Mathematica Bohemica

Tülay Yağmur

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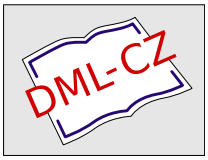
Mathematica Bohemica, Vol. 151 (2026), No. 4, 521–535

Persistent URL: http://dml.cz/dmlcz/153764

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ON DUAL AND HYPER-DUAL k-LEONARDO NUMBERS

Tülay Yağmur

Received March 22, 2025. Published online October 2, 2025.
Communicated by Clemens Fuchs

Abstract. We describe new classes of dual and hyper-dual numbers whose coefficients consist of the k-Leonardo numbers. We will call these numbers dual k-Leonardo numbers and hyper-dual k-Leonardo numbers, respectively. Then we examine several fundamental properties associated with the dual and hyper-dual k-Leonardo numbers, including recurrence relations, Binet-like formula, ordinary generating function, exponential generating function, and some identities such as Catalan and d'Ocagne.

Keywords: dual numbers; hyper-dual numbers; Leonardo numbers; k-Leonardo numbers

MSC 2020: 11B37, 11B39, 11B83, 11R52

1. Introduction

The sequence of Fibonacci numbers is the most famous of all integer sequences, which can be found on The Online Encyclopedia of Integer Sequences (OEIS) [13], identified as $A000045$. This sequence has been examined and studied in detail in several studies, for example, [8], [10], [18]. The sequence of Leonardo numbers, which can be found on the OEIS [13], is identified as $A001595$. The Leonardo sequence has a relationship with the Fibonacci sequence because its recurrence relation is a nonhomogeneous extension of the recurrence relation of the Fibonacci sequence. This makes the Leonardo sequence more intriguing to researchers. For example, in [3], Catarino and Borges studied the Leonardo sequence and obtained some significant formulas for this sequence.

The Leonardo sequence, denoted by $\{Le_n\}$, is defined for all $n \geqslant 2$ by the nonhomogeneous recurrence relation

$$Le_n = Le_{n-1} + Le_{n-2} + 1$$

DOI: 10.21136/MB.2025.0036-25

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with $Le_0 = Le_1 = 1$. Moreover, for $n \geqslant 2$, the homogeneous recurrence relation of the Leonardo sequence is given by

$$Le_{n+1} = 2Le_n - Le_{n-2}$$

with $Le_0 = Le_1 = 1$ (see [3]).

Recently, some researchers have studied the distinctive characteristics of some special sets related to the Leonardo numbers, including dual numbers, Gaussian numbers, hybrid numbers, hyper-dual numbers, and quaternions. Some examples of these studies can be found in [1], [2], [9], [11], [12], [14], [15], [16], [17], [19], [20]. In particular, Karataş defined the dual Leonardo numbers in [9]. For $n \geqslant 0$, the nth dual Leonardo number, denoted by DLe_n, is defined as

$$DLe_n = Le_n + \varepsilon Le_{n+1},$$

where ε is the dual unit. Furthermore, as an extension of dual Leonardo numbers, Özkaldı Karakuş et al. introduced the hyper-dual Leonardo numbers in [12]. For $n \geqslant 0$, the nth hyper-dual Leonardo number, denoted by $HDLe_n$, is defined as

$$HDLe_n = Le_n + \varepsilon Le_{n+1} + \varepsilon^* Le_{n+2} + \varepsilon\varepsilon^* Le_{n+3},$$

where ε and ε^* are the dual units.

Here, it is useful to briefly remind the concepts of dual and hyper-dual numbers. The dual numbers were introduced by Clifford [4]. Any dual number $\tilde{d}$ is defined as

$$\tilde{d} = d_0 + \varepsilon d_1,$$

where d_0 and d_1 are real numbers, and ε is the dual unit such that $\varepsilon^2 = 0$ and $\varepsilon \neq 0$. The hyper-dual numbers were introduced by Fike and Alonso [5], [6]. Any hyper-dual number $\widetilde{hd}$ is defined as

$$\widetilde{hd} = d_0 + \varepsilon d_1 + \varepsilon^* d_2 + \varepsilon\varepsilon^* d_3,$$

where d_0, d_1, d_2, and d_3 are real numbers, and ε and ε^* are the dual units that satisfy the rules

$$(1.1) \quad \varepsilon^2 = \varepsilon^{*2} = (\varepsilon\varepsilon^*)^2 = 0, \quad \varepsilon \neq \varepsilon^*, \quad \varepsilon\varepsilon^* = \varepsilon^*\varepsilon, \quad \varepsilon \neq 0, \quad \varepsilon^* \neq 0, \quad \varepsilon\varepsilon^* \neq 0.$$

In this paper, we consider the generalization of the Leonardo sequence called the k-Leonardo sequence. This newly generalized Leonardo sequence is defined by

Gökbaş in [7] as follows: Let k be a positive real number. The k-Leonardo numbers are defined by

$$Le_{k,n} = kLe_{k,n-1} + Le_{k,n-2} + 1, \quad n \geqslant 2 \tag{1.2}$$

with $Le_{k,0} = Le_{k,1} = 1$.

Moreover, the author of [7] gave the recurrence formula as

$$Le_{k,n+1} = (k+1)Le_{k,n} + (1-k)Le_{k,n-1} - Le_{k,n-2}, \quad n \geqslant 2$$

with $Le_{k,0} = Le_{k,1} = 1$ and $Le_{k,2} = k+2$, and the Binet formula of the k-Leonardo sequence as

$$Le_{k,n} = \alpha {t_1}^n + \beta {t_2}^n + \gamma {t_3}^n, \quad n \geqslant 0, \tag{1.3}$$

where

$$\alpha = \frac{(k^2+1)(k^2+4-(k-2)\sqrt{k^2+4})}{2k(k^2+4)}, \tag{1.4}$$

$$\beta = \frac{(k^2+1)(k^2+4+(k-2)\sqrt{k^2+4})}{2k(k^2+4)}, \quad \gamma = -\frac{1}{k},$$

$$t_1 = \frac{k+\sqrt{k^2+4}}{2}, \quad t_2 = \frac{k-\sqrt{k^2+4}}{2}, \quad t_3 = 1. \tag{1.5}$$

The main objective of this paper is to describe the dual k-Leonardo numbers, and hyper-dual k-Leonardo numbers as an extension of the dual k-Leonardo numbers. The dual k-Leonardo numbers is a generalization of the previously introduced dual Leonardo numbers, and the hyper-dual k-Leonardo numbers is a generalization of the previously introduced hyper-dual Leonardo numbers. Furthermore, we aim to investigate some fundamental properties of these numbers.

2. The dual k-Leonardo numbers

In this section, firstly, we give the definition of the general term of the dual k-Leonardo numbers. Then we present some formulas and identities for these numbers.

Definition 2.1. Let k be any positive real number. For a nonnegative integer n, the nth dual k-Leonardo number, denoted by $\widetilde{Le_{k,n}}$, is defined as

$$\widetilde{Le_{k,n}} = Le_{k,n} + \varepsilon Le_{k,n+1}, \tag{2.1}$$

where $Le_{k,n}$ is the nth k-Leonardo number, and ε is the dual unit.

R e m a r k 2.1. If we put $k = 1$ in (2.1), we get the nth dual Leonardo number in [9].

Theorem 2.1. *Let $n \geqslant 2$ be an integer. Then we have*

$$\widetilde{Le_{k,n}} = k\widetilde{Le_{k,n-1}} + \widetilde{Le_{k,n-2}} + (1+\varepsilon) \tag{2.2}$$

with $\widetilde{Le_{k,0}} = 1+\varepsilon$ and $\widetilde{Le_{k,1}} = 1+\varepsilon(k+2)$.

P r o o f. If $n = 2$, then we get

$$\begin{aligned} k\widetilde{Le_{k,1}} + \widetilde{Le_{k,0}} + (1+\varepsilon) &= k(1+\varepsilon(k+2)) + 2(1+\varepsilon) \\ &= (k+2) + \varepsilon(k^2+2k+2) \\ &= Le_{k,2} + \varepsilon Le_{k,3} \\ &= \widetilde{Le_{k,2}}, \end{aligned}$$

by virtue of (2.1).

If $n \geqslant 3$, then from (1.2) and (2.1) we get

$$\begin{aligned} \widetilde{Le_{k,n}} &= Le_{k,n} + \varepsilon Le_{k,n+1} \\ &= kLe_{k,n-1} + Le_{k,n-2} + 1 + \varepsilon(kLe_{k,n} + Le_{k,n-1} + 1) \\ &= k(Le_{k,n-1} + \varepsilon Le_{k,n}) + (Le_{k,n-2} + \varepsilon Le_{k,n-1}) + (1+\varepsilon) \\ &= k\widetilde{Le_{k,n-1}} + \widetilde{Le_{k,n-2}} + (1+\varepsilon). \end{aligned}$$

□

On the other hand, from the recurrence relation in (2.2), we can derive the following recurrence relation for the dual k-Leonardo numbers:

$$\widetilde{Le_{k,n+1}} = (k+1)\widetilde{Le_{k,n}} + (1-k)\widetilde{Le_{k,n-1}} - \widetilde{Le_{k,n-2}}, \quad n \geqslant 2. \tag{2.3}$$

Throughout this section, let ${t_1}^* := 1+\varepsilon t_1$, ${t_2}^* := 1+\varepsilon t_2$, and ${t_3}^* := 1+\varepsilon t_3$, where t_1, t_2, and t_3 are defined as in (1.5).

Theorem 2.2. *For $n \geqslant 0$, the Binet-like formula of the dual k-Leonardo numbers is*

$$\widetilde{Le_{k,n}} = \alpha {t_1}^* {t_1}^n + \beta {t_2}^* {t_2}^n + \gamma {t_3}^* {t_3}^n, \tag{2.4}$$

where α, β, γ, t_1, t_2, and t_3 are defined as in (1.4) and (1.5).

P r o o f. The proof is clear from (1.3) and (2.1). □

Theorem 2.3. *The ordinary generating function of the dual k-Leonardo numbers is*

$$f(x) = \sum_{n=0}^{\infty} \widetilde{Le_{k,n}} x^n = \frac{1 + \varepsilon - (k-\varepsilon)x + (k-\varepsilon)x^2}{1 - (k+1)x - (1-k)x^2 + x^3}.$$

P r o o f. From (2.3) we obtain

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \widetilde{Le_{k,n}} x^n = \widetilde{Le_{k,0}} + \widetilde{Le_{k,1}} x + \widetilde{Le_{k,2}} x^2 + \sum_{n=3}^{\infty} \widetilde{Le_{k,n}} x^n \\ &= \widetilde{Le_{k,0}} + \widetilde{Le_{k,1}} x + \widetilde{Le_{k,2}} x^2 + \sum_{n=3}^{\infty} ((k+1)\widetilde{Le_{k,n-1}} \\ &\quad + (1-k)\widetilde{Le_{k,n-2}} - \widetilde{Le_{k,n-3}}) x^n \\ &= \widetilde{Le_{k,0}} + \widetilde{Le_{k,1}} x + \widetilde{Le_{k,2}} x^2 + (k+1)x \sum_{n=3}^{\infty} \widetilde{Le_{k,n-1}} x^{n-1} \\ &\quad + (1-k)x^2 \sum_{n=3}^{\infty} \widetilde{Le_{k,n-2}} x^{n-2} - x^3 \sum_{n=3}^{\infty} \widetilde{Le_{k,n-3}} x^{n-3} \\ &= \widetilde{Le_{k,0}} + \widetilde{Le_{k,1}} x + \widetilde{Le_{k,2}} x^2 - (k+1)x(\widetilde{Le_{k,1}} x + \widetilde{Le_{k,0}}) - (1-k)x^2 \widetilde{Le_{k,0}} \\ &\quad + (k+1)xf(x) + (1-k)x^2 f(x) - x^3 f(x). \end{aligned}$$

Then we find

$$\begin{aligned} f(x)(1 - (k+1)x - (1-k)x^2 + x^3) &= \widetilde{Le_{k,0}} + (\widetilde{Le_{k,1}} - (k+1)\widetilde{Le_{k,0}})x \\ &\quad + (\widetilde{Le_{k,2}} - (k+1)\widetilde{Le_{k,1}} - (1-k)\widetilde{Le_{k,0}})x^2 \\ &= 1 + \varepsilon + (-k+\varepsilon)x + (k-\varepsilon)x^2, \end{aligned}$$

by virtue of $\widetilde{Le_{k,0}} = 1 + \varepsilon$, $\widetilde{Le_{k,1}} = 1 + \varepsilon(k+2)$, and $\widetilde{Le_{k,2}} = k + 2 + \varepsilon(k^2 + 2k + 2)$. Thus, we get

$$f(x) = \frac{1 + \varepsilon - (k-\varepsilon)x + (k-\varepsilon)x^2}{1 - (k+1)x - (1-k)x^2 + x^3}.$$

□

Theorem 2.4. *The exponential generating function of the dual k-Leonardo numbers is*

$$\sum_{n=0}^{\infty} \widetilde{Le_{k,n}} \frac{x^n}{n!} = \alpha {t_1}^* \mathrm{e}^{t_1 x} + \beta {t_2}^* \mathrm{e}^{t_2 x} + \gamma {t_3}^* \mathrm{e}^{t_3 x}.$$

P r o o f. Using (2.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{Le_{k,n}}\frac{x^n}{n!} &= \sum_{n=0}^{\infty}(\alpha {t_1}^* {t_1}^n + \beta {t_2}^* {t_2}^n + \gamma {t_3}^* {t_3}^n)\frac{x^n}{n!}\\ &= \alpha {t_1}^* \sum_{n=0}^{\infty}\frac{(t_1x)^n}{n!} + \beta {t_2}^* \sum_{n=0}^{\infty}\frac{(t_2x)^n}{n!} + \gamma {t_3}^* \sum_{n=0}^{\infty}\frac{(t_3x)^n}{n!}\\ &= \alpha {t_1}^* \mathrm{e}^{t_1x} + \beta {t_2}^* \mathrm{e}^{t_2x} + \gamma {t_3}^* \mathrm{e}^{t_3x}.\end{aligned}$$

□

The following result follows from Theorem 2.4.

Corollary 2.1. *The Poisson generating function of the dual k-Leonardo numbers is*

$$\sum_{n=0}^{\infty}\widetilde{Le_{k,n}}\frac{x^n}{n!}\mathrm{e}^{-x} = \alpha {t_1}^* \mathrm{e}^{(t_1-1)x} + \beta {t_2}^* \mathrm{e}^{(t_2-1)x} + \gamma {t_3}^* \mathrm{e}^{(t_3-1)x}.$$

From now on, throughout this paper, let $\Psi_1 := (k^2+1)/\sqrt{k^2+4}$, $\Psi_2 := (k-2) - \sqrt{k^2+4}$, and $\Psi_3 := \sqrt{k^2+4} + (k-2)$.

Theorem 2.5 (Catalan's Identity)**.** *For nonnegative integers n and r such that $r \leqslant n$, we have*

$$\begin{aligned}&\widetilde{Le_{k,n-r}}\widetilde{Le_{k,n+r}} - (\widetilde{Le_{k,n}})^2\\ &\quad= \frac{{\Psi_1}^2}{k}(1+\varepsilon k)(-1)^{n-r}({t_1}^r - {t_2}^r)^2 + \frac{\Psi_1\Psi_2}{4k^2}(2+\varepsilon(\Psi_3+4)){t_1}^{n-r}({t_1}^r - {t_3}^r)^2\\ &\quad\quad - \frac{\Psi_1\Psi_3}{4k^2}(2+\varepsilon(\Psi_2+4)){t_2}^{n-r}({t_2}^r - {t_3}^r)^2.\end{aligned}$$

P r o o f. By virtue of (2.4), we get

$$\begin{aligned}&\widetilde{Le_{k,n-r}}\widetilde{Le_{k,n+r}} - (\widetilde{Le_{k,n}})^2\\ &\quad= (\alpha {t_1}^* {t_1}^{n-r} + \beta {t_2}^* {t_2}^{n-r} + \gamma {t_3}^* {t_3}^{n-r})(\alpha {t_1}^* {t_1}^{n+r} + \beta {t_2}^* {t_2}^{n+r} + \gamma {t_3}^* {t_3}^{n+r})\\ &\quad\quad - (\alpha {t_1}^* {t_1}^n + \beta {t_2}^* {t_2}^n + \gamma {t_3}^* {t_3}^n)^2\\ &\quad= \alpha\beta {t_1}^* {t_2}^* (t_1t_2)^{n-r}({t_1}^r - {t_2}^r)^2 + \alpha\gamma {t_1}^* {t_3}^* (t_1t_3)^{n-r}({t_1}^r - {t_3}^r)^2\\ &\quad\quad + \beta\gamma {t_2}^* {t_3}^* (t_2t_3)^{n-r}({t_2}^r - {t_3}^r)^2.\end{aligned}$$

Since α, β, γ, t_1, t_2, and t_3 are defined as in (1.4) and (1.5), we can obtain the following:

$$
\begin{aligned}
(2.5)\qquad & \alpha\beta = \frac{(k^2+1)^2}{k(k^2+4)}, \quad \alpha\gamma = -\frac{(k^2+1)(k^2+4-(k-2)\sqrt{k^2+4})}{2k^2(k^2+4)}, \\
& \beta\gamma = -\frac{(k^2+1)(k^2+4+(k-2)\sqrt{k^2+4})}{2k^2(k^2+4)}, \\
& t_1t_2 = -1, \quad t_1t_3 = t_1, \quad t_2t_3 = t_2.
\end{aligned}
$$

Moreover, from ${t_1}^*$, ${t_2}^*$, and ${t_3}^*$, the following holds:

$$
\begin{aligned}
(2.6)\qquad & {t_1}^*{t_2}^* = 1+\varepsilon k, \quad {t_1}^*{t_3}^* = 1+\varepsilon\left(\frac{(k+2)+\sqrt{k^2+4}}{2}\right), \\
& {t_2}^*{t_3}^* = 1+\varepsilon\left(\frac{(k+2)-\sqrt{k^2+4}}{2}\right).
\end{aligned}
$$

Using the equalities in (2.5) and (2.6), we have

$$
\begin{aligned}
&\widetilde{Le_{k,n-r}}\widetilde{Le_{k,n+r}} - (\widetilde{Le_{k,n}})^2 \\
&\quad = \frac{(k^2+1)^2}{k(k^2+4)}(1+\varepsilon k)(-1)^{n-r}({t_1}^r - {t_2}^r)^2 \\
&\qquad - \frac{(k^2+1)(k^2+4-(k-2)\sqrt{k^2+4})}{2k^2(k^2+4)} \\
&\qquad \times \left(1+\varepsilon\left(\frac{(k+2)+\sqrt{k^2+4}}{2}\right)\right){t_1}^{n-r}({t_1}^r - {t_3}^r)^2 \\
&\qquad - \frac{(k^2+1)(k^2+4+(k-2)\sqrt{k^2+4})}{2k^2(k^2+4)} \\
&\qquad \times \left(1+\varepsilon\left(\frac{(k+2)-\sqrt{k^2+4}}{2}\right)\right){t_2}^{n-r}({t_2}^r - {t_3}^r)^2 \\
&\quad = \frac{(k^2+1)^2}{k(k^2+4)}(1+\varepsilon k)(-1)^{n-r}({t_1}^r - {t_2}^r)^2 \\
&\qquad - \frac{(k^2+1)(\sqrt{k^2+4}-(k-2))}{4k^2\sqrt{k^2+4}}(2+\varepsilon((k+2)+\sqrt{k^2+4})){t_1}^{n-r}({t_1}^r - {t_3}^r)^2 \\
&\qquad - \frac{(k^2+1)(\sqrt{k^2+4}+(k-2))}{4k^2\sqrt{k^2+4}}(2+\varepsilon((k+2)-\sqrt{k^2+4})){t_2}^{n-r}({t_2}^r - {t_3}^r)^2 \\
&\quad = \frac{{\Psi_1}^2}{k}(1+\varepsilon k)(-1)^{n-r}({t_1}^r - {t_2}^r)^2 + \frac{\Psi_1\Psi_2}{4k^2}(2+\varepsilon(\Psi_3+4)){t_1}^{n-r}({t_1}^r - {t_3}^r)^2 \\
&\qquad - \frac{\Psi_1\Psi_3}{4k^2}(2+\varepsilon(\Psi_2+4)){t_2}^{n-r}({t_2}^r - {t_3}^r)^2.
\end{aligned}
$$

□

Corollary 2.2 (Cassini's Identity). *For $n > 0$ we have*

$$\widetilde{Le_{k,n-1}}\widetilde{Le_{k,n+1}} - (\widetilde{Le_{k,n}})^2 = \frac{\Psi_1}{k}\bigg(\Psi_1(1+\varepsilon k)(-1)^{n-1}(k^2+4) - \frac{\Psi_3}{4}(2+\varepsilon(\Psi_3+4)){t_1}^{n-1} + \frac{\Psi_2}{4}(2+\varepsilon(\Psi_2+4)){t_2}^{n-1}\bigg).$$

P r o o f. Taking $r = 1$ in Theorem 2.5, we obtain

$$\widetilde{Le_{k,n-1}}\widetilde{Le_{k,n+1}} - (\widetilde{Le_{k,n}})^2 = \frac{{\Psi_1}^2}{k}(1+\varepsilon k)(-1)^{n-1}(t_1-t_2)^2 + \frac{\Psi_1\Psi_2}{4k^2}(2+\varepsilon(\Psi_3+4)){t_1}^{n-1}(t_1-t_3)^2 - \frac{\Psi_1\Psi_3}{4k^2}(2+\varepsilon(\Psi_2+4)){t_2}^{n-1}(t_2-t_3)^2.$$

From t_1, t_2, and t_3, we find

$$t_1 - t_2 = \sqrt{k^2+4}, \quad t_1 - t_3 = \frac{(k-2)+\sqrt{k^2+4}}{2}, \quad t_2 - t_3 = \frac{(k-2)-\sqrt{k^2+4}}{2}. \tag{2.7}$$

Using (2.7) and after some necessary calculations, the desired result can be obtained. □

Theorem 2.6 (d'Ocagne's Identity). *For nonnegative integers m and n such that $n \leqslant m$, we have*

$$\widetilde{Le_{k,n+1}}\widetilde{Le_{k,m}} - \widetilde{Le_{k,n}}\widetilde{Le_{k,m+1}} = \frac{{\Psi_1}^2}{k}(1+\varepsilon k)(-1)^{n+1}\sqrt{k^2+4}({t_1}^{m-n} - {t_2}^{m-n}) + \frac{\Psi_1}{2k}(2+\varepsilon(\Psi_3+4)){t_1}^n({t_1}^{m-n} - {t_3}^{m-n}) - \frac{\Psi_1}{2k}(2+\varepsilon(\Psi_2+4)){t_2}^n({t_2}^{m-n} - {t_3}^{m-n}).$$

P r o o f. From (2.4), we get

$$\begin{aligned}
&\widetilde{Le_{k,n+1}}\widetilde{Le_{k,m}} - \widetilde{Le_{k,n}}\widetilde{Le_{k,m+1}} \\
&\quad = (\alpha {t_1}^* {t_1}^{n+1} + \beta {t_2}^* {t_2}^{n+1} + \gamma {t_3}^* {t_3}^{n+1})(\alpha {t_1}^* {t_1}^m + \beta {t_2}^* {t_2}^m + \gamma {t_3}^* {t_3}^m) \\
&\qquad - (\alpha {t_1}^* {t_1}^n + \beta {t_2}^* {t_2}^n + \gamma {t_3}^* {t_3}^n)(\alpha {t_1}^* {t_1}^{m+1} + \beta {t_2}^* {t_2}^{m+1} + \gamma {t_3}^* {t_3}^{m+1}) \\
&\quad = \alpha\beta {t_1}^* {t_2}^* (t_1 t_2)^n (t_1 - t_2)({t_2}^{m-n} - {t_1}^{m-n}) \\
&\qquad + \alpha\gamma {t_1}^* {t_3}^* (t_1 t_3)^n (t_1 - t_3)({t_3}^{m-n} - {t_1}^{m-n}) \\
&\qquad + \beta\gamma {t_2}^* {t_3}^* (t_2 t_3)^n (t_2 - t_3)({t_3}^{m-n} - {t_2}^{m-n}).
\end{aligned}$$

Using (2.5), (2.6), and (2.7), we have

$$\begin{aligned}
&\widetilde{Le_{k,n+1}}\widetilde{Le_{k,m}} - \widetilde{Le_{k,n}}\widetilde{Le_{k,m+1}} \\
&\quad = \frac{(k^2+1)^2}{k(k^2+4)}(1+\varepsilon k)(-1)^{n+1}\sqrt{k^2+4}({t_1}^{m-n} - {t_2}^{m-n}) \\
&\qquad + \frac{(k^2+1)}{2k\sqrt{k^2+4}}(2+\varepsilon((k+2)+\sqrt{k^2+4})){t_1}^n({t_1}^{m-n} - {t_3}^{m-n}) \\
&\qquad - \frac{(k^2+1)}{2k\sqrt{k^2+4}}(2+\varepsilon((k+2)-\sqrt{k^2+4})){t_2}^n({t_2}^{m-n} - {t_3}^{m-n}) \\
&\quad = \frac{{\Psi_1}^2}{k}(1+\varepsilon k)(-1)^{n+1}\sqrt{k^2+4}({t_1}^{m-n} - {t_2}^{m-n}) \\
&\qquad + \frac{\Psi_1}{2k}(2+\varepsilon(\Psi_3+4)){t_1}^n({t_1}^{m-n} - {t_3}^{m-n}) \\
&\qquad - \frac{\Psi_1}{2k}(2+\varepsilon(\Psi_2+4)){t_2}^n({t_2}^{m-n} - {t_3}^{m-n}).
\end{aligned}$$

□

3. The hyper-dual k-Leonardo numbers

In this section, we define and study the hyper-dual k-Leonardo numbers.

Definition 3.1. Let k be any positive real number. For a nonnegative integer n, the nth hyper-dual k-Leonardo number, denoted by $\widetilde{HLe_{k,n}}$, is defined as

$$\widetilde{HLe_{k,n}} = Le_{k,n} + \varepsilon Le_{k,n+1} + \varepsilon^* Le_{k,n+2} + \varepsilon\varepsilon^* Le_{k,n+3}, \tag{3.1}$$

where $Le_{k,n}$ is the nth k-Leonardo number, and ε and ε^* are the dual units in (1.1).

Remark 3.1. If we put $k = 1$ in (3.1), we get the nth hyper-dual Leonardo number in [12].

Theorem 3.1. *Let $n \geqslant 2$ be an integer. Then we have*

$$\widetilde{HLe_{k,n}} = k\widetilde{HLe_{k,n-1}} + \widetilde{HLe_{k,n-2}} + (1+\varepsilon+\varepsilon^*+\varepsilon\varepsilon^*) \tag{3.2}$$

with $\widetilde{HLe_{k,0}} = 1+\varepsilon+(k+2)\varepsilon^*+(k^2+2k+2)\varepsilon\varepsilon^*$ and $\widetilde{HLe_{k,1}} = 1+(k+2)\varepsilon+(k^2+2k+2)\varepsilon^*+(k^3+2k^2+3k+3)\varepsilon\varepsilon^*$.

Proof. The proof is very similar to that of Theorem 2.1, so we omit it. □

From the recurrence relation in (3.2), the following recurrence relation for the hyper-dual k-Leonardo numbers can be deduced:

$$\widetilde{HLe_{k,n+1}} = (k+1)\widetilde{HLe_{k,n}} + (1-k)\widetilde{HLe_{k,n-1}} - \widetilde{HLe_{k,n-2}}, \quad n \geqslant 2. \tag{3.3}$$

Proposition 3.1. *Let $\widetilde{HLe_{k,n}}$ be the nth hyper-dual k-Leonardo number. Then*

$$\widetilde{HLe_{k,n}} - \varepsilon HLe_{k,n+1} - \varepsilon^* HLe_{k,n+2} - \varepsilon\varepsilon^* HLe_{k,n+3} = Le_{k,n} - 2\varepsilon\varepsilon^* Le_{k,n+3},$$
$$\widetilde{HLe_{k,n}} - \varepsilon HLe_{k,n+1} + \varepsilon^* HLe_{k,n+2} - \varepsilon\varepsilon^* HLe_{k,n+3} = Le_{k,n} + 2\varepsilon^* Le_{k,n+2}.$$

P r o o f. From (1.1) and (3.1), the proof is straightforward. □

Throughout the rest of this paper, let $\bar{t}_1 := 1 + t_1\varepsilon + {t_1}^2\varepsilon^* + {t_1}^3\varepsilon\varepsilon^*$, $\bar{t}_2 := 1 + t_2\varepsilon + {t_2}^2\varepsilon^* + {t_2}^3\varepsilon\varepsilon^*$, and $\bar{t}_3 := 1 + t_3\varepsilon + {t_3}^2\varepsilon^* + {t_3}^3\varepsilon\varepsilon^*$, where t_1, t_2, and t_3 are defined as in (1.5).

Theorem 3.2. *For a nonnegative integer n, the Binet-like formula of the hyper-dual k-Leonardo numbers is*

$$\widetilde{HLe_{k,n}} = \alpha\bar{t}_1{t_1}^n + \beta\bar{t}_2{t_2}^n + \gamma\bar{t}_3{t_3}^n, \tag{3.4}$$

where α, β, γ, t_1, t_2, and t_3 are defined as in (1.4) *and* (1.5).

P r o o f. The proof is clear from (1.3) and (3.1). □

Theorem 3.3. *The ordinary generating function of the hyper-dual k-Leonardo numbers is*

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} \widetilde{HLe_{k,n}} x^n \\ &= \frac{\widetilde{HLe_{k,0}} + (\widetilde{HLe_{k,1}} - (k+1)\widetilde{HLe_{k,0}})x}{1 - (k+1)x - (1-k)x^2 + x^3} \\ &\quad + \frac{(\widetilde{HLe_{k,2}} - (k+1)\widetilde{HLe_{k,1}} - (1-k)\widetilde{HLe_{k,0}})x^2}{1 - (k+1)x - (1-k)x^2 + x^3}. \end{aligned}$$

P r o o f. Let

$$g(x) = \widetilde{HLe_{k,0}} + \widetilde{HLe_{k,1}}x + \widetilde{HLe_{k,2}}x^2 + \ldots + \widetilde{HLe_{k,n}}x^n + \ldots \tag{3.5}$$

Then multiplying (3.5) by $-(k+1)x$, $-(1-k)x^2$, and x^3, respectively, we obtain

$$\begin{aligned}(3.6)\quad -(k+1)xg(x) = &-(k+1)\widetilde{HLe_{k,0}}x-(k+1)\widetilde{HLe_{k,1}}x^2\\ &-(k+1)\widetilde{HLe_{k,2}}x^3-\ldots-(k+1)\widetilde{HLe_{k,n-1}}x^n-\ldots,\\ (3.7)\quad -(1-k)x^2g(x) = &-(1-k)\widetilde{HLe_{k,0}}x^2-(1-k)\widetilde{HLe_{k,1}}x^3\\ &-(1-k)\widetilde{HLe_{k,2}}x^4-\ldots-(1-k)\widetilde{HLe_{k,n-2}}x^n-\ldots,\end{aligned}$$

and

$$(3.8)\qquad x^3g(x)=\widetilde{HLe_{k,0}}x^3+\widetilde{HLe_{k,1}}x^4+\widetilde{HLe_{k,2}}x^5+\ldots+\widetilde{HLe_{k,n-3}}x^n+\ldots$$

From (3.5), (3.6), (3.7), and (3.8), we get

$$\begin{aligned}&(1-(k+1)x-(1-k)x^2+x^3)g(x)\\ &\quad=\widetilde{HLe_{k,0}}+(\widetilde{HLe_{k,1}}-(k+1)\widetilde{HLe_{k,0}})x\\ &\qquad+(\widetilde{HLe_{k,2}}-(k+1)\widetilde{HLe_{k,1}}-(1-k)\widetilde{HLe_{k,0}})x^2\\ &\qquad+\sum_{n=3}^{\infty}(\widetilde{HLe_{k,n}}-(k+1)\widetilde{HLe_{k,n-1}}-(1-k)\widetilde{HLe_{k,n-2}}+\widetilde{HLe_{k,n-3}})x^n.\end{aligned}$$

Using the recurrence relation in (3.3), we have

$$\begin{aligned}&(1-(k+1)x-(1-k)x^2+x^3)g(x)\\ &\quad=\widetilde{HLe_{k,0}}+(\widetilde{HLe_{k,1}}-(k+1)\widetilde{HLe_{k,0}})x\\ &\qquad+(\widetilde{HLe_{k,2}}-(k+1)\widetilde{HLe_{k,1}}-(1-k)\widetilde{HLe_{k,0}})x^2.\end{aligned}$$

Then it follows that

$$\begin{aligned}g(x)=&\frac{\widetilde{HLe_{k,0}}+(\widetilde{HLe_{k,1}}-(k+1)\widetilde{HLe_{k,0}})x}{1-(k+1)x-(1-k)x^2+x^3}\\ &+\frac{(\widetilde{HLe_{k,2}}-(k+1)\widetilde{HLe_{k,1}}-(1-k)\widetilde{HLe_{k,0}})x^2}{1-(k+1)x-(1-k)x^2+x^3}.\end{aligned}$$

□

Theorem 3.4. *The exponential generating function of the hyper-dual k-Leonardo numbers is*

$$\sum_{n=0}^{\infty}\widetilde{HLe_{k,n}}\frac{x^n}{n!}=\alpha\bar{t}_1\mathrm{e}^{t_1x}+\beta\bar{t}_2\mathrm{e}^{t_2x}+\gamma\bar{t}_3\mathrm{e}^{t_3x}.$$

P r o o f. Using (3.4), the proof can be done in a similar manner to Theorem 2.4. □

The consequence of Theorem 3.4 is given in the following.

Corollary 3.1. *The Poisson generating function of the hyper-dual k-Leonardo numbers is*

$$\sum_{n=0}^{\infty} \widetilde{HLe_{k,n}} \frac{x^n}{n!} \mathrm{e}^{-x} = \alpha \bar{t}_1 \mathrm{e}^{(t_1-1)x} + \beta \bar{t}_2 \mathrm{e}^{(t_2-1)x} + \gamma \bar{t}_3 \mathrm{e}^{(t_3-1)x}.$$

Considering the multiplication rules of the dual units in (1.1), the next results can be derived.

Lemma 3.1. *For $\bar{t}_1$, $\bar{t}_2$, and $\bar{t}_3$, the following identities are true:*

$$\begin{aligned}
\bar{t}_1\bar{t}_2 &= (1+k\varepsilon)(1+(k^2+2)\varepsilon^*),\\
\bar{t}_1\bar{t}_3 &= \frac{(2+(\Psi_3+4)\varepsilon)(2+\sqrt{k^2+4}(\Psi_3+2)\varepsilon^*)}{4},\\
\bar{t}_2\bar{t}_3 &= \frac{(2+(\Psi_2+4)\varepsilon)(2-\sqrt{k^2+4}(\Psi_2+2)\varepsilon^*)}{4}.
\end{aligned}$$

Theorem 3.5 (Catalan's Identity)**.** *For nonnegative integers n and r such that $r \leqslant n$, we have*

$$\begin{aligned}
&\widetilde{HLe_{k,n-r}}\widetilde{HLe_{k,n+r}} - (\widetilde{HLe_{k,n}})^2\\
&\quad= \frac{{\Psi_1}^2}{k}(1+k\varepsilon)(1+(k^2+2)\varepsilon^*)(-1)^{n-r}({t_1}^r - {t_2}^r)^2\\
&\qquad+ \frac{\Psi_1\Psi_2}{8k^2}(2+(\Psi_3+4)\varepsilon)(2+\sqrt{k^2+4}(\Psi_3+2)\varepsilon^*){t_1}^{n-r}({t_1}^r - {t_3}^r)^2\\
&\qquad- \frac{\Psi_1\Psi_3}{8k^2}(2+(\Psi_2+4)\varepsilon)(2-\sqrt{k^2+4}(\Psi_2+2)\varepsilon^*){t_2}^{n-r}({t_2}^r - {t_3}^r)^2.
\end{aligned}$$

P r o o f. From (3.4), we find

$$\begin{aligned}
&\widetilde{HLe_{k,n-r}}\widetilde{HLe_{k,n+r}} - (\widetilde{HLe_{k,n}})^2\\
&\quad= (\alpha\bar{t}_1{t_1}^{n-r} + \beta\bar{t}_2{t_2}^{n-r} + \gamma\bar{t}_3{t_3}^{n-r})(\alpha\bar{t}_1{t_1}^{n+r} + \beta\bar{t}_2{t_2}^{n+r} + \gamma\bar{t}_3{t_3}^{n+r})\\
&\qquad- (\alpha\bar{t}_1{t_1}^{n} + \beta\bar{t}_2{t_2}^{n} + \gamma\bar{t}_3{t_3}^{n})^2\\
&\quad= \alpha\beta\bar{t}_1\bar{t}_2(t_1t_2)^{n-r}({t_1}^r - {t_2}^r)^2 + \alpha\gamma\bar{t}_1\bar{t}_3(t_1t_3)^{n-r}({t_1}^r - {t_3}^r)^2\\
&\qquad+ \beta\gamma\bar{t}_2\bar{t}_3(t_2t_3)^{n-r}({t_2}^r - {t_3}^r)^2.
\end{aligned}$$

By (2.5) and Lemma 3.1, we get

$$\begin{aligned}
&\widetilde{HLe_{k,n-r}}\widetilde{HLe_{k,n+r}} - (\widetilde{HLe_{k,n}})^2 \\
&\quad = \frac{(k^2+1)^2}{k(k^2+4)}(1+k\varepsilon)(1+(k^2+2)\varepsilon^*)(-1)^{n-r}({t_1}^r - {t_2}^r)^2 \\
&\qquad - \frac{(k^2+1)(\sqrt{k^2+4}-(k-2))(2+(\Psi_3+4)\varepsilon)(2+\sqrt{k^2+4}(\Psi_3+2)\varepsilon^*)}{8k^2\sqrt{k^2+4}} \\
&\qquad \times {t_1}^{n-r}({t_1}^r - {t_3}^r)^2 \\
&\qquad - \frac{(k^2+1)(\sqrt{k^2+4}+(k-2))(2+(\Psi_2+4)\varepsilon)(2-\sqrt{k^2+4}(\Psi_2+2)\varepsilon^*)}{8k^2\sqrt{k^2+4}} \\
&\qquad \times {t_2}^{n-r}({t_2}^r - {t_3}^r)^2 \\
&\quad = \frac{{\Psi_1}^2}{k}(1+k\varepsilon)(1+(k^2+2)\varepsilon^*)(-1)^{n-r}({t_1}^r - {t_2}^r)^2 \\
&\qquad + \frac{\Psi_1\Psi_2}{8k^2}(2+(\Psi_3+4)\varepsilon)(2+\sqrt{k^2+4}(\Psi_3+2)\varepsilon^*){t_1}^{n-r}({t_1}^r - {t_3}^r)^2 \\
&\qquad - \frac{\Psi_1\Psi_3}{8k^2}(2+(\Psi_2+4)\varepsilon)(2-\sqrt{k^2+4}(\Psi_2+2)\varepsilon^*){t_2}^{n-r}({t_2}^r - {t_3}^r)^2.
\end{aligned}$$

□

Taking $r = 1$ in Theorem 3.5, we have the following:

Corollary 3.2 (Cassini's Identity)**.** *For a positive integer n, we have*

$$\begin{aligned}
&\widetilde{HLe_{k,n-1}}\widetilde{HLe_{k,n+1}} - (\widetilde{HLe_{k,n}})^2 \\
&\quad = \frac{{\Psi_1}^2}{k}(1+k\varepsilon)(1+(k^2+2)\varepsilon^*)(-1)^{n-1}(k^2+4) \\
&\qquad - \frac{\Psi_1\Psi_3}{8k}(2+(\Psi_3+4)\varepsilon)(2+\sqrt{k^2+4}(\Psi_3+2)\varepsilon^*){t_1}^{n-1} \\
&\qquad + \frac{\Psi_1\Psi_2}{8k}(2+(\Psi_2+4)\varepsilon)(2-\sqrt{k^2+4}(\Psi_2+2)\varepsilon^*){t_2}^{n-1}.
\end{aligned}$$

Theorem 3.6 (d'Ocagne's Identity)**.** *For nonnegative integers m and n such that $n \leqslant m$, we have*

$$\begin{aligned}
&\widetilde{HLe_{k,n+1}}\widetilde{HLe_{k,m}} - \widetilde{HLe_{k,n}}\widetilde{HLe_{k,m+1}} \\
&\quad = \frac{{\Psi_1}^2}{k}(1+k\varepsilon)(1+(k^2+2)\varepsilon^*)\sqrt{k^2+4}(-1)^{n+1}({t_1}^{m-n} - {t_2}^{m-n}) \\
&\qquad + \frac{\Psi_1}{4k}(2+(\Psi_3+4)\varepsilon)(2+\sqrt{k^2+4}(\Psi_3+2)\varepsilon^*){t_1}^n({t_1}^{m-n} - {t_3}^{m-n}) \\
&\qquad - \frac{\Psi_1}{4k}(2+(\Psi_2+4)\varepsilon)(2-\sqrt{k^2+4}(\Psi_2+2)\varepsilon^*){t_2}^n({t_2}^{m-n} - {t_3}^{m-n}).
\end{aligned}$$

P r o o f. From (3.4) we get

$$\begin{aligned}
&\widetilde{HLe_{k,n+1}}\widetilde{HLe_{k,m}} - \widetilde{HLe_{k,n}}\widetilde{HLe_{k,m+1}} \\
&\quad = (\alpha\bar{t}_1{t_1}^{n+1} + \beta\bar{t}_2{t_2}^{n+1} + \gamma\bar{t}_3{t_3}^{n+1})(\alpha\bar{t}_1{t_1}^{m} + \beta\bar{t}_2{t_2}^{m} + \gamma\bar{t}_3{t_3}^{m}) \\
&\qquad - (\alpha\bar{t}_1{t_1}^{n} + \beta\bar{t}_2{t_2}^{n} + \gamma\bar{t}_3{t_3}^{n})(\alpha\bar{t}_1{t_1}^{m+1} + \beta\bar{t}_2{t_2}^{m+1} + \gamma\bar{t}_3{t_3}^{m+1}) \\
&\quad = \alpha\beta\bar{t}_1\bar{t}_2(t_1t_2)^n(t_1 - t_2)({t_2}^{m-n} - {t_1}^{m-n}) \\
&\qquad + \alpha\gamma\bar{t}_1\bar{t}_3(t_1t_3)^n(t_1 - t_3)({t_3}^{m-n} - {t_1}^{m-n}) \\
&\qquad + \beta\gamma\bar{t}_2\bar{t}_3(t_2t_3)^n(t_2 - t_3)({t_3}^{m-n} - {t_2}^{m-n}).
\end{aligned}$$

Using (2.5), (2.7), and Lemma 3.1, and after some necessary calculations, the desired result can be obtained. □

4. Conclusion

In the present paper, firstly, the dual k-Leonardo numbers are defined, and several properties involving these numbers are studied. These newly defined dual numbers are the generalized form of the previously introduced dual Leonardo numbers given in [9]. After that, as an extension of the dual k-Leonardo numbers, the hyper-dual k-Leonardo numbers are defined, and many formulas and identities for these numbers are obtained. The hyper-dual k-Leonardo numbers are the generalized form of the previously introduced hyper-dual Leonardo numbers given in [12].

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Author's address: *Tülay Yağmur*, Department of Mathematics, Aksaray University, 68100, Aksaray, Turkey, e-mail: `tulayyagmurr@gmail.com`.