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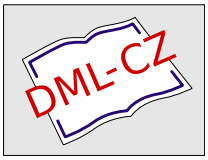
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# COEFFICIENT BOUNDS FOR THE FUNCTIONS AND INVERSE FUNCTIONS IN THE NEWLY CONSTRUCTED SUBCLASS OF BI-UNIVALENT FUNCTIONS

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*Abstract.* We introduce a novel subclass of bi-univalent functions with respect to symmetric points using generalized bivariate Fibonacci polynomials. Then we determine certain bounds for the initial coefficients of Maclaurin expansion of the functions belonging to this class of functions. In addition, we study the well-known Fekete-Szegö inequality for defined function class. We also discuss the same problems for the inverses of functions in this subclass. Further, we give a few corollaries for particular parameter values.

*Keywords*: generalized bivariate Fibonacci polynomial; analytic function; bi-univalent function; coefficient estimate; Fekete-Szegö functional; coefficients of inverse function

*MSC 2020*: 30C45, 33C50

## 1. Introduction

In the last few years, many mathematicians have focused their attention on the use of certain functions within the theory of geometric functions. As a result of this, many subclasses of bi-univalent functions which subordinate to special polynomials such as Fibonacci [13], [14], [16], [17], [37], Chebyshev [4], [5], [10], [36], Faber [32], [2], [18], Lucas [6], [3], [7], [26], Horadam [30], [34], [35], Gegenbauer [20], [1], Balancing [29], and their generalizations have been introduced. The subclasses of functions in question have diversified by the use of some differential operators, see [19], [28], [25], [33], [15]. In this paper, inspired by previous studies like [8], [9], [22], we introduce a new subclass of analytic bi-univalent functions using generalized bivariate Fibonacci polynomials. The use of generalized bivariate Fibonacci polynomials depends on their relationship with various special polynomials and number sequences.

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These polynomials and sequences are distinguished from each other by certain values of the parameters.

This paper is organized as follows: Section 1 is divided into two subsections. In the first subsection, some basic properties about univalent function theory are given, while the generalized bivariate Fibonacci polynomials that we used in subordination are reminded in the second subsection. Section 2 is dedicated to the upper bounds estimations for the initial Maclaurin coefficients and the Fekete-Szegö inequality of the functions within defined subclass, respectively. Finally in Section 3, we investigate the coefficient estimates and Fekete-Szegö problem for the inverse functions in the defined class. Also, some remarks and corollaries are given in Sections 2 and 3.

**1.1. Some basics in univalent function theory.** The set of all analytic functions in the open unit disk $\mathbb{D} = \{s\colon\ s \in \mathbb{C} \text{ and } |s| < 1\}$ is denoted by $\mathcal{A}$, which are of the form

$$\varrho(s) = s + a_2 s^2 + a_3 s^3 + \ldots = s + \sum_{n=2}^{\infty} a_n s^n, \quad s \in \mathbb{D}. \tag{1.1}$$

It is obvious that the function of the form (1.1) satisfies normalization conditions $\varrho(0) = 0$ and $\varrho'(0) = 1$. The following definition is used to express the concept of the subfamily $\mathcal{S}$ of $\mathcal{A}$. The subfamily $\mathcal{S}$ is described as consisting of functions that are univalent in $\mathcal{A}$. Also, $\mathcal{S}$ has several subsets with remarkable geometric properties which are starlikeness, convexity, close-to convexity and bi-univalence. A function $\varrho \in \mathcal{A}$ is called starlike (convex) in $\mathbb{D}$ if $\varrho(\mathbb{D})$ is a set that is starlike (convex) with respect to the origin. The class of starlike and convex functions in $\mathbb{D}$ are denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively. Analytic characterizations of these classes are as follows:

$$\mathcal{S}^* = \left\{\varrho\colon\ \varrho \in \mathcal{A} \text{ and } \Re\left(\frac{s\varrho'(s)}{\varrho(s)}\right) > 0 \quad \text{for } s \in \mathbb{D}\right\}, \tag{1.2}$$

$$\mathcal{C} = \left\{\varrho\colon\ \varrho \in \mathcal{A} \text{ and } \Re\left(1 + \frac{s\varrho''(s)}{\varrho'(s)}\right) > 0 \quad \text{for } s \in \mathbb{D}\right\}. \tag{1.3}$$

In the literature, Sakaguchi [27] defined the starlike function class $\mathcal{S}_S^*$ with respect to symmetric point which may be characterized by

$$\mathcal{S}_S^* = \left\{\varrho\colon\ \varrho \in \mathcal{S} \text{ and } \Re\left(\frac{s\varrho'(s)}{\varrho(s) - \varrho(-s)}\right) > 0 \quad \text{for } s \in \mathbb{D}\right\}. \tag{1.4}$$

Similarly, convex function class $\mathcal{C}_S$ with respect to symmetric point is defined by Das and Singh [11] as

$$\mathcal{C}_S = \left\{\varrho\colon\ \varrho \in \mathcal{S} \text{ and } \Re\left(\frac{(s\varrho'(s))'}{(\varrho(s) - \varrho(-s))'}\right) > 0 \quad \text{for } s \in \mathbb{D}\right\}. \tag{1.5}$$

The familiar Koebe quarter theorem (see [12]) ensures that the image of $\mathbb{D}$ under every function $\varrho \in \mathcal{S}$ contains a disk with radius $\frac{1}{4}$. Therefore, every function $\varrho \in \mathcal{S}$ has an inverse $\varrho^{-1} = \tau$ such that

$$\varrho^{-1}(\varrho(s)) = s, \quad s \in \mathbb{D} \quad \text{and} \quad \varrho(\varrho^{-1}(r)) = r, \quad |r| < r_0(\varrho),\ r_0(\varrho) \geqslant \frac{1}{4},$$

where

$$\varrho^{-1}(r) = r - a_2 r^2 + (2a_2^2 - a_3)r^3 - (5a_2^3 - 5a_2 a_3 + a_4)r^4 + \ldots := \tau(r). \tag{1.6}$$

Every inverse function does not need to be univalent in $\mathbb{D}$. If both $\varrho$ and $\varrho^{-1}$ are univalent in $\mathbb{D}$, then it is called bi-univalent function and the class of such functions is denoted by $\Sigma$. Examples of this class are

$$l_1(s) = \frac{s}{1-s} \quad \text{and} \quad l_2(s) = \frac{1}{2}\log\frac{1+s}{1-s} \tag{1.7}$$

such that

$$l_1^{-1}(r) = \frac{r}{1+r} \quad \text{and} \quad l_2^{-1}(r) = \frac{\mathrm{e}^{2r}-1}{\mathrm{e}^{2r}+1}. \tag{1.8}$$

In a recent study [31], Srivastava et al. provided a comprehensive overview of bi-univalent functions. Readers interested in further examples of the class $\Sigma$ may find them in the pioneering study [31]. In the literature, there exist numerous significant coefficient estimates on the initial coefficients of the analytic and bi-univalent functions. The first estimation on $|a_2|$ was conducted by Lewin [22] in 1967, while Netanyahu [24] proved that $\max |a_2| = \frac{4}{3}$ in 1969. Then, Brannan and Clunie [8] conjectured a bound for the same coefficient of the bi-univalent functions in 1980. There is an uncertainty for the coefficients $|a_n|$, where $n \in \mathbb{N}$, $n \geqslant 3$. The problem remains unsolved in geometric function theory. For this purpose, many researchers have conducted studies on non-sharp estimates of the initial coefficients of functions in the new subclasses. Generally the subordination principal was utilized in these papers. The following principle is hereby recalled:

If the functions $h_1$ and $h_2 \in \mathcal{A}$, then $h_1$ is said to be subordinate to $h_2$ if there exists a Schwarz function $r$ such that

$$r(0) = 0, \quad |r(s)| < 1 \quad \text{and} \quad h_1(s) = h_2(r(s)) \quad \text{for all } s \in \mathbb{D}.$$

The subordination is demonstrated by

$$h_1 \prec h_2 \quad \text{or} \quad h_1(s) \prec h_2(s) \quad \text{for all } s \in \mathbb{D}.$$

In particular, if the function $h_2$ is univalent in $\mathbb{D}$, then this subordination is equivalent to

$$h_1(0) = h_2(0), \quad h_1(\mathbb{D}) \subset h_2(\mathbb{D}).$$

Miller and Mocanu [23] give a comprehensive information about the subordination principle.

**1.2. Generalized bivariate Fibonacci polynomial.** In the literature there are many important special polynomials which we mentioned in Section 1. The Fibonacci polynomial is widely regarded as one of the most important special polynomials. It plays a vital role in the field of applied sciences due to their extensive utilization, and so this made possible defining its generalization. The authors in [21] gave a new generalization of Fibonacci polynomial which is defined as generalized bivariate Fibonacci polynomial.

Let $p(x, y)$ and $q(x, y)$ be polynomials with real coefficients. The generalization of Fibonacci polynomials is as follows (see [21]):

$$H(x, y, s) = \sum_{n=0}^{\infty} H_n(x, y)s^n = \frac{s}{1 - p(x, y)s - q(x, y)s^2}. \tag{1.9}$$

For the different special cases of $p(x, y)$ and $q(x, y)$, we obtain other important polynomials, which are given in Table 1.

| $p(x, y)$ | $q(x, y)$ | $H_n(x, y)$ |
|---|---|---|
| $2x$ | $-1$ | Chebyshev of the second kind, $U_{n-1}(x)$ |
| $x$ | $y$ | Bivariate Fibonacci, $F_n(x, y)$ |
| $x$ | 1 | Fibonacci, $F_n(x)$ |
| 1 | $2x$ | Jacobsthal, $J_n(x)$ |
| $3x$ | $-2$ | Fermat, $\mathcal{F}_n(x)$ |
| $2x$ | 1 | Pell, $P_n(x)$ |

Table 1. Special cases of the generalized bivariate Fibonacci polynomial.

It should be noted that we shall write $p = p(x, y)$ and $q = q(x, y)$, shortly.

## 2. Coefficient estimates and Fekete-Szegö inequality for the Class $\mathcal{G}_{\Sigma,\alpha}^{p,q}(g(s))$

In this section, we first introduce the function class $\mathcal{G}_{\Sigma,\alpha}^{p,q}(g(s))$ and present coefficient estimates for the functions in this class.

**Definition 2.1.** Let $\alpha \in [0,1]$. If a function $\varrho \in \sum$ of the form (1.1) satisfies the subordinations defined as

$$\frac{2s\varrho'(s)}{\varrho(s)-\varrho(-s)} + \frac{2(s\varrho'(s))'}{(\varrho(s)-\varrho(-s))'} - \frac{2\alpha s^2\varrho''(s)+2s\varrho'(s)}{\alpha s(\varrho(s)-\varrho(-s))'+(1-\alpha)(\varrho(s)-\varrho(-s))} \prec g(s) \tag{2.1}$$

and

$$\frac{2r\tau'(r)}{\tau(r)-\tau(-r)} + \frac{2(r\tau'(r))'}{(\tau(r)-\tau(-r))'} - \frac{2\alpha r^2\tau''(r)+2r\tau'(r)}{\alpha r(\tau(r)-\tau(-r))'+(1-\alpha)(\tau(r)-\tau(-r))} \prec g(r), \tag{2.2}$$

then it is said to be in the class $\mathcal{G}^{p,q}_{\Sigma,\alpha}(g(s))$, where

$$g(s) = \frac{H(x,y,s)}{s} = \frac{1}{1-ps-qs^2}, \quad s,r \in \mathbb{D}, \quad p^2+4q>0, \quad \text{and} \quad \tau = \varrho^{-1}.$$

For special values of parameters, we obtain new subclasses which are below:

R e m a r k 2.1. If we take $\alpha = 1$ in Definition 2.1, then we have $\mathcal{G}^{p,q}_{\Sigma,\alpha}(g(s)) \equiv \mathcal{G}^{p,q}_{\Sigma,1}(g(s))$, which satisfies

$$\frac{2s\varrho'(s)}{\varrho(s)-\varrho(-s)} \prec g(s) \tag{2.3}$$

and

$$\frac{2r\tau'(r)}{\tau(r)-\tau(-r)} \prec g(r). \tag{2.4}$$

R e m a r k 2.2. If we take $\alpha = 0$ in Definition 2.1, then we have $\mathcal{G}^{p,q}_{\Sigma,\alpha}(g(s)) \equiv \mathcal{G}^{p,q}_{\Sigma,0}(g(s))$, which satisfies

$$\frac{2(s\varrho'(s))'}{(\varrho(s)-\varrho(-s))'} \prec g(s) \tag{2.5}$$

and

$$\frac{2(r\tau'(r))'}{(\tau(r)-\tau(-r))'} \prec g(r). \tag{2.6}$$

**Theorem 2.1.** *Let $\varrho \in \mathcal{G}^{p,q}_{\Sigma,\alpha}(g(s))$ be of the form* (1.1). *Then*

$$|a_2| \leqslant \frac{|p|\sqrt{|p|}}{\sqrt{|2(3-2\alpha)p^2 - 4(2-\alpha)^2(p^2+q)|}} \tag{2.7}$$

*and*

$$|a_3| \leqslant \frac{|p|}{2|3-2\alpha|} + \frac{p^2}{4(2-\alpha)^2}, \tag{2.8}$$

*where $g(s) = 1/(1-ps-qs^2)$, $p^2+4q>0$, $s,r \in \mathbb{D}$.*

P r o o f. Let us suppose that $\varrho(s) \in \mathcal{G}^{p,q}_{\Sigma,\alpha}(g(s))$. By the subordination definiton, there exist two Schwarz functions $m,n\colon\ \mathbb{D}\to\mathbb{D}$, $m(s) = \sum\limits_{i=1}^{\infty} m_i s^i$ and $n(r) = \sum\limits_{i=1}^{\infty} n_i r^i$ such that

$$\begin{aligned} &\frac{2s\varrho'(s)}{\varrho(s)-\varrho(-s)} + \frac{2(s\varrho'(s))'}{(\varrho(s)-\varrho(-s))'} \\ &\qquad - \frac{2\alpha s^2\varrho''(s) + 2s\varrho'(s)}{\alpha s(\varrho(s)-\varrho(-s))' + (1-\alpha)(\varrho(s)-\varrho(-s))} = g(m(s)), \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\frac{2r\tau'(r)}{\tau(r)-\tau(-r)} + \frac{2(r\tau'(r))'}{(\tau(r)-\tau(-r))'} \\ &\qquad - \frac{2\alpha r^2\tau''(r) + 2r\tau'(r)}{\alpha r(\tau(r)-\tau(-r))' + (1-\alpha)(\tau(r)-\tau(-r))} = g(n(r)), \end{aligned} \tag{2.10}$$

where the function $\tau$ is of the form (1.6). According to the definition of the Schwarz function, we can easily say that

$$|m_i| \leqslant 1 \quad \text{and} \quad |n_i| \leqslant 1 \tag{2.11}$$

for all $i \in \mathbb{N}$. A basic calculation yields that

$$g(m(s)) = 1 + H_1(x,y)m_1 s + [H_1(x,y)m_2 + H_2(x,y)m_1^2]s^2 + \dots \tag{2.12}$$

and

$$g(n(r)) = 1 + H_1(x,y)n_1 r + [H_1(x,y)n_2 + H_2(x,y)n_1^2]r^2 + \dots \tag{2.13}$$

On the other hand, for the left sides of (2.9) and (2.10), we can write that

$$\begin{aligned} &\frac{2s\varrho'(s)}{\varrho(s)-\varrho(-s)} + \frac{2(s\varrho'(s))'}{(\varrho(s)-\varrho(-s))'} \\ &\qquad - \frac{2\alpha s^2\varrho''(s) + 2s\varrho'(s)}{\alpha s(\varrho(s)-\varrho(-s))' + (1-\alpha)(\varrho(s)-\varrho(-s))} \\ &\qquad = 1 + 2(2-\alpha)a_2 s + 2(3-2\alpha)a_3 s^2 + \dots \end{aligned} \tag{2.14}$$

and

$$\frac{2r\tau'(r)}{\tau(r)-\tau(-r)}+\frac{2(r\tau'(r))'}{(\tau(r)-\tau(-r))'} - \frac{2\alpha r^2\tau''(r)+2r\tau'(r)}{\alpha r(\tau(r)-\tau(-r))'+(1-\alpha)(\tau(r)-\tau(-r))} = 1-(4-2\alpha)a_2r+(2a_2^2-a_3)2(3-2\alpha)r^2+\dots \tag{2.15}$$

Now due to equation (2.9) we compare the coefficient in (2.12) and (2.14). So we obtain

$$H_1(x,y)m_1 = 2(2-\alpha)a_2 \tag{2.16}$$

and

$$H_1(x,y)m_2 + H_2(x,y)m_1^2 = 2(3-2\alpha)a_3. \tag{2.17}$$

Similarly, from equation (2.10) we can compare (2.13) and (2.15) and write the equalities

$$H_1(x,y)n_1 = -(4-2\alpha)a_2 \tag{2.18}$$

and

$$H_1(x,y)n_2 + H_2(x,y)n_1^2 = (2a_2^2-a_3)2(3-2\alpha). \tag{2.19}$$

From equations (2.16) and (2.18), we can write that

$$m_1 = -n_1 \tag{2.20}$$

and

$$a_2^2 = \frac{H_1^2(x,y)(m_1^2+n_1^2)}{8(2-\alpha)^2}. \tag{2.21}$$

If we sum equations (2.17) and (2.19), we obtain

$$H_1(x,y)(m_2+n_2) + H_2(x,y)(m_1^2+n_1^2) = 4a_2^2(3-2\alpha). \tag{2.22}$$

In addition, it can be easily seen from equation (2.21) that

$$m_1^2+n_1^2 = \frac{8(2-\alpha)^2}{H_1^2(x,y)}a_2^2. \tag{2.23}$$

Substituting equation (2.23) in (2.22) we get

$$a_2^2 = \frac{H_1^3(x,y)(m_2+n_2)}{4(3-2\alpha)H_1^2(x,y) - 8(2-\alpha)^2 H_2(x,y)}. \tag{2.24}$$

Here, setting $H_1(x,y) = p$ and $H_2(x,y) = p^2 + q$ in (2.24) and applying triangle inequality with the inequalities in (2.11), we obtain

$$|a_2| \leqslant \frac{|p|\sqrt{|p|}}{\sqrt{|2(3-2\alpha)p^2 - 4(2-\alpha)^2(p^2+q)|}}. \tag{2.25}$$

On the other hand, subtracting (2.19) from (2.17), it can be easily written that

$$a_3 = a_2^2 + \frac{H_1(x,y)(m_2-n_2)}{4(3-2\alpha)}. \tag{2.26}$$

Considering equation (2.21) in (2.26) and a straightforward calculation we obtain that

$$a_3 = \frac{H_1^2(x,y)(m_1^2+n_1^2)}{8(2-\alpha)^2} + \frac{H_1(x,y)(m_2-n_2)}{4(3-2\alpha)}. \tag{2.27}$$

By setting $H_1(x,y) = p$ in (2.27), and using triangle inequality with (2.11), we deduce

$$|a_3| \leqslant \frac{p^2}{4(2-\alpha)^2} + \frac{|p|}{2|(3-2\alpha)|}. \tag{2.28}$$

If we take $\alpha = 0$ in Theorem 2.1, we obtain the following result:

**Corollary 2.1.** *Let $\varrho(s) = s + \sum_{n=2}^{\infty} a_n s^n$ be in the class $\mathcal{G}_{\Sigma,0}^{p,q}(g(s))$. Then*

$$|a_2| \leqslant \frac{|p|\sqrt{|p|}}{\sqrt{|10p^2+16q|}}, \tag{2.29}$$

$$|a_3| \leqslant \frac{p^2}{16} + \frac{|p|}{6}. \tag{2.30}$$

If we take $\alpha = 1$ in Theorem 2.1, we obtain the following result:

**Corollary 2.2.** *Let $\varrho(s) = s + \sum_{n=2}^{\infty} a_n s^n$ be in the class $\mathcal{G}_{\Sigma,1}^{p,q}(g(s))$. Then*

$$|a_2| \leqslant \frac{|p|\sqrt{|p|}}{\sqrt{|2p^2+4q|}}, \tag{2.31}$$

$$|a_3| \leqslant \frac{p^2}{4} + \frac{|p|}{2}. \tag{2.32}$$

**Theorem 2.2.** *Let the function $\varrho$ belong to the class $\mathcal{G}^{p,q}_{\Sigma,\alpha}(g(s))$. Then for some $k \in \mathbb{R}$,*

$$|a_3 - ka_2^2| \leqslant \begin{cases} \dfrac{|p|}{|2(3-2\alpha)|}, & |1-k| \leqslant \dfrac{|T_1(\alpha,p,q)|}{|4(3-2\alpha)p^2|}, \\ \dfrac{2|p|^3|1-k|}{|T_1(\alpha,p,q)|}, & |1-k| \geqslant \dfrac{|T_1(\alpha,p,q)|}{|4(3-2\alpha)p^2|}, \end{cases} \tag{2.33}$$

*where* $T_1(\alpha,p,q) = 4(3-2\alpha)p^2 - 8(2-\alpha)^2(p^2+q)$.

P r o o f. For some $k \in \mathbb{R}$, using (2.26),

$$\begin{aligned} a_3 - ka_2^2 &= \frac{H_1(x,y)(m_2-n_2)}{4(3-2\alpha)} + a_2^2 - ka_2^2 \\ &= \frac{H_1(x,y)(m_2-n_2)}{4(3-2\alpha)} + (1-k)\frac{(m_2+n_2)H_1^3(x,y)}{4(3-2\alpha)H_1^2(x,y) - 8(2-\alpha)^2H_2} \\ &= H_1(x,y)\left[\left(h(k) + \frac{1}{4(3-2\alpha)}\right)m_2 + \left(h(k) - \frac{1}{4(3-2\alpha)}\right)n_2\right], \end{aligned}$$

where

$$h(k) = \frac{(1-k)H_1^2(x,y)}{4(3-2\alpha)H_1^2(x,y) - 8(2-\alpha)^2H_2(x,y)}.$$

Hence,

$$|a_3 - ka_2^2| = |p|\left|\left[\left(h(k) + \frac{1}{4(3-2\alpha)}\right)m_2 + \left(h(k) - \frac{1}{4(3-2\alpha)}\right)n_2\right]\right|. \tag{2.34}$$

If $0 \leqslant |h(k)| \leqslant 1/(4(3-2\alpha))$, namely $-1/(4(3-2\alpha)) \leqslant h(k) \leqslant 1/(4(3-2\alpha))$, then

$$\begin{aligned} |a_3 - ka_2^2| &\leqslant |p|\left\{\left|h(k) + \frac{1}{4(3-2\alpha)}\right| + \left|h(k) - \frac{1}{4(3-2\alpha)}\right|\right\} \\ &\leqslant \frac{|p|}{2(3-2\alpha)}. \end{aligned} \tag{2.35}$$

If $|h(k)| \geqslant |1/(4(3-2\alpha))|$, namely $h(k) \geqslant 1/(4(3-2\alpha))$ or $h(k) \leqslant -1/(4(3-2\alpha))$, we can write that

$$\begin{aligned} |a_3 - ka_2^2| &\leqslant |p|\left\{\left|h(k) + \frac{1}{4(3-2\alpha)}\right| + \left|h(k) - \frac{1}{4(3-2\alpha)}\right|\right\} \\ &\leqslant 2|p||h(k)|. \end{aligned} \tag{2.36}$$

So, the proof is completed. □

If we take $\alpha = 0$ in Theorem 2.2, we obtain the following result:

**Corollary 2.3.** *Let $p \neq 0$ and $p^2 + 4q \geqslant 0$. If $\varrho(s) = s + \sum\limits_{n=2}^{\infty} a_n s^n \in \mathcal{G}_{\Sigma,0}^{p,q}(g(s))$,*

$$|a_3 - ka_2^2| \leqslant \begin{cases} \dfrac{|p|}{6}, & |1-k| \leqslant \dfrac{|T_1(0,p,q)|}{12p^2}, \\ \dfrac{2|p|^3|1-k|}{|T_1(0,p,q)|}, & |1-k| \geqslant \dfrac{|T_1(0,p,q)|}{12p^2}. \end{cases} \tag{2.37}$$

If we take $\alpha = 1$ in Theorem 2.2, we obtain the following result:

**Corollary 2.4.** *Let $p \neq 0$ and $p^2 + 4q \geqslant 0$. If $\varrho(s) = s + \sum\limits_{n=2}^{\infty} a_n s^n \in \mathcal{G}_{\Sigma,1}^{p,q}(g(s))$,*

$$|a_3 - ka_2^2| \leqslant \begin{cases} \dfrac{|p|}{2}, & |1-k| \leqslant \dfrac{|T_1(1,p,q)|}{4p^2}, \\ \dfrac{2|p|^3|1-k|}{|T_1(1,p,q)|}, & |1-k| \geqslant \dfrac{|T_1(1,p,q)|}{4p^2}. \end{cases} \tag{2.38}$$

If we take $k = 1$ in Theorem 2.2, we obtain the following result:

**Corollary 2.5.** *Let $\alpha \in [0,1]$, and $p^2 + 4q \geqslant 0$. If $\varrho(s) \in \mathcal{G}_{\Sigma,\alpha}^{p,q}(g(s))$, then*

$$|a_3 - a_2^2| \leqslant \frac{|p|}{2|(3-2\alpha)|}. \tag{2.39}$$

## 3. Coefficient estimates and Fekete-Szegö inequality for the inverse functions in $\mathcal{G}_{\Sigma,\alpha}^{p,q}(g(s))$

In this section, we present upper bounds for the initial Taylor-Maclaurin coefficients of inverse functions in the class $\mathcal{G}_{\Sigma,\alpha}^{p,q}(g(s))$. Moreover, we handle the Fekete-Szegö problem for the same class.

**Theorem 3.1.** *Let $\tau \in \mathcal{G}_{\Sigma,\alpha}^{p,q}(g(s))$ be of the form* (1.6). *Then*

$$|d_2| \leqslant \frac{|p|\sqrt{|p|}}{\sqrt{|2(3-2\alpha)p^2 - 4(2-\alpha)^2(p^2+q)|}} \tag{3.1}$$

*and*

$$|d_3| \leqslant \begin{cases} \dfrac{|p|}{|2(3-2\alpha)|}, & 1 \leqslant \dfrac{|T_1(a,p,q)|}{|4(3-2\alpha)p^2|}, \\ \dfrac{2|p|^3}{|T_1(a,p,q)|}, & 1 \geqslant \dfrac{|T_1(a,p,q)|}{|4(3-2\alpha)p^2|}. \end{cases} \tag{3.2}$$

P r o o f. In view of equality (1.6), we have

$$d_2 = -a_2,$$
$$d_3 = 2a_2^2 - a_3.$$

Taking modulus of $d_2$, some computations yield that

$$|d_2| = |-a_2| = |a_2| \leqslant \frac{|p|\sqrt{|p|}}{\sqrt{|2(3-2\alpha)p^2 - 4(2-\alpha)^2(p^2+q)|}}.$$

Further, putting $k = 2$ in Theorem 2.2 and taking modulus, we deduce

$$|d_3| = |a_3 - 2a_2^2| \leqslant \begin{cases} \dfrac{|p|}{|2(3-2\alpha)|}, & 1 \leqslant \dfrac{|T_1(a,p,q)|}{|4(3-2\alpha)p^2|}, \\ \dfrac{2|p|^3}{|T_1(a,p,q)|}, & 1 \geqslant \dfrac{|T_1(a,p,q)|}{|4(3-2\alpha)p^2|}. \end{cases}$$

Therefore the proof is completed. □

**Theorem 3.2.** *Let the function $\tau$ be given in equality* (1.6). *Then for some $\mu \in \mathbb{R}$,*

$$|d_3 - \mu d_2^2| \leqslant \begin{cases} \dfrac{|p|}{|2(3-2\alpha)|}, & |1-\mu| \leqslant \dfrac{|T_1(a,p,q)|}{|4(3-2\alpha)p^2|}, \\ \dfrac{2|p|^3|1-\mu|}{|T_1(a,p,q)|}, & |1-\mu| \geqslant \dfrac{|T_1(a,p,q)|}{|4(3-2\alpha)p^2|}. \end{cases} \tag{3.3}$$

P r o o f. For some $\mu \in \mathbb{R}$, by using equations (2.24) and (2.26), we arrive at

$$\begin{aligned} d_3 - \mu d_2^2 &= (2-\mu)a_2^2 - a_2^2 - \frac{H_1(x,y)(m_2-n_2)}{4(3-2\alpha)} \\ &= (1-\mu)\frac{H_1^3(x,y)(m_2+n_2)}{4(3-2\alpha)H_1(x,y)^2 - 8(2-\alpha)^2H_2(x,y)} - \frac{H_1(x,y)(m_2-n_2)}{4(3-2\alpha)} \\ &= H_1(x,y)\Bigg[\left(\frac{(1-\mu)H_1^2(x,y)}{4(3-2\alpha)H_1^2(x,y) - 8(2-\alpha)^2H_2(x,y)} - \frac{1}{4(3-2\alpha)}\right)m_2 \\ &\quad + \left(\frac{(1-\mu)H_1^2(x,y)}{4(3-2\alpha)H_1^2(x,y) - 8(2-\alpha)^2H_2(x,y)} + \frac{1}{4(3-2\alpha)}\right)n_2\Bigg] \\ &= H_1(x,y)\Bigg[\left(t(\mu) - \frac{1}{4(3-2\alpha)}\right)m_2 + \left(t(\mu) + \frac{1}{4(3-2\alpha)}\right)n_2\Bigg], \end{aligned}$$

where

$$t(\mu) = \frac{(1-\mu)H_1^2(x,y)}{4(3-2\alpha)H_1^2(x,y) - 8(2-\alpha)^2H_2(x,y)}.$$

Now, setting $H_1(x,y) = p$, $H_2(x,y) = p^2 + q$ and using triangular inequality, we have

$$|d_3 - \mu d_2^2| = \left| p\left[\left(t(\mu) + \frac{1}{4(3-2\alpha)}\right)n_2 + \left(t(\mu) - \frac{1}{4(3-2\alpha)}\right)m_2\right]\right| \tag{3.4}$$
$$= |p|\left|\left[\left(t(\mu) + \frac{1}{4(3-2\alpha)}\right)n_2 + \left(t(\mu) - \frac{1}{4(3-2\alpha)}\right)m_2\right]\right|.$$

Thus, we conclude that

$$|d_3 - \mu d_2^2| \leqslant \begin{cases} \dfrac{|p|}{|2(3-2\alpha)|}, & 0 \leqslant |t(\mu)| \leqslant \dfrac{1}{4(3-2\alpha)}, \\ 2|p||t(\mu)|, & |t(\mu)| \geqslant \dfrac{1}{4(3-2\alpha)}. \end{cases} \tag{3.5}$$

□

If we take $\alpha = 0$ in Theorem 3.2, we obtain the following result:

**Corollary 3.1.** *Let $p \neq 0$ and $p^2 + 4q \geqslant 0$. If $\tau$ is of the form given by (1.6), then*

$$|d_3 - \mu d_2^2| \leqslant \begin{cases} \dfrac{|p|}{6}, & |1-\mu| \leqslant \dfrac{|T_1(0,p,q)|}{12p^2}, \\ \dfrac{2|p|^3|1-\mu|}{|T_1(0,p,q)|}, & |1-\mu| \geqslant \dfrac{|T_1(0,p,q)|}{12p^2}. \end{cases} \tag{3.6}$$

If we take $\alpha = 1$ in Theorem 3.2, we obtain the following result:

**Corollary 3.2.** *Let $p \neq 0$ and $p^2 + 4q \geqslant 0$. If $\tau$ is of the form given by (1.6), then*

$$|d_3 - \mu d_2^2| \leqslant \begin{cases} \dfrac{|p|}{2}, & |1-\mu| \leqslant \dfrac{|T_1(1,p,q)|}{4p^2}, \\ \dfrac{2|p|^3|1-\mu|}{|T_1(1,p,q)|}, & |1-\mu| \geqslant \dfrac{|T_1(1,p,q)|}{4p^2}. \end{cases} \tag{3.7}$$

If we take $\mu = 1$ in Theorem 3.2, we obtain the following result:

$$|d_3 - d_2^2| \leqslant \frac{|p|}{2|(3-2\alpha)|}. \tag{3.8}$$

## 4. Conclusions

In the present paper, we introduce a new function subclass by considering the subordination principle between analytic functions and generalized bivariate Fibonacci polynomials. In accordance with the subordination principle and fundamental rules of analysis, we have provided estimates for the first two coefficients of functions within the defined subclass $\mathcal{G}_{\Sigma,\alpha}^{p,q}(g(s))$ and their inverse. In addition, we handled the Fekete-Szegö problem for functions in the defined subclass and their inverse. In the future research, this study can be extended to the Hankel determinant problem. Also, new results can be found for the defined class by changing the subordination function.

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