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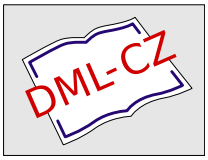
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ENTIRE FUNCTIONS SHARING ONE FINITE VALUE CM WITH THEIR SHIFTS AND DIFFERENCE OPERATORS

Boudaoud Miloudi

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Abstract. We investigate the uniqueness of an entire function $f(z)$ sharing one finite value CM with its shift and its difference operator, in each case we find that $f(z) = h(z)\beta^{\alpha z/c}$, where $\alpha \neq 0$, $\beta > 1$ and $h(z)$ is a periodic entire function of period c. Here, we say that two entire functions $f(z)$ and $g(z)$ share a value a CM if $f(z) - a$ and $g(z) - a$ have the same zeros with same multiplicities.

Keywords: uniqueness; entire function; difference operator

MSC 2020: 30D35

1. Introduction

In the field of complex analysis, using Nevanlinna theory, the study of entire functions is of great importance due to their definition on the entire complex plane. In this paper we will use $\rho(f)$ to denote the order of growth of f and $\lambda(f)$ (or $\lambda(1/f)$) to denote the exponent of convergence of zeros (or poles, respectively). For an entire function $f(z)$, we define its shift by

$$f_c(z) = f(z + c),$$

and its difference operators by

$$\Delta_c f(z) = f(z + c) - f(z),\ \Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z)), \quad n \in \mathbb{N},\ n \geqslant 2.$$

We say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicities) provided that $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros counting multiplicities. The uniqueness of meromorphic functions sharing values with their difference operators has been studied in many papers, see, e.g., [1], [2], [3], [4], [7], [10], and sharing values with their shifts has been investigated by many authors, see, e.g., [3], [5], [6], [8].

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In 2016, El Farissi, Latreuch and Asiri proved:

Theorem A ([3]). *Let $f(z)$ be a transcendental entire function of finite order such that $f(z) \not\equiv f(z+c)$. Then $f(z)$, $f(z+c)$ and $\Delta_c f(z)$ cannot share any finite value $a \neq 0$ CM. Furthermore; if $a = 0$, $f(z)$ must be of the form $f(z) = h(z)\mathrm{e}^{\alpha z/c}$, where $\alpha \neq 0$ and $h(z)$ is a periodic entire function of period c.*

It is natural to ask what happens if $\Delta_c f(z)$ is replaced by $\Delta_c f(z+nc)$ in Theorem A. Corresponding to this question, we obtain the following results.

Theorem 1.1. *Let $f(z)$ be a transcendental entire function of finite order such that $f(z) \not\equiv f(z+c)$. Then $f(z)$, $f(z+c)$ and $\Delta_c f(z+nc)$, $n \in \mathbb{N}$, cannot share any finite value $a \neq 0$ CM. Furthermore, if $a = 0$, $f(z)$ must be of the form*

$$f(z) = h(z)\beta^{\alpha z/c},$$

where $\alpha \neq 0$, $\beta > 1$ and $h(z)$ is a periodic entire function of period c.

Example 1.1. The entire function

$$f(z) = \mathrm{e}^z$$

such that

$$f(z+1) = \mathrm{e}f(z),\ \Delta_1 f(z+n) = \mathrm{e}^n(\mathrm{e}-1)f(z),$$

and hence, $f(z)$, $f(z+1)$ and $\Delta_1 f(z+n)$ share 0 CM.

In this corollary, we have replaced $f(z+c)$ with $\Delta_c f(z+(n+1)c)$ in Theorem 1.1, and we obtained the same result.

Corollary 1.1. *Let $f(z)$ be a transcendental entire function of finite order such that $f(z) \not\equiv f(z+c)$. Then $f(z)$, $\Delta_c f(z+nc)$ and $\Delta_c f(z+(n+1)c)$, $n \in \mathbb{N}$, cannot share any finite value $a \neq 0$ CM. Furthermore, if $a = 0$, $f(z)$ must be of the form*

$$f(z) = h(z)\beta^{\alpha z/c},$$

where $\alpha \neq 0$, $\beta > 1$ and $h(z)$ is a periodic entire function of period c.

Example 1.2. The entire function

$$f(z) = \mathrm{e}^{mz}, \quad m \in \mathbb{C}^*,$$

such that

$$\Delta_1 f(z+n) = \mathrm{e}^{mn}(\mathrm{e}^m - 1)f(z),\ \Delta_1 f(z+n+1) = \mathrm{e}^{m(n+1)}(\mathrm{e}^m - 1)f(z),$$

and hence, $f(z)$, $\Delta_1 f(z+n)$ and $\Delta_1 f(z+n+1)$ share 0 CM.

In this theorem, we generalized Theorem A by replacing $\Delta_c f(z)$ with $\Delta_c^n f(z)$, and we obtained the following results.

Theorem 1.2. *Let $f(z)$ be a transcendental entire function of finite order such that $f(z) \not\equiv f(z+c)$. Then $f(z)$, $f(z+c)$ and $\Delta_c^n f(z)$, $(n \geqslant 1)$, $n \in \mathbb{N}$, cannot share any finite value $a \neq 0$ CM. Furthermore, if $a = 0$, $f(z)$ must be of the form*

$$f(z) = h(z)\beta^{\alpha z/c},$$

where $\alpha \neq 0$, $\beta > 1$ and $h(z)$ is a periodic entire function of period c.

E x a m p l e 1.3. The entire function

$$f(z) = \mathrm{e}^z$$

such that

$$f(z+1) = \mathrm{e}f(z),\ \Delta_1^n f(z) = (\mathrm{e}-1)^n f(z), \quad n \in \mathbb{N}^*,$$

and hence, $f(z)$, $f(z+1)$ and $\Delta_1^n f(z)$ share 0 CM.

In this corollary, we have replaced $f(z+c)$ with $\Delta_c^n f(z+c)$ in Theorem 1.2, and we obtained the same result.

Corollary 1.2. *Let $f(z)$ be a transcendental entire function of finite order such that $f(z) \not\equiv f(z+c)$. Then $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^n f(z+c)$, $(n \geqslant 1)$, $n \in \mathbb{N}$, cannot share any finite value $a \neq 0$ CM. Furthermore, if $a = 0$, $f(z)$ must be of the form*

$$f(z) = h(z)\beta^{\alpha z/c},$$

where $\alpha \neq 0$, $\beta > 1$ and $h(z)$ is a periodic entire function of period c.

E x a m p l e 1.4. The entire function

$$f(z) = \mathrm{e}^z$$

such that

$$\Delta_1^n f(z) = (\mathrm{e}-1)^n f(z),\ \Delta_1^n f(z+1) = \mathrm{e}(\mathrm{e}-1)^n f(z), \quad n \in \mathbb{N}^*$$

and hence, $f(z)$, $\Delta_1^n f(z)$ and $\Delta_1^n f(z+1)$ share 0 CM.

2. Lemmas

For the proof of our results, we need the following lemmas.

Lemma 2.1. *Let n be a nonnegative integer, and let $f(z)$ be a meromorphic function*

$$\Delta_c^n f(z) + \Delta_c^{n+1} f(z) = \Delta_c^n f(z+c).$$

Lemma 2.2 ([9])**.** *Suppose $f_j(z)$ $(j = 1, 2, \ldots, n+1)$ and $g_j(z)$ $(j = 1, 2, \ldots, n)$ $(n \geqslant 1)$ are entire functions satisfying the following conditions:*

(i) $\sum\limits_{j=1}^{n} f_j(z)\mathrm{e}^{g_j(z)} \equiv f_{n+1}(z)$;

(ii) *The order of $f_j(z)$ is less than the order of $\mathrm{e}^{g_k(z)}$ for $1 \leqslant j \leqslant n+1$, $1 \leqslant k \leqslant n$. And furthermore, the order of $f_j(z)$ is less than the order of $\mathrm{e}^{g_h(z)-g_k(z)}$ for $n \geqslant 2$ and $1 \leqslant j \leqslant n+1$, $1 \leqslant h < k \leqslant n$.*

Then $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n+1)$.

Lemma 2.3. *Let $f(z)$ be an entire function of finite order, and let $a(z)$ be a periodic entire function of period c. Suppose that f is a solution of the difference equation*

$$\frac{f(z+c) - a(z)}{f(z) - a(z)} = K,$$

where K is a constant. Then

$$\frac{\Delta_c^n f(z) - a(z)}{f(z) - a(z)} = (K-1)^n - \frac{a(z)}{f(z) - a(z)}.$$

Lemma 2.4. *Let $f(z)$ and $g(z)$ be entire functions of finite order, and let $a(z)$ be a periodic entire function of period c. Suppose that f is a solution of the difference equation*

$$f(z+c) - a(z) = [f(z) - a(z)]\mathrm{e}^{g(z)}.$$

Then

$$f(z+nc) - a(z) = [f(z) - a(z)]\mathrm{e}^{\sum\limits_{i=0}^{n-1} g(z+ic)}.$$

3. Proofs of the lemmas, theorems, and corollaries

P r o o f of the Lemma 2.3. We prove by induction that for all $n \in \mathbb{N}^*$:

$$\frac{\Delta_c^n f(z) - a(z)}{f(z) - a(z)} = (K-1)^n - \frac{a(z)}{f(z) - a(z)}.$$

For $n = 1$: We have

$$\frac{f(z+c) - a(z)}{f(z) - a(z)} = K,$$

then

$$\frac{f(z+c) - a(z)}{f(z) - a(z)} - 1 = K - 1,$$

then

$$\frac{f(z+c) - f(z) - a(z)}{f(z) - a(z)} = \frac{\Delta_c^n f(z) - a(z)}{f(z) - a(z)} = K - 1 - \frac{a(z)}{f(z) - a(z)}.$$

Suppose the result is true for n. Then

$$\frac{\Delta_c^n f(z) - a(z)}{f(z) - a(z)} = (K-1)^n - \frac{a(z)}{f(z) - a(z)},$$

then

$$\Delta_c^n f(z) - a(z) = [f(z) - a(z)](K-1)^n - a(z),$$

then

$$\begin{aligned}
\Delta_c^{n+1} f(z) &= [f(z+c) - a(z)](K-1)^n - a(z) - [f(z) - a(z)](K-1)^n + a(z) \\
&= [f(z) - a(z)]K(K-1)^n - [f(z) - a(z)](K-1)^n \\
&= [f(z) - a(z)][K(K-1)^n - (K-1)^n] \\
&= [f(z) - a(z)](K-1)^{n+1} \\
\Delta_c^{n+1} f(z) - a(z) &= [f(z) - a(z)](K-1)^{n+1} - a(z),
\end{aligned}$$

then

$$\frac{\Delta_c^{n+1} f(z) - a(z)}{f(z) - a(z)} = (K-1)^{n+1} - \frac{a(z)}{f(z) - a(z)}.$$

Hence, the result is true for $n + 1$. □

P r o o f of Lemma 2.4. We prove by induction that for all $n \in \mathbb{N}^*$:

$$f(z + nc) - a(z) = [f(z) - a(z)]\mathrm{e}^{\sum_{i=0}^{n-1} g(z+ic)}.$$

For $n = 1$:

$$f(z+c) - a(z) = [f(z) - a(z)]\mathrm{e}^{g(z)}.$$

Suppose the result is true for n:

$$\begin{aligned} f(z+(n+1)c) - a(z) &= [f(z+nc) - a(z)]\mathrm{e}^{g(z+nc)} \\ &= [f(z) - a(z)]\mathrm{e}^{\sum_{i=0}^{n-1} g(z+ic)}\mathrm{e}^{g(z+nc)} \\ &= [f(z) - a(z)]\mathrm{e}^{\sum_{i=0}^{n} g(z+ic)}. \end{aligned}$$

Hence, the result is true for $n+1$. □

P r o o f of Theorem 1.1. Suppose that $f(z)$, $f(z+c)$ and $\Delta_c f(z+nc)$ share 0 CM. Then

$$\frac{f(z+c)}{f(z)} = \mathrm{e}^{p(z)} \tag{3.1}$$

and

$$\frac{\Delta_c f(z+nc)}{f(z)} = \frac{f(z+(n+1)c) - f(z+nc)}{f(z)} = \mathrm{e}^{q(z)}, \tag{3.2}$$

where p and q are polynomials. By (3.1) and Lemma (3.4), we deduce that

$$f(z+nc) = f(z)\mathrm{e}^{\sum_{i=0}^{n-1} p(z+ic)}. \tag{3.3}$$

From equations (3.2) and (3.3) we deduce that

$$\frac{f(z)\mathrm{e}^{\sum_{i=0}^{n} p(z+ic)} - f(z)\mathrm{e}^{\sum_{i=0}^{n-1} p(z+ic)}}{f(z)} = \mathrm{e}^{q(z)},$$

then

$$\mathrm{e}^{\sum_{i=0}^{n} p(z+ic)} - \mathrm{e}^{\sum_{i=0}^{n-1} p(z+ic)} = \mathrm{e}^{q(z)}. \tag{3.4}$$

In order to prove that $p(z)$ and $q(z)$ are constants, we need to treat the following cases:

(a) If $q(z) = Cst$, ($\mathrm{e}^{q(z)} = K$) and $p(z) \neq Cst$, then we deduce from (3.4) and Lemma 2.2, the contradiction

$$K = 0.$$

(b) If $p(z) = Cst$, $(e^{p(z)} = K)$ and $q(z) \neq Cst$:
If $e^{p(z)} = K = 1$. By (3.1) we deduce the contradiction

$$f(z + c) = f(z).$$

Then $K \neq 1$. By (3.4) we deduce

$$K^{n+1} - K^n = e^{q(z)},$$

then we deduce the contradiction

$$0 \neq \deg q(z) = \rho(e^{q(z)}) = \rho(K^{n+1} - K^n) = 0.$$

(c) If $p(z) \neq Cst$ and $q(z) \neq Cst$, we have two subcases:
(i) If $\deg p \neq \deg q$, then from (3.4) we have

$$e^{p(z+nc)} - e^{q(z) - \sum_{i=0}^{n-1} p(z+ic)} = 1.$$

Then by Lemma 2.2, we deduce the contradiction

$$1 = 0.$$

(ii) $\deg p = \deg q$
Case (A): If $p - q = Cst$, then by (3.4), we have

$$e^{Cst + \sum_{i=1}^{n} p(z+ic)} - e^{Cst + \sum_{i=1}^{n-1} p(z+ic)} = 1.$$

Then by Lemma 2.2, we deduce the contradiction

$$1 = 0.$$

Case (B): If $p - q = h \neq Cst$, then by (3.4) we have

$$e^{h(z) + \sum_{i=1}^{n} p(z+ic)} - e^{h(z) + \sum_{i=1}^{n-1} p(z+ic)} = 1.$$

Then by Lemma 2.2, we deduce the contradiction

$$1 = 0.$$

Suppose that $e^{p(z)} = \beta^{\alpha}$, where α is a complex number and $\beta > 1$. From (3.1) we have

$$f(z+c) = \beta^{\alpha} f(z). \tag{3.5}$$

If $f(z)$ and $g(z)$ are two different solutions of (3.5), then $h(z) = f(z)/g(z)$ is a periodic function of period c. Obviously, $g(z) = \beta^{\alpha z/c}$ is a solution of (3.5), hence, the entire solutions of (3.5) must be of the form $f(z) = h(z)\beta^{\alpha z/c}$, where $\alpha \neq 0$, $\beta > 1$, and $h(z)$ is a periodic entire function of period c.

Suppose now that $f(z)$, $f(z+c)$ and $\Delta_c f(z+nc)$ share $a \neq 0$ CM. Then

$$\frac{f(z+c)-a}{f(z)-a} = e^{p(z)} \tag{3.6}$$

and

$$\frac{\Delta_c f(z+nc)-a}{f(z)-a} = \frac{f(z+(n+1)c)-f(z+nc)-a}{f(z)-a} = e^{q(z)}, \tag{3.7}$$

where p and q are polynomials. From (3.6) and Lemma 2.4 we deduce that

$$f(z+nc)-a = (f(z)-a)e^{\sum_{i=0}^{n-1} p(z+ic)}. \tag{3.8}$$

From equations (3.7) and (3.8) we deduce that

$$\frac{(f(z)-a)e^{\sum_{i=0}^{n} p(z+ic)} - (f(z)-a)e^{\sum_{i=0}^{n-1} p(z+ic)} - a}{f(z)-a} = e^{q(z)}.$$

Then

$$\frac{a}{f(z)-a} = e^{\sum_{i=0}^{n} p(z+ic)} - e^{\sum_{i=0}^{n-1} p(z+ic)} - e^{q(z)}. \tag{3.9}$$

From (3.6) we deduce that

$$\frac{a}{f(z)-a} = \frac{ae^{p(z)}}{f(z+c)-a}. \tag{3.10}$$

By using equation (3.9), we deduce that

$$\frac{ae^{p(z)}}{f(z+c)-a} = e^{\sum_{i=0}^{n+1} p(z+ic)} - e^{\sum_{i=0}^{n} p(z+ic)} - e^{p(z)+q(z+c)}. \tag{3.11}$$

By using equations (3.9), (3.10) and (3.11), we deduce that

$$e^{\sum_{i=0}^{n+1} p(z+ic)} - 2e^{\sum_{i=0}^{n} p(z+ic)} + e^{\sum_{i=0}^{n-1} p(z+ic)} - e^{p(z)+q(z+c)} + e^{q(z)} = 0. \tag{3.12}$$

In order to prove that $p(z)$ and $q(z)$ are constants, we need to treat the following cases:

(a) If $p(z) = Cst$, ($\mathrm{e}^{p(z)} = K$) and $q(z) \neq Cst$.
If $\mathrm{e}^{p(z)} = K = 1$, then by (3.6), we deduce the contradiction

$$f(z+c) = f(z).$$

Then $\mathrm{e}^{p(z)} = K \neq 1$, and by (3.9), we deduce the contradiction

$$0 = \lambda\left(\frac{a}{f(z)-a}\right) = \lambda(K^{n+1} - K^n - \mathrm{e}^{q(z)}) = \rho(\mathrm{e}^{q(z)}) = \deg q(z) \neq 0.$$

(b) If $q(z) = Cst$, ($\mathrm{e}^{q(z)} = K$) and $p(z) \neq Cst$, then by (3.12) we have

$$\mathrm{e}^{\sum_{i=0}^{n+1} p(z+ic)} - 2\mathrm{e}^{\sum_{i=0}^{n} p(z+ic)} + \mathrm{e}^{\sum_{i=0}^{n-1} p(z+ic)} - K\mathrm{e}^{p(z)} = -K.$$

We deduce by Lemma 2.2 the contradiction

$$-K = 0.$$

(c) If $p(z) \neq Cst$ and $q(z) \neq Cst$, we have two subcases:
(i) If $\deg p \neq \deg q$, then from (3.12) we have

$$\mathrm{e}^{\sum_{i=0}^{n+1} p(z+ic)-q(z)} - 2\mathrm{e}^{\sum_{i=0}^{n} p(z+ic)-q(z)} + \mathrm{e}^{\sum_{i=0}^{n-1} p(z+ic)-q(z)} - \mathrm{e}^{p(z)+q(z+c)-q(z)} = -1.$$

We deduce by Lemma 2.2 the contradiction

(ii) $\deg p = \deg q$ $\qquad -1 = 0.$
Case (A): If $p - q = Cst$, then by (3.12) we have

$$\mathrm{e}^{Cst+\sum_{i=1}^{n+1} p(z+ic)} - 2\mathrm{e}^{Cst+\sum_{i=1}^{n} p(z+ic)} + \mathrm{e}^{Cst+\sum_{i=1}^{n-1} p(z+ic)} - \mathrm{e}^{Cst+q(z+c)} = -1.$$

We deduce by Lemma 2.2 the contradiction

$$-1 = 0.$$

Case (B): If $p - q = h \neq Cst$, then by (3.12), we have

$$\mathrm{e}^{\sum_{i=1}^{n+1} p(z+ic)+h(z)} - 2\mathrm{e}^{\sum_{i=1}^{n} p(z+ic)+h(z)} + \mathrm{e}^{\sum_{i=1}^{n-1} p(z+ic)+h(z)} - \mathrm{e}^{h(z)+q(z+c)} = -1.$$

Then by Lemma 2.2, we deduce the contradiction

$$-1 = 0.$$

Finally, we deduce that $p(z)$ and $q(z)$ are constants, which is also a contradiction with the fact that f is transcendental entire function. So, there is no entire function satisfying the hypothesis of Theorem 1.1 when $a \neq 0$. □

P r o o f of Corollary 1.1. Suppose that $f(z)$, $\Delta_c f(z+nc)$ and $\Delta_c f(z+(n+1)c)$ share 0 CM. Then

$$\frac{\Delta_c f(z+nc)}{f(z)} = \mathrm{e}^{p(z)} \tag{3.13}$$

and

$$\frac{\Delta_c f(z+(n+1)c)}{f(z)} = \frac{\Delta_c^2 f(z+nc) + \Delta_c f(z+nc)}{f(z)} = \mathrm{e}^{q(z)}, \tag{3.14}$$

where p and q are polynomials. By using equations (3.13) and (3.14),

$$\frac{\Delta_c^2 f(z+nc)}{f(z)} = \mathrm{e}^{q(z)} - \mathrm{e}^{p(z)}. \tag{3.15}$$

By using equation (3.13),

$$\Delta_c^2 f(z+nc) = f(z+c)\mathrm{e}^{p(z+c)} - f(z)\mathrm{e}^{p(z)}, \tag{3.16}$$

by using equations (3.15) and (3.16),

$$f(z)\mathrm{e}^{q(z)} - f(z)\mathrm{e}^{p(z)} = f(z+c)\mathrm{e}^{p(z+c)} - f(z)\mathrm{e}^{p(z)}.$$

Then

$$\frac{f(z+c)}{f(z)} = \mathrm{e}^{q(z)-p(z+c)}. \tag{3.17}$$

From equations (3.13) and (3.17) we deduce that $f(z)$, $f(z+c)$ and $\Delta_c f(z+nc)$ share 0 CM. By using Theorem 1.1, $f(z) = h(z)\beta^{\alpha z/c}$, where $\alpha \neq 0$, $\beta > 1$, and $h(z)$ is a periodic entire function of period c.

Suppose now that $f(z)$, $\Delta_c f(z+nc)$ and $\Delta_c f(z+(n+1)c)$ share $a \neq 0$ CM. Then

$$\frac{\Delta_c f(z+nc) - a}{f(z) - a} = \mathrm{e}^{p(z)} \tag{3.18}$$

and

$$\frac{\Delta_c f(z+(n+1)c) - a}{f(z) - a} = \frac{\Delta_c^2 f(z+nc) + \Delta_c f(z+nc) - a}{f(z) - a} = \mathrm{e}^{q(z)}, \tag{3.19}$$

where p and q are polynomials. By using (3.18) and (3.19)

$$\frac{\Delta_c^2 f(z+nc)}{f(z)-a} = \mathrm{e}^{q(z)} - \mathrm{e}^{p(z)}. \tag{3.20}$$

By using (3.18),

$$\Delta_c^2 f(z+nc) = f(z+c)\mathrm{e}^{p(z+c)} - a\mathrm{e}^{p(z+c)} - f(z)\mathrm{e}^{p(z)} + a\mathrm{e}^{p(z)}, \tag{3.21}$$

by using (3.20) and (3.21),

$$f(z)\mathrm{e}^{q(z)} - f(z)\mathrm{e}^{p(z)} - a\mathrm{e}^{q(z)} + a\mathrm{e}^{p(z)} = f(z+c)\mathrm{e}^{p(z+c)} - a\mathrm{e}^{p(z+c)} - f(z)\mathrm{e}^{p(z)} + a\mathrm{e}^{p(z)},$$

then

$$\frac{f(z+c)-a}{f(z)-a} = \mathrm{e}^{q(z)-p(z+c)}. \tag{3.22}$$

By (3.18), (3.22) and Theorem 1.1, we deduce a contradiction in that $f(z)$, $f(z+c)$ and $\Delta_c f(z+nc)$ cannot share $a \neq 0$ CM. Consequently, we deduce that $f(z)$, $\Delta_c f(z+nc)$ and $\Delta_c f(z+(n+1)c)$ cannot share $a \neq 0$ CM. □

P r o o f of Theorem 1.2. Suppose that $f(z)$, $f(z+c)$ and $\Delta_c^n f(z)$ share 0 CM, then

$$\frac{f(z+c)}{f(z)} = \mathrm{e}^{p(z)} \tag{3.23}$$

and

$$\frac{\Delta_c^n f(z)}{f(z)} = \frac{\sum\limits_{i=0}^{n} C_n^i (-1)^{n-i} f(z+ic)}{f(z)} = \mathrm{e}^{q(z)}, \tag{3.24}$$

where p and q are polynomials. By using equation (3.23) and Lemma 2.4,

$$f(z+nc) = f(z)\mathrm{e}^{\sum\limits_{i=0}^{n-1} p(z+ic)}. \tag{3.25}$$

By using (3.24) and (3.25),

$$\begin{aligned} \frac{\Delta_c^n f(z)}{f(z)} &= (-1)^n + n(-1)^{n-1}\mathrm{e}^{p(z)} + \ldots + n(-1)\mathrm{e}^{\sum\limits_{i=0}^{n-2} p(z+ic)} + \mathrm{e}^{\sum\limits_{i=0}^{n-1} p(z+ic)} \\ &= \mathrm{e}^{q(z)}. \end{aligned} \tag{3.26}$$

In order to prove that $p(z)$ and $q(z)$ are constants, we need to treat the following cases:

(a) If $p(z) = Cst$, ($e^{p(z)} = K$) and $q(z) \neq Cst$, then we deduce from (3.23) that

$$\frac{f(z+c)}{f(z)} = K. \tag{3.27}$$

If $K = 1$, then we deduce the contradiction

$$f(z+c) = f(z).$$

If $K \neq 1$, by using Lemma 2.3 and equations (3.24) and (3.27),

$$\frac{\Delta_c^n f(z)}{f(z)} = e^{q(z)} = (K-1)^n.$$

Then we deduce the contradiction

$$\deg e^{q(z)} = \rho(e^{q(z)}) = \rho[(K-1)^n] = 0,$$

where $q(z) \neq Cst$.

(b) If $q(z) = Cst$, ($e^{q(z)} = K$) and $p(z) \neq Cst$, then from (3.26) we deduce

$$\frac{\Delta_c^n f(z)}{f(z)} = (-1)^n + n(-1)^{n-1}e^{p(z)} + \ldots + n(-1)e^{\sum_{i=0}^{n-2} p(z+ic)} + e^{\sum_{i=0}^{n-1} p(z+ic)} = K.$$

Then

$$[(-1)^n - K]e^{-p(z)} + \ldots + n(-1)e^{\sum_{i=1}^{n-2} p(z+ic)} + e^{\sum_{i=1}^{n-1} p(z+ic)} = n(-1)^n.$$

By Lemma 2.2, we deduce the contradiction

$$n(-1)^n = 0.$$

(c) If $p(z) \neq Cst$ and $q(z) \neq Cst$, we have two subcases:

(i) $\deg p = \deg q$:

Case (A): If $p(z) = q(z)$, then from (3.26) we deduce

$$[n(-1)^{n-1} - 1]e^{p(z)} + \ldots + n(-1)e^{\sum_{i=0}^{n-2} p(z+ic)} + e^{\sum_{i=0}^{n-1} p(z+ic)} = -(-1)^n.$$

By Lemma 2.2, we deduce the contradiction

$$-(-1)^n = 0.$$

Case (B): If $p(z)-q(z)=Cst$, then from equation (3.26) we deduce

$$(-1)^n \mathrm{e}^{-\sum_{i=0}^{n-1} p(z+ic)}+[n(-1)^{n-1}-\mathrm{e}^{-Cst}]\mathrm{e}^{-\sum_{i=1}^{n-1} p(z+ic)}+\ldots+n(-1)\mathrm{e}^{-p(z+(n-1)c)}=-1.$$

Then by using Lemma 2.2 we deduce the contradiction

$$-1=0.$$

Case (C): If $p(z)-q(z)\neq Cst$, then from (3.26) we deduce

$$n(-1)^{n-1}\mathrm{e}^{p(z)}+\ldots+n(-1)\mathrm{e}^{\sum_{i=0}^{n-2} p(z+ic)}+\mathrm{e}^{\sum_{i=0}^{n-1} p(z+ic)}-\mathrm{e}^{q(z)}=-(-1)^n.$$

By Lemma 2.2, we deduce the contradiction

$$-(-1)^n=0.$$

(ii) $\deg p\neq \deg q$:
By using Lemma 2.2 and (3.26), we deduce the contradiction

$$(-1)^n=0.$$

Suppose that $\mathrm{e}^{p(z)}=\beta^{\alpha}$, where α is a complex number and $\beta>1$. From (3.23) we have

$$f(z+c)=\beta^{\alpha}f(z). \tag{3.28}$$

If $f(z)$ and $g(z)$ are two different solutions of (3.28), then $h(z)=f(z)/g(z)$ is a periodic function of period c. Obviously, $g(z)=\beta^{\alpha z/c}$ is a solution of (3.28), hence the entire solutions of (3.28) must be of the form $f(z)=h(z)\beta^{\alpha z/c}$, where $\alpha\neq 0$, $\beta>1$, and $h(z)$ is a periodic entire function of period c.

Suppose now that $f(z)$, $f(z+c)$ and $\Delta_c^n f(z)$ share $a\neq 0$ CM. Then

$$\frac{f(z+c)-a}{f(z)-a}=\mathrm{e}^{p(z)} \tag{3.29}$$

and

$$\frac{\Delta_c^n f(z)-a}{f(z)-a}=\mathrm{e}^{q(z)}, \tag{3.30}$$

where p and q are polynomials. From (3.29) and Lemma 2.4, we deduce that

$$f(z+nc)-a=(f(z)-a)\mathrm{e}^{\sum_{i=0}^{n-1}p(z+ic)}. \tag{3.31}$$

By using equations (3.30) and (3.31),

$$(-1)^n+n(-1)^{n-1}\mathrm{e}^{p(z)}+\ldots+n(-1)\mathrm{e}^{\sum_{i=0}^{n-2}p(z+ic)}+\mathrm{e}^{\sum_{i=0}^{n-1}p(z+ic)}-\frac{a}{f(z)-a}=\mathrm{e}^{q(z)},$$

then

$$\begin{aligned}\frac{a}{f(z)-a}&=(-1)^n+n(-1)^{n-1}\mathrm{e}^{p(z)}+\ldots\\&\quad+n(-1)\mathrm{e}^{\sum_{i=0}^{n-2}p(z+ic)}+\mathrm{e}^{\sum_{i=0}^{n-1}p(z+ic)}-\mathrm{e}^{q(z)}.\end{aligned} \tag{3.32}$$

In order to prove that $p(z)$ and $q(z)$ are constants, we need to treat the following cases:

(a) If $p(z)=Cst$, $(\mathrm{e}^{p(z)}=K)$ and $q(z)\neq Cst$ then, by using Lemma 2.3 and equations (3.29) and (3.30), we have

$$\frac{\Delta_c^n f(z)-a(z)}{f(z)-a(z)}=(K-1)^n-\frac{a(z)}{f(z)-a(z)}=\mathrm{e}^{q(z)}. \tag{3.33}$$

If $K=\mathrm{e}^{p(z)}=1$, then we deduce from equation (3.29) that

$$f(z+c)-a=f(z)-a.$$

Then we deduce the contradiction

$$f(z+c)=f(z).$$

Then

$$K\neq 1.$$

By using equation (3.33), we deduce the contradiction

$$0=\lambda\left(\frac{a}{f(z)-a}\right)=\lambda((K-1)^n-\mathrm{e}^{q(z)})=\rho(\mathrm{e}^{q(z)})\neq 0.$$

(b) If $q(z)=Cst$, $(\mathrm{e}^{q(z)}=K)$ and $p(z)\neq Cst$, then we deduce from equation (3.29) that

$$\frac{a}{f(z)-a}=\frac{a\mathrm{e}^{p(z)}}{f(z+c)-a}. \tag{3.34}$$

By using equation (3.32), we deduce

$$\frac{a\mathrm{e}^{p(z)}}{f(z+c)-a} = [(-1)^n - K]\mathrm{e}^{p(z)} + n(-1)^{n-1}\mathrm{e}^{p(z)+p(z+c)} + \ldots + n(-1)\mathrm{e}^{\sum_{i=0}^{n-1} p(z+ic)} + \mathrm{e}^{\sum_{i=0}^{n} p(z+ic)}. \tag{3.35}$$

By using equations (3.23), (3.34), and (3.35), we deduce

$$[(-1)^n - K] + [n(-1)^{n-1} - (-1)^n + K]\mathrm{e}^{p(z)} + \ldots + [1+n]\mathrm{e}^{\sum_{i=0}^{n-1} p(z+ic)} - \mathrm{e}^{\sum_{i=0}^{n} p(z+ic)} = 0,$$

then

$$[(-1)^n - K]\mathrm{e}^{-\sum_{i=0}^{n} p(z+ic)} + [n(-1)^{n-1} - (-1)^n + K]\mathrm{e}^{-\sum_{i=1}^{n} p(z+ic)} + \ldots + [1+n]\mathrm{e}^{-p(z+nc)} = 1.$$

By using Lemma 2.2, we deduce the contradiction

$$1 = 0.$$

(c) If $p(z) \neq Cst$ and $q(z) \neq Cst$, we need to treat the following cases:
By using equation (3.32), we deduce

$$\frac{a\mathrm{e}^{p(z)}}{f(z+c)-a} = (-1)^n\mathrm{e}^{p(z)} + n(-1)^{n-1}\mathrm{e}^{p(z)+p(z+c)} + \ldots + n(-1)\mathrm{e}^{\sum_{i=0}^{n-1} p(z+ic)} + \mathrm{e}^{\sum_{i=0}^{n} p(z+ic)} - \mathrm{e}^{p(z)+q(z+c)}. \tag{3.36}$$

By using equations (3.32), (3.34), and (3.36), we have

$$(-1)^n + [n(-1)^{n-1} - (-1)^n]\mathrm{e}^{p(z)} + \ldots + (1+n)\mathrm{e}^{\sum_{i=0}^{n-1} p(z+ic)} - \mathrm{e}^{\sum_{i=0}^{n} p(z+ic)} - \mathrm{e}^{q(z)} + \mathrm{e}^{p(z)+q(z+c)} = 0. \tag{3.37}$$

(i) $\deg p = \deg q$:
Case (A): If $q(z) - p(z) = Cst$, then we deduce from equation (3.37) that

$$(-1)^n + [n(-1)^{n-1} - (-1)^n - \mathrm{e}^{Cst}]\mathrm{e}^{p(z)} + \ldots + (1+n)\mathrm{e}^{\sum_{i=0}^{n-1} p(z+ic)} - \mathrm{e}^{\sum_{i=0}^{n} p(z+ic)} + \mathrm{e}^{p(z)+q(z+c)} = 0.$$

By using Lemma 2.2, we deduce the contradiction

$$(-1)^n = 0.$$

Case (B): If $q(z)-p(z)=h(z)\neq Cst$, then we deduce from equation (3.37) that

$$(-1)^n\mathrm{e}^{-p(z)}+\ldots+(1+n)\mathrm{e}^{\sum_{i=1}^{n-1}p(z+ic)}-\mathrm{e}^{\sum_{i=1}^{n}p(z+ic)}-\mathrm{e}^{h(z)}+\mathrm{e}^{q(z+c)}$$
$$=-n(-1)^{n-1}+(-1)^n=(-1)^n(n+1).$$

By using Lemma 2.2, we deduce the contradiction

$$(-1)^n(n+1)=0.$$

(ii) $\deg p\neq\deg q$:
By using Lemma 2.2 and equation (3.37), we deduce the contradiction

$$(-1)^n=0.$$

Finally, we deduce that $p(z)$ and $q(z)$ are constants, which is also a contradiction with the fact that f is a transcendental entire function. So, there is no entire function satisfying the hypothesis of Theorem 1.2 when $a\neq 0$. □

P r o o f of Corollary 1.2. Suppose that $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^n f(z+c)$ share 0 CM. Then

$$\frac{\Delta_c^n f(z)}{f(z)}=\mathrm{e}^{p(z)} \tag{3.38}$$

and

$$\frac{\Delta_c^n f(z+c)}{f(z)}=\mathrm{e}^{q(z)}, \tag{3.39}$$

where p and q are polynomials. By using Lemma 2.1 and from equation (3.39) we have

$$\frac{\Delta_c^n f(z)+\Delta_c^{n+1}f(z)}{f(z)}=\mathrm{e}^{q(z)}. \tag{3.40}$$

By using equations (3.38) and (3.40), we have

$$\frac{\Delta_c^{n+1}f(z)}{f(z)}=\mathrm{e}^{q(z)}-\mathrm{e}^{p(z)},$$

then

$$\Delta_c^{n+1}f(z)=f(z)\mathrm{e}^{q(z)}-f(z)\mathrm{e}^{p(z)}. \tag{3.41}$$

By using equation (3.38), we have

$$\Delta_c^n f(z) = f(z)\mathrm{e}^{p(z)},$$

then

$$\Delta_c^{n+1} f(z) = f(z+c)\mathrm{e}^{p(z+c)} - f(z)\mathrm{e}^{p(z)}. \tag{3.42}$$

By using equations (3.41) and (3.42), we have

$$f(z)\mathrm{e}^{q(z)} - f(z)\mathrm{e}^{p(z)} = f(z+c)\mathrm{e}^{p(z+c)} - f(z)\mathrm{e}^{p(z)},$$

then

$$\frac{f(z+c)}{f(z)} = \mathrm{e}^{q(z)-p(z+c)}. \tag{3.43}$$

From (3.38) and (3.43) we deduce $f(z)$, $f(z+c)$ and $\Delta_c^n f(z)$ share 0 CM. By using Theorem 1.2, $f(z) = h(z)\beta^{\alpha z/c}$, where $\alpha \neq 0$, $\beta > 1$, and $h(z)$ is a periodic entire function of period c.

Suppose now that $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^n f(z+c)$ share $a \neq 0$ CM. Then

$$\frac{\Delta_c^n f(z) - a}{f(z) - a} = \mathrm{e}^{p(z)} \tag{3.44}$$

and

$$\frac{\Delta_c^n f(z+c) - a}{f(z) - a} = \mathrm{e}^{q(z)}, \tag{3.45}$$

where p and q are polynomials. By using Lemma 2.1 and equation (3.45), we have

$$\frac{\Delta_c^{n+1} f(z) + \Delta_c^n f(z) - a}{f(z) - a} = \mathrm{e}^{q(z)}. \tag{3.46}$$

By using equations (3.44) and (3.46), we have

$$\frac{\Delta_c^{n+1} f(z)}{f(z) - a} = \mathrm{e}^{q(z)} - \mathrm{e}^{p(z)}.$$

Then

$$\Delta_c^{n+1} f(z) = f(z)\mathrm{e}^{q(z)} - f(z)\mathrm{e}^{p(z)} - a\mathrm{e}^{q(z)} + a\mathrm{e}^{p(z)}, \tag{3.47}$$

and by equation (3.44), we have

$$\Delta_c^{n+1} f(z) = f(z+c)\mathrm{e}^{p(z+c)} - f(z)\mathrm{e}^{p(z)} - a\mathrm{e}^{p(z+c)} + a\mathrm{e}^{p(z)}. \tag{3.48}$$

By using equations (3.47) and (3.48), we have

$$f(z+c)\mathrm{e}^{p(z+c)} - f(z)\mathrm{e}^{p(z)} - a\mathrm{e}^{p(z+c)} + a\mathrm{e}^{p(z)} = f(z)\mathrm{e}^{q(z)} - f(z)\mathrm{e}^{p(z)} - a\mathrm{e}^{q(z)} + a\mathrm{e}^{p(z)},$$

then

$$\frac{f(z+c)-a}{f(z)-a} = \mathrm{e}^{q(z)-p(z+c)}. \tag{3.49}$$

By (3.44), (3.49) and Theorem 1.2, we deduce a contradiction in that $f(z)$, $f(z+c)$ and $\Delta_c^n f(z)$ cannot share $a \neq 0$ CM. Consequently, we deduce that $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^n f(z+c)$ cannot share $a \neq 0$ CM. □

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