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SUMUDU TRANSFORM OF PREFUNCTIONS AND ITS APPLICATIONS TO CERTAIN INTEGRAL, INTEGRO DIFFERENTIAL EQUATIONS AND INITIAL VALUE PROBLEMS IN THE BICOMPLEX SPACE

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Abstract. An integral transform is a useful and simple tool represented by algebraic problems in the process of solving differential equations. Though not as familiar as the other transforms, Sumudu transform has interesting advantages. Because of its scale and unit preserving properties and the ability to solve differential equations without resorting to a new domain, it is considered to be advantageous over the other transforms. Prefunctions possess some of the properties of elementary functions but not periodicity. Extended prefunctions generalise prefunctions. After having obtained the Sumudu and its inverse transform of prefunctions and extended prefunctions, we have analysed them for their properties. The main objective of this article is to obtain closed form solutions for the integral, integro differential equations and initial value problems in terms of prefunctions and extended prefunctions. It is interesting to visualise their solutions using MATLAB for fixed parameters.

Keywords: bicomplex numbers; bicomplex Sumudu transform; prefunction; initial value problem

MSC 2020: 44A05, 30G35, 44A15

1. Introduction

In day to day life things keep changing and describing how often they change and at what rate, ends up in differential equations and have commendable ability to predict the world around us. Differential equations act as a channel between the developments of mathematics and applications and have gained importance in almost every area of application, viz medicine, economics, engineering, to name a few.

Khandeparkar et al. [7] have defined prefunctions for a real variable t, as a set of functions possessing a sequence $\{f_n(t), t \in \mathbb{R}\}$ which approaches to one of the

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elementary functions as $n \to 0$. Motivated by the work of Dhaigude [5], efforts have been made to define prefunctions of a complex variable z. Thirumalai et al. [8] have defined prefunctions of a complex variable z and studied their properties. Functions of a complex variable t that possess a sequence $\{f_n(t), t \in \mathbb{C}\}$ such that they tend to one of the elementary functions as $n \to 0$ are called prefunctions of a complex variable t. Prefunctions possess some of the properties that are possessed by the elementary functions but periodicity. The extended prefunctions are generalizations of elementary functions for which we can obtain sets of prefunctions.

The definitions of pre-exponential, pretrigonometric and prehyperbolic functions are listed in the table below:

S. No	Definition
1	$\operatorname{pexp}(t,\beta) = 1 + \sum\limits_{n=1}^{\infty} \frac{t^{n+\beta}}{\Gamma(n+1+\beta)}$, $t \in \mathbb{C}$ and $\beta \geqslant 0$
2	$\operatorname{pexp}(-t,\beta) = 1 + (-1)^{\beta} \sum\limits_{n=1}^{\infty} (-1)^n \frac{t^{n+\beta}}{\Gamma(n+1+\beta)}$, $t \in \mathbb{C}$ and $\beta \geqslant 0$
3	$\operatorname{pcos}(t,\beta) = 1 + \sum\limits_{n=1}^{\infty} (-1)^n \frac{t^{2n+\beta}}{\Gamma(2n+1+\beta)}$, $t \in \mathbb{C}$ and $\beta \geqslant 0$
4	$\operatorname{psin}(t,\beta) = \sum\limits_{n=0}^{\infty} (-1)^n \frac{t^{2n+\beta+1}}{\Gamma(2n+2+\beta)}$, $t \in \mathbb{C}$ and $\beta \geqslant 0$
5	$\operatorname{pcosh}(t,\beta) = 1 + \sum\limits_{n=1}^{\infty} \frac{t^{2n+\beta}}{\Gamma(2n+1+\beta)}$, $t \in \mathbb{C}$ and $\beta \geqslant 0$
6	$\operatorname{psinh}(t,\beta) = \sum\limits_{n=0}^{\infty} \frac{t^{2n+\beta+1}}{\Gamma(2n+2+\beta)}$, $t \in \mathbb{C}$ and $\beta \geqslant 0$

Table 1. Definition of prefunctions.

We obtain generalized trigonometric and hyperbolic extended prefunctions, by dividing the series expansions for $\operatorname{pexp}(-t,\beta)$ and $\operatorname{pexp}(t,\beta)$ into n absolutely convergent series. We fix $n = 3$ for our work in Table 2.

Sumudu transform was introduced in 1993 by Watugala [9] and used to solve ordinary differential equations in control engineering. It also plays a vital role in solving ordinary differential equations in the branches of Mathematics and Physics. The bicomplex Sumudu transform was defined in 2017 by Agarwal et al. [1].

Definition 1.1. The bicomplex Sumudu transform of $f(t)$, a bicomplex-valued piecewise continuous function of exponential order k, is defined as

$$S[f(t);\xi] = \frac{1}{\xi} \int_0^{\infty} \mathrm{e}^{-t/\xi} f(t)\,\mathrm{d}t = \widetilde{f}(\xi), \quad \xi \in \Omega, \tag{1.1}$$

S. No	Definition
7	$pM_{3,0}(t,\beta) = 1 + \sum\limits_{n=1}^{\infty} (-1)^n \frac{t^{3n+\beta}}{\Gamma(3n+1+\beta)}$, $t \in \mathbb{C}$ and $\beta \geqslant 0$
8	$pM_{3,1}(t,\beta) = \sum\limits_{n=0}^{\infty} (-1)^n \frac{t^{3n+1+\beta}}{\Gamma(3n+2+\beta)}$, $t \in \mathbb{C}$ and $\beta \geqslant 0$
9	$pM_{3,2}(t,\beta) = \sum\limits_{n=0}^{\infty} (-1)^n \frac{t^{3n+2+\beta}}{\Gamma(3n+3+\beta)}$, $t \in \mathbb{C}$ and $\beta \geqslant 0$
10	$pN_{3,0}(t,\beta) = 1 + \sum\limits_{n=1}^{\infty} \frac{t^{3n+\beta}}{\Gamma(3n+1+\beta)}$, $t \in \mathbb{C}$ and $\beta \geqslant 0$
11	$pN_{3,1}(t,\beta) = \sum\limits_{n=0}^{\infty} \frac{t^{3n+1+\beta}}{\Gamma(3n+2+\beta)}$, $t \in \mathbb{C}$ and $\beta \geqslant 0$
12	$pN_{3,2}(t,\beta) = \sum\limits_{n=0}^{\infty} \frac{t^{3n+2+\beta}}{\Gamma(3n+3+\beta)}$, $t \in \mathbb{C}$ and $\beta \geqslant 0$

Table 2. Definition of extended prefunctions.

where Ω is defined as

$$\Omega = \left\{ \xi \in C_2 \colon \operatorname{Re}\left(\frac{1}{\xi}\right) > k + \left| \operatorname{Im}_j\left(\frac{1}{\xi}\right) \right| \text{ and } \xi \notin O_2 \right\}, \tag{1.2}$$

$O_2 = \{\xi = z_1 + i_2 z_2 \colon z_1^2 + z_2^2 \neq 0\}$ being the set of singular elements in bicomplex space.

Setting β and the imaginary part of the bicomplex number, $\xi = z_1 + j z_2$ at 0, we obtain the complex Sumudu transform of prefunctions. In the complex Sumudu transform of prefunctions of a complex variable $z = x + \mathrm{i}y$, if we replace y by 0, we obtain the Sumudu transform of prefunctions of a real variable. Thus, the bicomplex Sumudu transform generalizes the Sumudu transforms in $\mathbb{C}$ and $\mathbb{R}$.

Let $f(t)$ represent a prefunction of a complex variable t. Then $f(t)$ is said to be of exponential order k, as $\lim\limits_{t\to\infty} f(t)\mathrm{e}^{-kt} = \lim\limits_{t\to\infty} f(t)/\mathrm{e}^{kt} = 0$, where k is real number and that guarantees the existence of Sumudu transform of prefunctions. For an extensive study about prefunctions, Sumudu transform and solutions of nonhomogeneous ordinary differential equations using Sumudu transform one can refer to [2], [3], [4], [6], [10].

We have obtained the Sumudu transform of prefunctions and extended prefunctions, discussed about the associated properties and solved certain nonhomogeneous ordinary differential equations, integral equations and integro differential equations whose solutions are pretrigonometric, prehyperbolic and extended prefunctions.

2. The bicomplex Sumudu transform of prefunctions

We know that

$$\operatorname{pexp}(t,\beta) = 1 + \sum_{n=1}^{\infty} \frac{t^{n+\beta}}{\Gamma(n+1+\beta)}, \quad t \in \mathbb{C} \text{ and } \beta \geqslant 0, \tag{2.1}$$

$$\begin{aligned} S[\operatorname{pexp}(t,\beta);\xi] &= \frac{1}{\xi}\int_0^{\infty} \mathrm{e}^{-t/\xi} \operatorname{pexp}(t,\beta)\,\mathrm{d}t \\ &= \frac{1}{\xi}\int_0^{\infty} \mathrm{e}^{-t/\xi}\left[1 + \sum_{n=1}^{\infty} \frac{t^{n+\beta}}{\Gamma(n+1+\beta)}\right]\mathrm{d}t \\ &= \frac{1}{\xi}\int_0^{\infty} \mathrm{e}^{-t/\xi}\,\mathrm{d}t + \frac{1}{\xi}\int_0^{\infty} \mathrm{e}^{-t/\xi} \sum_{n=1}^{\infty} \frac{t^{n+\beta}}{\Gamma(n+1+\beta)}\,\mathrm{d}t \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{\Gamma(n+1+\beta)}\Gamma(n+1+\beta)\xi^{n+\beta} \\ &= 1 + \sum_{n=1}^{\infty} \xi^{n+\beta} \\ &= 1 + \frac{\xi^{\beta+1}}{1-\xi}, \quad \xi \neq 1. \end{aligned}$$

Thus, the bicomplex Sumudu transform of the prefunction $\operatorname{pexp}(t,\beta)$ is

$$S[\operatorname{pexp}(t,\beta);\xi] = \widetilde{\operatorname{pexp}}(\xi,\beta) = 1 + \frac{\xi^{\beta+1}}{1-\xi},\ \xi \neq 1. \tag{2.2}$$

It is to be noticed that when $\beta = 0$, $\operatorname{pexp}(t,\beta) = \exp(t)$ and its Sumudu transform is given by $1/(1-\xi)$, $\xi \neq 1$.

Prefunction	Sumudu transform	Inverse Sumudu transform
$\operatorname{pexp}(-t,\beta)$	$S[p\exp(-t,\beta);\xi] = 1 - (-1)^{\beta}\frac{\xi^{\beta+1}}{1+\xi}$, $\xi \neq -1$	$S^{-1}\big[1 - (-1)^{\beta}\frac{\xi^{\beta+1}}{1+\xi}\big] = \operatorname{pexp}(-t,\beta)$
$\operatorname{pcos}(t,\beta)$	$S[\operatorname{pcos}(t,\beta);\xi] = 1 - \frac{\xi^{\beta+2}}{1+\xi^2}$, $\xi \neq \pm i$	$S^{-1}\big[1 - \frac{\xi^{\beta+2}}{1+\xi^2}\big] = \operatorname{pcos}(t,\beta)$
$\operatorname{psin}(t,\beta)$	$S[\operatorname{psin}(t,\beta);\xi] = \frac{\xi^{\beta+1}}{1+\xi^2}$, $\xi \neq \pm i$	$S^{-1}\big[\frac{\xi^{\beta+1}}{1+\xi^2}\big] = \operatorname{psin}(t,\beta)$
$\operatorname{pcosh}(t,\beta)$	$S[\operatorname{pcosh}(t,\beta);\xi] = 1 + \frac{\xi^{\beta+2}}{1-\xi^2}$, $\xi \neq \pm 1$	$S^{-1}\big[1 + \frac{\xi^{\beta+2}}{1-\xi^2}\big] = \operatorname{pcosh}(t,\beta)$
$\operatorname{psinh}(t,\beta)$	$S[\operatorname{psinh}(t,\beta);\xi] = \frac{\xi^{\beta+1}}{1-\xi^2}$, $\xi \neq \pm 1$	$S^{-1}\big[\frac{\xi^{\beta+1}}{1-\xi^2}\big] = \operatorname{psinh}(t,\beta)$

Table 3. Sumudu transform pairs of prefuntions.

Clearly, inverse bicomplex Sumudu transform of prefunction $\mathrm{pexp}(t,\beta)$ is

$$S^{-1}\left[1+\frac{\xi^{\beta+1}}{1-\xi}\right]=\mathrm{pexp}(t,\beta). \tag{2.3}$$

A similar working helps us to arrive at the bicomplex Sumudu transforms and the inverse transforms of prefunctions and they are listed below.

Extended prefunction	Sumudu transform	Inverse Sumudu transform
$pM_{3,0}(t,\beta)$	$S[pM_{3,0}(t,\beta);\xi]=1-\frac{\xi^{\beta+3}}{1+\xi^3}$, $\xi\neq -1$	$S^{-1}\big[1-\frac{\xi^{\beta+3}}{1+\xi^3}\big]=pM_{3,0}(t,\beta)$
$pM_{3,1}(t,\beta)$	$S[pM_{3,1}(t,\beta);\xi]=\frac{\xi^{\beta+1}}{1+\xi^3}$, $\xi\neq -1$	$S^{-1}\big[\frac{\xi^{\beta+1}}{1+\xi^3}\big]=pM_{3,1}(t,\beta)$
$pM_{3,2}(t,\beta)$	$S[pM_{3,2}(t,\beta);\xi]=\frac{\xi^{\beta+2}}{1+\xi^3}$, $\xi\neq -1$	$S^{-1}\big[\frac{\xi^{\beta+2}}{1+\xi^3}\big]=pM_{3,2}(t,\beta)$
$pN_{3,0}(t,\beta)$	$S[pN_{3,0}(t,\beta);\xi]=1+\frac{\xi^{\beta+3}}{1-\xi^3}$, $\xi\neq 1$	$S^{-1}\big[1+\frac{\xi^{\beta+3}}{1-\xi^3}\big]=pN_{3,0}(t,\beta)$
$pN_{3,1}(t,\beta)$	$S[pN_{3,1}(t,\beta);\xi]=\frac{\xi^{\beta+1}}{1-\xi^3}$, $\xi\neq 1$	$S^{-1}\big[\frac{\xi^{\beta+1}}{1-\xi^3}\big]=pN_{3,1}(t,\beta)$
$pN_{3,2}(t,\beta)$	$S[pN_{3,2}(t,\beta);\xi]=\frac{\xi^{\beta+2}}{1-\xi^3}$, $\xi\neq 1$	$S^{-1}\big[\frac{\xi^{\beta+2}}{1-\xi^3}\big]=pN_{3,2}(t,\beta)$

Table 4. Sumudu transform pairs of extended prefunctions.

3. Properties of bicomplex Sumudu transform of prefunctions

Theorem 3.1 (Linearity property)**.** *Let* $\widetilde{\mathrm{pexp}_1}(\xi,\alpha)=S[\mathrm{pexp}_1(t,\alpha);\xi]$ *and* $\widetilde{\mathrm{pexp}_2}(\xi,\beta)=S[\mathrm{pexp}_2(t,\beta);\xi]$ *be the bicomplex Sumudu transforms of prefunctions* $\mathrm{pexp}_1(t,\alpha)$ *and* $\mathrm{pexp}_2(t,\beta)$ *of exponential orders* K_1 *and* K_2*, respectively. Then*

$$S[c_1\mathrm{pexp}_1(t,\alpha;\xi)+c_2\mathrm{pexp}_2(t,\beta;\xi)]=c_1\widetilde{\mathrm{pexp}}_1(\xi,\alpha)+c_2\widetilde{\mathrm{pexp}}_2(\xi,\beta),\quad \xi\in\Omega, \tag{3.1}$$

where $K=\max\{K_1,K_2\}$.

P r o o f. We know that

$$\mathrm{pexp}(t,\alpha;\xi)=\left[1+\sum_{n=1}^{\infty}\frac{t^{n+\alpha}}{\Gamma(n+1+\alpha)}\right].$$

By applying the definition of bicomplex Sumudu transform of prefuntions we get

$$\begin{aligned}
&[c_1\,\mathrm{pexp}_1(t,\alpha;\xi)+c_2\,\mathrm{pexp}_2(t,\beta;\xi)]\\
&\qquad=\frac{1}{\xi}\int_0^\infty \mathrm{e}^{-t/\xi}[c_1\,\mathrm{pexp}_1(t,\alpha;\xi)+c_2\,\mathrm{pexp}_2(t,\beta;\xi)]\,\mathrm{d}t\\
&\qquad=\frac{1}{\xi}\int_0^\infty \mathrm{e}^{-t/\xi}\left[c_1\left(1+\sum_{n=1}^{\infty}\frac{t^{n+\alpha}}{\Gamma(n+1+\alpha)}\right)\right.\\
&\qquad\quad\left.+c_2\left(1+\sum_{n=1}^{\infty}\frac{t^{n+\beta}}{\Gamma(n+1+\beta)}\right)\right]\mathrm{d}z\\
&\qquad=c_1\left[1+\frac{\xi^{\alpha+1}}{1-\xi}\right]+c_2\left[1+\frac{\xi^{\beta+1}}{1-\xi}\right]\\
&\qquad=c_1S[\mathrm{pexp}_1(t,\alpha);\xi]+c_2S[\mathrm{pexp}_2(t,\beta);\xi]\\
&\qquad=c_1\widetilde{\mathrm{pexp}}_1(\xi,\alpha)+c_2\widetilde{\mathrm{pexp}}_2(\xi,\beta).
\end{aligned}$$

In the same manner, the linearity property can be proved for other prefunctions also. □

Theorem 3.2 (Change of scale property)**.** *Let* $\widetilde{\mathrm{pexp}}(\xi,\beta)=S[\mathrm{pexp}(t,\beta);\xi]$ *be the bicomplex Sumudu transform of prefunction* $\mathrm{pexp}(t,\beta)$*. Then*

$$S[\mathrm{pexp}(at,\beta);\xi]=\widetilde{\mathrm{pexp}}(a\xi,\beta),\quad a>0,\ \xi\in\Omega. \tag{3.2}$$

P r o o f. Consider

$$\begin{aligned}
S[\mathrm{pexp}(at,\beta);\xi]&=\frac{1}{\xi}\int_0^\infty \mathrm{e}^{-t/\xi}\,\mathrm{pexp}(at,\beta;\xi)\,\mathrm{d}t\\
&=\frac{1}{\xi}\int_0^\infty \mathrm{e}^{-t/\xi}\,\mathrm{pexp}(at,\beta;\xi)\,\mathrm{d}t\quad(\text{put } at=u)\\
&=\frac{1}{a\xi}\int_0^\infty \mathrm{e}^{-u/a\xi}\,\mathrm{pexp}(u,\beta;\xi)\,\mathrm{d}u\\
&=\frac{1}{a\xi}\int_0^\infty \mathrm{e}^{-u/a\xi}\left[1+\sum_{n=1}^{\infty}\frac{u^{n+\beta}}{\Gamma(n+1+\beta)}\right]\mathrm{d}u\\
&=\frac{1}{a\xi}\int_0^\infty \mathrm{e}^{-u/a\xi}\,\mathrm{d}u+\frac{1}{a\xi}\int_0^\infty \mathrm{e}^{-u/a\xi}\sum_{n=1}^{\infty}\frac{u^{n+\beta}}{\Gamma(n+1+\beta)}\,\mathrm{d}u\\
&=1+\frac{a\xi^{\beta+1}}{1-a\xi}\\
&=\widetilde{\mathrm{pexp}}(a\xi,\beta).
\end{aligned}$$

Likewise, the change of scale property can be proved for other prefunctions also. □

Theorem 3.3. *Let* $\widetilde{\text{pexp}}(\xi,\beta) = S[\text{pexp}(t,\beta);\xi]$ *be the Sumudu transform of bicomplex valued function* $\text{pexp}(t)$. *Then for* $\xi \in \Omega$

$$S[\text{pexp}'(t,\beta);\xi] = \frac{1}{\xi}[\widetilde{\text{pexp}}(\xi,\beta) - \text{pexp}(0,\beta)], \tag{3.3}$$

$$S[\text{pexp}''(t,\beta);\xi] = \frac{1}{\xi^2}[\widetilde{\text{pexp}}(\xi,\alpha) - \text{pexp}(0,\beta) - \xi\,\text{pexp}(0,\beta)],$$

$$S[\text{pexp}^{(m)}(t,\beta);\xi] = \frac{1}{\xi^m}\left[\widetilde{\text{pexp}}(\xi,\beta) - \text{pexp}(0,\beta) - \sum_{k=1}^{m-1}\xi^k\,\text{pexp}^k(0,\beta)\right].$$

P r o o f. Consider

$$\begin{aligned}
S[\text{pexp}'(t,\beta);\xi] &= \frac{1}{\xi}\int_o^\infty \mathrm{e}^{-t/\xi}\,\text{pexp}'(t,\beta)\,\mathrm{d}t \\
&= \frac{1}{\xi}\int_0^\infty \mathrm{e}^{-t/\xi}\left(\sum_{n=1}^\infty \frac{(n+\beta)t^{n+\beta-1}}{\Gamma(n+1+\beta)}\right)\mathrm{d}t \\
&= \frac{1}{\xi_1}\int_0^\infty \mathrm{e}^{-t/\xi_1}\left(\sum_{n=1}^\infty \frac{(n+\beta)t^{n+\beta-1}}{\Gamma(n+1+\beta)}\right)\mathrm{d}t\cdot e_1 \\
&\quad + \frac{1}{\xi_2}\int_0^\infty \mathrm{e}^{-t/\xi_2}\left(\sum_{n=1}^\infty \frac{(n+\beta)t^{n+\beta-1}}{\Gamma(n+1+\beta)}\right)\mathrm{d}t\cdot e_2.
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
S[\text{pexp}'(t,\beta);\xi] &= \left[-\frac{1}{\xi_1}\text{pexp}_1(0,\beta) + \frac{1}{\xi_1}\widetilde{\text{pexp}}(\xi_1,\beta)\right]e_1 \\
&\quad + \left[-\frac{1}{\xi_2}\text{pexp}_2(0,\beta) + \frac{1}{\xi_2}\widetilde{\text{pexp}}(\xi_2,\beta)\right]e_2 \\
&= \frac{1}{\xi_1 e_1 + \xi_2 e_2}[(\widetilde{\text{pexp}}(\xi_1,\beta)e_1 + \widetilde{\text{pexp}}(\xi_2,\beta)e_2) \\
&\quad - (\text{pexp}(0,\beta)e_1 + \text{pexp}(0,\beta)e_2)] \\
&= \frac{1}{\xi}[\widetilde{\text{pexp}}(\xi,\beta) - \text{pexp}(0,\beta)], \quad \xi \in \Omega.
\end{aligned}$$

By replacing $\text{pexp}_1'(t,\beta)$ using $\text{pexp}''(t,\beta)$, we have

$$\begin{aligned}
S[\text{pexp}''(t,\beta);\xi] &= S[\text{pexp}_1'(t,\beta);\xi] \\
&= \frac{1}{\xi}[\text{pexp}_1(\xi,\beta) - \text{pexp}_1(0,\beta)] \\
&= \frac{1}{\xi}\left[\frac{1}{\xi}[\widetilde{\text{pexp}}(\xi,\beta) - \text{pexp}(0,\beta)] - \text{pexp}'(0,\beta)\right] \\
&= \frac{1}{\xi^2}[\widetilde{\text{pexp}}(\xi,\beta) - \text{pexp}(0,\beta) - \xi\text{pexp}'(0,\beta)].
\end{aligned} \tag{3.4}$$

The above result is true for $n = 1$.
Assume that the result is true for $n = m$, that is,

$$S[\operatorname{pexp}^{(m)}(t,\beta);\xi] = \frac{1}{\xi^m}\left[\widetilde{\operatorname{pexp}}(\xi,\beta) - \operatorname{pexp}(0,\beta) - \sum_{k=1}^{m-1}\xi^k \operatorname{pexp}^{(k)}(0,\beta)\right]. \tag{3.5}$$

To prove the result for $n = m+1$, we replace $\operatorname{pexp}^{(m)}(t,\beta)$ using $\operatorname{pexp}^{(m+1)}(t,\beta)$. Then

$$\begin{aligned} S[\operatorname{pexp}^{(m+1)}(t,\beta);\xi] &= \frac{1}{\xi}\left[\frac{1}{\xi^m}\left[\widetilde{\operatorname{pexp}}(\xi,\beta) - \operatorname{pexp}(0,\beta) - \sum_{k=1}^{m-1}\xi^k \operatorname{pexp}^{(k)}(0,\beta)\right]\right] \\ &= \frac{1}{\xi^{m+1}}\left[\widetilde{\operatorname{pexp}}(\xi,\beta) - \operatorname{pexp}(0,\beta) - \sum_{k=1}^{m}\xi^k \operatorname{pexp}^{(k)}(0,\beta)\right]. \end{aligned}$$

Hence, by the principle of mathematical induction, the result is true for all $n \in N$. A similar proof holds true for other prefunctions also. □

Theorem 3.4. *Let* $\widetilde{\operatorname{pexp}}(\xi,\beta) = S[\operatorname{pexp}(t,\beta);\xi]$ *be the bicomplex Sumudu transform of prefunction* $\operatorname{pexp}(t,\beta)$. *Then*

$$S\left[\int_0^t \operatorname{pexp}(\eta,\beta)\,\mathrm{d}\eta;\xi\right] = \xi\,\widetilde{\operatorname{pexp}}(\xi,\beta), \quad \xi \in \Omega. \tag{3.6}$$

P r o o f. From the definition of bicomplex Sumudu transform, we have

$$S\left[\int_0^t \operatorname{pexp}(\eta,\beta)\,\mathrm{d}\eta;\xi\right] = \frac{1}{\xi}\int_0^\infty \mathrm{e}^{-t/\xi}\int_0^t \operatorname{pexp}(\eta,\beta)\,\mathrm{d}\eta\,\mathrm{d}t. \tag{3.7}$$

By changing the order of integration, we get

$$\begin{aligned} S\left[\int_0^t \operatorname{pexp}(\eta,\beta)\,\mathrm{d}\eta;\xi\right] &= \frac{1}{\xi}\int_0^\infty \operatorname{pexp}(\eta,\beta)\,\mathrm{d}\eta\int_\eta^\infty \mathrm{e}^{-t/\xi}\,\mathrm{d}t \\ &= \frac{1}{\xi}\int_0^\infty \xi\mathrm{e}^{-\eta/\xi}\operatorname{pexp}(\eta,\beta)\,\mathrm{d}\eta \\ &= \xi\,\widetilde{\operatorname{pexp}}(\xi,\beta). \end{aligned}$$

We can prove the same for other prefunctions using the same method. □

Theorem 3.5. *Let* $\widetilde{\operatorname{pexp}}(\xi,\beta) = S[\operatorname{pexp}(t,\beta);\xi]$ *be the bicomplex Sumudu transform of prefunction* $\operatorname{pexp}(t,\beta)$. *Then*

$$S[t\operatorname{pexp}(t,\beta);\xi] = \xi^2\frac{\mathrm{d}}{\mathrm{d}\xi}\widetilde{\operatorname{pexp}}(\xi,\beta) + \xi\,\widetilde{\operatorname{pexp}}(\xi,\beta), \quad \xi \in \Omega. \tag{3.8}$$

P r o o f. Since

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\widetilde{\mathrm{pexp}}(\xi,\beta)=\frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^\infty\frac{1}{\xi}\mathrm{e}^{-t/\xi}\left[1+\sum_{n=1}^\infty\frac{t^{n+\beta}}{\Gamma(n+1+\beta)}\right]\mathrm{d}t, \tag{3.9}$$

by using the idempotent representation of ξ,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}\widetilde{\mathrm{pexp}}(\xi,\beta)&=\frac{\mathrm{d}}{\mathrm{d}\xi_1}\int_0^\infty\frac{\mathrm{e}^{-t/\xi_1}}{\xi_1}\left[1+\sum_{n=1}^\infty\frac{t^{n+\beta}}{\Gamma(n+1+\beta)}\right]\mathrm{d}t\cdot e_1\\&\quad+\frac{\mathrm{d}}{\mathrm{d}\xi_2}\int_0^\infty\frac{\mathrm{e}^{-t/\xi_2}}{\xi_2}\left[1+\sum_{n=1}^\infty\frac{t^{n+\beta}}{\Gamma(n+1+\beta)}\right]\mathrm{d}t\cdot e_2.\end{aligned} \tag{3.10}$$

Using Leibniz rule for integration, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}\widetilde{\mathrm{pexp}}(\xi,\beta)&=\int_0^\infty\frac{\partial}{\partial\xi_1}\frac{\mathrm{e}^{-t/\xi_1}}{\xi_1}\left[1+\sum_{n=1}^\infty\frac{t^{n+\beta}}{\Gamma(n+1+\beta)}\right]\mathrm{d}t\cdot e_1\\&\quad+\int_0^\infty\frac{\partial}{\partial\xi_2}\frac{\mathrm{e}^{-t/\xi_2}}{\xi_2}\left[1+\sum_{n=1}^\infty\frac{t^{n+\beta}}{\Gamma(n+1+\beta)}\right]\mathrm{d}t\cdot e_2\\&=\left(\int_0^\infty\frac{1}{\xi_1^3}t\mathrm{e}^{-t/\xi_1}\left[1+\sum_{n=1}^\infty\frac{t^{n+\beta}}{\Gamma(n+1+\beta)}\right]\mathrm{d}t\cdot e_1\right.\\&\quad\left.-\int_0^\infty\frac{1}{\xi_1^2}t\mathrm{e}^{-t/\xi_1}\left[1+\sum_{n=1}^\infty\frac{t^{n+\beta}}{\Gamma(n+1+\beta)}\right]\mathrm{d}t\cdot e_1\right)\\&\quad+\left(\int_0^\infty\frac{1}{\xi_2^3}t\mathrm{e}^{-t/\xi_2}\left[1+\sum_{n=1}^\infty\frac{t^{n+\beta}}{\Gamma(n+1+\beta)}\right]\mathrm{d}t\cdot e_2\right.\\&\quad\left.-\int_0^\infty\frac{1}{\xi_2^2}t\mathrm{e}^{-t/\xi_2}\left[1+\sum_{n=1}^\infty\frac{t^{n+\beta}}{\Gamma(n+1+\beta)}\right]\mathrm{d}t\cdot e_2\right)\\&=\left(\frac{1}{\xi_1^2}S[t\mathrm{pexp}_1(t,\beta);\xi]-\frac{1}{\xi_1}S[\mathrm{pexp}_2(t,\beta);\xi]\right)e_1\\&\quad+\left(\frac{1}{\xi_2^2}S[t\mathrm{pexp}_1(t,\beta);\xi]-\frac{1}{\xi_2}S[\mathrm{pexp}_2(t,\beta);\xi]\right)e_2\\&=\frac{1}{(\xi_1e_1+\xi_2e_2)^2}S[t(\mathrm{pexp}_1(t,\beta)e_1+\mathrm{pexp}_2(t,\beta)e_2);\xi_1e_1+\xi_2e_2]\\&\quad-\frac{1}{(\xi_1e_1+\xi_2e_2)}S[\mathrm{pexp}_1(t,\beta)e_1+\mathrm{pexp}_2(t,\beta)e_2;\xi_1e_1+\xi_2e_2]\\&=\frac{1}{\xi^2}S[t\,\mathrm{pexp}(t,\beta);\xi]-\frac{1}{\xi}S[\mathrm{pexp}(t,\beta);\xi].\end{aligned}$$

Thus,

$$S[t\,\mathrm{pexp}(t,\beta);\xi]=\xi^2\frac{\mathrm{d}}{\mathrm{d}\xi}\widetilde{\mathrm{pexp}}(\xi,\beta)+\xi\,\widetilde{\mathrm{pexp}}(\xi,\beta). \tag{3.11}$$

Proofs for the other prefunctions follow the same method. □

Theorem 3.6. *Let* $\widetilde{\operatorname{pexp}}(\xi,\alpha) = S[\operatorname{pexp}(t,\alpha);\xi]$ *and* $\widetilde{\operatorname{pexp}}(\xi,\beta) = S[\operatorname{pexp}(t,\beta);\xi]$ *be the bicomplex Sumudu transforms of prefunctions* $\operatorname{pexp}(t,\alpha)$ *and* $\operatorname{pexp}(t,\beta)$ *of exponential orders* K_1 *and* K_2*, respectively. Then*

$$S[\operatorname{pexp}(t,\alpha) * \operatorname{pexp}(t,\beta);\xi] = \xi\,\widetilde{\operatorname{pexp}}(\xi,\alpha)\widetilde{\operatorname{pexp}}(\xi,\beta), \quad \xi \in \Omega, \tag{3.12}$$

where $K = \max\{K_1, K_2\}$.

P r o o f. We have

$$\begin{aligned} S[\operatorname{pexp}(t,\alpha) * \operatorname{pexp}(t,\beta);\xi] &= \frac{1}{\xi}\int_0^\infty \mathrm{e}^{-t/\xi}(\operatorname{pexp}(t,\alpha) * \operatorname{pexp}(t,\beta))\,\mathrm{d}t \\ &= \frac{1}{\xi}\int_0^\infty \mathrm{e}^{-t/\xi}\int_0^t \operatorname{pexp}(v,\alpha)\operatorname{pexp}(t-v,\beta)\,\mathrm{d}v\,\mathrm{d}t. \end{aligned}$$

By replacing $z - v$ using η and applying change of order of integration, we have

$$\begin{aligned} S[\operatorname{pexp}(t,\alpha) * \operatorname{pexp}(t,\beta);\xi] &= \frac{1}{\xi}\int_0^\infty \operatorname{pexp}(v,\alpha)\,\mathrm{d}v\int_\xi^\infty \mathrm{e}^{-z/\xi}\operatorname{pexp}(t-v,\beta)\,\mathrm{d}t \\ &= \frac{1}{\xi}\int_0^\infty \mathrm{e}^{-v/\xi}\operatorname{pexp}(v,\alpha)\,\mathrm{d}v\int_0^\infty \mathrm{e}^{-\eta/\xi}\operatorname{pexp}(\eta,\beta)\,\mathrm{d}\eta \\ &= \xi\,\widetilde{\operatorname{pexp}}(\xi,\alpha)\widetilde{\operatorname{pexp}}(\xi,\beta). \end{aligned}$$

Results on convolution for other prefunctions can be obatined in the same manner. □

Theorem 3.7. *Let* $\widetilde{\operatorname{pexp}}(\xi,\beta)$ *be the bicomplex Sumudu transform of prefunction* $\operatorname{pexp}(t,\beta)$ *of exponential order* K*, analytic in* Ω*. Then*

$$\operatorname{pexp}(t,\beta;\xi) = \frac{1}{2\pi i_1}\int_\Gamma \mathrm{e}^{t/\xi}\xi\,\widetilde{\operatorname{pexp}}(\xi,\beta)\,\mathrm{d}\xi = \sum \operatorname{Res}[\mathrm{e}^{t/\xi}\xi\,\widetilde{\operatorname{pexp}}(\xi,\beta)], \tag{3.13}$$

where $\Gamma = (\Gamma_1,\Gamma_2)$ *is a piecewise continuous differentiable closed contour in bicomplex space.*

P r o o f. We know that bicomplex Laplace transform of prefunction $\operatorname{pexp}(t,\beta)$ is

$$L\left[\operatorname{pexp}(t,\beta);\frac{1}{\xi}\right] = \int_0^\infty \mathrm{e}^{-t/\xi}\operatorname{pexp}(t,\beta)\,\mathrm{d}t, \xi \in \Omega, \tag{3.14}$$

which deduces to

$$\frac{1}{\xi}L\left[\operatorname{pexp}(t,\beta);\frac{1}{\xi}\right] = \frac{1}{\xi}\int_0^\infty \mathrm{e}^{-t/\xi}\operatorname{pexp}(t,\beta)\,\mathrm{d}t = \widetilde{\operatorname{pexp}}(\xi,\beta). \tag{3.15}$$

Then

$$L\left[\operatorname{pexp}(t,\beta);\frac{1}{\xi}\right] = \xi\,\widetilde{\operatorname{pexp}}(\xi,\beta). \tag{3.16}$$

Using inverse bicomplex Laplace transform, the above equation yields

$$\text{pexp}(t,\beta;\xi)=\frac{1}{2\pi i_1}\int_\Gamma \mathrm{e}^{t/\xi}\,\text{pexp}\left(\frac{1}{\xi},\alpha\right)\mathrm{d}\xi=\frac{1}{2\pi i_1}\int_\Gamma \mathrm{e}^{t/\xi}\,\xi\,\widetilde{\text{pexp}}(\xi,\beta)\,\mathrm{d}\xi. \tag{3.17}$$

For other prefunctions also we can find this result by using their respective definitions. □

4. Solutions to some integral and integro differential equations in terms of prefunctions using bicomplex Sumudu transform

Example 4.1. Consider the integral equation,

$$v(t,\beta)+\int_0^t v(t,\beta)\,\mathrm{d}t=1-(-1)^\beta\frac{t^{\beta+1}}{\Gamma(\beta+2)}+t. \tag{4.1}$$

Applying the bicomplex Sumudu transform,

$$S\{v(t,\beta)\}+S\left\{\int_0^t v(t,\alpha)\,\mathrm{d}z\right\}=S\{1\}-S\left\{(-1)^\beta\frac{t^{\beta+1}}{\Gamma(\beta+2)}\right\}+S\{t\},$$

$$V(\xi,\beta)+\xi V(\xi,\beta)=1-\frac{(-1)^\beta(\beta+1)!}{\Gamma(\beta+2)}\xi^{\beta+1}+\xi,$$

on simplification it gives

$$V(\xi,\beta)=1-\frac{(-1)^\beta\xi^{\beta+1}}{(1+\xi)}.$$

Taking inverse on both sides, we obtain $v(t,\beta)=\text{pexp}(-t,\beta)$.

Example 4.2. Consider the integro differential equation

$$v'(t,\beta)+\int_0^t v(t,\beta)\,\mathrm{d}t=t-\frac{t^{\beta+1}}{\Gamma(\beta+2)} \tag{4.2}$$

with initial condition

$$v(0,\beta)=1. \tag{4.3}$$

Taking bicomplex Sumudu transform on both sides, we have

$$S\{v'(t,\beta)\}+S\left\{\int_0^t v(t,\beta)\,\mathrm{d}t\right\}=S\{t\}-S\left\{\frac{t^{\beta+1}}{\Gamma(\beta+2)}\right\},$$

$$\frac{1}{\xi}[V(\xi,\beta)-v(0,\beta)]+\xi V(\xi,\beta)=\xi-\xi^{\beta+1}.$$

Using initial condition (4.3), we get

$$V(\xi,\beta)=1-\frac{\xi^{\beta+2}}{(\xi^2+1)}.$$

The inverse Sumudu transform fetches the following result:

$$v(t,\beta) = \mathrm{pcos}(t,\beta).$$

5. Prefunctions and extended prefunctions as solutions of initial value problems

In this section, we have applied bicomplex Sumudu transform techniques to obtain the solutions of some initial value problems involving linear nonhomogeneous first, second and third order ordinary differential equations with constant coefficients. The initial value problems, their solutions and the plots of the solutions are listed in Table 5.

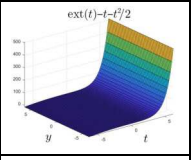
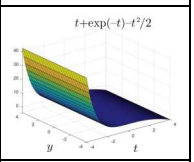
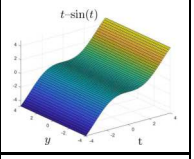
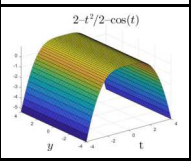
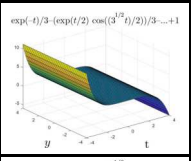
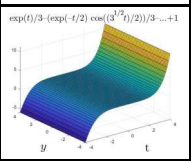
S. No	Initial value problem	Solution in terms of prefunction	Graph with parameter $\beta = 2$
1	$u'(t,\beta) - u(t,\beta) = \frac{t^\beta}{\Gamma(\beta+1)} - 1,$ $u(0,\beta) = 1$	$u(t,\beta;\xi) = \mathrm{pexp}(t,\beta;\xi)$	
2	$u'(t,\beta) + u(t,\beta) = 1 - (-1)^\beta \frac{t^\beta}{\Gamma(\beta+1)},$ $u(0,\beta) = 1$	$u(t,\beta;\xi) = \mathrm{pexp}(-t,\beta;\xi)$	
3	$u''(t,\beta) + u(t,\beta) = \frac{t^{\beta-1}}{\Gamma(\beta)},$ $u(0,\beta) = u'(0,\beta) = 0$	$u(t,\beta;\xi) = \mathrm{psin}(t,\beta;\xi)$	
4	$u''(t,\beta) + u(t,\beta) = 1 - \frac{t^\beta}{\Gamma(\beta+1)},$ $u(0,\beta) = 1, u'(0,\beta) = 0$	$u(t,\beta;\xi) = \mathrm{pcos}(t,\beta;\xi)$	
5	$u'''(t,\beta) + u(t,\beta) = 1 - \frac{t^\beta}{\Gamma(\beta+1)},$ $u(0,\beta) = 1, u'(0,\beta) = u''(0,\beta) = 0$	$u(z,\beta;\xi) = pM_{3,0}(z,\beta;\xi)$	
6	$u'''(t,\beta) - u(t,\beta) = \frac{t^\beta}{\Gamma(\beta+1)} - 1,$ $u(0,\beta) = 1, u'(0,\beta) = u''(0,\beta) = 0$	$u(t,\beta;\xi) = pN_{3,0}(t,\beta;\xi)$	

Table 5.

6. Conclusion

In this article we have obtained Sumudu and its inverse transforms for prefunctions. We also have analysed Sumudu transforms of prefunctions for its properties. It is interesting to see that the solutions of certain initial value problems upto third order are found to be the prefunctions and the extended prefunctions. Any study without visualization is meaningless. So we have plotted the graphs of the solutions to study the nature and behaviour.

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Use of AI tools declaration. We have not used Artificial Intelligence (AI) tools in the creation of the article. All authors are familiar with and agree with the contents of the manuscript.

Data availability statement. This study does not include any datasets generated or analyzed during the current study.

Conflict of interest. The authors declare that there is no conflict of interests regarding the publication of this paper.

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