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MULTIPLICITY RESULTS FOR A BOUNDARY VALUE PROBLEM OF p-KIRCHHOFF TYPE ON THE HALF-LINE VIA GENUS THEORY

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Abstract. We study the existence and multiplicity of infinitely distinct pairs of nontrivial solutions for a boundary value problem of Kirchhoff type posed on the half-line via the genus theory, which is a technique in critical point theory.

Keywords: genus theory; variational method; half-line; Kirchhoff equation; Palais-Smale condition

MSC 2020: 35J20, 35B38, 34B40, 74G35

1. Introduction

Kirchhoff equation was introduced in 1876 for the first time; it is an extension of the classical D'Alembert's wave equation by considering the effects of the changes in the length of a string during the vibrations. The existence and multiplicity of nontrivial solutions for a Kirchhoff type boundary value problem have been studied by several authors due to its huge importance and their applications in various fields, such as physics, electrical engineering and biology (see for example [10], [3] and the references therein).

Many authors have obtained solutions for Kirchhoff types equations exploited by variational methods. For more details, we refer the readers to [13], [1], [12] and the references therein.

Genus theory was introduced for the first time by Krasnoselskii [15], (see also [7]). Many authors were interested in the study of p-Kirchhoff type problems on bounded smooth domains of $\mathbb{R}^N$. For example, Corrêa and Figueiredo in [9], by using genus theory, proved existence and multiplicity of solutions for the following p-Kirchhoff

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boundary value problem:

$$-\left[M\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)\right]\Delta_p u = f(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain of $\mathbb{R}^N$, $1 < p < N$, M and f are continuous functions.

Also, in [5] and the references therein, some existence results are obtained for p-Kirchhoff type equation in the fractional case via the same technique.

Many authors generalized results for boundary value problems of p-Kirchhoff type equation via Krasnoselskii's genus, to $p(t)$-Kirchhoff type equation via the same technique on a bounded domain, where $p(t)$ is a function. For this, see for example [16].

The aim of this paper is to establish the existence and multiplicity of nontrivial solutions for a BVP of Kirchhoff type via genus theory on the half-line by using Clarke theorem.

We consider the following problem posed on $[0,\infty)$:

$$\begin{cases} (a + b\|u\|^p_{W_0^{1,p}})(-(|u'(t)|^{p-2}u'(t))' + |u(t)|^{p-2}u(t)) = \lambda h(t,u(t)), \\ u(0) = u(\infty) = 0, \end{cases} \tag{1.1}$$

$\lambda > 0$ is a numerical parameter, h is a Carathéodory function, p is a constant such that $p > 1$ and a, b are positive constants.

$L^p(0,\infty)$ is defined by

$$L^p(0,\infty) = \left\{u\colon\ (0,\infty) \to \mathbb{R} \text{ is measurable, } \int_0^\infty |u(t)|^p \,\mathrm{d}t < \infty\right\},$$

endowed with the norm

$$|u|_p = \left(\int_0^\infty |u(t)|^p \,\mathrm{d}t\right)^{1/p}.$$

The Sobolev space $W^{1,p}(0,\infty)$ is defined by

$$W^{1,p}(0,\infty) = \{u \in L^p(0,\infty)\colon\ u' \in L^p(0,\infty)\}$$

with a norm defined by $\|u\|_{1,p} = (|u|_p^p + |u'|_p^p)^{1/p}$.

Denote by $C([0,\infty))$ the space of continuous functions on $[0,\infty)$, $C_0^\infty([0,\infty))$ the space of indefinitely differentiable functions with compact support in $[0,\infty)$.

Now $W_0^{1,p}(0,\infty)$ denotes the closure of $C_0^\infty(0,\infty)$ in $W^{1,p}(0,\infty)$.

Let $r\colon\ [0,\infty) \to (0,\infty)$ be a continuously differentiable and bounded function with $M_1 = \max(\|r\|_{L^q}, \|r'\|_{L^q}) < \infty$.

$C_{l,r}(0;\infty)$ is defined by

$$C_{l,r}(0;\infty) = \left\{u \in C([0,\infty);\mathbb{R})\colon\ \lim_{t\to\infty} r(t)u(t) \text{ exists}\right\}$$

endowed with the norm

$$\|u\|_{\infty,r} = \sup_{t\in[0,\infty)} r(t)|u(t)|.$$

Proposition 1.1 (See [6]). *$L^p(0,\infty)$, $W^{1,p}(0,\infty)$ and $W_0^{1,p}(0,\infty)$ with $1 < p < \infty$ are separable, reflexive and uniformly convex Banach spaces.*

We now recall the Krasnoselskii genus and details on this may be found in ([2], [7], [14], [15]). Let E be a real Banach space. Let us denote by Σ the class of all closed subsets $A \subset E \setminus \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

Definition 1.1. Let $A \in \Sigma$. The Krasnoselskii genus $\gamma(A)$ is defined as being the least positive integer n such that there is an odd mapping $\varphi \in C(A, \mathbb{R}^n \setminus \{0\})$. If such n does not exist, we set $\gamma(A) = \infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.

Theorem 1.1 (See [14]). *Let $E = \mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

Note $\gamma(S^{N-1}) = N$. If E is of infinite dimension and separable and S is the unit sphere in E, then $\gamma(S) = \infty$.

Proposition 1.2 (See [14]). *Let $A, B \in \Sigma$. Then if there exists an odd map $f \in C(A, B)$, then $\gamma(A) \leqslant \gamma(B)$. Consequently, if there exists an odd homeomorphism $f\colon A \to B$, then $\gamma(A) = \gamma(B)$.*

Definition 1.2. Let $J \in C^1(E, \mathbb{R})$. If any sequence $(u_n) \subset E$ for which $(J(u_n))$ is bounded and $J'(u_n) \to 0$ when $n \to \infty$ in E' possesses a convergent subsequence, then we say that J satisfies the Palais-Smale condition (the (PS) condition).

We now state a theorem due to Clarke.

Theorem 1.2 (See [8], [18]). *Let $J \in C^1(E, \mathbb{R})$ be a functional satisfying the Palais-Smale condition. Also suppose that:*

- *J is bounded from below and even;*
- *there is a compact set $K \in \Sigma$ such that $\gamma(K) = k$ and $\sup_{x\in K} J(x) < J(0)$.*

Then J possesses at least k pairs of distinct critical points and their corresponding critical values are less than $J(0)$.

Definition 1.3. We say that $u \in W_0^{1,p}(0,\infty) = X$ is a weak solution of Problem (1.1) if and only if

$$(a + b\|u\|^p_{W_0^{1,p}})\left(\int_0^\infty |u'(t)|^{p-2}u'(t)v'(t)\,\mathrm{d}t + \int_0^\infty |u(t)|^{p-2}u(t)v(t)\,\mathrm{d}t\right) = \lambda\int_0^\infty h(t,u(t))v(t)\,\mathrm{d}t$$

for all $v \in X$.

2. Embedding

Similar reasoning as in [11] yields the following compactness criterion in the space $C_{l,r}([0,\infty),\mathbb{R})$.

Lemma 2.1. *Let $D \subset C_{l,r}([0,\infty),\mathbb{R})$ be a bounded set. Then D is relatively compact if the following conditions hold:*

(a) *D is equicontinuous on any compact sub-interval of $\mathbb{R}^+$, i.e., for all $J \subset [0,\infty)$ compact for all $\varepsilon > 0$, $\exists\,\delta > 0$ for all $t_1, t_2 \in J$: $|t_1 - t_2| < \delta \Rightarrow |r(t_1)u(t_1) - r(t_2)u(t_2)| \leqslant \varepsilon$ for all $u \in D$;*

(b) *D is equiconvergent at ∞ i.e., for all $\varepsilon > 0$, $\exists T = T(\varepsilon) > 0$ such that for all t: $t \geqslant T(\varepsilon) \Rightarrow |r(t)u(t) - (ru)(\infty)| \leqslant \varepsilon$ for all $u \in D$.*

Lemma 2.2. *$W_0^{1,p}(0,\infty)$ is continuously embedded in $C_{l,r}(0,\infty)$.*

Proof. For $u \in W_0^{1,p}(0,\infty)$ we have

$$\begin{aligned}
|r(t)u(t)| &= |r(t)u(t) - r(0)u(0)| \\
&= \left|\int_0^t (ru)'(s)\,\mathrm{d}s\right| \\
&\leqslant \left|\int_0^t (r'(s)u(s) + r(s)u'(s))\,\mathrm{d}s\right| \\
&\leqslant \int_0^\infty |r'(s)u(s)|\,\mathrm{d}s + \int_0^\infty |r(s)u'(s)|\,\mathrm{d}s \\
&\leqslant \left(\int_0^\infty (r'(s))^q\,\mathrm{d}s\right)^{1/q}\left(\int_0^\infty (u(s))^p\,\mathrm{d}s\right)^{1/p} \\
&\quad + \left(\int_0^\infty (r(s))^q\,\mathrm{d}s\right)^{1/q}\left(\int_0^\infty (u'(s))^p\,\mathrm{d}s\right)^{1/p} \\
&\leqslant |r'|_{L^q}\left(\int_0^\infty (u(s))^p\,\mathrm{d}s\right)^{1/p} + |r|_{L^q}\left(\int_0^\infty (u'(s))^p\,\mathrm{d}s\right)^{1/p} \\
&\leqslant \max(|r'|_{L^q}, |r|_{L^q})\|u\|_{W_0^{1,p}}.
\end{aligned}$$

Hence: $\|u\|_{\infty,r} \leqslant M\|u\|_{W_0^{1,p}}$, where $M = \max(|r'|_{L^q}, |r|_{L^q})$.

Then $W_0^{1,p} \hookrightarrow C_{l,r}[0;\infty)$. □

Lemma 2.3. *The embedding*

$$W_0^{1,p} \hookrightarrow\hookrightarrow C_{l,r}[0,\infty)$$

is compact.

P r o o f. Let $D \subset X = W_0^{1,p}(0,\infty)$ be a bounded set, then it is bounded in $C_{l,r}[0;\infty)$ $(X \hookrightarrow C_{l,r})$.

Let $R > 0$ be such that for all $u \in D$: $\|u\|_{W_0^{1,p}} \leqslant R$.

a) D is equicontinuous on every compact interval of $[0,\infty)$: indeed, let $u \in D$ and $t_1, t_2 \in J \subset [0,\infty)$, where J is a compact sub-interval; by using Holder's inequality, we have

$$\begin{aligned}
|r(t_1)u(t_1) - r(t_2)u(t_2)| &= \left| \int_{t_1}^{t_2} (ru)'(s)\,\mathrm{d}s \right| \\
&= \left| \int_{t_1}^{t_2} (r'(s)u(s) + r(s)u'(s))\,\mathrm{d}s \right| \\
&\leqslant \int_{t_1}^{t_2} |r'(s)u(s)|\,\mathrm{d}s + \int_{t_1}^{t_2} |r(s)u'(s)|\,\mathrm{d}s \\
&\leqslant \left(\int_{t_1}^{t_2} r'^q(s)\,\mathrm{d}s \right)^{1/q} \left(\int_{t_1}^{t_2} u^p(s)\,\mathrm{d}s \right)^{1/p} \\
&\quad + \left(\int_{t_1}^{t_2} r^q(s)\,\mathrm{d}s \right)^{1/q} \left(\int_{t_1}^{t_2} u'^p(s)\,\mathrm{d}s \right)^{1/p} \\
&\leqslant \max\left(\int_{t_1}^{t_2} r'^q(s)\,\mathrm{d}s; \int_{t_1}^{t_2} r^q(s)\,\mathrm{d}s \right) \|u\|_X \\
&\leqslant R \max\left(\int_{t_1}^{t_2} r'^q(s)\,\mathrm{d}s; \int_{t_1}^{t_2} r^q(s)\,\mathrm{d}s \right) \to 0
\end{aligned}$$

as $|t_1 - t_2| \to 0$.

b) D is equiconvergent at ∞; for $t \in [0,\infty)$ and $u \in D$

$$\begin{aligned}
|(ru)(t) - (ru)(\infty)| &= \left| \int_t^{\infty} (ru)'(s)\,\mathrm{d}s \right| \\
&\leqslant \left| \int_t^{\infty} (r'(s)u(s) + r(s)u'(s))\,\mathrm{d}s \right| \\
&\leqslant R \max\left(\int_t^{\infty} r'^q(s)\,\mathrm{d}s; \int_t^{\infty} r^q(s)\,\mathrm{d}s \right) \to 0
\end{aligned}$$

as $t \to \infty$.

Hence $X \hookrightarrow\hookrightarrow C_{l,r}[0,\infty)$. □

3. Main results

Theorem 3.1. *Assume the following are satisfied.*

(h_1) *There exist a constant $\gamma \in [0, 2p-1)$ and two positive functions α, β with α/r, $\beta/r^{\gamma+1} \in L^1(0,\infty)$ such that*

$$|h(t,u)| \leqslant \alpha(t) + \beta(t)|u|^\gamma \quad \text{for any } (t,u) \in [0,\infty) \times \mathbb{R}.$$

(h_2) *$h(t,u)$ is odd with respect to u and $H(t,u) = \int_0^u h(t,\xi)\,\mathrm{d}\xi > 0$ for every $(t,u) \in [0,\infty) \times \mathbb{R} \setminus \{0\}$.*

Then for any $k \in \mathbb{N}$, there exists λ_k such that when $\lambda > \lambda_k$, Problem (1.1) *has at least k distinct pairs of nontrivial solutions.*

P r o o f. The corresponding functional to Problem (1.1) is defined as

$$\varphi(u) = \frac{a}{p}\|u\|^p_{W_0^{1,p}} + \frac{b}{2p}\|u\|^{2p}_{W_0^{1,p}} - \lambda \int_0^\infty H(t,u(t))\,\mathrm{d}t, \tag{3.1}$$

where $H(t,u) = \int_0^u h(t,\xi)\,\mathrm{d}\xi$.

Claim 1. $\varphi \in C^1(X,\mathbb{R})$.

Firstly, we show that φ is well defined. Indeed, let $u \in X$. It follows from (h_1) and Lemma 2.2 that

$$\begin{aligned}
\int_0^\infty |H(t,u(t))|\,\mathrm{d}t &\leqslant \int_0^\infty \int_0^u |h(t,\xi)|\,\mathrm{d}\xi\,\mathrm{d}t \\
&\leqslant \int_0^\infty \int_0^u (\alpha(t) + \beta(t)|\xi|^\gamma)\,\mathrm{d}\xi\,\mathrm{d}t \\
&\leqslant \int_0^\infty \left(\alpha(t)u(t) + \frac{\beta(t)}{\gamma+1}|u(t)|^{\gamma+1}\right)\mathrm{d}t \\
&\leqslant \int_0^\infty \left(\frac{\alpha(t)}{r(t)} r(t)|u(t)| + \frac{\beta(t)}{(\gamma+1)(r(t))^{\gamma+1}}(r(t)|u(t)|)^{\gamma+1}\right)\mathrm{d}t \\
&\leqslant \|u\|_{\infty,r} \int_0^\infty \frac{\alpha(t)}{r(t)}\,\mathrm{d}t + \frac{\|u\|^{\gamma+1}_{\infty,r}}{\gamma+1} \int_0^\infty \frac{\beta(t)}{(r(t))^{\gamma+1}}\,\mathrm{d}t \\
&\leqslant c\|u\|_{W_0^{1,p}} \left|\frac{\alpha}{r}\right|_{L^1} + \frac{c^{\gamma+1}}{\gamma+1}\|u\|^{\gamma+1}_{W_0^{1,p}} \left|\frac{\beta}{r^{\gamma+1}}\right|_{L^1} < \infty.
\end{aligned}$$

Then φ is well defined on X.

Secondly, we show that φ is Gâteaux-differentiable. Indeed, we define the functional $\varphi_1 \colon W_0^{1,p} \mapsto \mathbb{R}$ by

$$\varphi_1(u) = \frac{a}{p}\|u\|^p_{W_0^{1,p}} + \frac{b}{2p}\|u\|^{2p}_{W_0^{1,p}}.$$

It is well known that φ_1 is Gâteaux-differentiable (see [4]) and

$$\varphi_1'(u)\cdot v = (a+b\|u\|^p_{W_0^{1,p}})\bigg(\int_0^\infty |u(t)|^{p-2}u(t)v(t)\,\mathrm{d}t + \int_0^\infty |u'(t)|^{p-2}u'(t)v'(t)\,\mathrm{d}t\bigg).$$

For $u\in X$, we consider the functional φ_2 defined by

$$\varphi_2(u) = \int_0^\infty H(t,u(t))\,\mathrm{d}t.$$

It suffices to prove that φ_2 is Gâteaux-differentiable.

For all $v\in X$, for small $s>0$ we have

$$\varphi_2(u+sv)-\varphi_2(u) = \int_0^\infty (H(t;u(t)+sv(t))-H(t;u(t)))\,\mathrm{d}t.$$

From the mean value theorem, we have

$$H(t,u(t)+sv(t))-H(t,u(t)) = sv(t)h(t,u(t)+s\theta v(t)),\quad \theta\in(0,1).$$

Then

$$\varphi_2(u+sv)-\varphi_2(u) = \int_0^\infty sv(t)h(t,u(t)+s\theta v(t))\,\mathrm{d}t,$$

hence,

$$\frac{\varphi_2(u+sv)-\varphi_2(u)}{s} = \int_0^\infty h(t,u(t)+s\theta v(t))v(t)\,\mathrm{d}t.$$

For all $u,v\in X$, and for any $t>0$ and small s, and for $\theta\in(0,1)$ we have

$$\begin{aligned}
|z_{u,v}| &= |u(t)+s\theta v(t)|r(t)\\
&\leqslant r(t)|u(t)| + s\theta|v(t)|r(t)\\
&\leqslant r(t)|u(t)| + r(t)|v(t)|\\
&\leqslant \|u\|_{\infty,r} + \|v\|_{\infty,r}\\
&\leqslant c\|u\|_{W_0^{1,p}} + c\|v\|_{W_0^{1,p}}\\
&\leqslant c(\|u\|_{W_0^{1,p}} + \|v\|_{W_0^{1,p}}) = R_{u,v}.
\end{aligned}$$

Moreover, from (h_1) we have:

$$\begin{aligned}|h(t;u+s\theta v)||v(t)| &\leqslant (\alpha(t)+\beta(t)|u+s\theta v|^\gamma)|v(t)|\\ &\leqslant \left(\alpha(t)+\frac{\beta(t)}{(r(t))^\gamma}|u+s\theta v|^\gamma (r(t))^\gamma\right)|v(t)|\\ &\leqslant \alpha(t)|v(t)|+\frac{\beta(t)}{(r(t))^\gamma}R_{u,v}^\gamma|v(t)|\\ &\leqslant \frac{\alpha(t)}{r(t)}r(t)|v(t)|+\frac{\beta(t)}{(r(t))^{\gamma+1}}R_{u,v}^\gamma|v(t)|r(t)\\ &\leqslant \frac{\alpha(t)}{r(t)}\|v\|_{\infty,r}+\frac{\beta(t)}{(r(t))^{\gamma+1}}R_{u,v}^\gamma\|v\|_{\infty,r}\\ &\leqslant c\left(\frac{\alpha(t)}{r(t)}+\frac{\beta(t)}{(r(t))^{\gamma+1}}R_{u,v}^\gamma\right)\|v\|_{W_0^{1,p}}\in L^1(0;\infty).\end{aligned}$$

So from the Lebesgue dominated convergence theorem, we have

$$\lim_{s\to 0}\int_0^\infty h(t,u(t)+s\theta v(t))v(t)\,\mathrm{d}t=\int_0^\infty h(t,u(t))v(t)\,\mathrm{d}t. \tag{3.2}$$

Consequently, φ_2 is Gâteaux-differentiable and

$$\varphi_2'(u)\cdot v=\int_0^\infty h(t,u(t))v(t)\,\mathrm{d}t.$$

Thirdly, we show that φ' is continuous. Indeed, let $\{u_n\}\subset X$ be such that $u_n\to u$ as $n\to\infty$ in X; since X embeds continuously in $C_{l,r}$, then $u_n\to u$ as $n\to\infty$ in $C_{l,r}$, consequently, (u_n) is bounded in $C_{l,r}$, and thus there is $R>0$ such that $\|u_n\|_{\infty,r}\leqslant R$ for all n.

Now we have

$$\begin{aligned}\frac{|h(t,u_n(t)|}{r(t)} &\leqslant \frac{(\alpha(t)+\beta(t)|u_n(t)|^\gamma)}{r(t)}\\ &\leqslant \frac{\alpha(t)}{r(t)}+\frac{\beta(t)}{(r(t))^{\gamma+1}}(r(t))^\gamma|u_n(t)|^\gamma\\ &\leqslant \frac{\alpha(t)}{r(t)}\ +\frac{\beta(t)}{(r(t))^{\gamma+1}}\|u_n\|_{\infty,r}^\gamma\\ &\leqslant \frac{\alpha(t)}{r(t)}+R^\gamma\frac{\beta(t)}{(r(t))^{\gamma+1}}\in L^1(0,\infty),\end{aligned}$$

then from the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_0^\infty\frac{h(t,u_n(t))}{r(t)}=\int_0^\infty\frac{h(t,u(t))}{r(t)}\,\mathrm{d}t. \tag{3.3}$$

Moreover, for all $v \in X$

$$(\varphi_2'(u_n) - \varphi_2'(u), v) = \int_0^\infty (h(t, u_n(t)) - h(t, u(t)))v(t)\,\mathrm{d}t.$$

Then

$$|(\varphi_2'(u_n) - \varphi_2'(u), v)| \leqslant \int_0^\infty |h(t, u_n(t)) - h(t, u(t))| r(t) \frac{|v(t)|}{r(t)}\,\mathrm{d}t,$$

hence

$$\begin{aligned} |(\varphi_2'(u_n) - \varphi_2'(u), v)| &\leqslant \|v\|_{\infty,r} \int_0^\infty \frac{|h(t, u_n(t)) - h(t, u(t))|}{r(t)}\,\mathrm{d}t \\ &\leqslant c\|v\|_{W_0^{1,p}} \int_0^\infty \frac{|h(t, u_n(t)) - h(t, u(t))|}{r(t)}\,\mathrm{d}t. \end{aligned}$$

We have

$$\begin{aligned} \|\varphi_2'(u_n) - \varphi_2'(u)\|_{X'} &= \sup_{\|v\| \leqslant 1} |(\varphi_2'(u_n) - \varphi_2'(u), v))| \\ &\leqslant c \int_0^\infty \frac{(h(t, u_n(t)) - h(t, u(t)))}{r(t)}\,\mathrm{d}t, \end{aligned}$$

passing to the limit when $n \to \infty$, and from (3.3) we obtain $\varphi_2'(u_n) \to \varphi_2'(u)$ as $n \to \infty$, hence, φ_2' is continuous.

Then $\varphi \in C^1(X, \mathbb{R})$ and for all $u, v \in X$

$$\begin{aligned} \varphi'(u) \cdot v = (a + b\|u\|^p_{W_0^{1,p}}) \bigg(&\int_0^\infty |u(t)|^{p-2}u(t)v(t)\,\mathrm{d}t \\ &+ \int_0^\infty |u'(t)|^{p-2}u'(t)v'(t)\,\mathrm{d}t \bigg) - \lambda \int_0^\infty h(t, u(t))v(t)\,\mathrm{d}t. \end{aligned} \tag{3.4}$$

Thus, the critical points of φ are the weak solutions of (1.1).

Claim 2. φ is bounded from below.

Indeed, from the continuous embedding of X in $C_{l,r}([0, \infty))$, for any $u \in X$ we have

$$r(t)|u(t)| \leqslant \|u\|_{\infty,r} \leqslant c\|u\|_{W_0^{1,p}}, \tag{3.5}$$

where c is the best constant of the continuous embedding. It follows from conditions (h_1), (h_2) and (3.5) that

$$
\begin{aligned}
\varphi(u) &= \frac{a}{p}\|u\|^p_{W_0^{1,p}} + \frac{b}{2p}\|u\|^{2p}_{W_0^{1,p}} - \lambda\int_0^\infty H(t,u(t))\,\mathrm{d}t \\
&\geqslant \frac{b}{2p}\|u\|^{2p}_{W_0^{1,p}} - \lambda\int_0^\infty H(t,u(t))\,\mathrm{d}t \\
&\geqslant \frac{b}{2p}\|u\|^{2p}_{W_0^{1,p}} - \lambda\int_0^\infty \left(\alpha(t)u(t) + \frac{\beta(t)}{\gamma+1}|u(t)|^{\gamma+1}\right)\mathrm{d}t \\
&\geqslant \frac{b}{2p}\|u\|^{2p}_{W_0^{1,p}} \\
&\quad - \lambda\int_0^\infty \left(\frac{\alpha(t)}{r(t)}r(t)|u(t)| + \frac{\beta(t)}{(\gamma+1)(r(t))^{\gamma+1}}(r(t)|u(t)|)^{\gamma+1}\right)\mathrm{d}t \\
&\geqslant \frac{b}{2p}\|u\|^{2p}_{W_0^{1,p}} - \lambda\|u\|_{\infty,r}\int_0^\infty \frac{\alpha(t)}{r(t)} - \lambda\frac{\|u\|^{\gamma+1}_{\infty,r}}{\gamma+1}\int_0^\infty \frac{\beta(t)}{(r(t))^{\gamma+1}}\,\mathrm{d}t \\
&\geqslant \frac{b}{2p}\|u\|^{2p}_{W_0^{1,p}} - \lambda c\|u\|_{W_0^{1,p}}\left|\frac{\alpha}{r}\right|_{L^1} - \lambda\frac{c^{\gamma+1}}{\gamma+1}\|u\|^{\gamma+1}_{W_0^{1,p}}\left|\frac{\beta}{r^{\gamma+1}}\right|_{L^1},
\end{aligned}
\tag{3.6}
$$

for any $u \in X$. Since $\gamma < 2p-1$, then $\lim\limits_{\|u\|_{W_0^{1,p}}\to\infty}\varphi(u) = \infty$ and consequently, φ is bounded from below.

Claim 3. The functional φ satisfies the (PS) condition.

Indeed, let $(u_n) \subset X$ be a Palais-Smale sequence for φ, i.e., $(\varphi(u_n))$ is a bounded sequence and $\lim\limits_{n\to\infty}\varphi'(u_n) = 0$. Thus, there exists a positive constant B such that

$$
\varphi(u_n) \leqslant B. \tag{3.7}
$$

From (3.6), (3.7) and since $\gamma < 2p-1$, in all cases we deduce that the sequence (u_n) is bounded in X. Thus, passing to a subsequence if necessary, there exists $u \in X$ such that $u_n \rightharpoonup u$ weakly in X. Moreover, by (3.4), we have

$$
\begin{aligned}
&(\varphi'(u_n) - \varphi'(u))\cdot(u_n - u) \\
&\quad = (a + b\|u_n\|^p_{W_0^{1,p}})\bigg(\int_0^\infty |u_n(t)|^{p-2}u_n(t)(u_n(t)-u(t))\,\mathrm{d}t \\
&\qquad + \int_0^\infty |u_n'(t)|^{p-2}u_n'(t)(u_n'(t)-u'(t))\,\mathrm{d}t\bigg) - \lambda\int_0^\infty h(t,u_n(t))(u_n(t)-u(t))\,\mathrm{d}t \\
&\qquad - (a + b\|u\|^p_{W_0^{1,p}})\bigg(\int_0^\infty |u(t)|^{p-2}u(t)(u_n(t)-u(t))\,\mathrm{d}t \\
&\qquad + \int_0^\infty |u'(t)|^{p-2}u'(t)(u_n'(t)-u'(t))\,\mathrm{d}t\bigg) + \lambda\int_0^\infty h(t,u(t))(u_n(t)-u(t))\,\mathrm{d}t.
\end{aligned}
$$

Since the embedding of X in $C_{l,r}([0,\infty))$ is compact, it follows that (u_n) converges to u in $C_{l,r}([0,\infty))$, and thus there exist $M_1, M_2 > 0$ such that

$$\|u_n - u\|_{\infty,r} < M_1 \quad \text{and} \quad \|u_n\|_{\infty,r} < M_2. \tag{3.8}$$

Now by using (h_1) we have

$$\begin{aligned}
&|h(t,u_n(t)) - h(t,u(t))||u_n(t) - u(t)| \\
&\qquad \leqslant |h(t,u_n(t)) - h(t,u(t))|\frac{1}{r(t)}r(t)|u_n(t) - u(t)| \\
&\qquad \leqslant \frac{1}{r(t)}\|u_n - u\|_{\infty,r}|h(t,u_n(t)) - h(t,u(t))| \\
&\qquad \leqslant \frac{M_1}{r(t)}(|h(t,u_n(t))| + |h(t,u(t))|) \\
&\qquad \leqslant \frac{M_1}{r(t)}(\alpha(t) + \beta(t)|u_n(t)|^\gamma + \alpha(t) + \beta(t)|u(t)|^\gamma) \\
&\qquad \leqslant 2M_1\frac{\alpha(t)}{(r(t))} + M_1\frac{\beta(t)}{(r(t))}(|u_n(t)|^\gamma + |u(t)|^\gamma) \\
&\qquad \leqslant 2M_1\frac{\alpha(t)}{(r(t))} + M_1\frac{\beta(t)}{(r(t))^{\gamma+1}}(r(t))^\gamma(|u_n(t)|^\gamma + |u(t)|^\gamma) \\
&\qquad \leqslant 2M_1\frac{\alpha(t)}{(r(t))} + M_1\frac{\beta(t)}{(r(t))^{\gamma+1}}(\|u_n\|_{\infty,r}^\gamma + \|u\|_{\infty,r}^\gamma) \\
&\qquad \leqslant 2M_1\frac{\alpha(t)}{(r(t))} + M_1\frac{\beta(t)}{(r(t))^{\gamma+1}}(M_2^\gamma + \|u\|_{\infty,r}^\gamma) \in L^1(0,\infty).
\end{aligned}$$

From the Lebesgue Dominated Convergence Theorem we have

$$\lambda\int_0^\infty (h(t,u_n(t)) - h(t,u(t)))(u_n(t) - u(t))\,\mathrm{d}t \to 0. \tag{3.9}$$

Since $\varphi'(u_n) \to 0$ and $u_n \rightharpoonup u$ in X, we have $(\varphi'(u_n) - \varphi'(u)) \cdot (u_n - u) \to 0$ as $n \to \infty$. Consequently, from (3.9) we have $S_n \to 0$ as $n \to \infty$, where

$$\begin{aligned}
S_n = {} & (a + b\|u_n\|_{W_0^{1,p}}^p)\bigg(\int_0^\infty |u_n(t)|^{p-2}u_n(t)(u_n(t) - u(t))\,\mathrm{d}t \\
& + \int_0^\infty |u_n'(t)|^{p-2}u_n'(t)(u_n'(t) - u'(t))\,\mathrm{d}t\bigg) \\
& - (a + b\|u\|_{W_0^{1,p}}^p)\bigg(\int_0^\infty |u(t)|^{p-2}u(t)(u_n(t) - u(t))\,\mathrm{d}t \\
& + \int_0^\infty |u'(t)|^{p-2}u'(t)(u_n'(t) - u'(t))\,\mathrm{d}t\bigg).
\end{aligned}$$

We can rewrite S_n as

$$\begin{aligned}S_n &= (a+b\|u_n\|^p_{W_0^{1,p}}) \times \left[\int_0^\infty (|u_n'(t)|^{p-2}u_n'(t) - |u'(t)|^{p-2}u'(t))(u_n'(t)-u'(t))\,\mathrm{d}t\right.\\ &\quad \left. + \int_0^\infty (|u_n(t)|^{p-2}u_n(t) - |u(t)|^{p-2}u(t))(u_n(t)-u(t))\,\mathrm{d}t\right]\\ &\quad + (a+b\|u_n\|^p_{W_0^{1,p}}) \times \left[\int_0^\infty |u'(t)|^{p-2}u'(t)(u_n'(t)-u'(t))\,\mathrm{d}t\right.\\ &\quad \left. + \int_0^\infty |u(t)|^{p-2}u(t)(u_n(t)-u(t))\,\mathrm{d}t\right] - (a+b\|u\|^p_{W_0^{1,p}})\\ &\quad \times \left(\int_0^\infty |u(t)|^{p-2}u(t)(u_n(t)-u(t))\,\mathrm{d}t + \int_0^\infty |u'(t)|^{p-2}u'(t)(u_n'(t)-u'(t))\,\mathrm{d}t\right).\end{aligned}$$

From the weak convergence of (u_n) in X and since $|u'|^{p-2}u'$, $|u|^{p-2}u \in L^{p/(p-1)}(0,\infty)$,

$$\int_0^\infty |u'(t)|^{p-2}u'(t)(u_n'(t)-u'(t))\,\mathrm{d}t + \int_0^\infty |u(t)|^{p-2}u(t)(u_n(t)-u(t))\,\mathrm{d}t \to 0 \tag{3.10}$$

as $n\to\infty$, hence from (3.10) and $S_n \to 0$, we get

$$\begin{aligned}(a+b\|u_n\|^p_{W_0^{1,p}}) &\times \left[\int_0^\infty (|u_n'(t)|^{p-2}u_n'(t) - |u'(t)|^{p-2}u'(t))(u_n'(t)-u'(t))\,\mathrm{d}t\right.\\ &\left. + \int_0^\infty (|u_n(t)|^{p-2}u_n(t) - |u(t)|^{p-2}u(t))(u_n(t)-u(t))\,\mathrm{d}t\right] \to 0\end{aligned}$$

as $n\to\infty$.

Since (u_n) is bounded in $W_0^{1,p}$,

$$\begin{aligned}&\int_0^\infty (|u_n'(t)|^{p-2}u_n'(t) - |u'(t)|^{p-2}u'(t))(u_n'(t)-u'(t))\,\mathrm{d}t\\ &\quad + \int_0^\infty (|u_n(t)|^{p-2}u_n(t) - |u(t)|^{p-2}u(t))(u_n(t)-u(t))\,\mathrm{d}t \to 0\end{aligned}$$

as $n\to\infty$.

To complete the proof of the Palais-Smale condition we follow the arguments used in the lecture notes [17]. We distinguish two cases: $p \geqslant 2$ and $1 < p < 2$.

The first case: $p \geqslant 2$.

From the inequality

$$(|x|^{p-2}x - |y|^{p-2}y)(x-y) \geqslant \frac{1}{2^p}|x-y|^p \quad \text{if } p \geqslant 2,$$

we get

$$\int_0^\infty |u_n'(t) - u'(t)|^p \,\mathrm{d}t \leqslant 2^p \int_0^\infty (|u_n'(t)|^{p-2}u_n'(t) - |u'(t)|^{p-2}u'(t))(u_n'(t) - u'(t)) \,\mathrm{d}t,$$

and

$$\int_0^\infty |u_n(t) - u(t)|^p \,\mathrm{d}t \leqslant 2^p \int_0^\infty (|u_n(t)|^{p-2}u_n(t) - |u(t)|^{p-2}u(t))(u_n(t) - u(t)) \,\mathrm{d}t.$$

Then

$$\begin{aligned}\int_0^\infty |u_n'(t) - u'(t)|^p \,\mathrm{d}t + \int_0^\infty |u_n(t) - u(t)|^p \,\mathrm{d}t \\ \leqslant 2^p \bigg[\int_0^\infty (|u_n'(t)|^{p-2}u_n'(t) - |u'(t)|^{p-2}u'(t))(u_n'(t) - u'(t)) \,\mathrm{d}t \\ + \int_0^\infty (|u_n(t)|^{p-2}u_n(t) - |u(t)|^{p-2}u(t))(u_n(t) - u(t)) \,\mathrm{d}t \bigg] \to 0\end{aligned}$$

as $n \to \infty$.

So

$$\int_0^\infty |u_n'(t) - u'(t)|^p \,\mathrm{d}t + \int_0^\infty |u_n(t) - u(t)|^p \,\mathrm{d}t \to 0$$

as $n \to \infty$.

The second case: $1 < p < 2$.

From the inequality

$$[(|x|^{p-2}x - |y|^{p-2}y)(x - y)]^{p/2} \geqslant \frac{p-1}{\sqrt{2}} \frac{|x - y|^p}{(|x|^p + |y|^p)^{(2-p)/p}} \quad \text{if } 1 < p < 2,$$

we have

$$\begin{aligned}\int_0^\infty |u_n'(t) - u'(t)|^p \,\mathrm{d}t \leqslant \frac{\sqrt{2}}{p-1} \int_0^\infty [(|u_n'(t)|^{p-2}u_n'(t) - |u'(t)|^{p-2}u'(t)) \\ \times (u_n'(t) - u'(t))]^{p/2} (|u_n'(t)|^p + |u'(t)|^p)^{(2-p)/p} \,\mathrm{d}t.\end{aligned}$$

By using Holder's inequality, we have

$$\begin{aligned}\int_0^\infty |u_n'(t) - u'(t)|^p \,\mathrm{d}t \leqslant \frac{\sqrt{2}}{p-1} \|((|u_n'(t)|^{p-2}u_n'(t) - |u'(t)|^{p-2}u'(t))(u_n' - u'))^{p/2}\|_{2/p} \\ \times \|(|u_n'(t)|^p + |u'(t)|^p)^{(2-p)/p}\|_{p/(2-p)}.\end{aligned}$$

In the same way we obtain

$$\int_0^\infty |u_n(t)-u(t)|^p\,\mathrm{d}t \leqslant \frac{\sqrt{2}}{p-1}\|((|u_n(t)|^{p-2}u_n(t)-|u(t)|^{p-2}u(t))(u_n-u))^{p/2}\|_{2/p} \times \|(|u_n(t)|^p+|u(t)|^p)^{(2-p)/p}\|_{p/(2-p)}.$$

So,

$$\begin{aligned}\int_0^\infty |u_n'(t)-u'(t)|^p\,\mathrm{d}t &+ \int_0^\infty |u_n(t)-u(t)|^p\,\mathrm{d}t \\ &\leqslant \frac{\sqrt{2}}{p-1}[\|((|u_n'(t)|^{p-2}u_n'(t)-|u'(t)|^{p-2}u'(t))(u_n'-u'))^{p/2}\|_{2/p} \\ &\quad\times \|(|u_n'(t)|^p+|u'(t)|^p)^{(2-p)/p}\|_{p/(2-p)} \\ &\quad+\|((|u_n(t)|^{p-2}u_n(t)-|u(t)|^{p-2}u(t))(u_n-u))^{p/2}\|_{2/p} \\ &\quad\times \|(|u_n(t)|^p+|u(t)|^p)^{(2-p)/p}\|_{p/(2-p)}].\end{aligned}$$

Since (u_n) is bounded in $W_0^{1,p}$, the constants M_1' and M_2' can be chosen such that

$$\begin{aligned}\|(|u_n'(t)|^p+|u'(t)|^p)^{(2-p)/p}\|_{p/(2-p)} &\leqslant M_1', \\ \|(|u_n(t)|^p+|u(t)|^p)^{(2-p)/p}\|_{p/(2-p)} &\leqslant M_2'.\end{aligned}$$

Then

$$\begin{aligned}\|u_n-u\|^p \leqslant \frac{\sqrt{2}}{p-1}\max(M_1',M_2')[&\|((|u_n'(t)|^{p-2}u_n'(t)-|u'(t)|^{p-2}u'(t))(u_n'-u'))^{p/2}\|_{2/p} \\ &+\|((|u_n(t)|^{p-2}u_n(t)-|u(t)|^{p-2}u(t))(u_n-u))^{p/2}\|_{2/p}] \to 0\end{aligned}$$

as $n\to\infty$.

We conclude that $\|u_n-u\|_{W_0^{1,p}}\to 0$ as $n\to\infty$, so φ satisfies the Palais-Smale condition.

Claim 4. Consider $\{e_1,e_2,\ldots\}$, a Schauder basis of the space $W_0^{1,p}(0,\infty)$ (see [19]), and for each $k\in\mathbb{N}$, consider X_k, the subset of $W_0^{1,p}(0,\infty)$ generated by the k vectors $\{e_1,e_2,\ldots,e_k\}$. Clearly X_k is a subspace of $W_0^{1,p}(0,\infty)$.

Consider, for $\rho>0$,

$$K_k(\rho)=\left\{u\in X_k\colon\ \|u\|_{W_0^{1,p}}^2=\sum_{i=1}^k\xi_i^2=\rho^2\right\}.$$

For any $\rho>0$, we consider the odd homeomorphism $\chi\colon K_k(\rho)\to S^{k-1}$ defined by $\chi(u)=(\xi_1,\xi_2,\ldots,\xi_k)$, where S^{k-1} is the sphere in $\mathbb{R}^k$. From Theorem 1.1 and Proposition 1.2, we conclude that $\gamma(K_k(\rho))=k$.

It follows from hypothesis (h_2) that $\int_0^\infty H(t,u(t))\,\mathrm{d}t > 0$ for any $u \in K_k(\rho)$. Then $\mu_k = \inf\limits_{u\in K_k(\rho)} \int_0^\infty H(t,u(t))\,\mathrm{d}t$ is strictly positive.

Let $\lambda_k = \mu_k^{-1}(a+b\rho^p)\rho^p$, and note $\lambda_k > 0$. Then when $\lambda > \lambda_k$, then for any $u \in K_k(\rho)$ we have

$$\begin{aligned}\varphi(u) &= \frac{a}{p}\|u\|^p_{W_0^{1,p}} + \frac{b}{2p}\|u\|^{2p}_{W_0^{1,p}} - \lambda\int_0^\infty H(t,u(t))\,\mathrm{d}t\\ &\leqslant (a+b\|u\|^p_{W_0^{1,p}})\|u\|^p_{W_0^{1,p}} - \lambda\mu_k\\ &< (a+b\rho^p)\rho^p - \lambda_k\mu_k = 0.\end{aligned}$$

Theorem 1.2 guarantees that the functional φ has at least k pairs of different critical points. Hence, Problem (1.1) has at least k distinct pairs of nontrivial solutions. □

4. Example

We consider the problem

$$\text{(4.1)}\qquad \begin{cases}(1+\|u\|^4_{W_0^{1,4}})(-(|u'(t)|^2u'(t))' + |u(t)|^2u(t)) = \mathrm{e}^{-t}u|u|, & t\in[0,\infty)\\ u(0) = u(\infty) = 0,\end{cases}$$

where $h(t,x) = \mathrm{e}^{-t}x|x|$, $a = b = 1$ and $\lambda = 1$. We take

$$r(t) = \frac{1}{1+t},\quad \alpha(t) = \frac{1}{1+t^4},\quad \beta(t) = \mathrm{e}^{-t}\quad\text{and}\quad \gamma = 2.$$

It is easy to see that conditions (h_1) and (h_2) hold. Consequently, problem (4.1) has infinitely many nontrivial solutions.

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