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# ON THE CHARACTERIZATIONS OF SPECIAL ADJOINT CURVES AND SURFACE FAMILY FOR THE MODIFIED ORTHOGONAL FRAME WITH TORSION

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*Abstract.* In this context, the main interest of this paper is to construct a family of surfaces in $E^3$ space which is the adjoint pair of a given curve. Then, using the modified orthogonal frame with torsion, we express the necessary conditions for the surface family that recognize the adjoint curve as a geodesic curve, an asymptotic curve and a line of curvature. Also, the singular points of the surface given by the parametric equation are expressed. It is concluded that the adjoint curve of the alpha curve whose torsion is not constant can not be geodesic on the surface under any circumstances. In addition, the case where the adjoint curve of the unit velocity alpha curve with nonconstant torsion with respect to the modified orthogonal frame with torsion is the Salkowski curve, is also exemplified. By the way, this work is illustrated by several examples.

*Keywords*: the modified orthogonal frame with torsion; adjoint curve; Salkowski curve; geodesic curve; asymptotic curve; line of curvature; surface family

*MSC 2020*: 53A04, 53A05, 53Z50

## 1. Introduction

Curves and surfaces are two of the most important basic topics of differential geometry, and we encounter them in almost every differential geometry book ([13], [21], [24], [26], [27], [29], [30]). In addition, the construction of another curve from a curve with the help of Frenet vectors has an important place in differential geometry. An example is the adjoint (or conjugate) curve defined in ([11], [12], [23]) as the integral of the binormal vector of a curve with arbitrary parameter $s$. The adjoint curve is a concept in differential geometry, particularly in the study of curves in Euclidean space. Robotics, particularly path planning and motion analysis, is a notable application of adjoint curves. Adjoint curves can help determine optimal paths

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that minimise energy consumption or avoid obstacles in robot motion planning. By analysing the properties of an adjoint curve, it is possible to derive smooth trajectories that are in compliance with physical constraints. All in all, adjoint curves provide valuable geometry insights, which improve the efficiency and effectiveness of robot applications. In general relativity, the path of a particle (geodesic) can be described as an adjoint curve in a curved space time. The modified orthogonal frame would be used to describe how the space time around a massive body (like a star or black hole) distorts the particle's path. The adjoint curves can then be used to analyze the optimal trajectory of the particle under the influence of gravitational forces. The combination of adjoint curves and the modified orthogonal frames in physics provides powerful tools for analysing the dynamics and transformations of systems in a variety of contexts. Whether in classical mechanics, general relativity, electromagnetism or fluid dynamics, these concepts help to understand how physical systems evolve under changes in reference frames, rotations, accelerations and other transformations. They enable precise calculations and theoretical models of motion, energy and forces in different physical environments. In geometric mechanics, the motion of a system (like a particle or rigid body) can often be described by curves in a manifold. The adjoint curves are often used in Lagrangian mechanics to describe the evolution of a system's configuration space, ensuring that the system's behavior is understood in relation to other constraints. Nurkan and Güven have recently defined the integral curves of Smarandache curves with respect to the Frenet frame in 3-dimensional Euclidean space; in particular, the characterisations of these adjoint curves when the given curve is a general helix and a slant helix [22]. However, the use of other frame instead of Frenet's gives the mathematician a new idea. Using the Frenet frame, Sasai defined the modified orthogonal frame at points where the curvature is different from zero [25]. Bukcu and Karacan expressed Sasai's work in 3-dimensional Minkowski space. They also gave a new version of the modified orthogonal frame with torsion in 3-dimensional Euclidean and Minkowski space [10].

A specific example of using the modified orthogonal frame is in planning robotic arm movements. In industrial robots, robot arms are often required to perform complex tasks such as assembly or welding, where precision and coordination of movements are critical. This application demonstrates how the modified orthogonal frames simplify the complex calculations involved in robot motion, leading to efficient and reliable performance in industrial environments. Considering all these studies, the idea of constructing the adjoint curve of a curve according to the modified orthogonal frame formed by torsion was born by Arıkan and Nurkan in [2]. In addition, the problem of finding a family of surfaces passing through a given curve and accepting this curve as a special curve is one of the important topics studied recently. In this sense, this problem was first formulated by Wang et al. [28]. They

gave necessary and sufficient conditions for the given curve to be geodesic on the surface. Li et al. [19] studied the problem of finding a family of surfaces that accepts a given curve as a line of curvature. For recent studies on the problem of finding a surface family that contains known special curve pairs and accepts these curves as an asymptotic and line of curvature, see ([5], [7], [9], [18]). In addition, recently, Atalay [3] expressed a new approach to special families of curved surfaces according to the modified orthogonal frame; Atalay and Şimşek [6] have expressed the necessary and sufficient conditions for polynomial curves, which are widely used in various fields of physics (such as projectile motion, kinematics, optics, quantum mechanics, relativity, electromagnetism, etc.) due to their ability to model and approximate a wide variety of physical phenomena, to be isogeodesic, isoasymptotic, and curvature line on the surface, and carried the subject to ruled surfaces in $E^3$. Yakut and Kızılay [31] studied the curve evolution with a new modified orthogonal Sabban frame. In addition to these, there have been several studies on special curve pairs and special surfaces based on the modified orthogonal frame ([1], [4], [8], [14], [15], [16], [17], [20]).

The concepts of adjoint curves and the modified orthogonal frames have specialized applications in physics, particularly in fields involving dynamics, mechanics, and geometry. While both concepts are advanced, they play key roles in the analysis of motion, rotations, and transformations in various physical contexts.

In this work, we define conditions for the adjoint curve of the given curve with respect to the modified frame with torsion to be both parametric on the parametric surface and geodesic, asymptotic or line of curvature in $E^3$, respectively.

## 2. Preliminaries

In this section, we will explain some basic definitions and two types of the modified orthogonal frames defined for curves with nonzero curvature and torsion.

Let $\alpha(s)$ be a $C^3$ space curve of arc length parameter $s$ in the Euclidean 3-space. Then the Frenet frame $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ of the curve $\alpha$, where $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ are the tangents of the principal, normal and binormal vectors of $\alpha$, respectively, is defined by

$$\mathbf{t}(s) = \alpha'(s), \quad \mathbf{n}(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \quad \mathbf{b}(s) = t(s) \times n(s).$$

Derivatives of the Frenet frame are given by the relations

$$\begin{pmatrix} \mathbf{t}'(s) \\ \mathbf{n}'(s) \\ \mathbf{b}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix},$$

where $\kappa(s) = \|\alpha''(s)\|$ and $\tau(s) = -\langle \mathbf{b}'(s), \mathbf{n}(s)\rangle$ are called the curvature and torsion of the curve $\alpha$, respectively.

Let $\alpha(s)$ be an analytic curve. Then this curve can be reparameterised by its arc length $s$. For this curve we will assume that the torsion function $\tau(s)$ is not identically zero. Thus, an orthogonal frame $\{\mathbf{T}_\tau(s), \mathbf{N}_\tau(s), \mathbf{B}_\tau(s)\}$ can be defined as follows:

$$\mathbf{T}_\tau = \frac{\mathrm{d}\alpha}{\mathrm{d}s}, \quad \mathbf{N}_\tau = \frac{\mathrm{d}\mathbf{T}_\tau}{\mathrm{d}s}, \quad \mathbf{B}_\tau = \mathbf{T}_\tau \times \mathbf{N}_\tau.$$

Taking into account the above equations and the Frenet equations, we obtain the following relations linking the Frenet frame and a new frame at nonzero points of $\tau$:

$$\mathbf{T}_\tau = \mathbf{t}, \quad \mathbf{N}_\tau = \tau\mathbf{n}, \quad \mathbf{B}_\tau = \tau\mathbf{b}.$$

A new frame makes it easier to include the following equations using the Frenet frame $\langle \mathbf{T}_\tau, \mathbf{N}_\tau\rangle = \langle \mathbf{T}_\tau, \mathbf{B}_\tau\rangle = \langle \mathbf{N}_\tau, \mathbf{B}_\tau\rangle = 0$, $\langle \mathbf{T}_\tau, \mathbf{T}_\tau\rangle = 1$, $\langle \mathbf{N}_\tau, \mathbf{N}_\tau\rangle = \langle \mathbf{B}_\tau, \mathbf{B}_\tau\rangle = \tau^2$. The derivative equations of the new orthogonal frame $\{\mathbf{T}_\tau(s), \mathbf{N}_\tau(s), \mathbf{B}_\tau(s)\}$ are derived from fundamental equations

$$\begin{cases} \mathbf{T}'_\tau(s) = \dfrac{\kappa}{\tau}\mathbf{N}_\tau(s), \\ \mathbf{N}'_\tau(s) = -\kappa\tau\mathbf{T}_\tau(s) + \dfrac{\tau'}{\tau}\mathbf{N}_\tau(s) + \tau\mathbf{B}_\tau(s), \\ \mathbf{B}'_\tau(s) = -\tau\mathbf{N}_\tau(s) + \frac{\tau'}{\tau}\mathbf{B}_\tau(s), \end{cases} \tag{2.1}$$

where $\tau = \det(\alpha', \alpha'', \alpha''')/\kappa^2$ is the torsion of the curve $\alpha$.

Thus, the orthogonal frame $\{\mathbf{T}_\tau(s), \mathbf{N}_\tau(s), \mathbf{B}_\tau(s)\}$ is called the modified orthogonal frame with nonzero torsion [10].

**Definition 2.1.** Let $\alpha$ be a unit speed regular curve with curvature $\kappa \neq 0$ and torsion $\tau \neq 0$. If the modified orthogonal frame with torsion of $\alpha$ is $\{\mathbf{T}_\tau(s), \mathbf{N}_\tau(s), \mathbf{B}_\tau(s)\}$, then the adjoint curve of $\alpha$ is defined as [2]

$$\beta(s) = \int B_\tau(s)\,\mathrm{d}s. \tag{2.2}$$

**Theorem 2.1.** *Let $\alpha$ be a unit speed regular curve in $E^3$ and $\beta$ be the adjoint curve of $\alpha$ according to the modified orthogonal frame with torsion. If the modified orthogonal frames of $\alpha$ and $\beta$ are $\{\mathbf{T}_\tau(s), \mathbf{N}_\tau(s), \mathbf{B}_\tau(s)\}$ and $\{\overline{\mathbf{T}}_\tau(s), \overline{\mathbf{N}}_\tau(s), \overline{\mathbf{B}}_\tau(s)\}$, the curvature and torsion are $\kappa$, $\tau$ and $\overline{\kappa}$, $\overline{\tau}$ respectively, then the following relations*

*hold* [2]*:*

$$
\begin{cases}
\overline{\mathbf{T}}_\tau = \dfrac{1}{\tau}\mathbf{B}_\tau, \\
\overline{\mathbf{N}}_\tau = -\dfrac{\kappa}{\tau^2}\mathbf{N}_\tau, \\
\overline{\mathbf{B}}_\tau = \dfrac{\kappa}{\tau}\mathbf{T}_\tau, \\
\overline{\kappa} = 1, \quad \overline{\tau} = \dfrac{\kappa}{\tau}.
\end{cases} \tag{2.3}
$$

**Corollary 2.1.** *If $\alpha$ is a general helix (or circular helix) parametrized by arc-length parameter $s$, then the adjoint curve of $\beta$ according to the modified orthogonal frame with torsion $\alpha$ is a general helix (or circular helix)* [2].

**Corollary 2.2.** *If $\beta$ is the adjoint curve of $\alpha$ (not a general helix) according to the modified orthogonal frame, then $\beta$ is a Salkowski curve* [2].

An isoparametric curve $\alpha(s)$ is a curve on a surface $\Phi = \Phi(s, v)$ that has a constant $s$ or $v$ parameter value. In other words, there exists a parameter $s_0$ or $v_0$ such that $\alpha(s) = \Phi(s, v_0)$ or $\alpha(v) = \Phi(s_0, v)$ [24]. A curve $\alpha(s)$ in $\Phi \subset I\mathbb{R}^3$ is a geodesic of $\Phi$ provided that its acceleration $\alpha''$ is always normal to $\Phi$ [24]. A regular curve $\alpha$ in $\Phi \subset I\mathbb{R}^3$ is an asymptotic curve provided that its velocity $\alpha'$ always points in an asymptotic direction, i.e., the direction in which the normal curvature is zero [24]. The simplest criterion for a curve in $\Phi$ to be asymptotic is that its acceleration $\alpha''$ is always tangent to $\Phi$. A regular curve $\alpha$ in $\Phi$ is a line of curvature provided that its velocity $\alpha'$ always points in a principal direction [24].

**Theorem 2.2.** *A surface curve is a line of curvature if and only if the surface normals along the curve form a developable surface* [13].

## 3. Special adjoint curved surface families according to the modified orthogonal frame with torsion

Suppose we are given a unit speed parametric curve $\alpha(s)$ with unit velocity such that $\|\alpha''(s)\| \neq 0$ and $\tau \neq 0$ in 3-dimensional space. Let $\beta$ be the adjoint curve of $\alpha$. The surface family that possesses $\beta$ as a common curve is given in the parametric form as

$$
\begin{cases}
\Phi(s, v) = \beta(s) + [x(s, v)\overline{\mathbf{T}}_\tau(s) + y(s, v)\overline{\mathbf{N}}_\tau(s) + z(s, v)\overline{\mathbf{B}}_\tau(s)], \\
L_1 \leqslant s \leqslant L_2, \quad K_1 \leqslant v \leqslant K_2,
\end{cases} \tag{3.1}
$$

where $x(s,v)$, $y(s,v)$ and $z(s,v)$ are called differentiable marching-scale functions and $\{\overline{\mathbf{T}}_\tau(s), \overline{\mathbf{N}}_\tau(s), \overline{\mathbf{B}}_\tau(s)\}$ is the modified orthogonal frame with nonzero torsion of the curve $\beta$.

R e m a r k 3.1. Note that choosing different marching scale functions gives different surfaces which have $\beta(s)$ as a common curve.

Our goal is to find the conditions for which the given curve $\beta(s)$ is an isoparametric, and geodesic, asymptotic or line of curvature on the surface $\Phi(s,v)$. Firstly, as $\beta(s)$ is an isoparametric curve on the surface $\Phi(s,v)$, there exists a parameter $v_0 \in [K_1, K_2]$ such that

$$x(s,v_0) = y(s,v_0) = z(s,v_0) \equiv 0, \quad L_1 \leqslant s \leqslant L_2. \tag{3.2}$$

Using equations (2.2) and (2.3) in (3.1),

$$\Phi(s,v) = \int B_\tau(s)\,\mathrm{d}s + \left[\frac{\kappa(s)}{\tau(s)} z(s,v)\mathbf{T}_\tau(s) - \frac{\kappa(s)}{\tau^2(s)} y(s,v)\mathbf{N}_\tau(s) + \frac{1}{\tau(s)} x(s,v)\mathbf{B}_\tau(s)\right],$$

$L_1 \leqslant s \leqslant L_2$, $K_1 \leqslant s \leqslant K_2$ is obtained.

The normal vector field of the surface $\Phi$ is given by

$$U(s,v) = \frac{\partial \Phi(s,v)}{\partial s} \times \frac{\partial \Phi(s,v)}{\partial v},$$

where "$\times$" is the vector product. We calculate

$$\begin{aligned} U(s,v) = {} & \left[\begin{array}{l} \dfrac{1}{\tau(s)} \dfrac{\partial x(s,v)}{\partial v} \Bigg( - x(s,v) - \dfrac{\kappa(s)}{\tau^2(s)} \dfrac{\partial y(s,v)}{\partial s} \\ - \left(\dfrac{\kappa(s)}{\tau^2(s)}\right)' y(s,v) - \kappa(s) \dfrac{\tau'(s)}{\tau^3(s)} y(s,v) + \dfrac{\kappa^2(s)}{\tau^2(s)} z(s,v)\Bigg) \\ + \dfrac{\kappa(s)}{\tau^2(s)} \dfrac{\partial y(s,v)}{\partial v} \Bigg(1 + \dfrac{1}{\tau(s)} \dfrac{\partial x(s,v)}{\partial s} \\ + \dfrac{\tau'(s)}{\tau^2(s)} x(s,v) - \dfrac{\kappa(s)}{\tau(s)} y(s,v)\Bigg) \end{array}\right] \mathbf{T}(s) \\ & + \left[\begin{array}{l} \dfrac{\kappa(s)}{\tau(s)} \dfrac{\partial z(s,v)}{\partial v} \Bigg(1 + \dfrac{1}{\tau(s)} \dfrac{\partial x(s,v)}{\partial s} + \dfrac{\tau'(s)}{\tau^2(s)} x(s,v) - \dfrac{\kappa(s)}{\tau(s)} y(s,v)\Bigg) \\ - \dfrac{1}{\tau(s)} \dfrac{\partial x(s,v)}{\partial s} \Bigg(\dfrac{\kappa(s)}{\tau(s)} \dfrac{\partial z(s,v)}{\partial s} + \left(\dfrac{\kappa(s)}{\tau(s)}\right)' z(s,v) + \dfrac{\kappa^2(s)}{\tau(s)} y(s,v)\Bigg) \end{array}\right] \mathbf{N}(s) \end{aligned}$$

$$+\left[\begin{array}{l}\dfrac{\kappa(s)}{\tau(s)}\dfrac{\partial z(s,v)}{\partial v}\left(x(s,v)+\dfrac{\kappa(s)}{\tau^2(s)}\dfrac{\partial y(s,v)}{\partial s}+\left(\dfrac{\kappa(s)}{\tau^2(s)}\right)' y(s,v)\right.\\ \left.+\kappa(s)\dfrac{\tau'(s)}{\tau^3(s)}y(s,v)+\dfrac{\kappa^2(s)}{\tau^2(s)}z(s,v)\right)\\ -\dfrac{\kappa(s)}{\tau^2(s)}\dfrac{\partial y(s,v)}{\partial v}\left(\dfrac{\kappa(s)}{\tau(s)}\dfrac{\partial z(s,v)}{\partial s}+\left(\dfrac{\kappa(s)}{\tau^2(s)}\right)' z(s,v)\right.\\ \left.+\dfrac{\kappa^2(s)}{\tau(s)}y(s,v)\right)\end{array}\right]\mathbf{B}(s).$$

Along the curve $\beta(s)$ we have

$$U(s,v_0)=\frac{\kappa(s)}{\tau(s)}\left(\frac{1}{\tau(s)}\frac{\partial y(s,v_0)}{\partial s}\mathbf{T}(s)+\frac{\partial z(s,v_0)}{\partial v}\mathbf{N}(s)\right). \tag{3.3}$$

If the unit normal vector vanishes at any point of a surface $\Phi(s,v)$, i.e., at any point, then these points are called the singular points of the surface. So the following result is obvious.

**Corollary 3.1.** *Since $\kappa\neq 0$, $\partial z(s,v_0)/\partial v$ or $\partial y(s,v_0)/\partial v$ must be different from zero for the normal to be defined. If $\partial z(s,v_0)/\partial v\neq 0$ and $\partial y(s,v_0)/\partial v\neq 0$, then there is no singular point on the surface $\Phi(s,v)$.*

**Theorem 3.1.** *Let $\alpha(s)$ be a unit speed curve with nonzero torsion and $\beta(s)$ be the adjoint curve of $\alpha(s)$. $\beta(s)$ is parametric and geodesic on the surface $\Phi(s,v)$ if*

$$\begin{cases}x(s,v_0)=y(s,v_0)=z(s,v_0)=\dfrac{\partial y(s,v_0)}{\partial v}\equiv 0, \quad L_1\leqslant s\leqslant L_2,\\ K_1\leqslant v,\ v_0\leqslant K_2\ (v_0\ \text{fixed}),\\ \dfrac{\partial z(s,v_0)}{\partial v}=-\lambda(s)\dfrac{\tau^2(s)}{\kappa(s)}, \quad \lambda(s)\neq 0,\\ \tau(s)=\text{constant}.\end{cases} \tag{3.4}$$

**Theorem 3.2.** *Let $\alpha(s)$ be a unit speed curve with nonzero torsion and $\beta(s)$ be the adjoint curve of $\alpha(s)$. $\beta(s)$ is parametric and asymptotic on the surface $\Phi(s,v)$ if*

$$\begin{cases}x(s,v_0)=y(s,v_0)=z(s,v_0)=\dfrac{\partial z(s,v_0)}{\partial v}\equiv 0, \quad L_1\leqslant s\leqslant L_2,\\ K_1\leqslant v,\ v_0\leqslant K_2\ (v_0\ \text{fixed}),\\ \dfrac{\partial y(s,v_0)}{\partial v}=\lambda(s)\tau(s), \quad \lambda(s)\neq 0.\end{cases} \tag{3.5}$$

**Corollary 3.2.** *The curve given according to the modified orthogonal frame with curvature cannot be geodesic and asymptotic at the same time.*

**Theorem 3.3.** *Let $\alpha(s)$ be a unit speed curve with nonzero torsion and $\beta(s)$ be the adjoint curve of $\alpha(s)$. $\beta(s)$ is parametric and line of curvature on the surface $\Phi(s,v)$ if*

$$\begin{cases} x(s,v_0) = y(s,v_0) = z(s,v_0) \equiv 0, \quad L_1 \leqslant s \leqslant L_2, \\ \theta(s) = -\int \kappa(s)\,\mathrm{d}s, \quad K_1 \leqslant v, v_0 \leqslant K_2 \ (v_0 \text{ fixed}), \\ \dfrac{\partial z(s,v_0)}{\partial v} = -\dfrac{1}{\tau(s)}\gamma(s)\cos\theta(s), \quad \gamma(s) \neq 0, \\ \dfrac{\partial y(s,v_0)}{\partial v} = \tau(s)\gamma(s)\sin\theta(s), \\ \dfrac{\partial y(s,v_0)}{\partial v} = \lambda(s)\tau(s), \quad \lambda(s) \neq 0. \end{cases} \tag{3.6}$$

P r o o f. Let $e(s)$ be a vector field orthogonal to the curve $\beta(s)$. Then we can write

$$e(s) = \cos\theta(s)\overline{\mathbf{N}}_\tau(s) + \sin\theta(s)\overline{\mathbf{B}}_\tau(s),$$

where $\theta(s)$ is the angle between $e(s)$ and the vector field $\overline{\mathbf{N}}_\tau(s)$ of the curve $\beta(s)$. According to Theorem 2.2, a line of curvature on the surface is given by (3.1) if and only if $e(s)//U(s,v_0)$ and the ruled surface

$$\Omega(s,v) = \beta(s) + ve(s)$$

is developable. Using (3.3) we have

$$e(s)//U(s,v_0) \Longleftrightarrow \frac{\partial z(s,v_0)}{\partial v} = \frac{1}{\tau(s)}\gamma(s)\cos\theta(s) \text{ and } \frac{\partial y(s,v_0)}{\partial v} = \tau(s)\gamma(s)\sin\theta(s)$$

and $\Omega(s,v)$ is developable if and only if $\det(\beta'(s), e(s), e'(s)) = 0$. Since

$$\det(\beta'(s), e(s), e'(s)) = \frac{\kappa^2(s)}{\tau^3(s)}(\kappa(s) + \theta'(s)),$$

$\kappa \neq 0$, we get

$$\det(\beta'(s), e(s), e'(s)) = 0 \Longleftrightarrow \theta(s) = -\int \kappa(s)\,\mathrm{d}s,$$

which completes the proof. □

## 4. Examples of generating special adjoint curved surfaces according to the modified orthogonal frame with torsion

Example 4.1. Let $\alpha(s) = (\frac{3}{5}\sin(s), -\frac{3}{5}\cos(s), \frac{4}{5}s)$ be a unit speed curve. The Frenet apparatuses of $\alpha(s)$ are

$$\begin{aligned}
\mathbf{t}(s) &= \left(\frac{3}{5}\cos(s), \frac{3}{5}\sin(s), \frac{4}{5}\right),\\
\mathbf{n}(s) &= (-\sin(s), -\cos(s), 0),\\
\mathbf{b}(s) &= \left(\frac{4}{5}\cos(s), -\frac{4}{5}\sin(s), -\frac{3}{5}\right),\\
\boldsymbol{\kappa}(s) &= \frac{3}{5},\ \boldsymbol{\tau}(s) = -\frac{4}{5}.
\end{aligned}$$

The modified orthogonal frame with torsion of the units speed curve derived elements $\alpha(s)$ are as follows:

$$\begin{aligned}
\mathbf{T}_\tau(s) &= \left(\frac{3}{5}\cos(s), \frac{3}{5}\sin(s), \frac{4}{5}\right),\\
\mathbf{N}_\tau(s) &= \left(\frac{4}{5}\sin(s), \frac{4}{5}\cos(s), 0\right),\\
\mathbf{B}_\tau(s) &= \left(-\frac{16}{25}\cos(s), \frac{16}{25}\sin(s), \frac{12}{25}\right).
\end{aligned}$$

Therefore, the adjoint curve of the curve $\alpha(s)$ is

$$\beta(s) = \int B_\tau(s)\,\mathrm{d}s = \left(-\frac{16}{25}\sin(s), \frac{16}{25}\cos(s), \frac{12}{25}s\right).$$

In Figure 1, for $-2\pi \leqslant s \leqslant 2\pi$, we have shown $\alpha(s)$, and the adjoint curve of $\alpha(s)$.

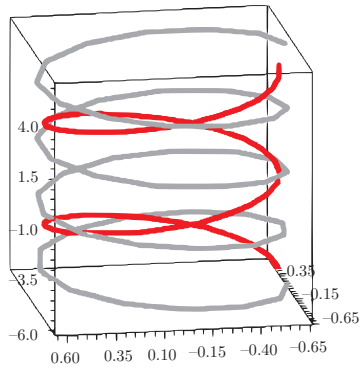



Figure 1. $\alpha(s)$ (in red) and the $\beta(s)$ adjoint curve (in gray).

R e m a r k 4.1. As can be seen from the equations and the figures, the alpha curve and the beta adjoint curve of alpha are circular helices.

Choosing $x(s,v) = y(s,v) \equiv 0$, $z(s,v) = -\frac{16}{5}sv$, $\lambda(s) = s$, $v_0 = 0$, we obtain the surface

$$\Phi_1(s,v) = \left(-\frac{16}{25}\sin(s) + \frac{36}{25}sv\cos(s), -\frac{16}{25}\cos(s) + \frac{36}{25}sv\sin(s), \frac{12}{25}s + \frac{48}{25}sv\right)$$

$-2\pi \leqslant s \leqslant 2\pi$, $-2\pi \leqslant v \leqslant 2\pi$, satisfying (3.4) and accepting $\beta(s)$ as an isoparametric and geodesic curve (Figure 2).

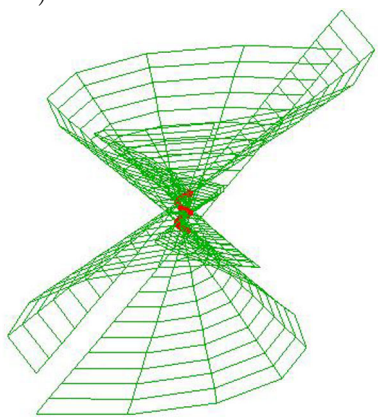


Figure 2. A member of the surface family accepting the curve $\beta(s)$ as an isoparametric, and geodesic curve.

Choosing $x(s,v) = z(s,v) \equiv 0$, $y(s,v) = -\frac{4}{5}sv$, $v_0 = 0$ we obtain the surface

$$\Phi_2(s,v) = \left(-\frac{16}{25}\sin(s) + \frac{3}{5}sv\sin(s), -\frac{16}{25}\cos(s) + \frac{3}{5}sv\cos(s), \frac{12}{25}s\right)$$

$-\pi \leqslant s \leqslant \pi$, $-2\pi \leqslant v \leqslant 2\pi$, satisfying (3.5) and accepting $\beta(s)$ as an isoparametric and asymptotic curve (Figure 3).

Choosing $x(s,v) \equiv 0$, $y(s,v) - \frac{4}{5}sv\sin(\frac{3}{5}s)$, $z(s,v) = \frac{5}{4}sv\cos(\frac{3}{5}s)$, $\gamma(s) = s$, $\theta(s) = \frac{3}{5}s$, $v_0 = 0$ we obtain the surface

$$\Phi_3(s,v) = \begin{pmatrix} -\frac{16}{25}\sin(s) + \frac{3}{5}sv\sin(s)\sin\left(\frac{3}{5}s\right) - \frac{9}{16}sv\cos(s)\cos\left(\frac{3}{5}s\right) \\ -\frac{16}{25}\cos(s) + \frac{3}{5}sv\cos(s)\sin\left(\frac{3}{5}s\right) - \frac{9}{16}sv\sin(s)\cos\left(\frac{3}{5}s\right) \\ \frac{12}{25}s - \frac{3}{4}sv\cos\left(\frac{3}{5}s\right) \end{pmatrix},$$

$-2\pi \leqslant s \leqslant 2\pi$, $-2\pi \leqslant v \leqslant 2\pi$, satisfying (3.6) and accepting $\beta(s)$ as a line of curvature (Figure 4).

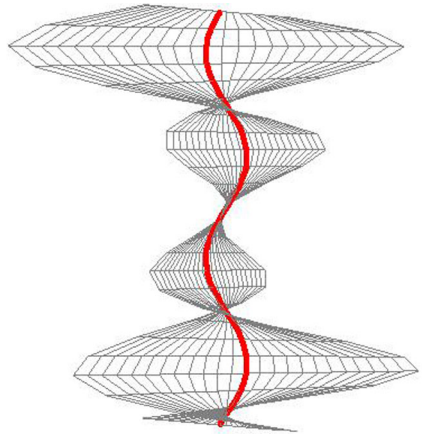


Figure 3. A member of the surface family $\beta(s)$ as an isoparametric, and asymptotic curve.

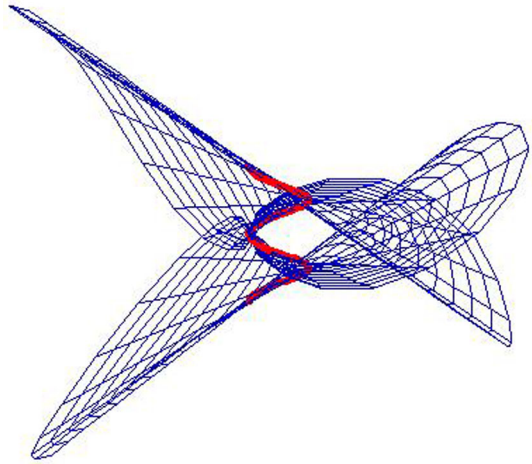


Figure 4. A member of the surface family $\beta(s)$ as an isoparametric, and line of curvature.

E x a m p l e 4.2. Let

$$\alpha(s) = \frac{3}{4}\left(\cos(s) + \frac{\cos(3s)}{9}, \sin(s) + \frac{\sin(3s)}{9}, -\frac{2}{\sqrt{3}}\cos(s)\right)$$

be a unit speed curve. The Frenet apparatuses of $\alpha(s)$ are

$$\mathbf{t}(s) = \frac{3}{4}\left(-\sin(s) - \frac{\sin(3s)}{3}, \cos(s) + \frac{\cos(3s)}{3}, \frac{2}{\sqrt{3}}\sin(s)\right),$$
$$\mathbf{n}(s) = \left(-\frac{\sqrt{3}}{2}\cos(2s), -\frac{\sqrt{3}}{2}\sin(2s), \frac{1}{2}\right),$$

$$\mathbf{b}(s) = \left(\frac{3\cos(s) - \cos(3s)}{5}, \sin^3(s), \frac{\sqrt{3}}{2}\cos(s)\right),$$
$$\boldsymbol{\kappa}(s) = \sqrt{3}\cos(s),\ \tau(s) = -\sqrt{3}\sin(s).$$

The modified orthogonal frame with torsion of the units speed curve derived elements $\alpha(s)$ is as follows:

$$\mathbf{T}_\tau(s) = \frac{3}{4}\left(-\sin(s) - \frac{\sin(3s)}{3}, \cos(s) + \frac{\cos(3s)}{3}, \frac{2}{\sqrt{3}}\sin(s)\right),$$

$$\mathbf{N}_\tau(s) = \left(\frac{3}{2}\sin(s)\cos(2s), \frac{3}{2}\sin(s)\sin(2s), -\frac{\sqrt{3}}{2}\sin(s)\right),$$

$$\mathbf{B}_\tau(s) = -\frac{\sqrt{3}}{4}\sin(s)\left(3\cos(s) - \cos(3s), -\sqrt{3}\sin^4(s), \frac{3}{2}\cos(s)\sin(s)\right).$$

Therefore, the adjoint curve of the curve $\alpha(s)$ is

$$\begin{aligned}\beta(s) &= \int B_\tau(s)\,\mathrm{d}s \\ &= \left(-\frac{\sqrt{3}}{4}(\sin^4(s) + \sin^2(s)), -\frac{\sqrt{3}}{32}(12s - 8\sin(2s) + \sin(4s)), \frac{3}{4}\cos^2(s)\right).\end{aligned}$$

In Figure 5, for $-2\pi \leqslant s \leqslant 2\pi$ we have shown $\alpha(s)$ and the adjoint curve of $\alpha(s)$.

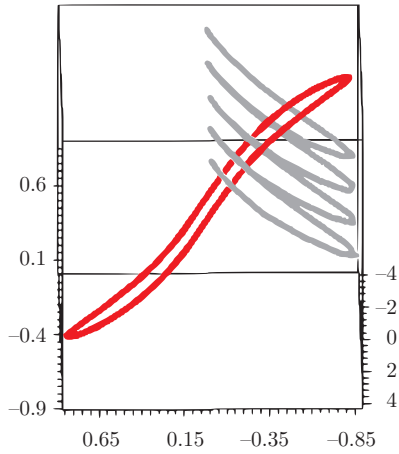



Figure 5. $\alpha(s)$ (in red) and the $\beta(s)$ adjoint curve (in gray).

**Corollary 4.1.** *From equations* (2.8), *for* $\overline{\kappa}(s) = 1$ *and* $\overline{\tau}(s) = -\cot(s)$ *(not constant) the beta adjoint curve of the alpha with respect to the modified frame with torsion is the Salkowski curve.*

**Corollary 4.2.** *Since the torsion of the given alpha curve is not constant, the adjoint curve of this alpha curve cannot be geodesic on the surface* $\Phi(s,v)$*, no matter which marching scale functions* $x$*,* $y$ *and* $z$ *are chosen.*

Choosing $x(s,v) = z(s,v) \equiv 0$, $y(s,v) - v\sqrt{3}\tan(s)\sin(s)$, $\lambda(s) = \tan(s)$, $v_0 = 0$ we obtain the surface

$$\Phi_4(s,v) = \begin{pmatrix} -\dfrac{\sqrt{3}}{4}(\sin^4(s) + \sin^2(s)) + \dfrac{3}{2}v\sin(s)\cos(2s) \\ -\dfrac{\sqrt{3}}{32}(12s - 8\sin(2s) + \sin(4s)) + \dfrac{3}{2}v\sin(s)\sin(2s) \\ \dfrac{3}{4}\cos^2(s) - \dfrac{\sqrt{3}}{4}v\sin(s) \end{pmatrix},$$

$-\pi \leqslant s \leqslant \pi$, $-2 \leqslant v \leqslant 2$, satisfying (3.5) and accepting is $\beta(s)$ as an isoparametric, and asymptotic curve (Figure 6).

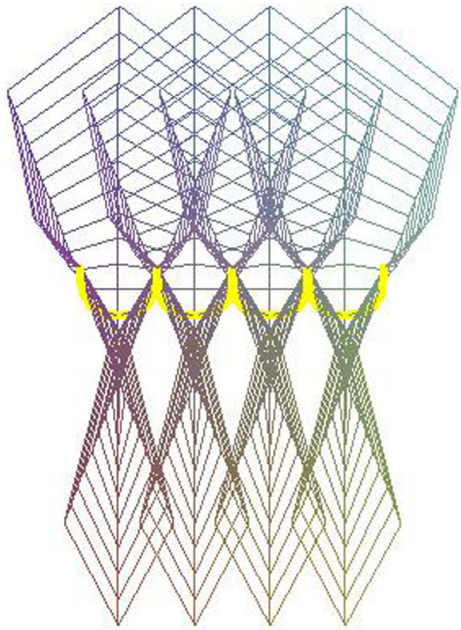


Figure 6. A member of the surface family $\beta(s)$ as an isoparametric, and asymptotic curve.

Choosing $\theta(s) = \sqrt{3}\sin(s)$, $\gamma(s) = \tan(s)$, $v_0 = 0$, $x(s,v) \equiv 0$, $y(s,v) - v\sqrt{3}\sin(s)\sin(\sqrt{3}\sin(s))$, $z(s,v) = -v(\sqrt{3}\sin(s))^{-1}\tan(s)\cos(\sqrt{3}\sin(s))$, we obtain

the surface

$$\Phi_5(s,v)=\begin{pmatrix} -\frac{\sqrt{3}}{4}(\sin^4(s)+\sin^2(s))+\frac{3}{2}v\sin(s)\cos(2s)\sin(\sqrt{3}\sin(s)) \\ +\frac{\sqrt{3}}{4}v\cos(\sqrt{3}\sin(s))+\frac{v}{4\sqrt{3}}\frac{\sin(3s)}{\sin(s)}\cos(\sqrt{3}\sin(s)), \\ -\frac{\sqrt{3}}{32}(12s-8\sin(2s)+\sin(4s))+\frac{3}{2}v\sin(s)\sin(2s)\sin(\sqrt{3}\sin(s)) \\ -\frac{\sqrt{3}}{4}v\cot(s)\cos(\sqrt{3}\sin(s))-\frac{v}{4\sqrt{3}}\frac{\cos(3s)}{\sin(s)}\cos(\sqrt{3}\sin(s)) \\ \frac{3}{4}\cos^2(s)-\frac{\sqrt{3}}{4}v\sin(s)\sin(\sqrt{3}\sin(s))-\frac{v}{2}\cos(\sqrt{3}\sin(s)) \end{pmatrix},$$

$-2\leqslant s\leqslant 2$, $-\pi\leqslant v\leqslant\pi$, satisfying (3.6) and accepting $\beta(s)$ as a line of curvature (Figure 7).

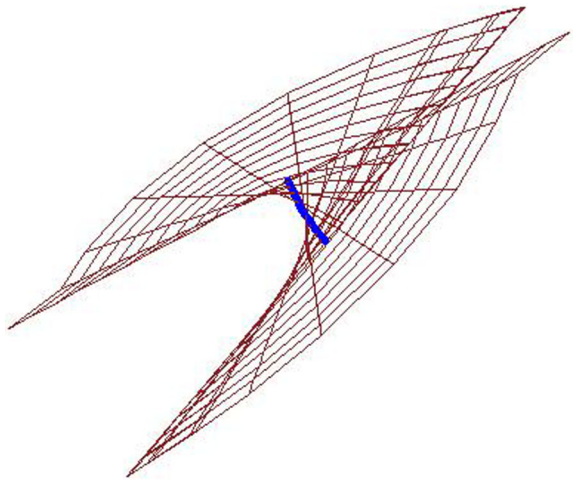


Figure 7. A member of the surface family $\beta(s)$ as an isoparametric, and line of curvature.

## 5. Conclusion

In this study, the parametric equation of the adjoint curve of a given curve according to the modified orthogonal frame defined in the case of nonzero torsion in 3-dimensional Euclidean space and the conditions for both parametric and geodesic, asymptotic and line of curvature, respectively, on the given surface are given. Also, the singular points of the surface given by the parametric equation are expressed. It is obvious that the results obtained here will be used in many areas where curves and surfaces are applied. Computer imaging, virtual reality, aerospace engineering, mechanical and kinematic analyses are prime examples. Another application

of the adjoint curve and the modified orthogonal frame is related to models that show how they simplify complex computations related to robotic motion and lead to more efficient and reliable performance in industrial environments. These models are determined by applying the modified orthogonal frame transformations to compute smooth paths that take into account the robot's joint boundaries and workspace constraints. In this work, since the adjoint curve of the given curve according to the modified orthogonal frame is used, it is expected to contribute to further research in different spaces in these areas.

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*References*

[1] *M. Akyigit, K. Eren, H. H. Kosal*: Tubular surfaces with modified orthogonal frame in Euclidean 3-space. Honam Math. J. *43* (2021), 453–463. zbl MR doi
[2] *M. Arıkan, S. K. Nurkan*: Adjoint curve according to the modified orthogonal frame with torsion in 3-space. Uşak Univ. J. Sci. Natural Sci. *4* (2020), 54–64. doi
[3] *G. Ş. Atalay*: A new approach to special curved surface families according to the modified orthogonal frame. AIMS Math. *9* (2024), 20662–20676. MR doi
[4] *G. S. Atalay*: A study of tubular surfaces according to the modified orthogonal frame with torsion. Honam Math. J. *46* (2024), 279–290. zbl MR doi
[5] *G. S. Atalay, H. Murat*: Surface families with common Mannheim-B curvature line. J. Sci. Arts *24* (2024), 357–368. doi
[6] *G. Ş. Atalay, K. H. A. Şimşek*: Family of surfaces with a common special polynomial curves. Bol. Soc. Parana. Mat. (3) *43* (2025), 14 pages. MR doi
[7] *K. H. Ayvacı, G. Ş. Atalay*: Surface family with a common Mannheim B-pair asymptotic curve. Int. J. Geom. Methods Mod. Phys. *20* (2023), Article ID 2350229, 13 pages. zbl MR doi
[8] *A. Z. Azak*: Involute-evolute curves according to the modified orthogonal frame. J. Sci. Arts *55* (2021), 385–394. doi
[9] *E. Bayram, M. Bilici*: Surface family with a common involute asymptotic curve. Int. J. Geom. Methods Mod. Phys. *13* (2016), Article ID 1650062, 9 pages. zbl MR doi
[10] *B. Bukcu, M. K. Karacan*: On the modified orthogonal frame with curvature and torsion in 3-space. Math Sci. Appl. E-Notes *4* (2016), 184–188. zbl MR
[11] *J. H. Choi, Y. H. Kim*: Associated curves of a Frenet curve and their applications. Appl. Math. Comput. *218* (2012), 9116–9124. zbl MR doi
[12] *S. Desmukh, B.-Y. Chen, A. Alghanemi*: Natural mates of Frenet curves in Euclidean 3-space. Turk. J. Math. *42* (2018), 2826–2840. zbl MR doi
[13] *M. P. do Carmo*: Differential Geometry of Curves and Surfaces. Prentice-Hall, Englewood Cliffs, 1976. zbl MR
[14] *K. Eren*: A study of the evolution of space curves with modified orthogonal frame in Euclidean 3-space. Appl. Math. E-Notes *22* (2022), 281–286. zbl MR
[15] *K. Eren*: New representation of Hasimoto surfaces with the modified orthogonal frame. Konuralp J. Math. *10* (2022), 69–72. MR
[16] *K. Eren, S. Ersoy*: On characterization of Smarandache curves constructed by modified orthogonal frame. Math. Sci. Appl. E-Notes *12* (2024), 101–112. doi
[17] *K. Eren, H. H. Kosal*: Evolution of space curves and the special ruled surfaces with the modified orthogonal frame. AIMS Math. *5* (2020), 2027–2039. zbl MR doi

[18] *X. Jiang, P. Jiang, J. Meng, K. Wang*: Surface pencil pair interpolating Bertrand pair as common asymptotic curves in Galilean space $G_3$. Int. J. Geom. Methods Mod. Phys. *18* (2021), Article ID 2150114, 11 pages. zbl MR doi
[19] *C.-Y. Li, R.-H. Wang, C.-G. Zhu*: Parametric representation of a surface pencil with a common line of curvature. Comput. Aided Des. *43* (2011), 1110–1117. doi
[20] *Y. Li, K. Eren, S. Ersoy, A. Savić*: Modified sweeping surfaces in Euclidean 3-space. Axioms *13* (2024), 800–815. doi
[21] *M. M. Lipschutz*: Theory and Problems of Differential Geometry. Schaum's Outline Series. McGraw-Hill, New York, 1969. zbl
[22] *S. K. Nurkan, I. Güven*: A new approach for Smarandache curves. Turk. J. Math. Comput. Sci. *14* (2022), 155–165. doi
[23] *S. K. Nurkan, I. A. Güven, M. K. Karacan*: Characterizations of adjoint curves in Euclidean 3-space. Proc. Natl. Acad. Sci. India, Sect. A, Phys. Sci. *89* (2019), 155–161. MR doi
[24] *B. O'Neill*: Elementary Differential Geometry. Elsevier/Academic Press, Amsterdam, 2006. zbl MR
[25] *T. Sasai*: The fundamental theorem of analytic space curves and apparent singularities of Fuchsian differential equations. Tohoku Math. J., II. Ser. *36* (1984), 17–24. zbl MR doi
[26] *M. Şenatalar*: Diferansiyel Geometri Eğriler ve Yüzeyler Teorisi. İstanbul Devlet Mühendislik Ve Mimarlık Akademisi Yayınları, 1977. (In Turkish.)
[27] *D. J. Struik*: Lectures on Classical Differential Geometry. Dover Books on Advanced Mathematics. Dover, New York, 1988. zbl MR
[28] *G.-J. Wang, K. Tang, C.-L. Tai*: Parametric representation of a surface pencil with a common spatial geodesic. Comput. Aided Des. *36* (2004), 447–459. zbl doi
[29] *G. E. Weatherburn*: Differential Geometry of Three Dimensions. Cambridge University Press, Cambridge, 1927. zbl
[30] *T. J. Willmore*: An Introduction to Differential Geometry. Clarendon Press, Oxford, 1959. zbl MR
[31] *A. T. Yakut, A. Kızılay*: On the curve evolution with a new the modified orthogonal Saban frame. AIMS Math. *9* (2024), 29573–29586. MR doi

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