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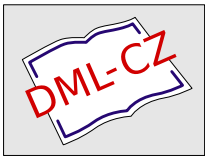
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ORBIT QUOTIENTS IN THE RADICAL RING OF MULTIPLICATIVE ARITHMETIC FUNCTIONS

Emil Daniel Schwab

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Abstract. We extend the role of algebraic structures to the study of multiplicative arithmetic functions by associating the corresponding radical ring and a left-group action with the quasi-field (two-sided brace) of multiplicative arithmetic functions. The orbit quotient of two compatible multiplicative arithmetic functions, defined using the orbit set of these functions, brings a new approach to expressing the fundamental properties of multiplicative arithmetic functions. In the last section, we show that some of the basic results concerning Fibonacci and Lucas numbers can be carried over to the quotient setting. Fibonacci numbers connect better with Dirichlet convolution, and Lucas's with unitary convolution. This fact can be perceived within the language united with that of arithmetic functions. The connections between the two integer sequences are highlighted through the two multiplicative prime-independent arithmetic functions, namely the Fibonacci function and the Lucas function. Orbit quotients of these functions and their right-shifted functions are all expressed only with Liouville's lambda function.

Keywords: multiplicative arithmetic function; two-sided brace; radical ring; group action; Fibonacci numbers; Lucas numbers

MSC 2020: 11A25, 16N20, 11B39

1. Introduction

The set of all elements of an associative ring $(R, +, \cdot)$ is a monoid under the *circle operation* defined by

$$a \circ b = a \cdot b + a + b, \quad a, b \in R.$$

The identity element of this monoid is the zero element 0 of the ring R. The group of all invertible elements of the monoid $(R, \circ)$ is called the *adjoint group* of R and is denoted by R°. An associative ring R without identity is a *radical ring* if $R^\circ = R$. Links between radical rings and their adjoint groups are an interesting topic in the abstract algebra.

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Braces, as a generalization of radical rings, were introduced by Rump [10], Definition 2 to study set-theoretical solutions of the Yang-Baxter equation, one of the fundamental equations in mathematical physics. An equivalent definition of a right brace was given in [2], Definition 2.2, and according to [2], Definition 2.5 a two-sided brace is a set G with two operations $+$ and $\cdot$ such that $(G,+)$ and $(G,\cdot)$ are both groups, the first being abelian, satisfying the "quasi-distributive" laws $(a+b)\cdot c+c = a\cdot c+a\cdot b$ and $a\cdot(b+c)+a = a\cdot b+a\cdot c$, for all $a,b,c\in G$.

In 1963, Kesava Menon [6] defined a nonempty set Q with two operations $+$ and $\cdot$ to be a quasi-field if $(Q,+)$ and $(Q,\cdot)$ are both abelian groups satisfying the quasi-distributive law above. The identity element under the addition is the identity element under the multiplication. An example of a quasi-field is the set of multiplicative arithmetic functions $\mathfrak{M}$ (see [14], p. 53) under the unitary convolution ([14], I.2) as addition and the Dirichlet convolution ([14], I.1) as multiplication, denoted here by $\sqcup$ and $*$, respectively. So, $(\mathfrak{M},\sqcup,*)$ is a two-sided brace which brings to our attention the specific tools used in the theory of braces as a radical ring, the lambda map as a left action ([1], Definition 2.7), linear cycle set ([10], Section 2), etc. We will move in this direction, bringing to the fore (alongside $\sqcup$ and $*$) the radical ring multiplication $\odot$ and a left group action of $(\mathfrak{M},*)$ on the group $(\mathfrak{M},\sqcup)$. A wider area opens up (including orbit quotients of compatible functions) to study the algebraic properties of multiplicative arithmetic functions and their applications.

All the arithmetic functions involved in this paper are considered to be multiplicative. A complex-valued function f defined on the set of positive integers (that is, an arithmetic function) is called multiplicative if $f(1)=1$ and $f(mn)=f(m)f(n)$ whenever $\gcd(m,n)=1$. A multiplicative function f is said to be completely multiplicative if $f(mn)=f(m)f(n)$ for all positive integers m and n. Since a multiplicative function f is uniquely determined by its values on the prime powers (if $p_1,p_2\dots p_r$ are distinct primes, then $f(p_1^{k_1}\dots p_r^{k_r})=\prod_{i=1}^{r} f(p_i^{k_i})$), the unitary convolution $\sqcup$ and the Dirichlet convolution $*$ of two multiplicative arithmetic functions f and g can be expressed as

$$(f\sqcup g)(p^m)=\begin{cases}1 & \text{if } m=0,\\ f(p^m)+g(p^m) & \text{if } m>0,\end{cases}\quad\text{and}\quad (f*g)(p^m)=\sum_{i=0}^{m} f(p^i)g(p^{m-i}),$$

respectively. Note that throughout this article, p is a prime, m is a non-negative integer and n is a positive integer. The iota function ι defined by

$$\iota(n)=\begin{cases}1 & \text{if } n=1,\\ 0 & \text{if } n>1\end{cases}$$

is the identity element of both abelian groups $(\mathfrak{M}, *)$ and $(\mathfrak{M}, \sqcup)$. Throughout the article, f^{-1} denotes the inverse of the element f in the group $(\mathfrak{M}, *)$, and $f^{(-)}$ denotes the inverse of f in $(\mathfrak{M}, \sqcup)$.

For basic notions on multiplicative arithmetic functions and of Fibonacci and Lucas numbers used in this paper we refer the reader to Sivaramakrishnan's and Koshy's books [14] and [7], respectively. Many authors used different techniques from the theory of multiplicative arithmetic functions, to investigate Fibonacci and generalized Fibonacci sequences (see for example [4], [5], [8], [9], [11], [12], [13]). Related to braces (to which we referred above), [3] includes an extensive bibliographic reference, a source for brace theory, and its connections with fundamental research areas.

2. The radical ring $(\mathfrak{M}, \sqcup, \odot)$ and the group homomorphism Λ

Since $(\mathfrak{M}, \sqcup, *)$ is a two-sided brace, according to [2], Proposition 2.6 it follows:

Theorem 2.1. *The set of all multiplicative arithmetic functions $\mathfrak{M}$ under the unitary convolution $\sqcup$ as the addition and the operation $\odot$ defined by*

$$(f \odot g)(p^m) = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m = 1, \\ \displaystyle\sum_{i=1}^{m-1} f(p^i) g(p^{m-i}) & \text{if } m > 1 \end{cases}$$

is a radical ring. The Dirichlet group $(\mathfrak{M}, *)$ is the adjoint group of the radical ring $(\mathfrak{M}, \sqcup, \odot)$.

Independent of [2], Proposition 2.6, it is easy to verify the ring axioms and it is straightforward to see that the Dirichlet convolution $*$ is the circle operation $\circ$ on the ring $(\mathfrak{M}, \sqcup, \odot)$. Now, the ring $(\mathfrak{M}, \sqcup, \odot)$ is a radical ring since $(\mathfrak{M}, *)$ is a group. The ring $(\mathfrak{M}, \sqcup, \odot)$ is without identity (if f is a multiplicative such that $f(p) \neq 0$, then $(f \odot g)(p) = 0 \neq f(p)$ for any $g \in \mathfrak{M}$).

Theorem 2.2. *The map $\Lambda \colon \mathfrak{M} \to \mathrm{Aut}(\mathfrak{M}, \sqcup)$ defined by*

$$\Lambda(f)(g) = (f \odot g) \sqcup g$$

is an embedding of the Dirichlet group $(\mathfrak{M}, *)$ into the group of automorphisms of the group $(\mathfrak{M}, \sqcup)$.

P r o o f. We will guide you through each step of the theorem's proof.

First, we will prove that $\Lambda(f)$ is an automorphism of the group $(\mathfrak{M}, \sqcup)$ for every $f \in \mathfrak{M}$. We have

$$\Lambda(f)(g \sqcup h) = f \odot (g \sqcup h) \sqcup g \sqcup h = (f \odot g) \sqcup (f \odot h) \sqcup g \sqcup h = \Lambda(f)(g) \sqcup \Lambda(f)(h).$$

Now, if $\Lambda(f)(g) = \Lambda(f)(h)$ (that is $(f \odot g) \sqcup g = (f \odot h) \sqcup h$), then

$$g(p) = [(f \odot g) \sqcup g](p) = [(f \odot h) \sqcup h](p) = h(p)$$

for every prime p. Assume that $g(p^n) = h(p^n)$ for all $n \in \{1, 2, \ldots, m\}$. Then from $[(f \odot g) \sqcup g](p^{m+1}) = [(f \odot h) \sqcup h](p^{m+1})$ it follows that

$$\sum_{i=1}^{m} f(p^i) g(p^{m+1-i}) + g(p^{m+1}) = \sum_{i=1}^{m} f(p^i) h(p^{m+1-i}) + h(p^{m+1}).$$

This tells us that $g(p^{m+1}) = h(p^{m+1})$ and we conclude that $\Lambda(f)$ is injective for every $f \in \mathfrak{M}$.

Let $h \in \mathfrak{M}$ be a given multiplicative arithmetic function and $g \in \mathfrak{M}$ defined by

$$g(p^m) = \begin{cases} 1 & \text{if } m = 0, \\ \displaystyle\sum_{i=0}^{m-1} (-1)^i (f^i \odot h)(p^m) & \text{if } m \geqslant 1, \end{cases}$$

where $f^0 \odot h = h$ (although the ring $(\mathfrak{M}, \sqcup, \odot)$ has no unit), and $f^i = f \odot f \odot \ldots \odot f$ (i-times, i being a positive integer). Then

$$\begin{aligned} g(p) &= (f^0 \odot h)(p) = h(p) \\ \text{and} \quad [\Lambda(f)(g)](p) &= [(f \odot g) \sqcup g](p) = 0 + g(p) = h(p), \end{aligned}$$

$$\begin{aligned} g(p^2) &= (f^0 \odot h)(p^2) - (f \odot h)(p^2) = h(p^2) - f(p)h(p) \\ \text{and} \quad [\Lambda(f)(g)](p^2) &= [(f \odot g) \sqcup g](p^2) = f(p)g(p) + g(p^2) = h(p^2). \end{aligned}$$

Now, if $m > 2$, then

$$\begin{aligned} [\Lambda(f)(g)](p^m) &= [(f \odot g) \sqcup g](p^m) = (f \odot g)(p^m) + g(p^m) \\ &= \sum_{i=1}^{m-1} f(p^i) g(p^{m-i}) + \sum_{i=0}^{m-1} (-1)^i (f^i \odot h)(p^m) = S_1 + h(p^m) + S_2, \end{aligned}$$

where

$$S_1 = \sum_{i=1}^{m-1} f(p^i)g(p^{m-i}) = \sum_{i=1}^{m-1} [f(p^i) \sum_{j=0}^{m-1-i} (-1)^j (f^j \odot h)(p^{m-i})]$$

and

$$S_2 = \sum_{i=1}^{m-1} (-1)^i (f^i \odot h)(p^m) = -f(p^{m-1})h(p) + \sum_{i=1}^{2} (-1)^i f(p^{m-2})(f^{i-1} \odot h)(p^2)$$
$$+ \ldots + \sum_{i=1}^{m-3} (-1)^i f(p^2)(f^{i-1} \odot h)(p^{m-2}) + \sum_{i=1}^{m-2} (-1)^i f(p)(f^{i-1} \odot h)(p^{m-1}).$$

Since $S_1 + S_2 = 0$, it follows that $[\Lambda(f)(g)](p^m) = h(p^m)$, that is, $\Lambda(f)$ is surjective for every $f \in \mathfrak{M}$. So, $\Lambda(f)$ is an automorphism of the group $(\mathfrak{M}, \sqcup)$.

With this last remark in mind, we will prove that

$$\Lambda(f_1 * f_2)(g) = \Lambda(f_1)(\Lambda(f_2)(g)) \quad \text{and} \quad \operatorname{Ker} \Lambda = \{\iota\}.$$

Since

$$\Lambda(f_1 * f_2)(g) = [(f_1 * f_2) \odot g] \sqcup g = [(f_1 \sqcup (f_1 \odot f_2) \sqcup f_2) \odot g] \sqcup g$$
$$= (f_1 \odot g) \sqcup (f_1 \odot f_2 \odot g) \sqcup (f_2 \odot g) \sqcup g$$

and

$$\Lambda(f_1)(\Lambda(f_2)(g)) = \Lambda(f_1)((f_2 \odot g) \sqcup g) = (f_1 \odot f_2 \odot g) \sqcup (f_1 \odot g) \sqcup (f_2 \odot g) \sqcup g,$$

it follows that $\Lambda(f_1 * f_2)(g) = \Lambda(f_1)(\Lambda(f_2)(g))$.

Now, if $f \in \operatorname{Ker} \Lambda$, then $(f \odot g) \sqcup g = g$ for every $g \in \mathfrak{M}$, that is, $(f \odot g)(p^m) = 0$ for every positive integer m, $g \in \mathfrak{M}$ and p prime. From $0 = (f \odot g)(p^2) = f(p)g(p)$ it follows that $f(p) = 0$ and using the mathematical induction we get $f = \iota$. The proof is now complete. □

3. A left group action of $(\mathfrak{M},*)$ on the group $(\mathfrak{M},\sqcup)$

The group homomorphism $\Lambda\colon (\mathfrak{M},*)\to \mathrm{Aut}(\mathfrak{M},\sqcup)$ defines a left group action of $(\mathfrak{M},*)$ on the group $(\mathfrak{M},\sqcup)$ by

$$f\cdot g=\Lambda(f)(g)=f\odot g\sqcup g.$$

In the brace theory, this action is well-known under "lambda action". So, we have

(a) $\iota\cdot f=f$ (identity),
(b) $(f*g)\cdot h=f\cdot(g\cdot h)$ (compatibility),

added with the homomorphism properties:

(c) $f\cdot\iota=\iota$,
(d) $f\cdot(g\sqcup h)=f\cdot g\sqcup f\cdot h$.

Some other elementary properties are given below.

Proposition 3.1. *The following assertions are true.*

(e) $f\cdot f^{-1}=f^{(-)}$.
(f) $f\cdot g\sqcup f=f*g=g\cdot f\sqcup g$.
(g) $f^{(-)}\odot f^{-1}=f\sqcup f^{-1}$ *and* $f^{(-)}*f^{-1}=f^{-1}\sqcup f^{-1}$.
(h) $f\sqcup f=(f^{(-)}\odot f)\sqcup(f*f)$.
(i) *If* $f=g\sqcup h$, *then* $g\cdot f\sqcup h\cdot f=f*f$ *and* $f\cdot g\sqcup f\cdot h=f\cdot f$.

P r o o f. We will use that $f\odot g\sqcup g=f^{(-)}\sqcup(f*g)$.

(e) $f\cdot f^{-1}=f^{(-)}\sqcup(f*f^{-1})=f^{(-)}$.
(f) $f\cdot g\sqcup f=f^{(-)}\sqcup(f*g)\sqcup f=g^{(-)}\sqcup(g*f)\sqcup g=g\cdot f\sqcup g$.
(g) Since $f\sqcup f^{-1}\sqcup f\odot f^{-1}=f\sqcup(f\odot f^{-1}\sqcup f^{-1})=f\sqcup f^{(-)}\sqcup(f*f^{-1})=\iota$, it follows that $f\sqcup f^{-1}=[f\odot f^{-1}]^{(-)}=f^{(-)}\odot f^{-1}$. Thus, $f^{(-)}*f^{-1}=f^{(-)}\sqcup(f^{(-)}\odot f^{-1})\sqcup f^{-1}=f^{-1}\sqcup f^{-1}$.
(h) $(f^{(-)}\odot f)\sqcup(f*f)=(f^{(-)}\odot f)\sqcup f\sqcup(f\odot f)\sqcup f=f\sqcup f$.
(i) $g\cdot f\sqcup h\cdot f=g^{(-)}\sqcup(g*f)\sqcup h^{(-)}\sqcup(h*f)=f^{(-)}\sqcup(f*g)\sqcup(f*h)=f*(g\sqcup h)=f*f$.

□

It is straightforward to see that this left action is not transitive. Indeed, if g and h are two multiplicative arithmetic functions such that $g(p)\neq h(p)$ for a prime p, then $f\cdot g\neq h$ for all $f\in\mathfrak{M}$ since $(f\cdot g)(p)=g(p)\neq h(p)$. The Dirichlet group $(\mathfrak{M},*)$ acts faithfully on $(\mathfrak{M},\sqcup)$ (i.e., if $f\neq g$, then there exists h such that $f\cdot h\neq g\cdot h$), since the group homomorphism $\Lambda\colon(\mathfrak{M},*)\to\mathrm{Aut}(\mathfrak{M},\sqcup)$ has a trivial kernel.

Definition 3.1. We will say that an arithmetic function g is degenerate if there is a prime p such that $g(n)=0$ for every positive integer n divisible by p.

If g is a nondegenerate multiplicative arithmetic function, then we will denote by $n(g,p)$ the smallest positive integer m for which $g(p^m) \neq 0$.

Theorem 3.1. *Given a nondegenerate multiplicative arithmetic function g, the following assertions are true.*

(i) *The orbit of g (i.e., all its possible destinations under the group action) is given by*

$$\mathrm{Orb}(g) = \{f \in \mathfrak{M}\colon\ n(f,p) = n(g,p) \text{ and } f(p^{n(f,p)}) = g(p^{n(g,p)}) \text{ for every prime } p\}.$$

(ii) *If $f \in \mathrm{Orb}(g)$, then $h \in \mathfrak{M}$, denoted in what follows by $\frac{f}{g}$, is uniquely determined by $h \cdot g = f$.*

P r o o f. (i) It is clear that if $f = h \cdot g$ for some $h \in \mathfrak{M}$, then

$$f(p) = (h \odot g)(p) + g(p) = g(p),$$

and

$$f(p^m) = \sum_{i=1}^{m-1} h(p^i)g(p^{m-i}) + g(p^m) = g(p^m) \quad \text{if } 1 \leqslant m \leqslant n(g,p).$$

Thus, $n(f,p) = n(g,p)$ and $f(p^{n(f,p)}) = g(p^{n(g,p)})$ for every prime p.

Conversely, let $f \in \mathfrak{M}$ such that $n(f,p) = n(g,p)$ and $f(p^{n(f,p)}) = g(p^{n(g,p)})$ for every prime p, and let h be the multiplicative arithmetic function defined inductively by

$$h(p) = \frac{f(p^{n(g,p)+1}) - g(p^{n(g,p)+1})}{g(p^{n(g,p)})}, \tag{3.1}$$

and

$$h(p^m) = \frac{f(p^{m+n(g,p)}) - \sum\limits_{i=n(g,p)+1}^{m+n(g,p)} g(p^i)h(p^{m+n(g,p)-i})}{g(p^{n(g,p)})}, \quad m > 1. \tag{3.2}$$

Then

$$1. \qquad (h \cdot g)(p) = (h \odot g)(p) + g(p) = f(p);$$

$$2. \qquad (h \cdot g)(p^m) = \sum_{i=1}^{m-1} g(p^i)h(p^{m-i}) + g(p^m) = f(p^m) \quad \text{if } 1 < m \leqslant n(g,p);$$

$$3. \ (h \cdot g)(p^{n(g,p)+1}) = \sum_{i=1}^{n(g,p)} g(p^i)h(p^{m-i}) + g(p^{n(g,p)+1})$$
$$= g(p^{n(g,p)})h(p) + g(p^{n(g,p)+1}) = f(p^{n(g,p)+1});$$

$$4. \ (h \cdot g)(p^{n(g,p)+k}) =_{(k>1)} \sum_{i=1}^{n(g,p)+k-1} g(p^i)h(p^{n(g,p)+k-i}) + g(p^{n(g,p)+k})$$
$$= \sum_{i=1}^{n(g,p)} g(p^i)h(p^{n(g,p)+k-i})$$
$$+ \sum_{i=n(g,p)+1}^{n(g,p)+k-1} g(p^i)h(p^{n(g,p)+k-i}) + g(p^{n(g,p)+k})$$
$$= g(p^{n(g,p)})h(p^k) + \sum_{i=n(g,p)+1}^{n(g,p)+k-1} g(p^i)h(p^{n(g,p)+k-i}) + g(p^{n(g,p)+k})$$
$$= f(p^{n(g,p)+k}) - \sum_{i=n(g,p)+1}^{n(g,p)+k} g(p^i)h(p^{n(g,p)+k-i})$$
$$+ \sum_{i=n(g,p)+1}^{n(g,p)+k-1} g(p^i)h(p^{n(g,p)+k-i}) + g(p^{n(g,p)+k}) = f(p^{n(g,p)+k}).$$

The proof of (i) is complete.

(ii) The uniqueness of h results from the fact that (3.1) and (3.2) are satisfied for any $h' \in \mathfrak{M}$ by the condition $h' \cdot g = f$. □

R e m a r k 3.1. If $g \in \mathfrak{M}$ is degenerate, i.e., $g(q^m) = 0$ for a prime q end every positive integer m, then $h \cdot g = h' \cdot g$ if $h(p^m) = h'(p^m)$ for any $p \neq q$ and $h(q^m) \neq h'(q^m)$ for a positive integer m. In the following sections nondegenerate multiplicative arithmetic functions will be taken into the foreground.

4. The orbit quotient of two compatible multiplicative arithmetic functions

Definition 4.1. We say that a pair (f, g) of multiplicative arithmetic functions is compatible if g is nondegenerate and $f \in \mathrm{Orb}(g)$.

Definition 4.2. The orbit quotient $\frac{f}{g}$ of a compatible pair (f, g) is the only multiplicative arithmetic function h for which $h \cdot g = f$.

Throughout the paper, the notation $\frac{f}{g}$ will imply (without mentioning it each time) that (f, g) is a compatible pair.

The simplest properties of orbit quotients are listed below.

Proposition 4.1. *The following assertions are true.*

(i) $\frac{f}{f} = \iota$;

(ii) $\frac{f^{(-)}}{f^{-1}} = f$ *(if f is nondegenerate);*

(iii) $\frac{f}{g} = \frac{f'}{g} \Rightarrow f = f'$ *and* $\frac{f}{g} = \frac{f}{g'} \Rightarrow g = g'$;

(iv) $\frac{f}{g} \sqcup \frac{f'}{g} = \frac{f \sqcup f' \sqcup g^{(-)}}{g}$;

(v) $\frac{f}{g} * \frac{g}{h} = \frac{f}{h}$ *and* $\frac{f}{g} * \frac{g}{f} = \iota$;

(vi) $\frac{f}{g} = \left(\frac{g}{f}\right)^{-1}$ *and* $\frac{f}{g} = \frac{f^{(-)}}{g^{(-)}}$;

(vii) *if* $\frac{f}{g} = \frac{f'}{g'}$ *and if* $\frac{f \sqcup f'}{g \sqcup g'}$, $\frac{f \sqcup f'^{(-)}}{g \sqcup g'^{(-)}}$ *make sense, then* $\frac{f \sqcup f'}{g \sqcup g'} = \frac{f}{g} = \frac{f \sqcup f'^{(-)}}{g \sqcup g'^{(-)}}$;

(viii) $\frac{f}{g} = \frac{f'}{g'}$ *if and only if* $f \odot g' = f' \odot g$ *(under the assumption that (f, g) and (f', g') are compatible pairs).*

(ix) $\frac{f * g \sqcup f^{(-)}}{g} = f = \frac{f * g}{[f^{-1}]^{(-)} \sqcup g}$;

(x) *(The Möbius inversion theorem)* $\frac{f \sqcup g^{(-)}}{\zeta} = g$ *if and only if* $\frac{f^{(-)} \sqcup g}{\mu} = f$, *where* $\zeta(n) = 1$ *for every positive integer n, and μ is the Möbius function defined by*

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes}, \\ 0 & \text{otherwise}; \end{cases}$$

(xi) *(The general inversion theorem)* $\frac{f \sqcup g^{(-)}}{h} = g$ *if and only if* $\frac{f^{(-)} \sqcup g}{h^{-1}} = f$.

P r o o f. (i) See (a) from the previous section.

(ii) It is straightforward to check that $n(f^{-1}, p) = n(f, p) = n(f^{(-)}, p)$ and $f^{-1}(p^{n(f^{-1},p)}) = -f(p^{n(f,p)}) = f^{(-)}(p^{n(f^{(-)},p)})$. So, $(f^{(-)}, f^{-1})$ is a compatible pair (only if f is nondegenerate), and by Proposition 3.1 (e) it follows $\frac{f^{(-)}}{f^{-1}} = f$.

(iii) It is clear that

$$\frac{f}{g} = \frac{f'}{g} = h \Rightarrow f = h \cdot g = f'$$

and

$$\frac{f}{g} = \frac{f}{g'} = h \Rightarrow \Lambda(h)(g) = \Lambda(h)(g') \Rightarrow g = g'.$$

(iv) We have $f \sqcup f' \sqcup g^{(-)} = \frac{f}{g} \cdot g \sqcup \frac{f'}{g} \cdot g \sqcup g^{(-)} = \frac{f}{g} \odot g \sqcup g \sqcup \frac{f'}{g} \odot g \sqcup g \sqcup g^{(-)} = \left(\frac{f}{g} \sqcup \frac{f'}{g}\right) \odot g \sqcup g = \left(\frac{f}{g} \sqcup \frac{f'}{g}\right) \cdot g$, and therefore $\frac{f}{g} \sqcup \frac{f'}{g} = \frac{f \sqcup f' \sqcup g^{(-)}}{g}$.

(v) Using (b) of the previous section, we get

$$\left(\frac{f}{g} * \frac{g}{h}\right) \cdot h = \frac{f}{g} \cdot \left(\frac{g}{h} \cdot h\right) = \frac{f}{g} \cdot g = f,$$

that is,

$$\frac{f}{g} * \frac{g}{h} = \frac{f}{h}.$$

Invoking this simplification rule, it follows that

$$\frac{f}{g} * \frac{g}{f} = \frac{f}{f} = \iota.$$

(vi) Now, from $\frac{f}{g} * \frac{g}{f} = \iota$ it follows that $\frac{f}{g} = \left(\frac{g}{f}\right)^{-1}$.

If $h = \frac{f}{g}$, then $f = h \odot g \sqcup g$ implies $f^{(-)} = h \odot g^{(-)} \sqcup g^{(-)}$, that is, $h = \frac{f^{(-)}}{g^{(-)}}$.

(vii) Using (d) of the previous section, we get

$$\frac{f}{g} \cdot (g \sqcup g') = \frac{f}{g} \cdot g \sqcup \frac{f'}{g'} \cdot g' = f \sqcup f',$$

and

$$\frac{f}{g} \cdot (g \sqcup g'^{(-)}) = \frac{f}{g} \cdot g \sqcup \frac{f'}{g'} \cdot g'^{(-)} = f \sqcup \frac{f'^{(-)}}{g'^{(-)}} \cdot g'^{(-)} = f \sqcup f'^{(-)}.$$

(viii) The hypothesis $\frac{f}{g} = \frac{f'}{g'}$ includes that (f, g) and (f', g') are compatible pairs. Further,

$$\begin{aligned} f \odot g' &= \frac{f}{g} \cdot g \odot g' = \left(\frac{f}{g} \odot g \sqcup g\right) \odot g' = \frac{f'}{g'} \odot g \odot g' \sqcup g \odot g' \\ &= \left(\frac{f'}{g'} \odot g' \sqcup g'\right) \odot g = \frac{f'}{g'} \cdot g' \odot g = f' \odot g. \end{aligned}$$

Conversely, assume that (f, g) and (f', g') are compatible pairs and $f \odot g' = f' \odot g$. We denote by h and h' the orbit quotients $\frac{f}{g}$ and $\frac{f'}{g'}$, respectively. Then

$$f = h \odot g \sqcup g \quad \text{and} \quad f' = h' \odot g' \sqcup g'.$$

Since $f \odot g' = f' \odot g$, it follows that

$$h \odot g \odot g' \sqcup g \odot g' = h' \odot g' \odot g \sqcup g' \odot g,$$

that is,

$$h \cdot (g \odot g') = h' \cdot (g \odot g').$$

So, $\frac{f}{g} = \frac{f'}{g'}$ if $g \odot g'$ is nondegenerate. Since,

$$(g \odot g')(p^m) = \begin{cases} 0 & \text{if } m < n(g,p) + n(g',p), \\ g(p^{n(g,p)})g'(p^{n(g',p)}) & \text{if } m = n(g,p) + n(g',p) \end{cases}$$

for every prime p, the multiplicative arithmetic function $g \odot g'$ is nondegenerate and the proof is complete.

(ix) The orbit quotient $\frac{f*g \sqcup f^{(-)}}{g} = f$ follows from Proposition 3.1 (f). Since $f = \frac{f^{(-)}}{f^{-1}} = \frac{f}{[f^{-1}]^{(-)}}$, that is $\frac{f*g \sqcup f^{(-)}}{g} = \frac{f}{[f^{-1}]^{(-)}}$, then by (vii) we get $f = \frac{f*g}{[f^{-1}]^{(-)} \sqcup g}$.

(x) It is straightforward to check that $\frac{f \sqcup g^{(-)}}{\zeta} = g$ if and only if $f = g * \zeta$ and $\frac{f^{(-)} \sqcup g}{\mu} = f$ if and only if $g = f * \mu$. Since $f = g * \zeta$ if and only if $g = f * \mu$, the assertion of (x) is valid.

(xi) The proof is the same as for (x). □

Examples. We supplement the above results with a few simple examples.

Let $\lambda = (-1)^{\Omega}$ and $\mu_u = (-1)^{\omega}$ be Liouville's lambda and Möbius' unitary functions, respectively, where $\Omega(n)$ is the number of prime factors of n counted with multiplicity, and $\omega(n)$ is the number of distinct prime factors of n ($\Omega(1) = \omega(1) = 0$). Since $\zeta^{-1} = \mu$, $\zeta^{(-)} = \mu_u$, $\mu^{-1} = \zeta$ and $\mu^{(-)} = \lambda^{-1}$, using (ii) and (vi) for $f = \zeta$, $f = \mu$ and $f = \mu^{(-)}$, we get the following orbit quotients:

$$\frac{\mu_u}{\mu} = \zeta = \frac{\zeta}{\lambda^{-1}}, \quad \frac{\lambda^{-1}}{\zeta} = \mu = \frac{\mu}{\mu_u}, \quad \text{and} \quad \frac{\mu}{\lambda} = \mu^{(-)}, \quad \frac{\lambda}{\mu} = \lambda,$$

respectively. The same orbit quotients follow by (ix) for $f = \zeta$, $g = \mu$; $f = \mu$, $g = \zeta$; and $f = \mu^{(-)}$, $g = \lambda$, respectively.

Proposition 4.1 becomes particularly useful in the description of orbit quotients. Below we mention some of them, this time involving the Euler totient Φ, the Id, τ and σ functions ($\mathrm{Id}(n) = n$, $\tau(n)$= the number of divisors of n, $\sigma(n)$= the sum of divisors of n):

$$\frac{\tau \sqcup \mu_u}{\zeta} = \zeta = \frac{\tau}{\lambda^{-1} \sqcup \zeta}; \quad \frac{\Phi \sqcup \lambda^{-1}}{\mathrm{Id}} = \mu = \frac{\Phi}{\mu_u \sqcup \mathrm{Id}}; \quad \frac{\sigma \sqcup \mu_u}{\mathrm{Id}} = \zeta = \frac{\sigma}{\lambda^{-1} \sqcup \mathrm{Id}}.$$

R e m a r k 4.1. It is clear that

$$\frac{\frac{\mu}{\frac{\mu}{\mu_u}}}{\frac{\frac{\lambda}{\mu}}{\frac{\mu_u}{\mu}}} = \frac{\frac{\mu}{\lambda}}{\zeta} = \frac{\mu^{(-)}}{\zeta} = \mu,$$

but $\frac{\frac{\mu}{\mu_u}}{\frac{\frac{\lambda}{\mu}}{\frac{\mu_u}{\mu}}}$ does not make sense because the denominator $\frac{\frac{\lambda}{\mu}}{\frac{\mu_u}{\mu}}$ does not make sense (the pair $(\frac{\lambda}{\mu} = \lambda, \frac{\mu_u}{\mu} = \zeta)$ is not compatible).

5. The Fibonacci and the Lucas prime-independent multiplicative arithmetic functions and Liouville's lambda function

The multiplicative arithmetic functions ζ, μ, τ, $\lambda = (-1)^{\Omega}$, $\mu_u = (-1)^{\omega}$, and $\lambda^{-1} = \mu^{(-)}$ from the previous section's examples satisfy the following two special properties:

(1) they are prime-independent (a multiplicative arithmetic function f is prime-independent if $f(p^m) = f(q^m)$ for every pair of primes (p, q) and every non-negative integer m);
(2) $n(f, p) = 1$.

We will denote by $\mathfrak{M}^*$ the set of all multiplicative arithmetic functions satisfying the two conditions above. If $g \in \mathfrak{M}^*$ and $g(p) = a\,(\neq 0)$, then, in what follows, $\widehat{a}$ will denote the orbit-class $\mathrm{Orb}(g) = \{f \in \mathfrak{M}^*\colon\ f(p) = a\}$. So, ζ, $\lambda^{-1} \in \widehat{1}$ and μ, μ_u, $\lambda \in \widehat{-1}$.

Now, for each such orbit-class $\widehat{a}$ $(a \neq 0)$ we will choose two representatives, namely the completely multiplicative (prime-independent) arithmetic function $\mathcal{G}_a = a^{\Omega}$, and the almost-zero multiplicative prime-independent arithmetic function $\mathcal{H}_a$ defined by $\mathcal{H}(p) = a$ and $\mathcal{H}(p^m) = 0$ if $m > 1$. Obviously, $\mathcal{G}_1 = \zeta$, $\mathcal{H}_1 = \lambda^{-1} = \widetilde{\mu}$, $\mathcal{G}_{-1} = \lambda$, and $\mathcal{H}_{-1} = \mu$.

Definition 5.1. If $f \in \mathfrak{M}^*$, then we will say that the prime-independent multiplicative arithmetic function $\overline{f}$ defined by $\overline{f}(p^m) = \frac{f(p^{m+1})}{f(p)}$, $m \geqslant 0$ is the associated function of f. If $f(p) = 1$ (that is if $f \in \widehat{1}$), then $\overline{f}$ is the left-shifted function of f and will be denoted by $\overleftarrow{f}$.

Proposition 5.1. *The following assertions are true.*

(i) $\mathcal{H}_a^{(-)} = \mathcal{G}_a^{-1}$.
(ii) $\frac{\mathcal{H}_a}{\mathcal{G}_a} = \mathcal{H}_a^{(-)}$ *and* $\frac{\mathcal{G}_a}{\mathcal{H}_a} = \mathcal{G}_a$.
(iii) $\overline{\mathcal{H}_a} = \iota$ *and* $\overline{\mathcal{G}_a} = \mathcal{G}_a$.

(iv) *If* $f \in \mathfrak{M}^*$, *then* $\frac{f}{\mathcal{H}_{f(p)}} = \overline{f}$ *and* $\frac{f}{\mathcal{G}_{f(p)}} = \overline{f} * \mathcal{G}_{f(p)}^{-1}$.

(v) *If* $f, g \in \widehat{a}$ $(a \neq 0)$, *then* $\frac{f}{g} = \overline{f} * \overline{g}^{-1}$.

P r o o f. (i) $(\mathcal{H}_{-a} * \mathcal{G}_a)(p^m) = 1$ if $m = 0$ and $(\mathcal{H}_{-a} * \mathcal{G}_a)(p^m) = -aa^{m-1} + a^m = 0$ if $m > 0$.

(ii) $\mathcal{H}_{-a} \cdot \mathcal{G}_a = \mathcal{G}_a^{-1} \cdot \mathcal{G}_a = [\mathcal{G}_a^{-1}]^{(-)} \sqcup \iota = \mathcal{H}_a$ and $\frac{\mathcal{G}_a}{\mathcal{H}_a} = \left(\frac{\mathcal{H}_a}{\mathcal{G}_a}\right)^{-1} = \mathcal{H}_{-a}^{-1} = \mathcal{G}_a$.

(iii) Obvious.

(iv) $(\overline{f} \cdot \mathcal{H}_{f(p)})(p) = (\overline{f} \odot \mathcal{H}_{f(p)} \sqcup \mathcal{H}_{f(p)})(p) = \mathcal{H}_{f(p)})(p) = f(p)$, and if $m > 1$, then

$$(\overline{f} \cdot \mathcal{H}_{f(p)})(p^m) = (\overline{f} \odot \mathcal{H}_{f(p)} \sqcup \mathcal{H}_{f(p)})(p^m) = \overline{f}(p^{m-1}) f(p) = f(p^m).$$

(v) $\frac{f}{g} = \frac{f}{\mathcal{H}_a} * \frac{\mathcal{H}_a}{g} = \overline{f} * \overline{g}^{-1}$. □

Now two new representatives of the orbit class $\widehat{1}$ will draw our attention, namely the Fibonacci $\mathcal{F}$ and the Lucas $\mathcal{L}$ prime-independent multiplicative arithmetic functions defined by

$$\mathcal{F}(p^m) = F_{m+1} \quad \text{and} \quad \mathcal{L}(p^m) = \begin{cases} 1 & \text{if } m = 0, \\ L_m & \text{if } m > 0, \end{cases}$$

respectively, where $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3, \ldots$ are the Fibonacci numbers and $L_1 = 1$, $L_2 = 3$, $L_3 = 4$, $L_4 = 7, \ldots$ are the Lucas numbers ($F_{n+2} = F_{n+1} + F_n$; $L_{n+2} = L_{n+1} + L_n$ whenever $n > 0$).

At the beginning, we will state the formulas that were in a central place (within a much more general framework) in [11], [12], [13].

Theorem 5.1. *The following equalities hold.*

(i) (*Binet's formulas in the convolution setting* – [11], *Theorem* 1, [12], *Theorem* 3.2 *and Theorem* 4.2 (i))

$$\mathcal{F} = \mathcal{G}_\varphi * \mathcal{G}_\psi \quad \text{and} \quad \mathcal{L} = \mathcal{G}_\varphi \sqcup \mathcal{G}_\psi;$$

(ii) (*Inversion formulas* – [13], *p.* 7 *and* [12], *Theorem* 4.2 (ii))

$$\mathcal{F}^{-1} = \mathcal{H}_\varphi^{(-)} * \mathcal{H}_\psi^{(-)} \quad \text{and} \quad \zeta^{-1} = \mathcal{H}_\varphi^{(-)} \sqcup \mathcal{H}_\psi^{(-)} = \mu,$$

where $\varphi = (1 + \sqrt{5})/2$ *is the golden ratio, and* $\psi = -1/\varphi = 1 - \varphi = (1 - \sqrt{5})/2$.

Corollary 5.1. *We have*

(i) $(\mathcal{F} * \mu) \sqcup \mathcal{F} = \mathcal{L}$;
(ii) $(\mathcal{F}^{-1} * \mathcal{L}) \sqcup \mathcal{F}^{-1} = \mu$;
(iii) $\mathcal{F} * \lambda^{-1} = \overleftarrow{\mathcal{F}}$;
(iv) $\mathcal{L} * \lambda^{-1} \sqcup \lambda^{-1} = \overleftarrow{\mathcal{L}}$
(v) $(\mathcal{L} \sqcup \mathcal{F}^{-1}) \sqcup (\mathcal{L} \odot \mathcal{F}^{-1}) = \lambda \odot \mathcal{F}^{-1}$;
(vi) $\mathcal{L} \sqcup \lambda^{(-)} = \mathcal{F} * \lambda^{(-)}$.

P r o o f. (i)

$$(\mathcal{F} * \mu) \sqcup \mathcal{F} = \mathcal{F} * (\mathcal{G}_\varphi^{-1} \sqcup \mathcal{G}_\psi^{-1}) \sqcup \mathcal{F} = (\mathcal{F} * \mathcal{G}_\varphi^{-1}) \sqcup (\mathcal{F} * \mathcal{G}_\psi^{-1}) = \mathcal{G}_\psi \sqcup \mathcal{G}_\varphi = \mathcal{L}.$$

(ii)

$$\begin{aligned}(\mathcal{F}^{-1} * \mathcal{L}) \sqcup \mathcal{F}^{-1} &= \mathcal{F}^{-1} * (\mathcal{G}_\varphi \sqcup \mathcal{G}_\psi) \sqcup \mathcal{F}^{-1} = (\mathcal{F}^{-1} * \mathcal{G}_\varphi) \sqcup (\mathcal{F}^{-1} * \mathcal{G}_\psi) \\ &= \mathcal{G}_\psi^{-1} \sqcup \mathcal{G}_\varphi^{-1} = \mathcal{H}_\psi^{(-)} \sqcup \mathcal{H}_\varphi^{(-)} = \mu.\end{aligned}$$

(iii) Since $\lambda^{-1}(p) = 1$ and $\lambda^{-1}(p^m) = 0$ if $m > 1$, it follows that

$$(\mathcal{F} * \lambda^{-1})(p) = \mathcal{F}(p) + \lambda^{-1}(p) = 2 = \overleftarrow{\mathcal{F}}(p)$$

and

$$(\mathcal{F} * \lambda^{-1})(p^m) = \mathcal{F}(p^m) + \mathcal{F}(p^{m-1}) = F_{m+1} + F_m = F_{m+2} = \overleftarrow{\mathcal{F}}(p^m), \quad m > 1.$$

Hence $\mathcal{F} * \lambda^{-1} = \overleftarrow{\mathcal{F}}$.

(iv) The proof of (iv) follows the same lines.

(v) Since $\mathcal{F}^{-1} = \mathcal{H}_\varphi^{(-)} * \mathcal{H}_\psi^{(-)}$, we get

$$\mathcal{F}^{-1}(p^m) = \begin{cases} 1 & \text{if } m = 0, \\ -1 & \text{if } m = 1, 2, \\ 0 & \text{if } m > 2. \end{cases}$$

It follows that

$$(\lambda \odot \mathcal{F}^{-1})(p^m) = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m = 1, \\ 1 & \text{if } m = 2, \\ 0 & \text{if } m > 2, \end{cases} \qquad (\mathcal{L} \sqcup \mathcal{F}^{-1})(p^m) = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m = 1, \\ L_2 - 1 & \text{if } m = 2, \\ L_m & \text{if } m > 2, \end{cases}$$

and

$$(\mathcal{L} \odot \mathcal{F}^{-1})(p^m) = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m = 1, \\ -L_1 & \text{if } m = 2, \\ -L_{m-1} - L_{m-2} & \text{if } m > 2. \end{cases}$$

The result is that

$$(\mathcal{L} \sqcup \mathcal{F}^{-1}) \sqcup (\mathcal{L} \odot \mathcal{F}^{-1}) = \lambda \odot \mathcal{F}^{-1}.$$

(vi) Using (v), we get

$$\begin{aligned} \lambda^{(-)} &= \lambda^{(-)} \sqcup \mathcal{L} \sqcup \mathcal{F}^{-1} \sqcup (\mathcal{L} \odot \mathcal{F}^{-1}) \sqcup (\lambda^{(-)} \odot \mathcal{F}^{-1}) \\ &= \lambda^{(-)} \sqcup \mathcal{L} \sqcup \mathcal{F}^{-1} \sqcup (\mathcal{L} \sqcup \lambda^{(-)}) \odot \mathcal{F}^{-1} = (\mathcal{L} \sqcup \lambda^{(-)}) * \mathcal{F}^{-1}, \end{aligned}$$

and the proof is complete. □

Definition 5.2. We define the right shift multiplicative arithmetic function $\overrightarrow{f}$ of the multiplicative arithmetic function f by

$$\overrightarrow{f}(p^m) = \begin{cases} 1 & \text{if } m = 0, \\ f(p^{m-1}) & \text{if } m > 0, \end{cases}$$

for every prime p.

It is straightforward to see that if f is multiplicative and prime-independent, then $\overrightarrow{f} \in \mathfrak{M}^*$ and $\overrightarrow{f} \in \widehat{1}$. Since $\mathcal{F}, \mathcal{L} \in \mathfrak{M}^*$ and $\mathcal{F}, \mathcal{L} \in \widehat{1}$, it follows that

$$\{(\overrightarrow{\mathcal{F}}, \mathcal{F}),\ (\mathcal{L}, \mathcal{F}),\ (\overrightarrow{\mathcal{L}}, \overrightarrow{\mathcal{F}}),\ (\mathcal{L}, \overrightarrow{\mathcal{F}}),\ (\overrightarrow{\mathcal{L}}, \mathcal{F}),\ (\overrightarrow{\mathcal{L}}, \mathcal{L})\}$$

is the set of all nontrivial distinct compatible pairs with the Fibonacci and Lucas prime-independent multiplicative functions and their right shift functions (we do not consider the compatible pairs (f, g) and (g, f) to be distinct; the orbit quotients $\frac{f}{g}$ and $\frac{g}{f}$ are linked by $\frac{f}{g} = (\frac{g}{f})^{-1}$). In what follows, we will show that each of the orbit quotients of these Fibonacci and Lucas functions include Liouville's lambda function in some form.

Theorem 5.2. *The following assertions are true:*

(i) $\frac{\overrightarrow{\mathcal{F}}}{\mathcal{F}} = \lambda$, $\frac{\mathcal{L}}{\mathcal{F}} = \lambda^{(-)}$, $\frac{\overrightarrow{\mathcal{L}}}{\overrightarrow{\mathcal{F}}} = \widetilde{\lambda}$,

(ii) $\frac{\mathcal{L}}{\overrightarrow{\mathcal{F}}} = \lambda^{(-)} * \lambda^{-1}$, $\frac{\overrightarrow{\mathcal{L}}}{\mathcal{F}} = \widetilde{\lambda} * \lambda$, $\frac{\overrightarrow{\mathcal{L}}}{\mathcal{L}} = \widetilde{\lambda} * \lambda * (\lambda^{(-)})^{-1}$, *where* $\widetilde{\lambda} = (\lambda^{-1})^{(-)} \sqcup \overrightarrow{\lambda^{-1}}$.

P r o o f. (i) Since $\lambda^{-1}(p) = 1$ and $\lambda^{-1}(p^m) = 0$ if $m > 1$, it follows that

$$(\lambda^{-1} \cdot \overrightarrow{\mathcal{F}})(p) = (\lambda^{-1} \odot \overrightarrow{\mathcal{F}})(p) + \overrightarrow{\mathcal{F}}(p) = 1 = \mathcal{F}(p)$$

and

$$(\lambda^{-1} \cdot \overrightarrow{\mathcal{F}})(p^m) = (\lambda^{-1} \odot \overrightarrow{\mathcal{F}})(p^m) + \overrightarrow{\mathcal{F}}(p^m) = F_{m-1} + F_m = \mathcal{F}(p^m), \quad m > 1,$$

that is, $\frac{\mathcal{F}}{\overrightarrow{\mathcal{F}}} = \lambda^{-1}$ and therefore

$$\frac{\overrightarrow{\mathcal{F}}}{\mathcal{F}} = \lambda.$$

Now, using Proposition 5.1 (v), the left-shifted formulas Corollary 5.1 (iii) and (iv), the quasi-distributive law of Dirichlet convolution over unitary convolution, and finally Corollary 5.1 (vi), in that order, we get:

$$\begin{aligned}\frac{\mathcal{L}}{\mathcal{F}} &= \overleftarrow{\mathcal{L}} * \overleftarrow{\mathcal{F}}^{-1} = (\mathcal{L} * \lambda^{-1} \sqcup \lambda^{-1}) * \lambda * \mathcal{F}^{-1} \\ &= [\lambda^{(-)} \sqcup \lambda \sqcup (\mathcal{L} * \lambda^{-1} \sqcup \lambda^{-1}) * \lambda] * \mathcal{F}^{-1} = (\lambda^{(-)} \sqcup \mathcal{L}) * \mathcal{F}^{-1} \\ &= \lambda^{(-)}.\end{aligned}$$

Since $\widetilde{\lambda}(p^m) = 0$ for every positive integer $m \neq 2$ and $\widetilde{\lambda}(p^2) = 1$, it follows that

$$\begin{aligned}(\widetilde{\lambda} \cdot \overrightarrow{\mathcal{F}})(p) &= (\widetilde{\lambda} \odot \overrightarrow{\mathcal{F}})(p) + \overrightarrow{\mathcal{F}}(p) = 1 = \overrightarrow{\mathcal{L}}(p), \\ (\widetilde{\lambda} \cdot \overrightarrow{\mathcal{F}})(p^2) &= (\widetilde{\lambda} \odot \overrightarrow{\mathcal{F}})(p^2) + \overrightarrow{\mathcal{F}}(p^2) = 1 = \overrightarrow{\mathcal{L}}(p^2),\end{aligned}$$

and

$$(\widetilde{\lambda} \cdot \overrightarrow{\mathcal{F}})(p^m) = (\widetilde{\lambda} \odot \overrightarrow{\mathcal{F}})(p^m) + \overrightarrow{\mathcal{F}}(p^m) = F_{m-2} + F_m = L_{m-1} = \overrightarrow{\mathcal{L}}(p^m), \quad m > 2,$$

and therefore

$$\frac{\overrightarrow{\mathcal{L}}}{\overrightarrow{\mathcal{F}}} = \widetilde{\lambda}.$$

(ii) All three formulas follow from (i) and Proposition 4.1 (v) by setting

$$\frac{\mathcal{L}}{\overrightarrow{\mathcal{F}}} = \frac{\mathcal{L}}{\mathcal{F}} * \frac{\mathcal{F}}{\overrightarrow{\mathcal{F}}}, \quad \frac{\overrightarrow{\mathcal{L}}}{\mathcal{F}} = \frac{\overrightarrow{\mathcal{L}}}{\overrightarrow{\mathcal{F}}} * \frac{\overrightarrow{\mathcal{F}}}{\mathcal{F}}, \quad \text{and} \quad \frac{\overrightarrow{\mathcal{L}}}{\mathcal{L}} = \frac{\overrightarrow{\mathcal{L}}}{\mathcal{F}} * \frac{\mathcal{F}}{\mathcal{L}},$$

respectively. □

At the end of the section, we proceed by placing some of the above results into play. We illustrate with an example from each of the last three sentences (Theorem 5.1, Corollary 5.1, and Theorem 5.2).

(1) $\mathcal{F} = \mathcal{G}_\varphi * \mathcal{G}_\psi$ leads to the classical Binet's formula:

$$F_m = \mathcal{F}(p^{m-1}) = (\mathcal{G}_\varphi * \mathcal{G}_\psi)(p^{m-1}) = \sum_{i=0}^{m-1} \varphi^i \psi^{m-1-i} = \frac{\varphi^m - \psi^m}{\varphi - \psi}.$$

(2) $(\mathcal{F} * \mu) \sqcup \mathcal{F} = \mathcal{L}$ leads us to the simplest connected identity:

$$L_m = \mathcal{L}(p^m) = (\mathcal{F} * \mu)(p^m) + \mathcal{F}(p^m) = F_{m-1} + F_{m+1}.$$

(3) $\frac{\mathcal{L}}{\mathcal{F}} = \lambda^{(-)}$ is more technical (see its proof) and a bit harder to read.

$$L_m = (\mathcal{L})(p^m) = (\lambda^{(-)} \cdot \mathcal{F})(p^m) = (\lambda^{(-)} \odot \mathcal{F})(p^m) + \mathcal{F}(p^m).$$

Thus, if $m > 1$, then

$$L_m = \sum_{i=1}^{m-1} F_{i+1}(-1)^{m-i-1} + F_{m+1} = (-1)^m \sum_{j=2}^{m} (-1)^j F_j + F_{m+1}$$

or, in equivalent terms,

$$(-1)^{m+1} F_{m-1} + 1 = \sum_{i=1}^{m} (-1)^{i+1} F_i.$$

Afterword to the last section. We list some number-theoretic formulas, highlighting them in the convolution and the orbit quotient settings. Some Fibonacci number formulas also find their place on this short list.

▷ The number-theoretic Möbius inversion formula:

$$f(n) = \sum_{d|n} g(d) \Leftrightarrow g(n) = \sum_{d|n} f(d)\mu\left(\frac{n}{d}\right),$$
$$f = g * \zeta \Leftrightarrow g = f * \mu \text{ (in the convolution setting)},$$
$$\frac{f \sqcup g^{(-)}}{\zeta} = g \Leftrightarrow \frac{f^{(-)} \sqcup g}{\mu} = f \quad \text{(in the orbit quotient setting)};$$

▷ Binet's Fibonacci number formula:

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi},$$
$$\mathcal{F} = \mathcal{G}_\varphi * \mathcal{G}_\psi \text{ (in the convolution setting)},$$
$$\frac{\mathcal{F} \sqcup \mathcal{G}_\varphi^{(-)}}{\mathcal{G}_\psi} = \mathcal{G}_\varphi \text{ (in the orbit quotient setting)};$$

▷ Gauss's formula that uses Euler's totient function:

$$\sum_{d|n} \Phi(d) = n,$$
$$\Phi * \zeta = \mathrm{Id} \text{ (in the convolution setting)},$$
$$\frac{\Phi \sqcup \lambda^{-1}}{\mathrm{Id}} = \mu \text{ (in the orbit quotient setting)};$$

▷ The simplest connected formula between Fibonacci and Lucas numbers:

$$L_n = F_{n-1} + F_{n+1},$$
$$\mathcal{L} \sqcup \mathcal{F}^{(-)} = \mathcal{F} * \mu \text{ (in the convolution setting)},$$
$$\frac{\overrightarrow{\mathcal{L}}}{\overrightarrow{\mathcal{F}}} = \widetilde{\lambda} \text{ (in the orbit quotient setting)};$$

▷ The sum formula of Fibonacci numbers with alternate negatives:

$$\sum_{i=1}^{n} (-1)^{i+1} F_i = 1 + (-1)^{n+1} F_{n-1},$$
$$\mathcal{L} \sqcup \lambda^{(-)} = \mathcal{F} * \lambda^{(-)} \text{ (in the convolution setting)},$$
$$\frac{\mathcal{L}}{\mathcal{F}} = \lambda^{(-)} \text{ (in the orbit quotient setting)}.$$

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