}$, $\mathrm{f}(\zeta)\in\mathcal{T}$.

Theorem 2.1. *The function $\widehat{\mathfrak{O}}_\imath(\varrho,\rho;\zeta)\in\widehat{\mathcal{M}}(\eta,\mu)$ if and only if*

$$\begin{cases}\eta\varrho\psi'_{\imath+3}(\rho)+(\varrho+\eta(\varrho+1))\psi'_{\imath+2}(\rho)+(1+\eta+\varrho)\psi'_{\imath+1}(\rho)+\psi'_\imath(\rho)\leqslant 1-\mu,\ \imath\geqslant 1,\\ \eta\varrho\rho^3+(\varrho+\eta(4\varrho+1))\rho^2+(1+2\eta(1+\varrho)+2\varrho)\rho+(1-\mathrm{e}^{-\rho})\leqslant 1-\mu,\qquad \imath=0.\end{cases}$$

P r o o f. According to Lemma 1.1, we have to prove that

$$\begin{aligned}&\sum_{\jmath=2}^{\infty}[\jmath(1+\eta(\jmath-1))][1+\varrho(\jmath-1)]\mathrm{e}^{-\rho}\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{(\jmath-1)!}\\ &=\mathrm{e}^{-\rho}\Bigg\{\sum_{\jmath=2}^{\infty}[(1-\eta)\jmath+\eta\jmath^2+(1-\eta)\varrho\jmath(\jmath-1)+\eta\varrho\jmath^2(\jmath-1)]\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{(\jmath-1)!}\Bigg\}\\ &=\mathrm{e}^{-\rho}\Bigg\{\sum_{\jmath=2}^{\infty}[\eta\varrho(\jmath-1)^3+(\varrho+\eta(\varrho+1))(\jmath-1)^2\\ &\quad+(1+\eta+\varrho)(\jmath-1)+1]\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{(\jmath-1)!}\Bigg\}\\ &=\mathrm{e}^{-\rho}\Bigg\{\eta\varrho\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^{\imath+3}}{(\jmath-1)!}+(\varrho+\eta(\varrho+1))\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^{\imath+2}}{(\jmath-1)!}\\ &\quad+(1+\eta+\varrho)\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^{\imath+1}}{(\jmath-1)!}+\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{(\jmath-1)!}\Bigg\}\end{aligned}$$

$$= \mathrm{e}^{-\rho}\Bigg\{\eta\varrho\sum_{\jmath=1}^{\infty}\frac{\rho^{\jmath}\jmath^{\imath+3}}{\jmath!} + (\varrho+\eta(\varrho+1))\sum_{\jmath=1}^{\infty}\frac{\rho^{\jmath}\jmath^{\imath+2}}{\jmath!}$$
$$+ (1+\eta+\varrho)\sum_{\jmath=1}^{\infty}\frac{\rho^{\jmath}\jmath^{\imath+1}}{\jmath!} + \sum_{\jmath=1}^{\infty}\frac{\rho^{\jmath}\jmath^{\imath}}{\jmath!}\Bigg\}$$
$$= \begin{cases} \eta\varrho\psi'_{\imath+3} + (\varrho+\eta(\varrho+1))\psi'_{\imath+2}(\rho) + (1+\eta+\varrho)\psi'_{\imath+1}(\rho) + \psi'_{\imath}(\rho) \leqslant 1-\mu, & \imath \geqslant 1, \\ \eta\varrho\rho^3 + (\varrho+\eta(4\varrho+1))\rho^2 + (1+2\eta(1+\varrho)+2\varrho)\rho + (1-\mathrm{e}^{-\rho}) \leqslant 1-\mu, & \imath = 0. \end{cases}$$

Thus, Theorem 2.1's proof is completed. □

We emphasized the next three special cases that generalize some results previously obtained by different authors:

Corollary 2.1. *Replacing $\eta = 0$ in the above theorem, $\widehat{\mathfrak{O}}_{\imath}(\varrho,\rho;\zeta) \in \widehat{\mathcal{M}}(\mu)$ if and only if*

$$\begin{cases} \varrho\psi'_{\imath+2}(\rho) + (1+\varrho)\psi'_{\imath+1}(\rho) + \psi'_{\imath}(\rho) \leqslant 1-\mu, & \imath \geqslant 1, \\ \varrho\rho^2 + (1+2\varrho)\rho + (1-\mathrm{e}^{-\rho}) \leqslant 1-\mu, & \imath = 0. \end{cases}$$

Corollary 2.2. *Replacing $\varrho = 0$ in the above theorem, $\mathcal{A}_{\imath}(\rho;\zeta) \in \widehat{\mathcal{M}}(\eta,\mu)$ if and only if*

$$\begin{cases} \eta\psi'_{\imath+2}(\rho) + (1+\eta)\psi'_{\imath+1}(\rho) + \psi'_{\imath}(\rho) \leqslant 1-\mu, & \imath \geqslant 1, \\ \eta\rho^2 + (1+2\eta)\rho + (1-\mathrm{e}^{-\rho}) \leqslant 1-\mu, & \imath = 0. \end{cases}$$

Corollary 2.3. *Replacing $\eta = 0$ and $\varrho = 0$ in the above theorem, $\mathcal{A}_{\imath}(\rho;\zeta) \in \widehat{\mathcal{M}}(\mu)$ if and only if*

$$\begin{cases} \psi'_{\imath+1}(\rho) + \psi'_{\imath}(\rho) \leqslant 1-\mu, & \imath \geqslant 1, \\ \rho + (1-\mathrm{e}^{-\rho}) \leqslant 1-\mu, & \imath = 0. \end{cases}$$

Theorem 2.2. *If $\mathrm{f}(\zeta) \in \mathfrak{U}_{\sigma}(D,E)$, $\mathfrak{C}_{\imath}(\varrho,\rho;\zeta)$ defined by (1.7) is in $\widehat{\mathcal{M}}(\eta,\mu)$ if and only if*

$$\begin{cases} (D-E)|\sigma|[\eta\varrho\psi'_{\imath+2}(\rho) + (\eta+\varrho)\psi'_{\imath+1}(\rho) + \psi'_{\imath}(\rho)] \leqslant 1-\mu, & \imath \geqslant 1, \\ (D-E)|\sigma|[\eta\varrho\rho^2 + (\varrho+\eta(\varrho+1))\rho + (1-\mathrm{e}^{-\rho})] \leqslant 1-\mu, & \imath = 0. \end{cases}$$

P r o o f. According to Lemma 1.1, 1.5 and (1.7), we have to prove that

$$\begin{aligned}
&\sum_{\jmath=2}^{\infty}[\jmath(1+\eta(\jmath-1))][1+\varrho(\jmath-1)]\left[\frac{(D-E)|\sigma|}{\jmath}\right]\mathrm{e}^{-\rho}\frac{\rho^{\jmath-1}(\jmath-1)^{\imath}}{(\jmath-1)!}\\
&=[(D-E)|\sigma|]\mathrm{e}^{-\rho}\left\{\sum_{\jmath=2}^{\infty}[(1-\eta)+\varrho(1-\eta)(\jmath-1)+\eta\jmath+\eta\varrho\jmath(\jmath-1)]\frac{\rho^{\jmath-1}(\jmath-1)^{\imath}}{(\jmath-1)!}\right\}\\
&=[(D-E)|\sigma|]\mathrm{e}^{-\rho}\left\{\sum_{\jmath=2}^{\infty}[\eta\varrho(\jmath-1)^2+(\varrho+\eta)(\jmath-1)+1]\frac{\rho^{\jmath-1}(\jmath-1)^{\imath}}{(\jmath-1)!}\right\}\\
&=[(D-E)|\sigma|]\mathrm{e}^{-\rho}\left\{\eta\varrho\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^{\imath+2}}{(\jmath-1)!}+(\varrho+\eta)\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^{\imath+1}}{(\jmath-1)!}\right.\\
&\qquad\qquad\left.+\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^{\imath}}{(\jmath-1)!}\right\}\\
&=[(D-E)|\sigma|]\mathrm{e}^{-\rho}\left\{\eta\varrho\sum_{\jmath=1}^{\infty}\frac{\rho^{\jmath}\jmath^{\imath+2}}{\jmath!}+(\varrho+\eta)\sum_{\jmath=1}^{\infty}\frac{\rho^{\jmath}\jmath^{\imath+1}}{\jmath!}+\sum_{\jmath=1}^{\infty}\frac{\rho^{\jmath}\jmath^{\imath}}{\jmath!}\right\}\\
&=\begin{cases}(D-E)|\sigma|[\eta\varrho\psi'_{\imath+2}(\rho)+(\eta+\varrho)\psi'_{\imath+1}(\rho)+\psi'_{\imath}(\rho)]\leqslant 1-\mu, & \imath\geqslant 1,\\ (D-E)|\sigma|[\eta\varrho\rho^2+(\varrho+\eta(\varrho+1))\rho+(1-\mathrm{e}^{-\rho})]\leqslant 1-\mu, & \imath=0.\end{cases}
\end{aligned}$$

Thus, Theorem 2.2's proof is completed. □

For special choices of the parameters we get the next special cases:

Corollary 2.4. *Replacing $\eta = 0$ in the above theorem, if $\mathrm{f}(\zeta) \in \mathfrak{U}_{\sigma}(D,E)$, $\mathfrak{C}_{\imath}(\varrho,\rho;\zeta)$ defined by* (1.7) *is in $\widehat{\mathcal{M}}(\mu)$ if and only if*

$$\begin{cases}(D-E)|\sigma|[\varrho\psi'_{\imath+1}(\rho)+\psi'_{\imath}(\rho)]\leqslant 1-\mu, & \imath\geqslant 1,\\ (D-E)|\sigma|[\varrho\rho+(1-\mathrm{e}^{-\rho})]\leqslant 1-\mu, & \imath=0.\end{cases}$$

Corollary 2.5. *Replacing $\varrho = 0$ in the above theorem, if $\mathrm{f}(\zeta) \in \mathfrak{U}_{\sigma}(D,E)$, $\mathfrak{Q}_{\imath}(\rho;\zeta)$ defined by* (1.7) *is in $\widehat{\mathcal{M}}(\eta,\mu)$ if and only if*

$$\begin{cases}(D-E)|\sigma|[\eta\psi'_{\imath+1}(\rho)+\psi'_{\imath}(\rho)]\leqslant 1-\mu, & \imath\geqslant 1,\\ (D-E)|\sigma|[\eta\rho+(1-\mathrm{e}^{-\rho})]\leqslant 1-\mu, & \imath=0.\end{cases}$$

Corollary 2.6. *Replacing $\eta = 0$ and $\varrho = 0$ in the above theorem, if $\mathrm{f}(\zeta) \in \mathfrak{U}_{\sigma}(D,E)$, $\mathfrak{Q}_{\imath}(\rho;\zeta)$ defined by* (1.7) *is in $\widehat{\mathcal{M}}(\mu)$ if and only if*

$$\begin{cases}(D-E)|\sigma|\psi'_{\imath}(\rho)\leqslant 1-\mu, & \imath\geqslant 1,\\ (D-E)|\sigma|(1-\mathrm{e}^{-\rho})\leqslant 1-\mu, & \imath=0.\end{cases}$$

Theorem 2.3. *The function*

$$\mathfrak{I}_\imath(\rho;\zeta)=\int_0^\zeta\left[2-\frac{\wp_\imath(\rho;s)}{s}\right]\mathrm{d}s$$

is in $\widehat{\mathcal{M}}(\eta,\mu)$ *if and only if*

$$\begin{cases}\eta\psi'_{\imath+1}(\rho)+\psi'_\imath(\rho)\leqslant 1-\mu, & \imath\geqslant 1,\\ \eta\rho+(1-\mathrm{e}^{-\rho})\leqslant 1-\mu, & \imath=0.\end{cases}$$

P r o o f. It is simple to observe that

$$\mathfrak{I}_\imath(\rho;\zeta)=\zeta-\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{\jmath!}\mathrm{e}^{-\rho}\zeta^\jmath.$$

According to Lemma 1.1, we have to prove that

$$\begin{aligned}\sum_{\jmath=2}^{\infty}[\jmath(1+\eta(\jmath-1))]\mathrm{e}^{-\rho}\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{\jmath!}&=\mathrm{e}^{-\rho}\left\{\sum_{\jmath=2}^{\infty}[1+\eta(\jmath-1)]\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{(\jmath-1)!}\right\}\\&=\mathrm{e}^{-\rho}\left\{\eta\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^{\imath+1}}{(\jmath-1)!}+\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{(\jmath-1)!}\right\}\\&=\begin{cases}\eta\psi'_{\imath+1}(\rho)+\psi'_\imath(\rho)\leqslant 1-\mu, & \imath\geqslant 1,\\ \eta\rho+(1-\mathrm{e}^{-\rho})\leqslant 1-\mu, & \imath=0.\end{cases}\end{aligned}$$

Thus, Theorem 2.3's proof is completed. □

Theorem 2.4. *The function* $\widehat{\mathfrak{O}}_\imath(\varrho,\rho;\zeta)\in\widehat{\mathcal{W}}(\eta,\mu)$ *if and only if*

$$\begin{cases}\eta\varrho\psi'_{\imath+3}(\rho)+(\varrho(1+\eta)+\eta)\psi'_{\imath+2}(\rho)+(1+\eta+\varrho(1-\mu))\psi'_{\imath+1}(\rho)\\ \qquad+(1-\mu)\psi'_\imath(\rho)\leqslant 1-\mu, & \imath\geqslant 1,\\ \eta\varrho\rho^3+(\eta+\varrho(4\eta+1))\rho^2+[(1+2\eta)(1+\varrho)+\varrho(1-\mu)]\rho\leqslant(1-\mu)\mathrm{e}^{-\rho}, & \imath=0.\end{cases}$$

P r o o f. According to Lemma 1.2, we have to prove that

$$\begin{aligned}&\sum_{\jmath=2}^{\infty}[(\jmath-1)(\eta\jmath+1)+1-\mu][1+\varrho(\jmath-1)]\mathrm{e}^{-\rho}\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{(\jmath-1)!}\\&\quad=\mathrm{e}^{-\rho}\left\{\sum_{\jmath=2}^{\infty}[(\eta\jmath+1)\varrho(\jmath-1)^2+(\eta\jmath+1)(\jmath-1)\right.\\&\qquad\left.+(1-\mu)\varrho(\jmath-1)+(1-\mu)]\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{(\jmath-1)!}\right\}\\&\quad=\mathrm{e}^{-\rho}\left\{\sum_{\jmath=2}^{\infty}[\eta\varrho(\jmath-1)^3+(\varrho(1+\eta)+\eta)(\jmath-1)^2\right.\end{aligned}$$

$$
\begin{aligned}
&+(1+\eta+\varrho(1-\mu))(\jmath-1)+(1-\mu)]\frac{\rho^{\jmath-1}(\jmath-1)^{\imath}}{(\jmath-1)!}\Big\}\\
&=\mathrm{e}^{-\rho}\Big\{\eta\varrho\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^{\imath+3}}{(\jmath-1)!}+(\varrho(1+\eta)+\eta)\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^{\imath+2}}{(\jmath-1)!}\\
&\quad+(1+\eta+\varrho(1-\mu))\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^{\imath+1}}{(\jmath-1)!}\\
&\quad+(1-\mu)\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^{\imath+1}}{(\jmath-1)!}\Big\}\\
&=\mathrm{e}^{-\rho}\Big\{\eta\varrho\sum_{\jmath=1}^{\infty}\frac{\rho^{\jmath}\jmath^{\imath+3}}{\jmath!}+(\varrho(1+\eta)+\eta)\sum_{\jmath=1}^{\infty}\frac{\rho^{\jmath}\jmath^{\imath+2}}{\jmath!}\\
&\quad+(1+\eta+\varrho(1-\mu))\sum_{\jmath=1}^{\infty}\frac{\rho^{\jmath}\jmath^{\imath+1}}{\jmath!}+(1-\mu)\sum_{\jmath=1}^{\infty}\frac{\rho^{\jmath}\jmath^{\imath}}{\jmath!}\Big\}\\
&=\begin{cases}\eta\varrho\psi'_{\imath+3}(\rho)+(\varrho(1+\eta)+\eta)\psi'_{\imath+2}(\rho)+(1+\eta+\varrho(1-\mu))\psi'_{\imath+1}(\rho)\\
\qquad+(1-\mu)\psi'_{\imath}(\rho)\leqslant 1-\mu, & \imath\geqslant 1,\\
\eta\varrho\rho^3+(\eta+\varrho(4\eta+1))\rho^2\\
\qquad+[(1+2\eta)(1+\varrho)+\varrho(1-\mu)]\rho\leqslant(1-\mu)\mathrm{e}^{-\rho}, & \imath=0.\end{cases}
\end{aligned}
$$

Thus, Theorem 2.4's proof is completed. □

The following two special cases of Theorem 2.4 show that this theorem generalises the mentioned recent results.

Corollary 2.7. *Letting $\eta=0$ in Theorem 2.4, $\widehat{\mathfrak{O}}_{\imath}(\varrho,\rho;\zeta)\in\mathcal{S}^{*}(\mu)$ if and only if*

$$
\begin{cases}\varrho\psi'_{\imath+2}(\rho)+(1+\varrho(1-\mu))\psi'_{\imath+1}(\rho)+(1-\mu)\psi'_{\imath}(\rho)\leqslant 1-\mu, & \imath\geqslant 1,\\ \varrho\rho^2+[1+\varrho(2-\mu)]\rho\leqslant(1-\mu)\mathrm{e}^{-\rho}, & \imath=0.\end{cases}
$$

Corollary 2.8. *Letting $\varrho=0$ in Theorem 2.4, $\mathcal{A}_{\imath}(\rho;\zeta)\in\widehat{\mathcal{W}}(\eta,\mu)$ if and only if*

$$
\begin{cases}\eta\psi'_{\imath+2}(\rho)+(1+\eta)\psi'_{\imath+1}(\rho)+(1-\mu)\psi'_{\imath}(\rho)\leqslant 1-\mu, & \imath\geqslant 1,\\ \eta\rho^2+(1+2\eta)\rho\leqslant(1-\mu)\mathrm{e}^{-\rho}, & \imath=0.\end{cases}
$$

R e m a r k 2.1.

(i) For $\imath=0$, Theorem 2.4 reduces to the result obtained by Lashin et al. [18], Theorem 7 (see also [15], Theorem 2.2, with $t_{\ell}=m^{\ell}/\ell!$).

(ii) For $\imath=0$, Corollary 2.7 reduces to the result obtained by Lashin et al. [18], Corollary 8 (see also [15], Corollary 2.3, with $t_{\ell}=m^{\ell}/\ell!$).

(iii) For $\imath = 0$, Corollary 2.8 reduces to the result obtained by Kota and El-Ashwah [15], Corollary 2.5, with $t_\ell = m^\ell/\ell!$.

Theorem 2.5. *The function* $\widehat{\mathfrak{O}}_\imath(\varrho, \rho; \zeta) \in \widehat{\mathcal{O}}(\eta, \mu)$ *if and only if*

$$\begin{cases} \eta\varrho\psi'_{\imath+2}(\rho) + (\varrho+\eta)\psi'_{\imath+1}(\rho) + \psi'_\imath(\rho) \leqslant 1-\mu, & \imath \geqslant 1, \\ \eta\varrho\rho^2 + (\eta + \varrho(\eta+1))\rho + (1-\mathrm{e}^{-\rho}) \leqslant 1-\mu, & \imath = 0. \end{cases}$$

P r o o f. According to Lemma 1.3, we have to prove that

$$\begin{aligned} &\sum_{\jmath=2}^{\infty} [1+\eta(\jmath-1)][1+\varrho(\jmath-1)]\mathrm{e}^{-\rho}\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{(\jmath-1)!} \\ &= \mathrm{e}^{-\rho}\left\{\sum_{\jmath=2}^{\infty}[\eta\varrho(\jmath-1)^2 + (\varrho+\eta)(\jmath-1)+1]\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{(\jmath-1)!}\right\} \\ &= \mathrm{e}^{-\rho}\left\{\eta\varrho\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^{\imath+2}}{(\jmath-1)!} + (\varrho+\eta)\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^{\imath+1}}{(\jmath-1)!} + \sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{(\jmath-1)!}\right\} \\ &= \mathrm{e}^{-\rho}\left\{\eta\varrho\sum_{\jmath=1}^{\infty}\frac{\rho^\jmath \jmath^{\imath+2}}{\jmath!} + (\varrho+\eta)\sum_{\jmath=1}^{\infty}\frac{\rho^\jmath \jmath^{\imath+1}}{\jmath!} + \sum_{\jmath=1}^{\infty}\frac{\rho^\jmath \jmath^\imath}{\jmath!}\right\} \\ &= \begin{cases} \eta\varrho\psi'_{\imath+2}(\rho) + (\varrho+\eta)\psi'_{\imath+1}(\rho) + \psi'_\imath(\rho) \leqslant 1-\mu, & \imath \geqslant 1, \\ \eta\varrho\rho^2 + (\eta+\varrho(\eta+1))\rho + (1-\mathrm{e}^{-\rho}) \leqslant 1-\mu, & \imath = 0. \end{cases} \end{aligned}$$

Thus, Theorem 2.5's proof is completed. □

For special choices of the parameters we get the next special cases:

R e m a r k 2.2.

(i) For $\imath = 0$, Theorem 2.5 reduces to the result obtained by Lashin et al. [18], Theorem 9 (see also [15], Theorem 2.3, with $t_\ell = m^\ell/\ell!$).

(ii) For $\imath = 0$ and setting $\eta = 1$ in Theorem 2.5, we get the result obtained in Corollary 2.1 (see also [15], Corollary 2.1, with $t_\ell = m^\ell/\ell!$).

(iii) For $\imath = 0$ and setting $\eta = 1$ and $\varrho = 0$ in Theorem 2.5, we get the result obtained in Corollary 2.3 (see also [15], Corollary 2.2, with $t_\ell = m^\ell/\ell!$).

Corollary 2.9. *Setting* $\varrho = 0$ *in Theorem* 2.5, $\mathcal{A}_\imath(\rho; \zeta) \in \widehat{\mathcal{O}}(\eta, \mu)$ *if and only if*

$$\begin{cases} \eta\psi'_{\imath+1}(\rho) + \psi'_\imath(\rho) \leqslant 1-\mu, & \imath \geqslant 1, \\ \eta\rho + (1-\mathrm{e}^{-\rho}) \leqslant 1-\mu, & \imath = 0. \end{cases}$$

R e m a r k 2.3. For $\imath = 0$, Corollary 2.9 reduces to the result obtained by Frasin [13], Corollary 5.1 (see also [15], Corollary 2.6, with $t_\ell = m^\ell/\ell!$).

Theorem 2.6. *The function* $\widehat{\mathfrak{D}}_\imath(\varrho,\rho;\zeta)\in\widehat{\mathcal{P}}(\eta,\mu)$ *if and only if*

$$\begin{cases}\eta\varrho\psi'_{\imath+3}(\rho)+[\eta(1+2\varrho)-\varrho\mu]\psi'_{\imath+2}(\rho)+[2\eta-\mu+\varrho(1-\mu)]\psi'_{\imath+1}(\rho)\\ \qquad\qquad +(1-\mu)\psi'_\imath(\rho)\leqslant 1-\mu, & \imath\geqslant 1,\\ \eta\varrho\rho^3+[\eta(5\varrho+1)-\varrho\mu]\rho^2+[(3\eta-\mu)(\varrho+1)+\varrho(1-\mu)]\rho\\ \qquad\qquad +(1-\mu)(1-\mathrm{e}^{-\rho})\leqslant 1-\mu, & \imath=0.\end{cases}$$

P r o o f. According to Lemma 1.4, we have to prove that

$$\begin{aligned}&\sum_{\jmath=2}^{\infty}[(\jmath-1)(\eta(\jmath+1)-1)+\jmath(1-\mu)][1+\varrho(\jmath-1)]\mathrm{e}^{-\rho}\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{(\jmath-1)!}\\ &=\mathrm{e}^{-\rho}\Bigg\{\sum_{\jmath=2}^{\infty}[\eta\varrho(\jmath-1)^2(\jmath+1)+[\eta+\varrho(1-\mu)](\jmath-1)\jmath-\varrho(\jmath-1)^2\\ &\quad+(1-\mu)\jmath-(\jmath-1)]\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{(\jmath-1)!}\Bigg\}\\ &=\mathrm{e}^{-\rho}\Bigg\{\sum_{\jmath=2}^{\infty}[\eta\varrho(\jmath-1)^3+[\eta(1+2\varrho)-\varrho\mu](\jmath-1)^2\\ &\quad+[2\eta-\mu+\varrho(1-\mu)](\jmath-1)+(1-\mu)]\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{(\jmath-1)!}\Bigg\}\\ &=\mathrm{e}^{-\rho}\Bigg\{\eta\varrho\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^{\imath+3}}{(\jmath-1)!}+[\eta(1+2\varrho)-\varrho\mu]\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^{\imath+2}}{(\jmath-1)!}\\ &\quad+[2\eta-\mu+\varrho(1-\mu)]\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^{\imath+1}}{(\jmath-1)!}\\ &\quad+(1-\mu)\sum_{\jmath=2}^{\infty}\frac{\rho^{\jmath-1}(\jmath-1)^\imath}{(\jmath-1)!}\Bigg\}\\ &=\mathrm{e}^{-\rho}\Bigg\{\eta\varrho\sum_{\jmath=1}^{\infty}\frac{\rho^\jmath\jmath^{\imath+3}}{\jmath!}+[\eta(1+2\varrho)-\varrho\mu]\sum_{\jmath=1}^{\infty}\frac{\rho^\jmath\jmath^{\imath+2}}{\jmath!}\\ &\quad+[2\eta-\mu+\varrho(1-\mu)]\sum_{\jmath=1}^{\infty}\frac{\rho^\jmath\jmath^{\imath+1}}{\jmath!}+(1-\mu)\sum_{\jmath=1}^{\infty}\frac{\rho^\jmath\jmath^\imath}{\jmath!}\Bigg\}\\ &=\begin{cases}\eta\varrho\psi'_{\imath+3}(\rho)+[\eta(1+2\varrho)-\varrho\mu]\psi'_{\imath+2}(\rho)\\ \qquad\qquad +[2\eta-\mu+\varrho(1-\mu)]\psi'_{\imath+1}(\rho)+(1-\mu)\psi'_\imath(\rho)\leqslant 1-\mu, & \imath\geqslant 1,\\ \eta\varrho\rho^3+[\eta(5\varrho+1)-\varrho\mu]\rho^2\\ \qquad\qquad +[(3\eta-\mu)(\varrho+1)+\varrho(1-\mu)]\rho+(1-\mu)(1-\mathrm{e}^{-\rho})\leqslant 1-\mu, & \imath=0.\end{cases}\end{aligned}$$

Thus, Theorem 2.6's proof is completed. □

For special choices of the parameters we get the next special cases:

R e m a r k 2.4.

(i) For $\imath = 0$, Theorem 2.6 reduces to the result obtained by Lashin et al. [18], Theorem 12 (see also [15], Theorem 2.5, with $t_\ell = m^\ell/\ell!$).

(ii) For $\imath = 0$ and setting $\eta = 1$ in Theorem 2.6, we get the result obtained by Kota and El-Ashwah [15], Corollary 2.8, with $t_\ell = m^\ell/\ell!$.

Corollary 2.10. *Setting $\varrho = 0$ in Theorem* 2.6, $\mathcal{A}_\imath(\rho;\zeta) \in \widehat{\mathcal{P}}(\eta,\mu)$ *if and only if*

$$\begin{cases} \eta\psi'_{\imath+2}(\rho) + (2\eta-\mu)\psi'_{\imath+1}(\rho) + (1-\mu)\psi'_\imath(\rho) \leqslant 1-\mu, & \imath \geqslant 1, \\ \eta\rho^2 + (3\eta-\mu)\rho + (1-\mu)(1-\mathrm{e}^{-\rho}) \leqslant 1-\mu, & \imath = 0. \end{cases}$$

3. Conclusions

Several applications on analytic functions have been studied by several authors. The Touchard polynomials, commonly referred to as the exponential polynomials, are a class of binomial-type polynomials. Numerous researchers are interested in the Touchard polynomials. In the theory of geometric functions, they are highly applicable. In 2020, Murugusundraramoorthy and Porwal [21] investigated some conditions for Touchard polynomial to be in the subclasses of analytic functions. This paper discusses certain criteria that must be satisfied for a Touchard polynomial to belong to subclasses of analytic functions. Also, we obtain some inclusion relations between the classes $\mathfrak{U}_\sigma(D,E)$ and $\widehat{\mathcal{M}}(\eta,\mu)$. We also go over the geometric characteristics of an integral operator that has to do with the Touchard polynomial. Some particular instances of our primary discoveries are also remarked upon and discussed.

An analysis of Touchard polynomial theory is conducted within the context of operational procedures. Given their explicit relations, integral representations, and summation equations, the interest in such polynomials is highlighted. It is possible to expand this research and establish new relations for generalized polynomials by utilizing the other special families of polynomials.

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