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FURTHER RESULTS ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. This paper focuses on the uniform stability of the zero solution and the asymptotic behavior of all solutions in the context of the neutral differential equation

$$(\star) \qquad \frac{\mathrm{d}}{\mathrm{d}t}[x(t)-px(t-\tau)]+q(t)x(t-\sigma)=0 \quad \text{for } t\geqslant t_0,$$

where $\tau,\sigma\in\mathbb{R}^+$, $p\in[0,1)$ and $q\in\mathrm{C}([t_0,\infty),\mathbb{R}_0^+)$. These results differ from most of the existing results in that our results extend to include values of $p\in[\frac{1}{2},1)$ for $(\star)$. Besides, the practical applicability of the given results is shown by a numerical example.

Keywords: neutral differential equation; uniform stability; globally attractivity; globally asymptotic stability

MSC 2020: 34K20, 34K25, 34K40

1. Introduction

Over the last ten years, stability theory, as a way of describing the dynamics present in a differential equation, has become much more adaptable and powerful. During the last few decades, many scholars actively worked on the development and establishment of sufficient conditions that guarantee the property that every solution of a delay differential equation converges to zero as the time variable tends to infinity. Later on, some researchers extended and generalized these tests for neutral differential equations. The thing worth mentioning here is that both delay and neutral differential equations involve delays. The latter includes added complexity in that there are neutral terms with derivatives, which makes their analysis and solution more complicated.

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In this paper, we consider the following neutral differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - p(t)x(t-\tau)] + q(t)x(t-\sigma) = 0 \quad \text{for } t \geqslant t_0, \tag{1.1}$$

where $\tau, \sigma \in \mathbb{R}^+$, $p \in \mathrm{C}([t_0, \infty), \mathbb{R})$ and $q \in \mathrm{C}([t_0, \infty), \mathbb{R}_0^+)$. When $p(t) \equiv p \in \mathbb{R}$, (1.1) can be simplified to

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - px(t-\tau)] + q(t)x(t-\sigma) = 0 \quad \text{for } t \geqslant t_0. \tag{1.2}$$

Additionally, if $\tau = 0$, then (1.2) can be written as

$$x'(t) + \frac{q(t)}{1-p}x(t-\sigma) = 0 \quad \text{for } t \geqslant t_0.$$

Furthermore, for $p(t) \equiv 0$, the asymptotic behavior of the solutions for simplified DDEs by (1.1), which can be expressed as

$$x'(t) + q(t)x(t-\sigma) = 0 \quad \text{for } t \geqslant t_0, \tag{1.3}$$

has been thoroughly examined in many literature, refer to [1]–[10], [12]–[15] for more details. To commence, we will provide an overview that summarizes the most striking results concerning (1.1), (1.2) and (1.3), which is relevant to the framework of this work. In 1983, Ladas, Sficas and Stavroulakis [9] established sufficient conditions for every solution of delay differential equations to tend to zero at infinity. Additionally, these conditions imply an important point regarding the behavior of solutions of delay differential equations and carry serious consequences in view of stability theory and related fields of mathematical research and application.

Theorem A (Cf. [9], Theorem 3)**.** *Assume that*

$$\int_{t_0}^{\infty} q(\xi)\,\mathrm{d}\xi = \infty \tag{1.4}$$

and

$$\limsup_{t\to\infty} \int_{t}^{t+\sigma} q(\xi)\,\mathrm{d}\xi < 1. \tag{1.5}$$

Then every solution of (1.3) *tends to zero as* $t \to \infty$.

Subsequently, in 1987, Yoneyama [13] expanded and generalized the previous result by replacing (1.5) with a weaker condition.

Theorem B (Cf. [13], Theorem 4.1). *Assume* (1.4) *and*

$$\limsup_{t\to\infty} \int_t^{t+\sigma} q(\xi)\,\mathrm{d}\xi < \frac{3}{2}. \tag{1.6}$$

Then every solution of (1.3) *tends to zero as* $t \to \infty$.

Notably, condition (1.5) holds greater strength than condition (1.6); however, the upper bound $\frac{3}{2}$ in condition (1.6) is considered as the optimal possible stability for (1.3), as mentioned in references [10], [13], [14]. This highlights the significance and strength of Theorem B in maintaining significant stability in the absence of neutral elements in the equation.

Following this, Ladas and Sficas [8] extended Theorem A when neutral coefficient p is constant to encompass (1.2) and devised sufficient conditions for the convergence of every solution $x(t)$ of (1.2) to zero as $t \to \infty$ in 1988.

Theorem C (Cf. [8], Corollary 3). *Assume* $p \in [0, \frac{1}{2})$, (1.4) *and*

$$\limsup_{t\to\infty} \int_t^{t+\sigma} q(\xi)\,\mathrm{d}\xi < 1 - 2p. \tag{1.7}$$

Then every solution of (1.2) *tends to zero as* $t \to \infty$.

It should be noted that in [8] the authors assume $p \in [0, 1)$. Additionally, condition (1.7) reduces to (1.5) for (1.3), i.e., when $p = 0$. One can see that Theorem C is not the best result of asymptotic behavior for (1.1).

Hence, in 1996, Yu [16] attempted to achieve better results when the neutral coefficient p is a variable of (1.1) in the following theorem.

Theorem D (Cf. [16], Theorem 2). *Assume that there exists* $p \in [0, \frac{1}{2})$ *such that* $|p(t)| \leqslant p$ *for* $t \geqslant t_0$. *Further, assume* (1.4) *and*

$$\limsup_{t\to\infty} \int_t^{t+\sigma} q(\xi)\,\mathrm{d}\xi < \frac{3}{2} - 2p(2-p). \tag{1.8}$$

Then every solution of (1.1) *tends to zero as* $t \to \infty$.

It should be noted that in [16] the author assumes $p \in [0, 1)$. Moreover, when $p = 0$, condition (1.8) simplifies to (1.5) for (1.3), thus enhances (1.7). As observed, Theorem D is still far from achieving the optimal possible.

Several years later, Tang and Zou [11] found the opportunity to fill some of this gap. They made improvements upon Theorem B and provided alternative result of Theorem B, where $p \in [0, 1)$ is admissible. This endeavor ultimately led to the attainment of currently best result for (1.1), as quoted below.

Theorem E (Cf. [11], Theorem 2.2)**.** *Assume that there exists* $p \in [0, \frac{1}{2})$ *such that* $|p(t)| \leqslant p$ *for* $t \geqslant t_0$*. Further, assume* (1.4) *and*

$$\limsup_{t\to\infty} \int_t^{t+\sigma} q(\xi)\,\mathrm{d}\xi < T(p) := \begin{cases} \dfrac{3}{2} - 2p, & p \in \Big[0, \dfrac{1}{4}\Big), \\ \sqrt{2(1-2p)}, & p \in \Big[\dfrac{1}{4}, \dfrac{1}{2}\Big). \end{cases} \tag{1.9}$$

Then every solution of (1.1) *tends to zero as* $t \to \infty$*.*

Figure 1 illustrates the upper bounds for Theorem C, Theorem D, and Theorem E.

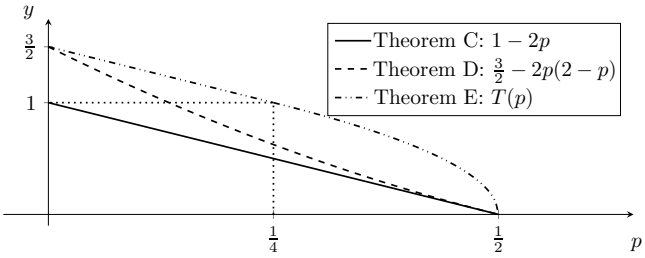


Figure 1. Upper bounds of Theorem C, Theorem D and Theorem E when $p \in [0, \frac{1}{2})$.

Theorem F (Cf. [11], Theorem 3.2)**.** *Assume that* $p \in [0, 1)$ *and* (1.4) *holds. Further, assume*

$$\limsup_{t\to\infty} \int_t^{t+\sigma+(m-1)\tau} q(\xi)\,\mathrm{d}\xi < \frac{1-p}{1-p^m}\,\frac{3-4p^m}{2}, \tag{1.10}$$

where $m \in \mathbb{N}$ *satisfies* $4p^m \leqslant 1$*. Then every solution of* (1.2) *tends to zero as* $t \to \infty$*.*

Let $\nu^* = \max\{\tau, \sigma\}$ and $\nu_* = \min\{\tau, \sigma\}$. A function $x \in \mathrm{C}([t_0 - \nu^*, \infty), \mathbb{R})$ is said to be a solution of (1.1) for some $s \in [t_0, \infty)$ if $x - px(\cdot - \tau)$ is continuously differentiable on $[s, \infty)$ and satisfies (1.1) identically on $[s, \infty)$. It can be concluded that for any given value of $s \in [t_0, \infty)$ and any initial function $\varphi \in \mathrm{C}([s - \nu^*, s], \mathbb{R})$, (1.1) has a unique solution $x = x(\cdot; s, \varphi)$ over the interval $[s, \infty)$.

Next, we list a few essential definitions that will be utilized in the proof of our main results.

Definition 1.1. The zero solution of (1.1) is said to be

(i) *uniformly stable* if for each $\varepsilon \in \mathbb{R}^+$ and each $s \in [t_0, \infty)$, there exists $\delta = \delta(\varepsilon) \in \mathbb{R}^+$ such that $|\varphi(t)| < \delta$ for all $t \in [s - \nu^*, s]$ implies $|x(t)| < \varepsilon$ for all $t \in [s, \infty)$,

(ii) *globally attractive* if $\lim\limits_{t\to\infty} x(t; s, \varphi) = 0$ for any $\varphi \in \mathrm{C}([s - \nu^*, s], \mathbb{R})$, where $s \in [t_0, \infty)$,

(iii) *globally asymptotically stable* if (i) and (ii) hold.

Uniform asymptotic stability and uniform exponential stability are also studied in the stability theory of differential equations. While uniform asymptotic stability ensures that solutions converge to an equilibrium point over time regardless of initial conditions, uniform exponential stability not only guarantees convergence but also provides a rate at which this convergence occurs, exhibiting a stronger form of stability. Global asymptotic stability complements uniform asymptotic stability by ensuring that all solutions, starting from any initial condition, converge to the equilibrium point, thus reinforcing the notion that uniform asymptotic stability applies uniformly across a broader range of initial conditions.

The goal of this study is to improve and expand Theorem F, with the aim of deriving new results. Our theorems should give enhanced understandings to both the uniform stability of zero solution and the asymptotic behavior of all solutions of (1.2).

The paper is organized as follows. In Section 2, we give results on the uniform stability of zero solution and asymptotic behavior of all solutions of (1.2). In Section 3, we present a numerical example to demonstrate applicability and novelty of our main results. Section 4 includes detailed proofs of the results in Section 2. Finally, in Section 5, we give a few comments and examine a few thoughts for Theorem F, Theorem 2.3 and Theorem 2.4 in [11] to finish up the paper.

2. Main results

In this section, we are now presenting our main results on the uniform stability of zero solution and asymptotic behavior of all solutions of (1.2).

Theorem 2.1. *Assume that $p \in [0,1)$. Further, assume*

$$\int_t^{t+\sigma+(m-1)\tau} q(\xi)\,\mathrm{d}\xi \leqslant \frac{1-p}{1-p^m}\sqrt{2(1-2p^m)} \quad \text{for } t \geqslant t_0, \tag{2.1}$$

where $m \in \mathbb{N}$ satisfies $2p^m < 1$. Then the zero solution of (1.2) *is uniformly stable.*

Theorem 2.2. *Assume that $p \in [0,1)$ and* (1.4) *holds. Further, assume*

$$\limsup_{t\to\infty} \int_t^{t+\sigma+(m-1)\tau} q(\xi)\,\mathrm{d}\xi < \frac{1-p}{1-p^m}\sqrt{2(1-2p^m)}, \tag{2.2}$$

where $m \in \mathbb{N}$ satisfies $2p^m < 1$. Then the zero solution of (1.2) *is globally attractive.*

Theorem 2.3. *Assume that* $p \in [0,1)$ *and* (1.4) *holds. Further, assume*

$$\sup_{t\geqslant t_0}\left\{\int_t^{t+\sigma+(m-1)\tau} q(\xi)\,\mathrm{d}\xi\right\} < \frac{1-p}{1-p^m}\sqrt{2(1-2p^m)},$$

where $m \in \mathbb{N}$ *satisfies* $2p^m < 1$. *Then the zero solution of* (1.2) *is globally asymptotically stable.*

3. A numerical example

In this section, we present a neutral differential equation where none of the results (to the best of our knowledge) can deliver an answer for the asymptotic nature of its solutions but our result Theorem 2.2 can be applied to get a positive answer.

E x a m p l e 3.1. Consider the neutral differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-\frac{2}{3}x(t-2)\right]+\frac{1}{15}\left[1+\frac{\pi}{4}\sin\left(\frac{2}{3}\pi t\right)\right]x(t-1)=0 \quad \text{for } t\geqslant 0. \tag{3.1}$$

Observe that $p=\frac{2}{3}$, $\tau=2$, $q(t)=\frac{1}{15}[1+\frac{\pi}{4}\sin(\frac{2}{3}\pi t)]\geqslant 0$ and $\sigma=1$ for $t\geqslant 0$.

▷ **Theorem F**: As an initial analysis, it becomes evident that the inequality

$$4p^m=\frac{2^{m+2}}{3^m}\leqslant 1 \quad \text{for } m=4,5,\ldots$$

holds. Upon calculation, we determine for $m=1,2,\ldots$ that

$$\begin{aligned}\ell_m := \limsup_{t\to\infty}\int_t^{t+\sigma+(m-1)\tau} q(\xi)\,\mathrm{d}\xi &= \limsup_{t\to\infty}\int_t^{t+2m-1}\frac{1}{15}\left(1+\frac{\pi}{4}\sin\left(\frac{2}{3}\pi\xi\right)\right)\mathrm{d}\xi\\ &= \frac{2m-1}{15}+\left\{\begin{array}{ll}0, & \mathrm{mod}(m,3)\equiv 0\\ \dfrac{\sqrt{3}}{40}, & \mathrm{mod}(m,3)\not\equiv 0\end{array}\right\}.\end{aligned}$$

It can be demonstrated that the sequence $u_m := (1/\ell_m)\cdot((1-p)/(1-p^m))\cdot(3-4p^m)/2$ satisfies the inequality $u_{m+1}\leqslant u_m$ for $m=1,2,\ldots$ and $u_4 = 716/(728+39\sqrt{3})\approx 0.9<1$. This observation immediately shows that

$$\ell_m > \frac{1-p}{1-p^m}\frac{3-4p^m}{2} \quad \text{for } m=4,5,\ldots$$

Consequently, Theorem F cannot deliver any answer here.

▷ **Theorem 2.2**: On the other hand, we observe with $m := 2$ that

$$2p^2 = \frac{2^3}{3^2} = \frac{8}{9} \leqslant 1.$$

Furthermore, upon computation, it is revealed that

$$\ell_2 = \frac{1}{5} = 0.2 < \frac{1-p}{1-p^m}\sqrt{2(1-2p^m)} = \frac{\sqrt{2}}{5} \approx 0.28.$$

Furthermore, it is easy to note that (1.4) holds. Thus, the conditions outlined in Theorem 2.2 are satisfied. As a result, Theorem 2.2 applies here, thereby establishing the global attractivity of the zero solution of (3.1). Two solutions of (3.1) are plotted in Figure 2.

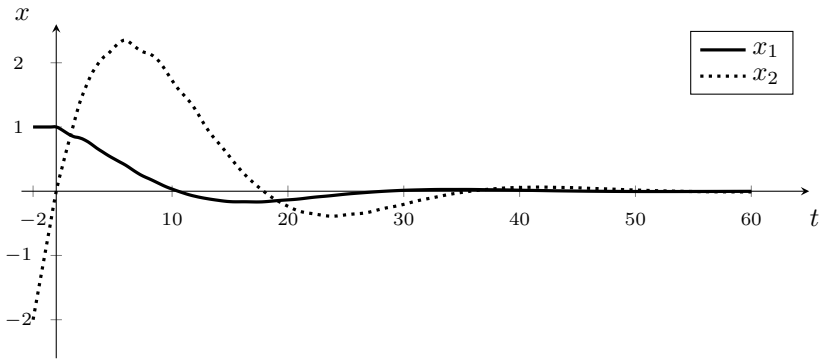


Figure 2. Graphics of two solutions of (3.1) are plotted. Here, x_1 and x_2 have the initial functions $\varphi_1(t) = 1$ and $\varphi_2(t) = t$ for $t \in [-2, 0]$, respectively.

4. Proof of the main results

This section is devoted to the proofs of our main results.

P r o o f of Theorem 2.1. If $p = 0$, we obtain equation (1.3), and as a result, Theorem 2.1 has already been shown in [13]. Hereafter, we consider the case where $p \in (0, 1)$. Now, pick that $\ell \in \mathbb{N}$ such that $\ell\nu_* \geqslant 2(\sigma + m\tau)$. By Definition 1.1 (i), for any $\varepsilon > 0$, define $\delta := (1-p)\varepsilon/((1+p)(2p+3)^\ell)$. We will prove that for any $s \in [t_0, \infty)$, $\varphi \in \mathrm{C}([s-\nu^*, s], (-\delta, \delta))$ implies that $x(t)$ satisfies

$$|x(t)| < \varepsilon \quad \text{for } t \in [s, \infty). \tag{4.1}$$

Here, we intend to establish the validity of (4.1) through two stage examination, initially focusing on the interval $[s, s+\ell\nu_*]$ and subsequently extending our examination to the interval $(s+\ell\nu_*, \infty)$. The first part of the proof is similar to the proof of Theorem 1 in [16]. We can prove the following inequality holds:

$$|x(t)| < (2p+3)^{\ell}\delta \quad \text{for } t \in [s, s+\ell\nu_*]. \tag{4.2}$$

Obviously, (4.2) implies that

$$|x(t)| < \varepsilon \quad \text{for } t \in [s, s+\ell\nu_*] \tag{4.3}$$

holds. Subsequently, we need to establish the validity of (4.3) on $(s+\ell\nu_*, \infty)$. To do so, we assume the contrary, suggesting the existence of a certain $t_1 \in (s+\ell\nu_*, \infty)$ such that

$$|x(t_1)| = \varepsilon \quad \text{and} \quad |x(t)| < \varepsilon \quad \text{for } t \in [s, t_1). \tag{4.4}$$

Without loss of generality, we may suppose that $x(t_1) = \varepsilon$. For simplicity and convenience, throughout this proof, we define a function z by

$$z(t) := x(t) - px(t-\tau) \quad \text{for } t \in [s, \infty). \tag{4.5}$$

Using (1.2), (4.4) and (4.5), it follows that

$$z(s+\ell\nu_*) \leqslant |x(s+\ell\nu_*)| + p|x(s+\ell\nu_*-\tau)| < (2p+3)^{\ell}(1+p)\delta = (1-p)\varepsilon$$

and

$$z(t_1) \geqslant |x(t_1)| - p|x(t_1-\tau)| > (1-p)\varepsilon, \tag{4.6}$$

and thus $z(s+\ell\nu_*) < z(t_1)$. Therefore, we can find $t_2 \in (s+\ell\nu_*, t_1]$ such that

$$z(t_2) = \max\{z(t):\ s+\ell\nu_* \leqslant t \leqslant t_1\} \text{ and } z(t) < z(t_2) \quad \text{for } t \in [s+\ell\nu_*, t_2). \tag{4.7}$$

Define a function y by

$$y(t) := z(t) - p^m \frac{1-p}{1-p^m}\varepsilon \quad \text{for } t \in [s, \infty). \tag{4.8}$$

Then, we deduce from (4.5) and (4.8) that

$$
\begin{aligned}
(4.9) \qquad -x(t-\sigma) &= -\sum_{i=0}^{m-1} p^i z(t-\sigma-i\tau) - p^m x(t-\sigma-m\tau) \\
&\leqslant -\sum_{i=0}^{m-1} p^i z(t-\sigma-i\tau) + p^m \varepsilon \\
&= -\sum_{i=0}^{m-1} p^i y(t-\sigma-i\tau) \quad \text{for } t \in [t_2 - [\sigma + (m-1)\tau], t_2].
\end{aligned}
$$

It follows from (1.2), (4.5), (4.8), and (4.9) that

$$
(4.10) \qquad y'(t) \leqslant -q(t) \sum_{i=0}^{m-1} p^i y(t-\sigma-i\tau) \quad \text{for } t \in [t_2 - [\sigma + (m-1)\tau], t_2].
$$

Thereafter, we obtain by (1.2), (4.4), (4.5) and (4.8) that

$$
(4.11) \qquad y'(t) \leqslant \varepsilon q(t) \quad \text{for } t \in [s, t_2].
$$

It is evident by (4.6), (4.7) and (4.8) that

$$
(4.12) \qquad y(t_2) > (1-p)\frac{1-2p^m}{1-p^m}\varepsilon > 0.
$$

Afterward, it is imperative to establish that there exists $j \in \{0, 1, \ldots, m-1\}$ such that $y(t_2-\sigma-j\tau) \leqslant 0$ holds. Otherwise, we have $y(t_2-\sigma-i\tau) > 0$ for $i \in \{0, 1, \ldots, m-1\}$. Thus, there is a left neighborhood $(t_2-\sigma-i\tau-h, t_2-\sigma-i\tau)$ of $t_2-\sigma-i\tau$ for some $h > 0$ such that $y(t) > 0$ for $(t_2-\sigma-i\tau-h, t_2-\sigma-i\tau)$, and so $y(t-\sigma-i\tau) > 0$ for (t_2-h, t_2) and $i \in \{0, 1, \ldots, m-1\}$, and therefore by (4.6) and (4.9), we see that $z(t)$ is not increasing on (t_2-h, t_2). This contradicts the definition of t_2 and so $y(t_2-\sigma-j\tau) \leqslant 0$ for some $j \in \{0, 1, \ldots, m-1\}$. Hence, there exists $t_3 \in [t_2 - (\sigma + (m-1)\tau), t_2)$ such that

$$
(4.13) \qquad y(t_3) = 0 \quad \text{and} \quad y(t) > 0 \quad \text{for } t \in (t_3, t_2].
$$

Currently, we deliberate upon the following two possibilities.

(i) If $t - \sigma - i\tau \leqslant t_3$ for $t \in [t_2 - (\sigma + (m-1)\tau), t_2]$, then we estimate from integrating (4.13) that

$$
-y(t-\sigma-i\tau) \leqslant \varepsilon \int_{t-(\sigma+(m-1)\tau)}^{t_3} q(\xi)\,\mathrm{d}\xi \quad \text{for } t - \sigma - i\tau \leqslant t_3.
$$

(ii) If $t_3 < t-\sigma-i\tau \leqslant t_2$ for $t \in [t_2-[\sigma+(m-1)\tau], t_2]$, then we deduce

$$-y(t-\sigma-i\tau) < 0 \leqslant \varepsilon \int_{t-(\sigma+(m-1)\tau)}^{t_3} q(\xi)\,\mathrm{d}\xi \quad \text{for } t_3 < t-\sigma-i\tau \leqslant t_2.$$

Combining (i) and (ii), we have

$$-y(t-\sigma-i\tau) \leqslant \varepsilon \int_{t-(\sigma+(m-1)\tau)}^{t_3} q(\xi)\,\mathrm{d}\xi \quad \text{for } t-\sigma-i\tau \in [t_3, t_2]. \tag{4.14}$$

Substituting (4.14) into (4.10) and using (2.1), we have

$$y'(t) \leqslant \frac{1-p^m}{1-p}\varepsilon q(t)\left[\frac{1-p}{1-p^m}\sqrt{2(1-2p^m)} - \int_{t_3}^{t} q(\xi)\,\mathrm{d}\xi\right] \quad \text{for } t \in [t_3, t_2]. \tag{4.15}$$

By integrating (4.15) over $[t_3, t_2]$ and using (4.13), we obtain

$$\begin{aligned} y(t_2) &\leqslant \frac{1-p^m}{1-p}\varepsilon \int_{t_3}^{t_2} q(\eta)\left[\frac{1-p}{1-p^m}\sqrt{2(1-2p^m)} - \int_{t_3}^{\eta} q(\xi)\,\mathrm{d}\xi\right]\mathrm{d}\eta \\ &= \frac{1-p^m}{1-p}\varepsilon\left[\frac{1-p}{1-p^m}\sqrt{2(1-2p^m)}\int_{t_3}^{t_2} q(\xi)\,\mathrm{d}\xi - \frac{1}{2}\left(\int_{t_3}^{t_2} q(\xi)\,\mathrm{d}\xi\right)^2\right] \\ &\leqslant \frac{1-p^m}{1-p}\varepsilon\left[\left(\frac{1-p}{1-p^m}\sqrt{2(1-2p^m)}\right)^2 - \frac{1}{2}\left(\frac{1-p}{1-p^m}\sqrt{2(1-2p^m)}\right)^2\right] \\ &= (1-p)\frac{1-2p^m}{1-p^m}\varepsilon, \end{aligned}$$

which contradicts the evidence established in (4.12). Hence, (4.3) holds for $t \in [s+\ell\nu_*, \infty)$, as desired. □

P r o o f of Theorem 2.2. To establish the desired result, it is sufficient to prove that

$$\lim_{t\to\infty} x(t) = 0. \tag{4.16}$$

Let us consider the following two cases for the function x.

Case 1. If x is nonoscillatory, implying it eventually settles into either a positive or negative, for $s \in [t_0, \infty)$, we can presume x becomes eventually positive on $s \in [t_0, \infty)$. Let z be defined by (4.5). By utilizing the definition of z as provided in (4.5) and subsequently referring to (1.2), it becomes evident that z eventually decreases on $[s, \infty)$. Let $L_z := \lim_{t\to\infty} z(t)$. Since x is bounded, L_z remains finite. Integrating (1.2) over $[t_1, \infty)$, where $t_1 \in [s, \infty)$ is sufficiently large, yields

$$\int_{t_1}^{\infty} q(\xi)x(\xi-\sigma)\,\mathrm{d}\xi = z(t_1) - L_z < \infty,$$

indicating from (1.4) that $\liminf_{t\to\infty} x(t) = 0$. Denoting $L_x := \limsup_{t\to\infty} x(t)$, we pick two increasing divergent sequences $\{\xi_k\}_{k\in\mathbb{N}}, \{\varsigma_k\}_{k\in\mathbb{N}} \subset [s,\infty)$ such that $x(\xi_k) \to 0$ and $x(\varsigma_k) \to L_x$ as $k \to \infty$. It is worth noting that the sequences $\{x(\xi_k - \tau)\}_{k\in\mathbb{N}}$ and $\{x(\varsigma_k - \tau)\}_{k\in\mathbb{N}}$ are bounded, and by the Bolzano-Weierstrass theorem, they possess convergent subsequences. Without loss of generality, we denote these convergent subsequences as $\{x(\xi_k - \tau)\}_{k\in\mathbb{N}}$ and $\{x(\varsigma_k - \tau)\}_{k\in\mathbb{N}}$, respectively. Furthermore, it is important to observe that the upper bounds of the sequences $\{x(\xi_k - \tau)\}_{k\in\mathbb{N}}$ and $\{x(\varsigma_k - \tau)\}_{k\in\mathbb{N}}$ cannot exceed L_x. Consequently, we infer from (4.5) that

$$\begin{aligned} z(\varsigma_k) - z(\xi_k) &= [x(\varsigma_k) - px(\varsigma_k - \tau)] - [x(\xi_k) - px(\xi_k - \tau)] \\ &= x(\varsigma_k) - px(\varsigma_k - \tau) - x(\xi_k) + px(\xi_k - \tau) \\ &\geqslant x(\varsigma_k) - px(\varsigma_k - \tau) - x(\xi_k) \quad \text{for } k \in \mathbb{N}. \end{aligned}$$

Letting $k \to \infty$ yields $0 = L_z - L_z \geqslant (1-p)L_x \geqslant 0$, which implies $L_x = 0$, i.e., (4.16) holds. Thus, the proof for Case 1 is finalized.

Case 2. If x is oscillatory, we define

$$L_x := \limsup_{t\to\infty} |x(t)|.$$

Our assertion is that $L_x = 0$. Assuming the contrary, suppose $L_x > 0$. Let $\{\varsigma_k\}_{\mathbb{N}} \subset [s,\infty)$ be an increasing divergent sequence such that $x(\varsigma_k) \to L_x$ as $k \to \infty$. It is noteworthy that the sequence $\{x(\varsigma_k - \tau)\}_{k\in\mathbb{N}}$ is bounded, and by the Bolzano-Weierstrass theorem, it possesses a convergent subsequence. Without loss of generality, we denote this convergent subsequence as $\{x(\varsigma_k - \tau)\}_{k\in\mathbb{N}}$. Furthermore, it is evident that the limit of the sequence $\{x(\varsigma_k - \tau)\}_{k\in\mathbb{N}}$ cannot exceed L_x. As a result,

$$|z(\varsigma_k)| \geqslant |x(\varsigma_k)| - p|x(\varsigma_k - \tau)| \quad \text{for } k \in \mathbb{N}.$$

As $k \to \infty$, we conclude that

$$L_z \geqslant (1-p)L_x > 0, \tag{4.17}$$

where $L_z := \limsup_{t\to\infty} |z(t)|$.

According to (2.2), for any given $\varepsilon \in (0, (1-2p^m)L_x)$, there exist $t_1 \in [t_0,\infty)$ and $B > 0$ such that

$$|x(t)| \leqslant L_x + \varepsilon \quad \text{for } t \in [t_1 - \nu^*, \infty), \tag{4.18}$$

and

$$\int_{t-(\sigma+(m-1)\tau)}^{t} q(\xi)\,\mathrm{d}\xi \leqslant B < \frac{1-p}{1-p^m}\sqrt{2(1-2p^m)} \quad \text{for } t \in [t_1,\infty).$$

Subsequently, we define a function y as

$$y(t) := z(t) - p^m \frac{1-p}{1-p^m}(L_x + \varepsilon) \quad \text{for } t \in [t_1 - [\sigma + (m-1)\tau], \infty). \tag{4.19}$$

Then, by utilizing (4.5), (4.18), and (4.19), we derive that

$$\begin{aligned} -x(t-\sigma) &= -\sum_{i=0}^{m-1} p^i z(t-\sigma-i\tau) - p^m x(t-\sigma-m\tau) \\ &\leqslant -\sum_{i=0}^{m-1} p^i z(t-\sigma-i\tau) + p^m(L_x+\varepsilon) \\ &= -\sum_{i=0}^{m-1} p^i y(t-\sigma-i\tau) \quad \text{for } t \in [t_1, \infty). \end{aligned} \tag{4.20}$$

It is evident by (1.2), (4.19), and (4.20) that

$$y'(t) = z'(t) = -q(t)x(t-\sigma) \leqslant -q(t)\sum_{i=0}^{m-1} p^i y(t-\sigma-i\tau) \quad \text{for } t \in [t_1, \infty). \tag{4.21}$$

It should be emphasized that z' is oscillatory, and hence there exists an increasing divergent sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset [s, \infty)$ such that $\xi_k > t_1 + 2(\sigma + (m-1)\tau) + \tau$ for $k \in \mathbb{N}$, $\xi_k \to \infty$ and $|z(\xi_k)| \to L_z$ as $k \to \infty$ with $|z(\xi_k)| > (1-p)(L_x - \varepsilon)$ for each $k \in \mathbb{N}$, and also the sequence $\{\xi_k\}_{k\in\mathbb{N}}$ consists of the left local maximums of $|z|$. Without loss of generality, we may assume that $z(\xi_k) > 0$. Then, for $k \in \mathbb{N}$, we can deduce

$$\begin{aligned} y(\xi_k) &= z(\xi_k) - p^m \frac{1-p}{1-p^m}(L_x+\varepsilon) \\ &> (1-p)(L_x-\varepsilon) - p^m \frac{1-p}{1-p^m}(L_x+\varepsilon) \\ &= (1-p)\frac{(1-2p^m)L_x - \varepsilon}{1-p^m} > 0. \end{aligned}$$

In view of the definition of the sequence $\{\xi_k\}_{k\in\mathbb{N}}$, we see that $y(\xi_k - \sigma - i_0\tau) \leqslant 0$ for some $i_0 \in \{0, 1, \ldots, m-1\}$ and all $k \in \mathbb{N}$. Hence, for each $k \in \mathbb{N}$, there exists $s_k \in [\xi_k - (\sigma + (m-1)\tau), \xi_k)$ such that

$$y(s_k) = 0 \quad \text{and} \quad y(t) > 0 \quad \text{for } t \in (s_k, \xi_k].$$

It follows from (1.2), (4.18), and (4.19) that

$$y'(t) \leqslant (L_x + \varepsilon)q(t) \quad \text{for } t \in [t_1, \xi_k] \text{ and } k \in \mathbb{N}.$$

We can show that

$$-y(t-\sigma-i_0\tau) \leqslant (L_x+\varepsilon)\bigg(B-\int_{s_k}^{t} q(\xi)\,\mathrm{d}\xi\bigg) \quad \text{for } t\in[s_k,\xi_k] \text{ and } k\in\mathbb{N}. \tag{4.22}$$

Substituting (4.22) into (4.21), we have

$$y'(t) \leqslant \frac{1-p^m}{1-p}(L_x+\varepsilon)q(t)\bigg(B-\int_{s_k}^{t} q(\xi)\,\mathrm{d}\xi\bigg) \quad \text{for } t\in[s_k,\xi_k] \text{ and } k\in\mathbb{N}.$$

Replacing ε by $(L_x+\varepsilon)$ and using similar arguments to that in the proof of Theorem 2.1, we can conclude that

$$y(\xi_k) \leqslant \frac{1-p^m}{2(1-p)}B^2(L_x+\varepsilon) \quad \text{for } k\in\mathbb{N}.$$

Thus, for $k\in\mathbb{N}$ we estimate

$$z(\xi_k) = y(\xi_k) + p^m\frac{1-p}{1-p^m}(L_x+\varepsilon) \leqslant \bigg(\frac{1-p^m}{2(1-p)}B^2 + p^m\frac{1-p}{1-p^m}\bigg)(L_x+\varepsilon). \tag{4.23}$$

Letting $k\to\infty$ and $\varepsilon\to 0$ in (4.23), we have

$$L_z = \limsup_{k\to\infty} z(\xi_k) \leqslant \bigg(\frac{1-p^m}{2(1-p)}B^2 + p^m\frac{1-p}{1-p^m}\bigg)L_x < (1-p)L_x,$$

which contradicts the assertion (4.17), i.e., $L_x = 0$. Hence, the proof for Case 2 is completed. □

P r o o f of Theorem 2.3. The proof immediately follows from Theorem 2.1 and Theorem 2.2. □

5. Final comments and discussion

In this section, our attention is directed towards exploring some ideas. Initially, we outline that replacing the condition $p+\frac{3}{2}p^m \leqslant 1$ in [11], Theorem 2.3 and Theorem 2.4 with a weaker one $p+(1+(1-p)^2)p^m \leqslant 1$ in the following theorems will lead to improved results.

Theorem 5.1. *Assume that there exists $p\in[0,1)$ such that $0\leqslant p(t)\leqslant p$ for all $t\geqslant t_0$. Further, assume*

$$\int_{t}^{t+3\sigma+(m-1)\tau} q(\xi)\,\mathrm{d}\xi \leqslant (1-p)\bigg(1+\frac{1-p}{2}\bigg) \quad \textit{for } t\geqslant t_0,$$

where $m\in\mathbb{N}$ satisfies $p+(1+(1-p)^2)p^m \leqslant 1$. Then the zero solution of (1.1) *is uniformly stable.*

Theorem 5.2. *Assume that there exists $p \in [0,1)$ such that $0 \leqslant p(t) \leqslant p$ for all $t \geqslant t_0$. Further, assume* (1.4) *and*

$$\limsup_{t\to\infty} \int_t^{t+3\sigma+(m-1)\tau} q(\xi)\,\mathrm{d}\xi < (1-p)\left(1+\frac{1-p}{2}\right),$$

where $m \in \mathbb{N}$ satisfies $p + (1+(1-p)^2)p^m \leqslant 1$. Then the zero solution of (1.1) *is globally attractive.*

By combining [11], Theorem 3.1 with Theorem 2.1, we derive a more general result extending [11], Theorem 2.1.

Theorem 5.3. *Assume that $p \in [0,1)$. Further, assume*

$$\int_t^{t+\sigma+(m-1)\tau} q(\xi)\,\mathrm{d}\xi \leqslant \begin{cases} \dfrac{1-p}{1-p^m}\dfrac{3-4p^m}{2}, & p \in \Big[0, \dfrac{1}{\sqrt[m]{4}}\Big) \\ \dfrac{1-p}{1-p^m}\sqrt{2(1-2p^m)}, & p \in \Big[\dfrac{1}{\sqrt[m]{4}}, \dfrac{1}{\sqrt[m]{2}}\Big) \end{cases} \quad \text{for } t \geqslant t_0,$$

where $m \in \mathbb{N}$ satisfies $2p^m < 1$. Then the zero solution of equation (1.2) *is uniformly stable.*

Additionally, by combining Theorem F with Theorem 2.2, we establish a broader generalization of Theorem E.

Theorem 5.4. *Assume that $p \in [0,1)$ and* (1.4) *holds. Further, assume*

$$\limsup_{t\to\infty} \int_t^{t+\sigma+(m-1)\tau} q(\xi)\,\mathrm{d}\xi < \begin{cases} \dfrac{1-p}{1-p^m}\dfrac{3-4p^m}{2}, & p \in \Big[0, \dfrac{1}{\sqrt[m]{4}}\Big), \\ \dfrac{1-p}{1-p^m}\sqrt{2(1-2p^m)}, & p \in \Big[\dfrac{1}{\sqrt[m]{4}}, \dfrac{1}{\sqrt[m]{2}}\Big), \end{cases}$$

where $m \in \mathbb{N}$ satisfies $2p^m < 1$. Then the zero solution of (1.2) *is globally attractive.*

It is evident that when $m = 1$ and $p(t) \equiv p$, the previous theorems reduce to [11], Theorem 2.1 and Theorem 2.2.

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